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QUANTUM q-LANGLANDS CORRESPONDENCE

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Dedicated to Ernest Vinberg
on the occasion of his 80th birthday

Abstract. We conjecture, and prove for all simply-laced Lie algebras, an identification between the spaces of q-deformed conformal blocks for the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ and quantum affine algebras of $\widehat{^L\mathfrak{g}}$, where $^L\mathfrak{g}$ is the Langlands dual Lie algebra to $\mathfrak{g}$. We argue that this identification may be viewed as a manifestation of a q-deformation of the quantum Langlands correspondence. Our proof relies on expressing the q-deformed conformal blocks for both algebras in terms of the quantum K-theory of the Nakajima quiver varieties. The physical origin of the isomorphism between them lies in the 6d little string theory. The quantum Langlands correspondence emerges in the limit in which the 6d little string theory becomes the 6d conformal field theory with $(2,0)$ supersymmetry.

Contents

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1. Introduction

1.1. **Overview.** In the 50 years of its existence, the Langlands program and the Langlands philosophy have grown to encompass many objects of central importance to both mathematics and mathematical physics.

In particular, the geometric Langlands correspondence starts with a complex projective algebraic curve $\mathcal{C}$ with the goal, as it is usually understood today, to prove an equivalence between certain categories associated to a pair G, ${}^LG$ of Langlands dual connected reductive complex Lie groups. These are certain categories of sheaves (of $\mathcal{D}$-modules and $\mathcal{O}$-modules, respectively) on the moduli stack $\mathrm{Bun}_{{}^LG}$ of ${}^LG$-bundles on $\mathcal{C}$ and the moduli stack Loc_G of flat G-bundles on $\mathcal{C}$.[1] Kapustin and Witten have shown [65] that this equivalence is closely related to S-duality of maximally supersymmetric 4d gauge theories with gauge groups being the compact forms of G and ${}^LG$.

Beilinson and Drinfeld have constructed in [17] an important part of the geometric Langlands correspondence using the isomorphism [36] between the center of the (chiral) affine Kac–Moody algebra $\widehat{{}^L\mathfrak{g}}$ at the critical level ${}^Lk = -{}^Lh^\vee$ and the classical $\mathcal{W}$-algebra $\mathcal{W}_\infty(\mathfrak{g})$. Their construction is closely connected to the 2d conformal field theory and the theory of chiral (or vertex) algebras (see [44] for a survey; and also [122] in which an analogy between 2d CFT and the theory of automorphic representations was first observed and investigated).

Since the level of $\widehat{{}^L\mathfrak{g}}$ may be deformed away from the critical value, and at the same time $\mathcal{W}_\infty(\mathfrak{g})$ may be deformed to the quantum $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$, one is naturally led to look for a quantum deformation of the geometric Langlands correspondence.

Many interesting structures have emerged in the studies under the umbrella of "quantum geometric Langlands" (from the point of view of 2d CFT [41, 42, 55, 56, 104, 110, 112, 118]; in the framework of 4d gauge theory [47, 64, 65]; and, in the abelian case, as a deformation of the Fourier–Mukai transform [98]).

1.1.1. For us, the main feature of the quantum geometric Langlands correspondence is an isomorphism between the spaces of conformal blocks of certain representations of two chiral algebras:

$$\widehat{{}^L\mathfrak{g}}_{{}^Lk} \longleftrightarrow \mathcal{W}_\beta(\mathfrak{g}), \tag{1.1}$$

the affine Kac–Moody algebra of ${}^L\mathfrak{g}$ at level ${}^Lk$ and the $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$. The algebra $\mathcal{W}_\beta(\mathfrak{g})$ is obtained by the quantum Drinfeld–Sokolov reduction [19, 35, 36] of the affine algebra $\widehat{\mathfrak{g}}$ at level k, where $\beta = m(k + h^\vee)$ in the notation of [50].[2]

We will establish this isomorphism and prove a stronger result in the case of simply-laced $\mathfrak{g}$ and genus zero curve $\mathcal{C}$: an identification of conformal blocks of the two algebras if the parameters are generic and related by the formula

$$\beta - m = \frac{1}{{}^L(k + h^\vee)}. \tag{1.2}$$

The relation between the corresponding chiral algebras may be viewed as a strong/weak coupling transformation. Indeed, if we define $\tau = \beta/m$ and ${}^L\tau = -{}^L(k + h^\vee)$, then (1.2) says that

$$\tau - 1 = -1/(m\,{}^L\tau), \tag{1.3}$$

and so ${}^L\tau$ near zero corresponds to large values of τ. The parameters τ and ${}^L\tau$ are related to the complexified coupling constants of the two S-dual 4d Yang-Mills theories. Note the shift $\tau \mapsto \tau - 1$, as compared to the $\mathcal{W}$-algebra duality formula of [36] (see Section 6 for more details). This is a shift of the theta angle from the 4d gauge theory perspective (see Section 9).

[1] The existence of such an equivalence, which may be viewed as a categorical non-abelian Fourier transform, was originally proposed by Beilinson and Drinfeld; later, a precise conjecture was formulated in [12]. We note that some of our notation bucks the usual conventions. In particular, the roles of G and ${}^LG$ are exchanged.

[2] Thus, what we denote here by $\mathcal{W}_\beta(\mathfrak{g})$ is $\mathcal{W}_k(\mathfrak{g})$ of [44, 46], where $\beta = m(k + h^\vee)$. In our present notation, the classical $\mathcal{W}$-algebra associated to $\mathfrak{g}$ is $\mathcal{W}_\infty(\mathfrak{g})$. See Section 6 for more details.

Here by identification of the spaces of conformal blocks we mean a canonical isomorphism between them. However, this canonical isomorphism arises only after we introduce one more parameter and perform one more deformation.

1.1.2. We consider a *two-parameter deformation* of the geometric Langlands correspondence: the q-deformation together with the deformation away from the critical level. This turns out to be a productive point of view.

Namely, we replace the above chiral algebras with their deformed counterparts: the quantum affine algebra $U_\hbar(\widehat{^L\mathfrak{g}})$, which is an $\hbar$-deformation of the universal enveloping algebra of $\widehat{^L\mathfrak{g}}$ introduced by Drinfeld and Jimbo [31, 63], and the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ introduced in [50] (see also [13, 38, 108] for $\mathfrak{g} = sl_n$), which is a deformation of $\mathcal{W}_\beta(\mathfrak{g})$. We will refer to both of these as "q-deformations", both for brevity and because q will appear as a step in difference equations that are of principal importance to us. (In our notation, the quantum affine algebra $U_\hbar(\widehat{^L\mathfrak{g}})$ becomes the enveloping algebra of $\widehat{^L\mathfrak{g}}$ in the limit $\hbar \to 1$; this agrees with the notation used in [93]. For a fixed non-critical value of ^Lk, this limit is the same as the limit $q \to 1$.)

We focus on the case that the curve $\mathcal{C}$ is an infinite cylinder,

$$\mathcal{C} \cong \mathbb{C}^\times \cong \text{infinite cylinder}.$$

It should be noted that integrable deformations away from the conformal point are unlikely to exist unless $\mathcal{C}$ is flat. The torus case should follow from the case of the cylinder, by imposing periodic identifications.[3] The case when $\mathcal{C}$ is a plane can be obtained from ours, by taking the radius of the cylinder to infinity.

We conjecture (and prove in the simply-laced case) a correspondence between q-deformed conformal blocks of the quantum affine algebra $U_\hbar(\widehat{^L\mathfrak{g}})$ and the deformed $\mathcal{W}$-algebra

$$U_\hbar(\widehat{^L\mathfrak{g}}) \longleftrightarrow \mathcal{W}_{q,t}(\mathfrak{g}), \tag{1.4}$$

where the parameters

$$q = \hbar^{-^L(k+h^\vee)}, \qquad t = q^\beta, \tag{1.5}$$

are generic and related by the formula

$$t = q^m/\hbar \tag{1.6}$$

which yields (1.3).

It is this identification of the deformed conformal blocks that we refer to as a "quantum q-Langlands correspondence" in the title of the present paper.

The physical setting for the correspondence is a 6d string theory, called the "$(2,0)$ little string theory". The little string theory [76, 107] is a one-parameter deformation of the ubiquitous 6d $(2,0)$ superconformal theory (see, e.g., [125]). The deformation corresponds to giving strings a non-zero characteristic size, and "converts" the relevant chiral algebras, such as $\widehat{\mathfrak{g}}$ and $\mathcal{W}_\beta(\mathfrak{g})$, into the corresponding deformed algebras.

1.1.3. Some preliminary remarks about deformed conformal blocks are in order. In the case of an affine Kac–Moody algebra and a cylinder $\mathcal{C}$, the space of conformal blocks is isomorphic to the space of solutions of the Kniznik-Zamolodchikov (KZ) equations, which behave well as the insertion points are taken to infinity. The space of q-deformed conformal blocks for quantum affine algebras can be similarly defined, following [51], as the space of solutions of the quantum Kniznik-Zamolodchikov (qKZ) equations. In

[3] To get the deformed conformal blocks on a torus $\mathcal{C} = \mathbb{C}^\times/p^{\mathbb{Z}}$, one would study with blocks on $\mathcal{C} = \mathbb{C}^\times$, but with insertions that are invariant under the action of $p^{\mathbb{Z}}$.

either case, there is a particular fundamental solution of the equations which comes from sewing chiral vertex operators. This solution is given by (1.7) in the case of deformed conformal blocks of $U_\hbar(\widehat{^L\mathfrak{g}})$.

To the best of our knowledge, the definition of the space of deformed conformal blocks for the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ was not available in the literature until now. The blocks formally equal correlation functions of free field vertex operators of the deformed $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra in (1.9), constructed in [50], however the definition is not complete. One has yet to specify the space of allowed contours of integration for screening charges. Further, the analogues of the qKZ equations were previously unknown for the deformed $\mathcal{W}$-algebras $\mathcal{W}_{q,t}(\mathfrak{g})$, as far as we know.

One of the results in this paper is a definition of the space of deformed $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra conformal blocks, and a characterization of the difference equations they satisfy. The key new insight is the geometric interpretation of these objects in terms of (quantum) K-theory of a Nakajima quiver variety X [93], whose quiver diagram is based on the Dynkin diagram of $\mathfrak{g}$.

1.2. **Statement of the correspondence.** Let x be a coordinate on $\mathcal{C} \cong \mathbb{C}^\times$. Fix a finite collection of distinct points on $\mathcal{C}$, with coordinates a_i. We propose, and prove in the simply-laced case, a correspondence between the following two types of q-conformal blocks on $\mathcal{C}$.

1.2.1. On the *electric* side, we consider the quantum affine algebra $U_\hbar(\widehat{^L\mathfrak{g}})$ blocks [51]

$$\langle \lambda' | \prod_i \Phi_{^L\rho_i}(a_i)\, |\lambda\rangle \tag{1.7}$$

where $\Phi_{^L\rho}(x)$ is a chiral vertex operator corresponding to a finite-dimensional $U_\hbar(\widehat{^L\mathfrak{g}})$-module $^L\rho$. The state $|\lambda\rangle$ is the highest weight vector in a level ^Lk Verma module. Its weight $\lambda \in {^L\mathfrak{h}}^*$ is an element of the dual of the Cartan subalgebra for $^L\mathfrak{g}$. This is illustrated in Figure 1.

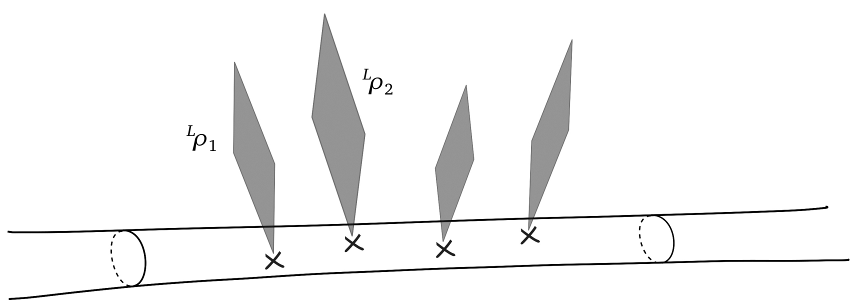

FIGURE 1. The cylinder $\mathcal{C}$ with the insertions of vertex operators corresponding to finite-dimensional $U_\hbar(\widehat{^L\mathfrak{g}})$-modules $^L\rho_i$ at the points $a_i \in \mathcal{C}$. Boundary conditions at infinity are the highest weight vectors $\langle\lambda'|$ and $|\lambda\rangle$.

It suffices to focus on vertex operators corresponding to the fundamental representations because all others may be generated from these, by fusion. The highest weight of a fundamental representation is one of the fundamental weights ^Lw_a of $^L\mathfrak{g}$. The conformal block (1.7) takes values in a weight subspace of

$$\bigotimes_i (^L\rho_i) = \bigotimes_a (^L\rho_a)^{\otimes m_a}\,,$$

namely, it has

$$\text{weight} = \lambda' - \lambda = \sum_a m_a\, {}^L w_a - \sum_a d_a\, {}^L e_a\,, \quad d_a \geq 0\,. \tag{1.8}$$

In (1.8), we write the weight as the difference of the highest weights and simple positive roots ${}^L e_a$ of ${}^L\mathfrak{g}$. The index a runs here from 1 to $\mathrm{rk}(\mathfrak{g})$.

1.2.2. On the *magnetic* side, we consider q-correlators of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra of the form

$$\langle \mu' |\, \prod_i V_i^\vee(a_i) \prod_a \Big(Q_a^\vee\Big)^{d_a}\, |\mu\rangle. \tag{1.9}$$

$V_a^\vee(x)$ and $Q_a^\vee$ are the vertex and the screening charge operators defined by E. Frenkel and N. Reshetikhin in [50]. They are labeled by coroots and coweights of $\mathfrak{g}$, respectively. Recall that Langlands duality maps coweights and coroots of $\mathfrak{g}$ to weights and roots of ${}^L\mathfrak{g}$, respectively. The screening charge operators are defined as integrals of screening current vertex operators $Q_a^\vee = \int dx\; S_a^\vee(x)$, so (1.9) is implicitly an integral formula for $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra blocks.

The coweights of $\mathfrak{g}$ labeling $V_a^\vee(x)$ are the highest weights of the fundamental representations of ${}^L\mathfrak{g}$. The operator $V_i^\vee(a_i)$, inserted at a point on $\mathcal{C}$ with the coordinate a_i, is associated to the same representation of ${}^L\mathfrak{g}$ as the corresponding vertex operator in (1.7). The state $|\mu\rangle$, labeled by an element $\mu \in \mathfrak{h}$ of the Cartan subalgebra of $\mathfrak{g}$, generates an irreducible Fock representation of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra [50]. The (co)weights μ and μ' are determined by λ and λ' (the exact formula depends on the chosen normalization).

1.2.3. The key result of the paper is the following theorem.

Theorem 1. *Let $\mathfrak{g}$ be a simply-laced Lie algebra. The deformed conformal blocks of $U_\hbar(\widehat{{}^L\mathfrak{g}})$ in* (1.7) *and the deformed conformal blocks of $\mathcal{W}_{q,t}(\mathfrak{g})$ in* (1.9) *are identified by the*

$$\textit{specific covector} \times\; U_\hbar(\widehat{{}^L\mathfrak{g}})\ \textit{conformal block} = \mathcal{W}_{q,t}(\mathfrak{g})\ \textit{algebra block}, \tag{1.10}$$

provided that the parameters of the two algebras are generic and related by equation (1.5).

The covector in (1.10), as well as other ingredients of Theorem 1 are best explained in geometric terms, namely, in terms of the (quantum) K-theory of a Nakajima quiver variety X; see below. Specifically, the covector in question corresponds to the insertion of the identity $\mathscr{O}_X \in K_\mathsf{T}(X)$ (more precisely, to no insertion) in a certain enumerative problem. In geometric representation theory literature, it is customary to characterize $\mathscr{O}_X$ by a certain Whittaker property under the action of lowering operators; see, e.g., [79] for a discussion in cohomology. While we did not pursue such characterization in the present paper, there is little doubt that it can be given.

We will also explain, following the predictions of string theory, what is the natural setting for the non-simply-laced cases, see Subsection (1.6). As certain crucial geometric representation theory ingredients are missing in this case, we propose the non-simply-laced analog of Theorem 1 as a conjecture.

1.3. **Geometry behind the correspondence.** The central ingredient of our proof is that for Lie algebras of simply-laced-type, when

$${}^L\mathfrak{g} = \mathfrak{g},$$

we can realize the q-conformal blocks (1.7) and (1.9) as *vertex functions* in equivariant quantum K-theory of a certain holomorphic symplectic variety X. The variety X is the Nakajima quiver variety with

$$\text{quiver } \mathcal{Q} = \text{Dynkin diagram of } \mathfrak{g}\,.$$

1.3.1. A Nakajima quiver variety X is a hyper-Kähler quotient (or a holomorphic symplectic reduction)

$$X = T^*\mathrm{Rep}\,\mathcal{Q}/\!\!/\!\!/\!\!/G_{\mathcal{Q}}\,, \tag{1.11}$$

where

$$\mathrm{Rep}\,\mathcal{Q} = \bigoplus_{a\to b} \mathrm{Hom}(V_a, V_b) \bigoplus_a \mathrm{Hom}(V_a, W_a) \tag{1.12}$$

and

$$G_{\mathcal{Q}} = \prod_a \mathrm{GL}(V_a), \qquad G_W = \prod_a \mathrm{GL}(W_a)\,. \tag{1.13}$$

The arrows in (1.12) are the arrows of the quiver. The dimensions of the vector spaces V_a and W_a correspond as follows:

$$\dim V_a = d_a\,, \quad \dim W_a = m_a,$$

to the weight space data in (1.8).

1.3.2. The quotient in (1.11) involves a geometric invariant theory (GIT) quotient, which depends on a choice of stability conditions. As a result, vertex functions also depend on a stability condition. This stability condition makes them analytic in a certain region of the Kähler moduli space of X. The transition matrix between vertex functions and q-conformal blocks will similarly depend on the stability condition. This dependence will be understood in what follows.

1.3.3. The majority of variables in (1.7) and (1.9) become *equivariant variables* in their geometric interpretation. We have

$$G_W \times \mathbb{C}^\times_\hbar \subset \mathrm{Aut}(X)$$

where $\mathbb{C}^\times_\hbar$ rescales the cotangent directions in (1.11) with weight $\hbar^{-1}$. This gives the symplectic form on X weight $\hbar$ under $\mathbb{C}^\times_\hbar$. We fix a maximal torus $A \subset G_W$ and denote

$$T = A \times \mathbb{C}^\times_\hbar\,.$$

The coordinates a_i of A are the positions at which the vertex operators are inserted in (1.7) and (1.9), while $\hbar$ is the quantum group deformation parameter in $U_\hbar(\widehat{^L\mathfrak{g}})$.

A multiplicative group $\mathbb{C}^\times_q$ acts on quasimaps $\mathbb{P}^1 \dashrightarrow X$ by automorphisms of the domain $\mathbb{P}^1$. The coordinate $q \in \mathbb{C}^\times_q$ is the q-difference parameter from the title of the paper.

1.3.4. In [81], Nakajima identified $K_T(X)$ with a space of weight (1.8) in a $U_\hbar(\widehat{^L\mathfrak{g}})$-module. This is an important result in geometric representation theory which generated a lot of further research. In [79] the authors suggested a somewhat different approach to constructing geometric actions of quantum groups. One of its advantages is its transparent connection with *quantum* cohomology and K-theory of X; see [79, 93].

By quantum cohomology and K-theory we mean enumerative theories of curves in X. The precise flavor of such computations depends on the exact setup of the enumerative problem, including the choice of the moduli spaces in question. Givental and collaborators developed a very general K-theoretic analog of quantum cohomology using the moduli spaces of stable rational maps; see e.g. [58]. This is not the theory that will be used here. The following features of the quantum K-theory used here will be important:

— it deals with quasimaps to a GIT quotient as in [27],

— the quotient (1.11) is a holomorphic symplectic reduction of a cotangent bundle; see [93] for an introduction.

1.3.5. The basic object of the theory of [93] is the vertex function $\mathbf{V}$. The vertex function is an equivariant K-theoretic count of quasimaps from $\mathbb{C}$ to X of all possible degrees. It is an analog of Givental's **I**-function. The variables z in this generating function are called *Kähler parameters*. They are related to the choice of the Fock vacuum $|\lambda\rangle$ in (1.7) and $|\mu\rangle$ in (1.9). Its definition and basic properties will be reviewed in Section 3.2 below.

1.3.6. A key geometric property of vertex functions are the q-difference equations that they satisfy, as functions of both equivariant and Kähler variables (see [93, Section 8], for an introduction). In particular, the q-difference equations in the variables a_i were identified in Section 10 of [93] with the quantum Knizhnik-Zamolodchikov (qKZ) equations of I. Frenkel and N. Reshetikhin [51]. In [51], these were introduced as the q-difference equations that determine the q-deformations of conformal blocks corresponding to $\widehat{{}^L\mathfrak{g}}$ in (1.7).

More precisely, the fundamental solutions of qKZ are vertex functions counting maps from $\mathbb{C}^\times$ to X together with relative insertions at $0 \in \mathbb{C}$ [93]. The relative insertions may be traded for descendent insertions [6, 109].

In this introductory discussion, we will call quasimap counts with a relative insertion at $0 \in \mathbb{C}$ the *vector* vertex functions. This is to distinguish them from the normal vertex functions counting quasimaps from $\mathbb{C}$ to X.

1.4. **Key points of the proof.** Theorem 1 follows from connections between (1.7), (1.9), and the vertex functions which, in broad strokes, go as follows.

1.4.1. *Vector vertex functions vs.* $U_\hbar(\widehat{{}^L\mathfrak{g}})$*-conformal blocks.* On the electric, that is, $U_\hbar(\widehat{{}^L\mathfrak{g}})$-algebra side, we have a characterization of deformed conformal blocks in (1.7) by the quantum Knizhnik-Zamolodchikov equations that they satisfy. Vector vertex functions provide a different basis of solutions of the same qKZ equation. The difference manifests itself through difference analytic dependence on the equivariant variables a_i and the Kähler variables z.

As correlation functions of chiral operators, conformal blocks are analytic in a region of schematic form

$$|a_5| \gg |a_1| \gg |a_3| \gg \dots, \tag{1.14}$$

corresponding to time ordering of operators. This is the ordering in which we sew together the chiral vertex operators on $\mathcal{C}$ to get the conformal block, and this basic analyticity is unaffected by q-deformation.

By contrast, vector vertex functions are born as convergent power series in the Kähler variables z, and they have poles in any region of the form (1.14). The variable z in which they are holomorphic enter as parameters in the qKZ equation, namely as an element of the Cartan torus for $\widehat{{}^L\mathfrak{g}}$.

The dichotomy between the two kinds of solutions may be axiomatized as in [5]. We have a flat q-difference connection on a product of two toric varieties (with coordinates a and z), which is regular in each group of variables separately, but is not regular jointly. Regions of the form (1.14) and $z \to 0$ are punctured neighborhoods of fixed points in the two toric varieties. The solutions analytic there are called the a- and z-solutions, respectively. With this terminology, we can say that

$$\text{vector vertex functions} = z\text{-solutions to } U_\hbar(\widehat{{}^L\mathfrak{g}}) \text{ qKZ equation}\,. \tag{1.15}$$

1.4.2. *Elliptic stable envelopes.* Like any two bases of meromorphic solutions to the same difference equations, the vector vertex functions and the $U_\hbar(\widehat{^L\mathfrak{g}})$-conformal blocks are connected by a q-periodic transition matrix. This q-periodic transition matrix may be called the *pole subtraction matrix*, because it literally cancels unwanted poles in one set of variables at the expense of introducing poles in another set of variables; see [5] for a detailed discussion.

This pole subtraction matrix was identified geometrically in [5] as the elliptic cohomology version of stable envelopes of the Nakajima variety X. In equivariant cohomology, stable envelopes were introduced in [79]. They are the main geometric input in the construction of quantum group actions suggested there; see Section 9 of [93] for an overview. This notion has a natural lift to equivariant K-theory, derived categories of coherent sheaves, and, as shown in [5], also to the equivariant elliptic cohomology.

Parallel to cohomology and K-theory, elliptic stable envelopes produce an action of a quantum group, namely an elliptic quantum group. The analysis of [5] equates the monodromy of qKZ with the braiding for this elliptic quantum group. First steps towards such identification were taken already in [51], with many subsequent developments, as discussed in [5].

In the enumerative problem, elliptic stable envelopes are inserted via the evaluation map at infinity of $\mathbb{C}^\times$, away from the point 0 where the relative conditions have been inserted. Later, when we discuss integral representation of the solutions, they will appear as elliptic functions multiplying the measure of integration as in Section 2.2.6. In either interpretation, they map vector vertex functions to $U_\hbar(\widehat{^L\mathfrak{g}})$-conformal blocks.

1.4.3. *Vertex function and $\mathcal{W}$-algebra correlators.* On the magnetic, that is, $\mathcal{W}$-algebra side we prove in Theorem 3.1 in Section 3 that the vertex functions $\mathbf{V}$ of X, counting quasimaps

$$\mathbb{C} \dashrightarrow X, \tag{1.16}$$

equal the integrals (1.9) for a specific choice of contours of integration.[4] The integral formulas for vertex functions of X arise as follows.

It is well-known (and reviewed in the appendix) that K-theoretic computations on a GIT-quotient by a reductive group G may be expressed as G-invariants in a G-equivariant computation on the prequotient. The projection onto G-invariants may be recast, by the Weyl integration formula, as an integral over a suitable cycle in a maximal torus in G. Generalizing this, it is not difficult to show, see Section 3.4, that for K-theoretic computations on the moduli spaces of quasimaps to a GIT-quotient, there are similar integral formulas. (In fact, such integral formulas are used routinely in supersymmetric gauge theory literature. There, they connect two different ways to compute the supersymmetric index of the 3d gauge theory on $\mathbb{C} \times S^1$, starting from either the Higgs or the Coulomb branch. Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories are studied in [22, 23, 82].)

To complete the match, it suffices to recognize in these formulas the integral formulas of [50] for the free field correlators of $\mathcal{W}_{q,t}(\mathfrak{g})$.

The same dichotomy arises in the discussion of the magnetic conformal blocks. Vertex functions are analytic as $z \to 0$, while the natural requirement for the $\mathcal{W}_{q,t}(\mathfrak{g})$-conformal blocks is to be analytic in regions of the forms (1.14). Very importantly, the very same elliptic stable envelopes transform the z-series into functions with the right analyticity

[4] Formal integral solutions of differential or q-difference equations use only the covariance of $\int dx$ with respect to affine linear transformations. For q-difference equations, $\int dx$ is indistinguishable from $\int g(x)dx$, where $g(x)$ is any elliptic function. So, by a choice of a contour of integration we really mean a choice of both $g(x)$ and γ in $\int_\gamma dx\, g(x) \dots$, where γ has to be constrained by the poles of both the integrand and of $g(x)$; see the discussion in Section 2.2.6.

in the a-variables. The geometry of the correspondence is tautologically the same, as the insertion of the elliptic stable envelope happens at infinity, away from the point 0 which distinguishes vertex functions from their vector analogs. In integral formulas, stable envelopes appear as elliptic functions multiplying the measure of integration.

1.4.4. *The match of conformal blocks.* It must be intuitively clear that vertex functions are a special case of the vector vertex functions, namely the one corresponding to no insertion at 0. Since the moduli spaces in question are not really identical, the correct technical way to see this is via the degeneration formula as in Section 4.1. In particular, it expresses vertex functions as vector vertex functions paired with a specific covector; see formula (4.8). Applying elliptic stable envelopes to both sides gives the statement of Theorem 1.

The above identification is a special case of a more general important problem in enumerative geometry — to match relative counts with the so-called descendent counts. By definition, the insertions in the descendent counts are pulled back via the evaluation map to the quotient stack, while the evaluation map from the relative moduli spaces goes to the Nakajima variety X. While, by the degeneration formula, the two kinds of counts formally contain the same enumerative information, it is very important to be able to control this equivalence explicitly. A very powerful result in this direction has been obtained by Smirnov in [109], and we use this result here. An alternative, and more convenient for our purposes, result has been obtained by two of the present authors in [6] after the present work had been completed.

1.5. **First applications and some further directions.**

1.5.1. *Difference equations for $\mathcal{W}_{q,t}(\mathfrak{g})$-conformal blocks.* The match of the q-conformal blocks can be used to transfer valuable information in both directions.

On the one hand, the equation (1.10) implies that the $\mathcal{W}_{q,t}(\mathfrak{g})$-algebra conformal block solves an explicit scalar q-difference equation gauge equivalent to the qKZ equations. The existence of such equations is not clear from the first principles of deformed $\mathcal{W}$-algebras as they exist today and their further investigation is surely a very interesting direction of research.

Note, in particular, that the monodromy of these difference equations is the same as the monodromy of the qKZ equations. The stable envelope analysis of [5] shows abstractly that it is given by the R-matrices of the corresponding elliptic quantum group, as predicted in [51].

1.5.2. *Integral solutions of qKZ.* In a different direction, any vertex with descendants has an integral representation and the match between descendant and relative counts gives an integral solutions to qKZ. Finding such solutions has been an area of very active research. The formulas of [6] give a uniform general answer that specializes to the results of [77, 78, 102, 114–116, 120] for $\mathfrak{g} = \mathfrak{sl}_n$.

1.5.3. *General quivers.* The geometric steps outlined above work for a Nakajima variety associated to a completely general quiver $\mathcal{Q}$, which may have loops at vertices, parallel edges,[5] etc. For any such quiver, there is a quantum loop group [79, 93, 94], and the corresponding qKZ equations, which form a part of the quantum difference equations.

[5]Note that the meaning of parallel edges is different in Nakajima theory and in the usual notation for Dynkin diagrams. In Nakajima theory

$$\text{Cartan matrix} = 2 - Q - Q^T,$$

where Q is the adjacency matrix of $\mathcal{Q}$. In particular, the Cartan matrix is always symmetric, and so the Lie algebra is always simply-laced, in this sense.

Our argument gives an integral solution to these difference equations in a form that may be interpreted as a $\mathcal{W}$-algebra conformal block.

A representation-theoretic study of these conformal blocks may be an interesting direction for further research. Note that $\mathcal{W}$-algebras associated to quivers appear in the work of Kimura and Pestun [67] in connection with Nekrasov's theory of qq-character constraints in quiver gauge theories [85–87].

1.6. **Non-simply-laced groups and folding.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra which is not simply-laced, that is,

$$^{L}\mathfrak{g} \neq \mathfrak{g}\,.$$

The Dynkin diagram of $\mathfrak{g}$ is a quotient of the Dynkin diagram of a simply-laced Lie algebra $\mathfrak{g}_0$ by an abelian group H of diagram automorphisms as tabulated in (7.3). This well-known procedure is called *folding*.

1.6.1. Let the quiver $\mathcal{Q}_0$ be the Dynkin diagram of $\mathfrak{g}_0$ and let X_0 be the corresponding Nakajima quiver variety, as before. We require the dimension vectors to be invariant under H. Such data is labeled by representations of $^{L}\mathfrak{g}$, the Langlands dual Lie algebra of $\mathfrak{g}$; see Section 7.

1.6.2. We consider H-invariant quasimaps to X_0, where H acts simultaneously on the target *and* the source $\mathbb{P}^1$ of the quasimaps. As usual, the H-invariant part of the obstruction theory defines a perfect obstruction theory for the moduli space $\mathsf{QM}(X_0)^H$ of H-invariant quasimaps. Thus we can define the folded vertex functions which we denote $\mathbf{V}^H$.

These folded vertex functions have an integral formula, just like the unfolded ones. By inspection, these match the integral formulas for the $\mathcal{W}_{q,t}(\mathfrak{g})$ deformed conformal blocks.

1.6.3. We conjecture that the steps from Sections 1.4.1 and 1.4.2 generalize. This requires the development of elliptic stable envelopes (and, as a consequence, K-theoretic stable envelopes) in the folded setting. If true, this would prove our conjectural correspondence in full generality.

1.7. **String theory origin.** The q-conformal blocks of $U_\hbar(\widehat{^{L}\mathfrak{g}})$ and $\mathcal{W}_{q,t}(\mathfrak{g})$ algebras are the partition functions of the 6d "little" string theory with $(2,0)$ supersymmetry.

Little string theory has a conformal limit, in which it becomes a point particle theory, the 6d $(2,0)$ superconformal field theory. This theory is sometimes denoted as theory $\mathcal{X}(\mathfrak{g})$; it has been related to quantum Langlands correspondence in [92, 125], following [64, 65].

The conformal limit of little string turns out to coincide with the conformal limit of the algebras, when q-deformations go away.

1.7.1. For $\mathfrak{g}$ a simply-laced Lie algebra, one takes the $\mathfrak{g}$-type little string theory on a six-manifold

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}. \tag{1.17}$$

Here $\mathcal{C}$ is the Riemann surface on which the chiral algebras live. The parameters q and t^{-1} are related to equivariant rotations of the two complex planes in (1.17); $\hbar$ is associated with an R-symmetry twist, and (1.6) is required to preserve supersymmetry.

The vertex operator insertions in (1.7) and (1.9) correspond to introducing codimension four defects of the little string theory, supported at points of $\mathcal{C}$ and the complex plane in (1.17) rotated by q. This is illustrated in Figure 2.

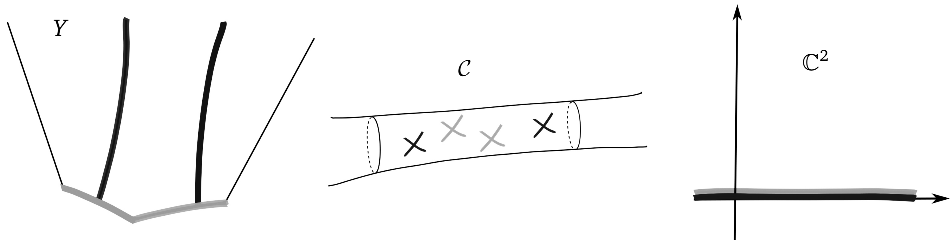

FIGURE 2. The 10d spacetime of the IIB string is the product of an ADE surface Y, the cylinder $\mathcal{C}$, and $\mathbb{C}^2$. The defects we consider are located at $\mathbb{C} \subset \mathbb{C}^2$ that is rotated by q, at points in $\mathcal{C}$, and at middle-dimensional cycles in Y. Compact cycles in Y, shown in gray, give rise to the screening operators, while the dual non-compact cycles $H_2(Y, \partial Y, \mathbb{Z})$ produce vertex operators in fundamental representations.

1.7.2. The $\mathfrak{g}$-type little string on (1.17) arises in a limit of IIB string theory on $Y \times M_6$ where Y is an ADE surface of type $\mathfrak{g}$. The defects of little string theory on $\mathbb{C}$ in M_6 lift to D3 branes of IIB string. In Y, the D3 branes are supported on 2-cycles whose homology class in $H_2(Y, \partial Y, \mathbb{Z})$ is identified with the weight in (1.8) using the identification of $H_2(Y, \partial Y, \mathbb{Z})$ with the weight lattice of $\mathfrak{g}$.

1.7.3. The partition function of the 6d little string theory on M_6 in (1.17) with the defect D3 branes turns out to localize, due to supersymmetry, to the partition function of the theory on the defects themselves. The theory on defects is [30] a 3d quiver gauge theory with quiver $\mathcal{Q}$ whose Higgs branch is the Nakajima variety X in (1.11) (the theory has $\mathcal{N} = 4$ supersymmetry). The 3d gauge theory is supported on

$$\mathbb{C} \times S^1, \tag{1.18}$$

where $\mathbb{C}$ is identified with the complex plane in M_6 supporting the defects. The fact that a defect on $\mathbb{C}$ in M_6 supports a 3d gauge theory is due to a stringy effect. Given a D3 brane at a point on $\mathcal{C}$, there are winding modes of strings which begin and end on the brane, and wind around the circle in $\mathcal{C} \cong \mathbb{C}^\times$. These winding modes are mapped to momentum modes on the (T-)dual circle, corresponding to the S^1 in (1.18).

The partition functions of the 3d gauge theory are the vertex functions of X, computed by quantum K-theory of [93]. They give either the electric or magnetic blocks, depending on the boundary conditions at infinity in $\mathbb{C}$.

Many other examples of relations between partition functions of supersymmetric gauge and string theories and ($q-$)conformal blocks (called BPS/CFT correspondence [83]) appeared in physics literature over the years; [1, 8, 88] are a few. One should note that the relation between the 6d $(2,0)$ theory and gauge theories we use here is *different* from that in [8]. We use supersymmetry to localize the 6d theory to the theory on its defects – and observe, following [2], that in little string theory, the theory on the defects is a Nakajima quiver gauge theory, for any $\mathfrak{g}$, and all possible defects.

1.7.4. To get non-simply-laced theories, we start with the little string corresponding to a simply-laced Lie algebra $\mathfrak{g}_0$ compactified on M_6 in (1.17), and add an H-twist. The H-action is represented by a simultaneous rotation around the origin of the $\mathbb{C}$-plane supporting the defects, and permutation of the modes of the theory induced from the action by generator h of H on the Dynkin diagram of $\mathfrak{g}_0$.

The theory on the D3 brane defects is described by starting with an H-invariant quiver gauge theory based on $\mathcal{Q}_0$, compactified on $S^1 \times \mathbb{C}$. The H-twist restricts the fields of the theory on the defects to be those invariant under the simultaneous rotation of $\mathbb{C}$ and h-action on the quiver.

1.7.5. All of our discussion so far corresponds to the *unramified* case of the geometric Langlands correspondence. An important generalization is to include *ramifications* at a number of points on $\mathcal{C}$.

From the string theory perspective, this is straightforward: ramifications correspond to another class of defects in little string theory on M_6 supported at points on $\mathcal{C}$ and filling $\mathbb{C} \times \mathbb{C}$. These defects were studied in [2]. They originate from D5 branes supported on 2-cycles in Y in IIB on $Y \times M_6$. Their effect on the D3 brane gauge theory is to introduce an additional sector, coming from D3-D5 strings, which breaks supersymmetry to $\mathcal{N} = 2$ in 3d. The partition function of this theory on $\mathbb{C} \times S^1$ is a $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra conformal block with vertex operators which are q-deformations of $\mathcal{W}(\mathfrak{g})$ algebra primaries. The mathematical implication of this is a precise statement of what the variety X becomes in the ramified case (the Higgs branch of the 3d $\mathcal{N} = 2$ quiver gauge theory); and a conjecture for ramified quantum q-Langlands correspondence. On the left hand side in (1.4), one considers q-conformal blocks of $U_\hbar(\widehat{{}^L\mathfrak{g}})$ with vertex operators labeled by the Verma module representations of $U_\hbar({}^L\mathfrak{g})$ inserted at ramification points; on the right, we get the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra blocks from [2].

1.7.6. Little string theory of $\mathfrak{g}_0$ on M_6/H is related to both of the 4d Yang-Mills theories with gauge groups based on Lie algebras $\mathfrak{g}$ and ${}^L\mathfrak{g}$. S-duality relating the 4d gauge theories is a consequence of T-duality in string theory. One views M_6/H as a T^2 fibration over $\mathcal{C} \times B$. The two gauge theories arise by T-duality on one or the other cycle of the T^2, after one takes the limit in which the characteristic size of the string and the size of the torus go to zero.

In the limit, the partition function of little string theory on the one hand computes conformal blocks of $\widehat{{}^L\mathfrak{g}}$ and $\mathcal{W}_\beta(\mathfrak{g})$ algebras; and on the other it computes the partition functions of the 4d gauge theories based on ${}^L\mathfrak{g}$ and $\mathfrak{g}$, respectively. We also derive from this the identification of the parameters of the two 2d CFT's with the parameters $\tau = m(\beta - 1)$ and ${}^L\tau ={}^L (k + h^\vee)$ of the two gauge theories. (See Section 9.)

We hope that our work will help provide a unified framework for the quantum geometric Langlands correspondence relating the 2d conformal field theory and the 4d gauge theory approaches of [41, 42, 55, 56, 104, 110, 112, 118] and [47, 60, 64, 65].

1.8. **Plan of the paper.** The paper is organized as follows. In Section 2 we review relevant aspects of the $U_\hbar(\widehat{{}^L\mathfrak{g}})$ and $\mathcal{W}_{q,t}(\mathfrak{g})$ algebras. In Sections 3 and 4, we specialize to the case of simply-laced $\mathfrak{g}$. In Section 3 we first review relevant aspects of quantum K-theory and of vertex functions. Then, we develop an integral representation of vertex functions and relate them to free field correlators of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra in (1.9). In Section 4, we review the results of [93] relating vertex functions to solutions of a qKZ equation corresponding to $U_\hbar(\widehat{{}^L\mathfrak{g}})$, and the role of elliptic stable envelopes of [5]. This completes the proof of the quantum Langlands correspondence for simply-laced $\mathfrak{g}$. The $\mathfrak{g} = A_1$ example is worked out in detail in Section 5. It should help the reader connect the results of the present paper to earlier work. In Section 6 we discuss various approaches to the quantum geometric Langlands correspondence and explain why the existence of isomorphisms between conformal blocks of the affine Kac–Moody algebra $\widehat{{}^L\mathfrak{g}}$ and the $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$ may be viewed as its manifestation. We relate this to a conjectural equivalence of two braided tensor categories associated to $\widehat{{}^L\mathfrak{g}}$ and $\mathcal{W}_\beta(\mathfrak{g})$. We also discuss the identification between these conformal blocks using the integral (free field) representation, and give explicit examples of what our results in the q-deformed case imply in the conformal limit. In Sections 7 and 8 we explain the relation to physics of 3d gauge theories, and to their string theory embedding. This leads us to the conjecture for the non-simply-laced cases. In Section 9 we explain the relation of little string theory to 4d gauge theories that were related to Langlands correspondence in [65]. The last section is the appendix which reviews the theory of GIT quotients in K-theory.

2. q-deformed conformal blocks

2.1. **Electric side.**

2.1.1. Let $\mathcal{C} \cong \mathbb{C}^\times$ be the Riemann surface from Section 1.2. For any simple Lie algebra ${}^L\mathfrak{g}$, I. Frenkel and N. Reshetikhin in [51] described the $\hbar$-deformation of the $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$ WZW model conformal blocks on $\mathcal{C}$ based on the quantum affine algebra $U_\hbar(\widehat{{}^L\mathfrak{g}})$. We briefly recall some of their results here. Throughout this subsection, the normalization of the bilinear form $(,)_{{}^L\mathfrak{g}}$ on the Lie algebra is chosen so the longest root has length squared equal to 2; these are the usual conventions for affine Lie algebras.

2.1.2. The deformed conformal blocks are correlators of chiral vertex operators:

$$\Psi(a_1, \dots, a_\ell, \dots, a_n) = \langle \lambda' | \Phi_1(a_1) \dots \Phi_\ell(a_\ell) \dots \Phi_n(a_n) | \lambda \rangle. \tag{2.1}$$

State $|\lambda\rangle$ is a highest weight vector of a Verma module $\rho_{\lambda, {}^Lk}$ for $U_\hbar(\widehat{{}^L\mathfrak{g}})$ at level ${}^Lk$. These are $\hbar$-deformations of Verma modules of $\widehat{{}^L\mathfrak{g}}$.

2.1.3. A chiral vertex operator $\Phi_\ell(a)$ is labeled by a finite-dimensional representation ρ_ℓ of $U_\hbar(\widehat{{}^L\mathfrak{g}})$ and acts as an intertwiner

$$\Phi_\ell(a) : \; \rho_{\lambda_i, {}^Lk} \to \rho_{\lambda_j, k} \otimes \rho_\ell(a) \, a^{h(\lambda_i) - h(\lambda_j)}, \tag{2.2}$$

where $\rho_{\lambda_{i,j}, {}^Lk}$ are Verma modules of $U_\hbar(\widehat{{}^L\mathfrak{g}})$ and

$$\begin{aligned} \rho_\ell(a) = {} & \text{the representation } \rho_\ell \\ & \text{precomposed by the action of } x \in \mathbb{C}^\times \\ & \text{by a loop rotation automorphism of } U_\hbar(\widehat{{}^L\mathfrak{g}}) \end{aligned}$$

is an analog of an evaluation representation for $U_\hbar(\widehat{{}^L\mathfrak{g}})$.

Above, $h(\lambda)$ is the same factor as for the affine Lie algebra, given by the conformal weight of the state $|\lambda\rangle$. The space of intertwiners with this data is

$$H^{\lambda_j, \rho_\ell}_{\lambda_i} = \mathrm{Hom}_{U_\hbar({}^L\mathfrak{g})}(\rho_{\lambda_i}, \rho_{\lambda_j} \otimes \rho_\ell),$$

where $U_\hbar({}^L\mathfrak{g})$ is the finite quantum group, and ρ's are the corresponding modules — a direct generalization of the WZW statement.

2.1.4. The deformed conformal block $\Psi(a)$ takes values in a weight subspace

$$\Psi(a_1, \dots, a_\ell, \dots, a_n) \in (\rho_1 \otimes \dots \rho_\ell \otimes \dots \otimes \rho_n)_{\lambda' - \lambda}, \tag{2.3}$$

the weight $\lambda' - \lambda$.

2.1.5. As a deformation of the Knizhnik-Zamolodchikov equations [68], I. Frenkel and N. Reshetikhin obtain a holonomic system of q-difference equations for the conformal block (2.1). It is called the quantum Knizhnik-Zamolodchikov equation and is the most powerful description of (2.1). It has the form:

$$\begin{aligned}\Psi(a_1,\ldots,qa_\ell,\ldots,a_n) =&\mathcal{R}_{\ell\ell-1}(qa_\ell/a_{\ell-1})\cdots\mathcal{R}_{\ell 1}(qa_\ell/a_{\ell-1})(\hbar^\rho)_\ell\\ &\times\mathcal{R}_{\ell n}(a_\ell/a_n)\ldots\mathcal{R}_{\ell\ell+1}(a_\ell/a_{\ell+1})\Psi(a_1,\ldots,a_\ell,\ldots,a_n)\end{aligned} \tag{2.4}$$

where

$$q = \hbar^{-{}^L(k+h^\vee)},$$

and

$$\mathcal{R}_{ij}(x) \subset \mathrm{End}(\rho_i \otimes \rho_j)$$

is the $U_\hbar(\widehat{{}^L\mathfrak{g}})$ R-matrix corresponding to a pair ρ_i, ρ_j finite-dimensional $U_\hbar(\widehat{{}^L\mathfrak{g}})$ modules. Furthermore, $(\hbar^\mu)_\ell$ acts on the ℓth component of the tensor, corresponding to representation ρ_ℓ. Its action on a vector v_w of weight w is

$$\hbar^\mu(v_w) = \hbar^{(\mu,w)}v_w.$$

The vector ρ is the Weyl vector, equal to half the sum of positive roots of ${}^L\mathfrak{g}$. Once we fix a specific ordering of vertex operators in (2.1) or, equivalently, a region of the form

$$|a_5| > |a_2| > |a_7| > \ldots,$$

the qKZ equation determines the q-conformal blocks completely. The solutions in each region are labeled by elements of

$$\bigoplus_{\lambda_1,\ldots,\lambda_{n-1}} H^{\lambda_0,\rho_1}_{\lambda_1} \otimes \ldots \otimes H^{\lambda_{\ell-1},\rho_\ell}_{\lambda_\ell} \otimes \ldots \otimes H^{\lambda_{n-1},\rho_n}_{\lambda_\infty}$$

where $\lambda_1,\ldots,\lambda_{n-1}$ are the highest weights of Verma modules in intermediate channels. Note that the dimension of this space equals the dimension of (2.3).

2.1.6. The notation here differs from that of I. Frenkel and N. Reshetikhin in [51] by:

$$(q)_{\text{ here}} = (p)_{\text{FR}}, \qquad (\hbar)_{\text{ here}} = (q^2)_{\text{FR}}, \tag{2.5}$$

and $(p = q^{-2^L(k+h)})_{\text{FR}}$.

2.1.7. In the conformal limit, when

$$\hbar,\ \ q \longrightarrow 1 \tag{2.6}$$

with ${}^L(k+h)$ fixed, the qKZ equation reduces to the KZ equation

$${}^L(k+h)\, a_\ell \frac{\partial}{\partial a_\ell}\,\Psi = \left(\sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j) + r_{\ell 0} + r_{\ell\infty}\right)\,\Psi, \tag{2.7}$$

derived in [68]. The matrix

$$r_{ij}(a_i/a_j) = \frac{r_{ij}a_i + r_{ji}a_j}{a_i - a_j}$$

with

$$r = \frac{1}{2}\sum_a h_a \otimes h_a + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha}$$

in the standard Lie theory notation, is the classical R-matrix of WZW on $\mathcal{C} = \mathbb{C}^\times$. This is referred to as the trigonometric R-matrix, as opposed to the rational one, corresponding to the case $\mathcal{C} = \mathbb{C}$.

2.2. Magnetic side.

2.2.1. Let $\mathcal{C}$ and $\mathfrak{g}$ be as before. The deformed $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra, and certain classes of vertex and screening operators, were constructed by E. Frenkel and N. Reshetikhin in [50] in terms of free fields, as a q-deformation of a $\mathcal{W}_\beta(\mathfrak{g})$ algebra, where $t = q^\beta$. (See also [13, 38, 108] for $\mathfrak{g} = sl_n$.) The free field realization implies that the q-conformal blocks on $\mathcal{C}$ of the form

$$\langle \mu' | V_1^\vee(a_1) \dots V_n^\vee(a_n) \prod_a (Q_a^\vee)^{d_a} | \mu \rangle \tag{2.8}$$

have a direct description in terms of certain contour integrals.

2.2.2. Let $\mathfrak{g}$ be a simple Lie algebra, and let C_{ab} be its Cartan matrix, defined as

$$C_{ab} = 2(e_a, e_b)/(e_a, e_a) = (e_a^\vee, e_b), \tag{2.9}$$

in terms of simple positive roots e_a, the coroots $e_a^\vee$, and the invariant inner product $(,)$ on the Lie algebra. Let m be the lacing number, the maximum number of arrows connecting a pair of nodes in the Dynkin diagram. Unless $m = 1$ and the theory is simply-laced, the Cartan matrix is not symmetric. Instead, the symmetric matrix is

$$B_{ab} = m_a C_{ab} = m(e_a, e_b), \tag{2.10}$$

where we defined

$$m_a = m(e_a, e_a)/2.$$

We choose the normalization of the inner product $(,)_\mathfrak{g}$ so that $m_a = m$ for long roots and $m_a = 1$ for short roots.

2.2.3. To define the deformed $\mathcal{W}_{q,t}(\mathfrak{g})$, one starts [50] with the q-deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathbf{g})$ in terms of "root"-type generators $e_a[k]$, for $k \in \mathbb{Z}$ where a labels the simple positive root of $\mathfrak{g}$. The generators satisfy commutation relations

$$[e_a[k], e_b[\ell]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}}) B_{ab}(q^k, t^k) \delta_{k,-\ell}.$$

Here, $B_{ab}(q,t)$ is a q-deformation[6] of (2.10),

$$B_{ab}(q,t) = [m_a]_q \, C_{ab}(q,t),$$

where $C_{ab}(q,t)$ is the q-deformed Cartan matrix

$$C_{ab}(q,t) = q^{\frac{m_a}{2}} t^{-\frac{1}{2}} + q^{-\frac{m_a}{2}} t^{\frac{1}{2}} - [I_{ab}]_q,$$

and I_{ab} is the classical incidence number, $I_{ab} = 2\delta_{ab} - C_{ab}$.

2.2.4. We get a Fock representation of the Heisenberg algebra, denoted by π_μ, by starting with the state $|\mu\rangle$, such that

$$e_a[k]|\mu\rangle = 0, k > 0, \qquad e_a[0]|\mu\rangle = (\mu, e_a)|\mu\rangle,$$

and acting by the algebra generators.

[6] The quantum number n is defined as $[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$.

2.2.5. One defines the magnetic and the electric screening currents following [50]:

$$S_a^\vee(x) = [\ldots] x^{-e_a[0]/m_a} : \exp\Big(\sum_{k\neq 0} \frac{e_a[k]}{q^{\frac{km_a}{2}} - q^{-\frac{km_a}{2}}} x^k\Big) : \tag{2.11}$$

and

$$S_a(x) = [\ldots] x^{e_a[0]/\beta} : \exp\Big(-\sum_{k\neq 0} \frac{e_a[k]}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}} x^k\Big) : . \tag{2.12}$$

The terms denoted by $[\ldots]$ above are operators responsible, in part, for shifts of momenta μ in π_μ in (2.14). If $\mathfrak{g}$ is simply-laced, there is a symmetry exchanging $S_a^\vee$ and S_a and swapping q and t.

The algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ is defined as the associative algebra generated by the (Fourier coefficients of) operators $T(x)$ which commute with the screening charges $S_a^\vee(x)$ and $S_a(x)$ up to a total difference, e.g.,

$$[T(x), S_a^\vee(x')] = \mathcal{D}_{x',q} f(x,x'), \qquad \mathcal{D}_{x,q} f(x) = \frac{f(x) - f(qx)}{x(1-q)} . \tag{2.13}$$

2.2.6. For the corresponding screening charges

$$\begin{aligned} Q_a^\vee &= \int S_a^\vee(x) \ : \pi_0 \to \pi_{-e_a\beta/m_a}, \\ Q_a &= \int S_a(x) \ : \pi_0 \to \pi_{e_a}, \end{aligned} \tag{2.14}$$

equation (2.13) implies

$$\Big[T(x), \int S_a^\vee(x')\Big] = 0 .$$

Here $f(x) \mapsto \int f(x)$ is any linear functional such that

$$\int \frac{f(x)}{x} = \int \frac{f(qx)}{x} .$$

For example, we can take

$$\int f(x) = \int_\gamma f(x)\, dx$$

for any path γ such that

$$q\cdot\gamma - \gamma = 0 \in H_1(\mathbb{C}^\times \setminus \text{singularities of the integrand}) .$$

More flexibly, we can take,

$$\int f(x) = \int_\gamma f(x)\, g(x)\, dx , \quad g(qx) = g(x) , \tag{2.15}$$

with the same assumption on the integration cycle γ.

As we will see below, the insertion of the right elliptic function $g(x)$ under the integral as in (2.15) corresponds geometrically to the insertion of an elliptic stable envelope in quasimap enumeration. These elliptic stable envelopes transform the z-solutions that appear naturally in the enumerative problem into a-solutions that correspond to conformal blocks.

For $S_a(x)$, the analysis is the same, except q-shifts are replaced by t-shifts.

2.2.7. The weight-type generators $w_a[k]$ are associated with fundamental weights of $\mathfrak{g}$. They are defined by

$$e_a[k] = \sum_b C_{ab}(q^k, t^k) w_b[k]$$

which satisfy

$$[e_a[k], w_b[\ell]] = \frac{1}{k}(q^{\frac{km_a}{2}} - q^{-\frac{km_a}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})\delta_{ab}\delta_{k,-\ell}.$$

2.2.8. Similarly, there are magnetic $V_a^\vee$ and electric V_a degenerate vertex operators, associated with fundamental coweights and weights (as defined in Section 9 of [50], with some slight relabeling):

$$V_a^\vee(x) = x^{w_a[0]/m_a} : \exp\Big(- \sum_{k\neq 0} \frac{w_a[k]}{q^{\frac{km_a}{2}} - q^{-\frac{km_a}{2}}}\, x^k\Big) : \tag{2.16}$$

and

$$V_a(x) = x^{-w_a[0]/\beta} : \exp\Big(\sum_{k\neq 0} \frac{w_a[k]}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}\, x^k\Big) : \tag{2.17}$$

for the electric vertex operators. The insertion at infinity is determined by charge conservation.

2.2.9. We denote

$$\varphi_q(s) = \prod_{n=0}^{\infty}(1 - q^n s)\,. \tag{2.18}$$

In the taxonomy of special function, the function (2.18) is best described as the reciprocal of the q-analog of the Γ-function. It is also known under many other names.

The infinite product (2.18) is a half of the odd genus one theta-function

$$\theta_q(s) = \varphi_q(s)\,\varphi_q(q/s)\,, \tag{2.19}$$

which is vanishing at $s = 1$ and normalized to be a single-valued function of s.

For simply-laced $\mathfrak{g}$ we will often drop the subscripts and then $\varphi(s) = \varphi_q(s)$, etc.

2.2.10. Collecting the definitions above, the $\mathcal{W}_{q,t}(\mathfrak{g})$ correlator in (2.8)

$$\langle \mu' | V_1^\vee(a_1) \dots V_n^\vee(a_n) \prod_a (Q_a^\vee)^{d_a} |\mu\rangle \tag{2.20}$$

is the integral

$$\int d_{\mathrm{Haar}} x\, x^\mu\ \Phi(x, a), \tag{2.21}$$

where we defined

$$x^\mu = \prod_{a,\alpha} x_{a,\alpha}^{(\mu,e_a)}$$

and $d_{\mathrm{Haar}}x = \prod_{a,\alpha} dx_{a,\alpha}/x_{a,\alpha}$. The integrand $\Phi(x, a)$ is a product of three terms, which come from normal (re)ordering of the operators in (2.20): The first comes from the screening currents associated to node a:

$$\prod_{\alpha<\alpha'} \langle S_a^\vee(x_{\alpha,a}) S_a^\vee(x_{\alpha',a})\rangle = \prod_{\alpha\neq\alpha'} \frac{\varphi_{q_a}(x_{\alpha,a}/x_{\alpha',a})}{\varphi_{q_a}(t\, x_{\alpha,a}/x_{\alpha',a})} \prod_{\alpha<\alpha'} \frac{\theta_{q_a}(t x_{\alpha,a}/x_{\alpha',a})}{\theta_{q_a}(\, x_{\alpha,a}/x_{\alpha',a})}. \tag{2.22}$$

Here and below, $\langle\rangle$ stands for expectation values computed in state $|0\rangle$. The second factor comes from screening charges associated to a pair of nodes a, b connected by a link to the Dynkin diagram of $\mathfrak{g}$, and equals:

$$\prod_{\alpha,\beta}\langle S_a^\vee(x_{\alpha,a})S_b^\vee(x_{\beta,b})\rangle = \prod_{\alpha,\beta}\frac{\varphi_{q_{ab}}(tv_{ab}\, x_{\alpha,a}/x_{\beta,b})}{\varphi_{q_{ab}}(v_{ab}\, x_{\alpha,a}/x_{\beta,b})}. \tag{2.23}$$

Above v_a, v_{ab} are defined as follows: $v_a = \sqrt{q_a/t}$ and $v_{ab} = \sqrt{q_{ab}/t}$, where $q_a = q^{m_a}$ and $q_{ab} = q^{\min(m_a,m_b)}$. (If either of the nodes a, b corresponds to a short root, then $q_{ab} = q$, and, if both of the nodes are long, then $q_{ab} = q^m$.) The third, and final factor comes from a normal ordering of the vertex operators $V_a^\vee(a_i)$'s with the screening currents $S_a^\vee$ coming from the same node:

$$\prod_{\alpha,a}\langle S_a^\vee(x_{\alpha,a})V_a^\vee(a_i)\rangle = \prod_{a,\alpha}\frac{\varphi_{q_a}(tv_a\, a_i/x_{\alpha,a})}{\varphi_{q_a}(v_a\, a_i/x_{\alpha,a})}. \tag{2.24}$$

The μ dependent factor in (2.21) accounts for the fact that the incoming stat in (2.20) is $|\mu\rangle$, and not the trivial vacuum $|0\rangle$.

2.2.11. In writing (2.22) we assume the argument x of $\varphi_q(x)$ is less than one, $|x| < 1$. Otherwise $\varphi_q(x)$ gets replaced by $1/\varphi_q(q/x) = \varphi_q(x)/\theta_q(x)$. This is a feature of deformed chiral algebras, as defined in [49].

2.3. **Conformal limit.** The conformal limit, in which one recovers the ordinary (conformal) $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$,

$$\mathcal{W}_{q,t}(\mathfrak{g}) \to \mathcal{W}_\beta(\mathfrak{g})$$

corresponds to taking

$$q,\ t = q^\beta \to 1, \tag{2.25}$$

keeping β fixed, as in [50]. The conformal $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$ with $\beta = m(k + h^\vee)$ is obtained from $\widehat{\mathfrak{g}}$ of level k via the quantum Drinfeld–Sokolov reduction (see [36, 46] and Section 6 below for details).

2.3.1. The limit (2.25) requires rescaling of the generators of the algebra. The generators of the algebra that stay finite in the limit are $e'_a[k] = e_a[k]/\log(q)$ and $w'_a[k] = w_a[k]/\log(q)$. In the limit, we get

$$\langle S_a^\vee(x)\, S_b^\vee(x')\rangle = (x - x')^{\frac{\beta}{m}(e_a^\vee, e_b^\vee)}, \qquad \langle S_a^\vee(x) V_b^\vee(x')\rangle = (x - x')^{-\frac{\beta}{m}(e_a^\vee, w_b^\vee)}, \tag{2.26}$$

and

$$\langle S_a(x)\, S_b(x')\rangle = (x - x')^{\frac{m}{\beta}(e_a, e_b)}, \qquad \langle S_a(x) V_b(x')\rangle = (x - x')^{-\frac{m}{\beta}(e_a, w_b)}, \tag{2.27}$$

where $e_a^\vee$, $w_a^\vee$ are the coroots, and the fundamental coweights, respectively.[7] The formulas reflect the fact that for a pair of Langlands dual Lie algebras, $\mathfrak{g}$ and ${}^L\mathfrak{g}$, there is an isomorphism of the corresponding $\mathcal{W}$-algebras [36]:

$$\mathcal{W}_\beta(\mathfrak{g}) = \mathcal{W}_{{}^L\beta}({}^L\mathfrak{g}), \qquad \beta\, {}^L\beta = m,$$

(see Section 6 for details). One recalls that, under exchanging $\mathfrak{g}$ and ${}^L\mathfrak{g}$, roots and coroots get exchanged, as well as the weights and coweights, and the inner product gets rescaled, $(,)_\mathfrak{g} = m(,)_{{}^L\mathfrak{g}}$.

[7] The fundamental coweights are defined by $(e_a, w_b^\vee) = \delta_{a,b}$.

3. Integral representation of vertex functions

3.1. Quasimaps to Nakajima varieties.

3.1.1. Let X be a Nakajima variety as in Section 1.3

$$X = T^*\mathrm{Rep}\,\mathcal{Q}/\!\!/\!\!/\!\!/G_{\mathcal{Q}} = \mu^{-1}(0) /\!\!/ G_{\mathcal{Q}}\,, \tag{3.1}$$

where μ is the complex moment map for the action of $G_{\mathcal{Q}}$. From (3.1), it is a GIT quotient of an affine algebraic variety by an action of a reductive group.

For such quotients, Ciocan-Fontanine, Kim, and Maulik define in [27] a notion of quasimap

$$f : C \dashrightarrow X\,. \tag{3.2}$$

These are maps to X with certain singularities. Informally, a *stable* quasimap f is allowed to take a GIT-unstable value at finitely many points of C. In what follows, all quasimaps are assumed stable.

Analogous notions are known in both supersymmetric gauge theory and mathematics literature. The precise definition of [27] is best suited for our goals here. An introductory discussion of quasimaps of [27] may be found in [93].

3.1.2. The vector spaces V_a in the quiver description of X descend to vector bundles on X. These are called *tautological*.

The data of a quasimap includes vector bundles $\mathscr{V}_a$ on the domain C; they coincide with the pullbacks f^*V_a of the tautological bundles for regular maps $f : C \to X$ and are part of the definition in general. Similarly, we have the trivial bundles $\mathscr{W}_a$ on C corresponding to the trivial bundles W_a on X. We denote by

$$\mathscr{M} = \bigoplus_{a\to b} \mathscr{H}om(\mathscr{V}_a, \mathscr{V}_b) \bigoplus_{a} \mathscr{H}om(\mathscr{V}_a, \mathscr{W}_a) \tag{3.3}$$

the bundle corresponding to (1.12).

By definition, a quasimap is a collection of bundles $\{\mathscr{V}_a, \mathscr{W}_a\}$ together with a stable section

$$f \in H^0(C, \mathscr{M} \oplus \hbar^{-1} \otimes \mathscr{M}^*)$$

satisfying the moment map equation

$$\mu(f) = 0 \in H^0(C, \bigoplus_a \mathscr{E}nd(\mathscr{V}_a))\,.$$

Stability of f means it evaluates to a GIT-stable point at all but finitely many points of C. This data is considered up to isomorphisms fixing C pointwise. In other words, we consider quasimaps from parametrized domains.

3.1.3. Let $\mathsf{QM}(X)$ be the moduli space of quasimaps from $C \cong \mathbb{P}^1$ to X. On the domain C, we fix marked points

$$p_1 = 0\,, \quad p_2 = \infty\,,$$

and denote

$$\mathbb{C}^\times_q = \mathrm{Aut}(C, p_1, p_2)\,.$$

Here the subscript is to distinguish this torus from other tori present; an element of $\mathbb{C}^\times_q$ will be denoted q. We take $T_{p_1}C$ as the defining (i.e., weight one) representation of $\mathbb{C}^\times_q$.

3.1.4. The degree of a quasimap is defined as follows:

$$\deg f = (\deg \mathscr{V}_1, \deg \mathscr{V}_2, \dots) \in H_2(X, \mathbb{Z})_{\text{effective}}\,;$$

see the discussion in Section 7.2 of [93]. This is a locally constant function on $\mathsf{QM}(X)$.

3.1.5. By definition, vertex functions for X are computed using $\mathbb{C}^\times_q$-equivariant K-theoretic localization on the open set

$$\mathsf{QM}_{\text{non-sing } \infty} \subset \mathsf{QM}$$

formed by quasimaps non-singular at $p_2 = \infty$. It is therefore important to understand the structure of the fixed locus $\left(\mathsf{QM}_{\text{non-sing } \infty}\right)^{\mathbb{C}^\times_q}$. It is discussed, in particular, in Section 7.2 of [93].

3.1.6. The analysis of the fixed loci may be summarized as follows. We define

$$\mathbb{V}_a = \bigoplus_{k\in\mathbb{Z}} \mathbb{V}_a[k] = H^0(\mathscr{V}_a\big|_{C\setminus\{p_2\}}),$$

where $\mathbb{V}_a[k]$ is the subspace of weight k with respect to $\mathbb{C}^\times_q$. By invariance, all quiver maps preserve this weight decomposition. We define the framing spaces $\mathbb{W}[k]$ in the same way and obtain

$$\mathbb{W}_a[k] = \begin{cases} W_a\,, & k \le 0\,,\\ 0\,, & k > 0\,, \end{cases} \tag{3.4}$$

because the bundles $\mathscr{W}_a$ are trivial.

Multiplication by the coordinate induces an embedding

$$\mathbb{V}_a[k] \hookrightarrow \mathbb{V}_a[k-1] \hookrightarrow \cdots \hookrightarrow \mathbb{V}_a[-\infty] = V_a\,, \tag{3.5}$$

compatible with quiver maps, where V_a is the quiver data for the point $f(\infty) \in X$. A $\mathbb{C}^\times_q$-fixed stable quasimap f takes a constant stable value on $C \setminus \{0,\infty\}$ and, since f is additionally assumed non-singular at infinity, $f(\infty)$ is that generic value of f.

We conclude

$$\left(\mathsf{QM}_{\text{non-sing } \infty}\right)^{\mathbb{C}^\times_q} = \left\{ \begin{array}{c} \text{a stable quiver representation} \\ +\text{ a flag of subrepresentations} \\ \text{satisfying (3.4)} \end{array} \right\} \Big/ \prod \mathrm{GL}(V_a)\,. \tag{3.6}$$

3.2. **Vertex functions.**

3.2.1. Vertex functions are defined as generating functions of equivariant counts of quasimaps of all degrees. Concretely, consider the evaluation map

$$\mathrm{ev} : \mathsf{QM}_{\text{non-sing } \infty}(X) \to X$$

that records the value $f(\infty)$ of a quasimap f. We introduce a weighting by $z^{\deg f}$, where z are the variables in the generating function, and define

$$\mathsf{Vertex} = \mathrm{ev}_*\left(\widehat{\mathscr{O}}_{\mathrm{vir}}\, z^{\deg f}\right) \in K_{\mathsf{T}\times\mathbb{C}^\times_q}(X)_{\text{localized}} \otimes \mathbb{Q}[[z]]\,, \tag{3.7}$$

where the symmetrized virtual structure sheaf $\widehat{\mathscr{O}}_{\mathrm{vir}}$ will be discussed below and $\mathbb{Q}[[z]]$ denotes formal power series in z with exponents supported in the effective cone.

3.2.2. The push-forward in (3.7) is defined using $\mathbb{C}^\times_q$-equivariant localization. (It is clear from the above description of the $\mathbb{C}^\times_q$-fixed quasimaps that the evaluation map is proper on these fixed loci.) Because of this, vertex functions are series in z with coefficients in localized equivariant cohomology. Their denominators are the source of their richness and complexity; analogous functions without denominators (called the *cap* in the professional lingo; see below) lose this complexity.

3.2.3. The symmetrized virtual structure sheaf is defined by [93]

$$\widehat{\mathscr{O}}_{\mathrm{vir}} = \mathscr{O}_{\mathrm{vir}} \otimes \left(\mathscr{K}_{\mathrm{vir}} \frac{\det \mathscr{T}^{1/2}\big|_{\infty}}{\det \mathscr{T}^{1/2}\big|_{0}} \right)^{1/2}, \tag{3.8}$$

where $\mathscr{O}_{\mathrm{vir}}$ is the virtual structure sheaf constructed in the standard way from the perfect obstruction theory of quasimaps, see [16,26,34], $\mathscr{K}_{\mathrm{vir}}$ is the virtual canonical bundle, that is, the determinant of the virtual cotangent bundle, and the remaining term involves a choice of polarization of X and is mainly needed to avoid square roots of q.

3.2.4. The virtual tangent bundle to $\mathsf{QM}(X)$ may be described as follows:

$$T_{\mathrm{vir}}\,\mathsf{QM}(X) = \mathrm{Def} - \mathrm{Obs} = H^{\bullet}(C, \mathscr{T})\,, \tag{3.9}$$

where

$$\mathscr{T} = \mathscr{M} \oplus \hbar^{-1} \otimes \mathscr{M}^* - (1+\hbar^{-1}) \oplus_a \mathscr{E}nd(\mathscr{V}_a)$$

is the virtual bundle on C corresponding to the tangent bundle TX of our Nakajima variety.

3.2.5. By definition, a polarization $T^{1/2} \in K_{\mathsf{T}}(X)$ is a choice of a half of the tangent bundle, that is, a choice of the solution of the equation

$$T^{1/2}X + \hbar^{-1} \otimes \left(T^{1/2}X\right)^{\vee} = TX$$

in $K_{\mathsf{T}}(X)$, where vee denotes dual. Natural polarization of Nakajima varieties correspond to choosing one out of every pair of quiver arrows. A polarization $T^{1/2}$ induces a virtual vector bundle $\mathscr{T}^{1/2}$ on the domain C of the quasimap such that

$$\mathscr{T} = \mathscr{T}^{1/2} + \hbar^{-1} \otimes \left(\mathscr{T}^{1/2}\right)^{\vee}. \tag{3.10}$$

The fibers of the line bundle $\det \mathscr{T}^{1/2}$ enter (3.8). The square root in (3.8) exists, perhaps after introducing $\hbar^{1/2}$. For quasimaps, it can be given explicitly in terms of a polarization. See Section 6.1 of [93] and also [89] for general results in this direction. Vertex functions defined using a different choice of polarization differ by a q-shift of the variables z only.

3.3. **Localization contributions.**

3.3.1. Recall that the push-forward in (3.7) is defined using $\mathbb{C}_q^{\times}$-equivariant localization. A general shape of virtual localization formulas, see [26, 59], is the following. Restricted to the fixed locus, the obstruction theory splits

$$T_{\mathrm{vir}}\Big|_{\mathsf{QM}(X)^{\mathbb{C}_q^{\times}}} = T_{\mathrm{vir,fixed}} \oplus T_{\mathrm{vir,moving}} \tag{3.11}$$

into trivial and non-trivial $\mathbb{C}_q^{\times}$-eigenspaces. The fixed part of the obstruction theory produces a perfect obstruction theory for the fixed locus, from which its own virtual structure sheaf and the symmetrized virtual structure sheaf are derived. The moving parts of the obstruction theory enter the localization formula as a K-theoretic analog of the Euler class or, more precisely, $\hat{\mathsf{a}}$-genus (3.20) for the virtual localization of $\widehat{\mathscr{O}}_{\mathrm{vir}}$.

3.3.2. Our plan for the analysis of the localization contributions is the following:

- first, we show (3.6) is a GIT quotient. This makes the techniques reviewed in the appendix applicable to these fixed loci.
- We identify the fixed part of the quasimap obstruction theory with the natural obstruction theory of (3.6).
- We include the moving contributions to derive an integral representation for the vertex functions.

3.3.3. In order to show (3.6) is a GIT quotient, one may analyze GIT-stability on the ambient space

$$(\mathsf{QM}_{\text{non-sing } \infty})^{\mathbb{C}_q^\times} \subset \left\{ \begin{matrix} \text{a stable quiver representation} \\ + \text{flags of subspaces in } V_a \end{matrix} \right\} \Big/ \prod \mathrm{GL}(V_a)\,, \tag{3.12}$$

where the flags of subspaces need not form a flag of subrepresentations. The required ample line bundles will be obtained by restriction from the ambient space in (3.12).

The ambient space in (3.12) is not a Nakajima quiver variety, but it may be presented as a quiver variety in which the data for X is extended by chains

$$V_a \leftarrow V_a' \leftarrow V_a'' \leftarrow \dots \tag{3.13}$$

attached to every V_a. The spaces in (3.13) correspond to subspaces in (3.5), excluding repetitions. This extended quiver data is taken modulo $G \times G'$ where

$$G = \prod \mathrm{GL}(V_a)\,, \quad G' = \prod \mathrm{GL}(V_a') \times \mathrm{GL}(V_a'') \times \cdots .$$

As a GIT-stability parameter, we need to specify a character $\bar{\chi}$ of $G \times G'$. We take

$$\bar{\chi} = \chi^m \, \chi'\,, \quad m \gg 0,$$

where χ is the stability parameter for X and χ' is the character of G' that forces the maps (3.13) to be injective.

Lemma 1. *The quotient in* (3.12) *is the GIT quotient of the extended quiver data with the stability parameter $\bar{\chi}$.*

Proof. We use King's reformulation of the GIT-stability of quiver representations in terms of slope stability; see for example the exposition in Section 2.3 of [57]. Namely, a representation R is semistable if and only if

$$\mathrm{slope}_{\bar{\chi}}(S) \le \mathrm{slope}_{\bar{\chi}}(R)$$

for every non-zero subrepresentation $S \subset R$, where

$$\mathrm{slope}_{\bar{\chi}}(R) = \frac{\bar{\chi} \cdot \dim R}{(1, \dots, 1) \cdot \dim R}. \tag{3.14}$$

In (3.14), we interpret $\bar{\chi}$ and $\dim R$ as dimension vectors for the extended quiver and take the usual dot product of dimension vectors. To include framing spaces in this formalism, one can replace them with arrows from an extra vertex $V_0 \cong \mathbb{C}$, as first suggested by Crawley–Boevey. See for example the discussion in Section 3.1 of [57].

Since $m \gg 0$, the $G \times G'$-semistability of the extended quiver data implies G-semistability of the original data, and hence its stability because in X there are no strictly semistable points. Because of this stability, any subrepresentation S passes the slope test automatically except when it contains all or none of the spaces V_a. In the latter case, we use our choice of χ' to conclude that the semistable representations of the extended quiver are the stable representations of the original quiver with a choice of injective maps (3.13). ☐

Corollary 1. *The stable points in* (3.6) *are GIT semistable* (=*stable*) *points for $\mathscr{L} = \mathscr{L}_0 \otimes \chi^m$, $m \gg 0$, where $\mathscr{L}_0$ is any ample line bundle pulled back from the product of partial flag varieties and χ is the character that gives the stability condition for X.*

3.3.4. The natural obstruction theory of quasimaps is constructed relative to the map

$$\mathsf{QM}(X) \to \text{stack of bundles } \{\mathscr{V}_a\}$$

to a smooth stack of bundles on the domain C. The terms in the relative obstruction theory are given by the cohomology of the bundles giving the quiver data and the moment map.

We have

$$\frac{\{\text{flags of subspaces in } V_a\}}{\prod \mathrm{GL}(V_a)} = \begin{array}{c}\text{stack of } \mathbb{C}_q^\times\text{-equivariant bundles } \{\mathscr{V}_a\} \\ \text{with trivial } \mathbb{C}_q^\times\text{-action on } \mathscr{V}_a\big|_\infty\end{array}. \tag{3.15}$$

The inclusion of Corollary 1

$$\left(\mathsf{QM}_{\text{nonsing }\infty}\right)^{\mathbb{C}_q^\times} \subset \left\{\begin{array}{c}\text{a quiver representation}\\ +\text{ flags of subspaces in } V_a\end{array}\right\} \Big/\!\!\!\Big/ \prod \mathrm{GL}(V_a)\,, \tag{3.16}$$

where the double slash denotes a GIT quotient, can be interpreted in quasimap terms as follows.

3.3.5. Observe that

$$H^0(\mathscr{G}(-\infty))^{\mathbb{C}_q^\times} = H^1(\mathscr{G})^{\mathbb{C}_q^\times} = 0 \tag{3.17}$$

for any equivariant bundle $\mathscr{G}$ on $\mathbb{P}^1$ such that $\mathscr{G}\big|_\infty$ is a trivial $\mathbb{C}_q^\times$-module. Here $\mathscr{G}(-\infty)$ denotes the twist by the divisor $\infty \in \mathbb{P}^1$. Therefore, from the exact sequence

$$0 \to \mathscr{G}(-\infty) \to \mathscr{G} \to \mathscr{G}|_\infty \to 0$$

we get

$$0 \to \chi(\mathscr{G})^{\mathbb{C}_q^\times} = H^0(\mathscr{G})^{\mathbb{C}_q^\times} \to \mathscr{G}|_\infty \to H^1(\mathscr{G}(-\infty))^{\mathbb{C}_q^\times} \to 0\,. \tag{3.18}$$

In particular, (3.18) applies to the bundles

$$\mathscr{G} = \mathscr{H}om(\mathscr{V}_a, \mathscr{V}_b)\,, \dots\,,$$

whose sections are the quiver maps. For them, the middle term in (3.18) gives the vector space $\mathrm{Hom}(V_a, V_b)$, while the zero locus of the map

$$\mathrm{Hom}(V_a, V_b) \to H^1(\mathscr{V}_a^\vee \otimes \mathscr{V}_b(-\infty))^{\mathbb{C}_q^\times} = \bigoplus_k \mathrm{Hom}(\mathbb{V}_a[k], V_b/\mathbb{V}_b[k])$$

defines, together with the moment map equation, the inclusion (3.16). Here the summation is over all $\mathbb{V}_a[k]$ excluding repetitions. The $\mathbb{C}_q^\times$-fixed part of the moment map equations is a section of

$$\hbar^{-1} \otimes \bigoplus_a \mathrm{Hom}_{\mathsf{flag}}(V_a, V_a) = \hbar^{-1} \otimes \bigoplus_a H^\bullet(\mathscr{V}_a^\vee \otimes \mathscr{V}_a)^{\mathbb{C}_q^\times}\,,$$

where the subscript in $\mathrm{Hom}_{\mathsf{flag}}$ denotes maps that preserve the filtration by $\mathbb{V}_a[k]$.

This completes 2/3 of the plan outlined in Section 3.3.2.

3.4. **Integral formulas.**

3.4.1. In the full vertex function, the (symmetrized) virtual structure sheaf of the fixed locus enters simultaneously with the contributions of the moving parts of the quasimap obstruction theory. This leads to the following formulas for the vertex functions. (The necessary background information about K-theoretic computation on GIT quotients is collected in the appendix.)

3.4.2. Let $\mathsf{S} \subset \prod \mathrm{GL}(V_a)$ be a maximal torus. The $\mathsf{S} \times \mathsf{T}$-fixed points on the prequotient in (3.16) correspond to coordinate flags and zero quiver maps. Coordinate flags mean that S appears as a group preserving a splitting

$$\mathscr{V}_a = \bigoplus_{\alpha} \mathscr{L}_{a,\alpha}\,, \quad \mathscr{L}_{a,\alpha} = \mathscr{O}(d_{a,\alpha}[0])\,, \quad d_{a,\alpha} = \deg \mathscr{L}_{a,\alpha}\,, \tag{3.19}$$

into a direct sum of line bundles. We denote by $s_{a,\alpha}$ the S-weight of $\mathscr{L}_{a,\alpha}$. These are the coordinates in S and the equivariant Chern roots of the bundles V_a over X. These very same variables were denoted by $s_{a,\alpha} = x_{a,\alpha}^{-1}$, elsewhere in the paper. With our conventions,

$$\text{weight}\left(\mathscr{L}_{a,\alpha}\big|_0\right) = q^{d_{a,\alpha}} s_{a,\alpha}\,, \quad \text{weight}\left(\mathscr{L}_{a,\alpha}\big|_\infty\right) = s_{a,\alpha}\,,$$

where the weights are for the torus $\mathsf{S} \times \mathsf{T} \times \mathbb{C}_q^\times$.

As we will see, integral formulas for the vertex will be more naturally written not in terms of the variables $s_{a,\alpha}$ but rather in terms of the weights of $\{\mathscr{V}_a\}$ over the point $0 \in C$.

3.4.3. The basic ingredient in the integration formulas is the function

$$\widehat{\mathsf{a}}(s) = s^{1/2} - s^{-1/2} \tag{3.20}$$

extended to equivariant K-theory as a genus, that is, so that

$$\widehat{\mathsf{a}}(\mathscr{G}_1 + \mathscr{G}_2) = \widehat{\mathsf{a}}(\mathscr{G}_1)\,\widehat{\mathsf{a}}(\mathscr{G}_2)\,. \tag{3.21}$$

The importance of this function is clear from the equality

$$\left(\mathscr{O}_{\mathrm{vir}} \otimes \mathscr{K}_{\mathrm{vir}}^{1/2}\right)_{\mathrm{moving}} = \widehat{\mathsf{a}}\left(T_{\mathrm{vir,moving}}\right)^{-1} \tag{3.22}$$

for the moving part of the virtual structure sheaf in localization formulas. Formula (3.22) is an immediate consequence of the localization formula for $\mathscr{O}_{\mathrm{vir}}$; see [26, 59]. Here and in what follows the *moving* terms are the terms of non-trivial weight with respect to $\mathsf{S} \times \mathsf{T} \times \mathbb{C}_q^\times$.

In particular, an algebraic consequence of the identification (3.15) is that

$$\frac{\Delta_{\mathrm{Weyl}}}{\widehat{\mathsf{a}}(T\mathrm{Flags\ in\ (3.15)})} = \widehat{\mathsf{a}}(\mathrm{Lie} \prod \mathrm{Aut}(\mathscr{V}_a)^{\mathbb{C}_q^\times}/\mathsf{S})\,.$$

Thus the integration measure in (A.13) in the specific setting of (3.16) naturally becomes a part of the localization weight (3.22), namely the part that comes from $\mathbb{C}_q^\times$-equivariant automorphisms of $\{\mathscr{V}_a\}$ other than those in S.

We conclude the following.

Proposition 1. *For any $\mathscr{F} \in K_{\mathsf{T}}(X)$, we have*

$$\chi(X, \mathsf{Vertex} \otimes \mathscr{F}) = \frac{1}{|W|} \sum_{\{d_{a,\alpha}\}} q^{-\frac{1}{2}\deg \mathscr{T}^{1/2}} \prod_{a,\alpha} z_a^{d_{a,\alpha}} \int_{\gamma_\chi} \frac{\mathscr{F}(s)\, d_{\mathrm{Haar}} s}{\widehat{\mathsf{a}}\left(T_{\mathrm{vir,moving}}\right)}, \tag{3.23}$$

where the summation is over all splittings (3.19)*, $\mathscr{T}^{1/2}$ denotes the virtual bundle on C induced by the chosen polarization, the cycle γ_χ corresponds to a choice of stability parameter χ as in* (A.12)*, and $\mathscr{F}(s)$ is the expression of $\mathscr{F}$ in the Chern roots of the tautological bundles.*

Note that by Lemma 4 in Section A.0.7 the integration $\int_{\gamma_\chi}$ in (3.23) is really an iterated residue of the integrand.

3.4.4. The summation over splittings in (3.23) can be treated in two complementary ways.

On the one hand, one can sum over the whole lattice of splittings, and this will be convenient for interpreting the eventual integral (3.32) as a linear functional invariant under q-shifts.

On the other hand, for most splittings, there are no stable quasimaps and hence those contribute zero to the sum in (3.23). We call splittings that correspond to stable $\mathbb{C}^\times_q$-fixed quasimaps *effective*. A necessary condition for a splitting to be effective is discussed in Section 7.2 of [93].

3.4.5. Formulas (3.9) and (3.10) show that the following lemma applies to the denominator in (3.23).

Lemma 2. *For any bundle $\mathscr{G}$ on $\mathbb{P}^1$ we have*

$$q^{-\deg \mathscr{G}/2}\, \frac{\widehat{\mathsf{a}}\left(\mathscr{G}|_\infty + \hbar^{-1}\mathscr{G}^\vee|_\infty\right)}{\widehat{\mathsf{a}}\left(H^\bullet\left(\mathscr{G}+\hbar^{-1}\mathscr{G}^\vee\right)\right)} = \left(-\hbar^{1/2}\right)^{-\deg \mathscr{G}} \; \frac{\varphi\left(\hbar\mathscr{G}|_\infty\right)\varphi\left(q\mathscr{G}|_0\right)}{\varphi\left(q\mathscr{G}|_\infty\right)\varphi\left(\hbar\mathscr{G}|_0\right)}\,, \tag{3.24}$$

where φ is the function (2.18) *extended multiplicatively as in* (3.21).

Proof. It is enough to prove (3.24) for a line bundle, in which case it reduces to an elementary identity. □

Note that for $\mathscr{G} = \mathscr{T}^{1/2}$ the prefactor in the LHS of (3.24) coincides with the power of q in (3.23), while the numerator in the LHS of (3.24) is nothing but $\widehat{\mathsf{a}}(TX)$. As in Section 8.3 of [93], we incorporate the prefactor in the RHS of (3.24) into a shift $z_\#$ of the variable z so that

$$z_\#^{\deg f} = \left(-\hbar^{1/2}\right)^{-\deg \mathscr{T}^{1/2}} \prod_{a,\alpha} z_a^{d_{a,\alpha}}\,.$$

With this notation, (3.23) may be restated as follows:

$$\begin{aligned}\chi(X, \mathsf{Vertex}\otimes\mathscr{F})\\ = \frac{1}{|W|}\sum_{\{d_{a,\alpha}\}} z_\#^{\deg f}\int_{\gamma_\chi}\left(\frac{\mathscr{F}(s)\, d_{\mathrm{Haar}}s}{\widehat{\mathsf{a}}\left(T\right)}\;\frac{\varphi\left(\hbar T^{1/2}\right)\varphi\left(q\mathscr{T}_0^{1/2}\right)}{\varphi\left(qT^{1/2}\right)\varphi\left(\hbar\mathscr{T}_0^{1/2}\right)}\right)^{\sim}_{\text{moving}}\,,\end{aligned} \tag{3.25}$$

where $\mathscr{T}_0^{1/2}$ denotes the fiber of $\mathscr{T}^{1/2}$ over $0\in C$, tilde refers to the computation on the prequotient, and only moving terms are retained from the product of $\widehat{\mathsf{a}}$- and φ-functions.

3.4.6. From the point of view of difference equations, a better normalization of the vertex functions is the following:

$$\mathbf{V} = \mathbf{e}(z_\#)\,\varphi((q-\hbar)T^{1/2})\,\mathsf{Vertex}\,; \tag{3.26}$$

see Section 8.3 in [93] and also Section 6.1 in [5] ($\mathbf{V}$ here was denoted by $\widetilde{\mathbf{V}}$, in those papers). It solves certain q-difference equations in both the equivariant variables and the Kähler variables z. Here

$$\mathbf{e}(z) = \exp\left(\frac{\boldsymbol{\lambda}(\ln z, \ln t)}{\ln q}\right)\,, \tag{3.27}$$

where

$$\boldsymbol{\lambda} : H^2(X,\mathbb{C})\otimes \mathrm{Lie}\,\mathsf{T} \to \mathrm{End}\,K(X^{\mathsf{T}})\otimes\mathbb{C}$$

extends by linearity the map that takes a line bundle $\mathscr{L}\in \mathrm{Pic}(X)$ to the logarithm of the operator $\mathscr{L}\otimes$—; see the discussion in Section 8.2 of [93]. This function has an elementary description in terms of the prequotient. Indeed, the line bundle $\det V_a$ associated to the variable z_a has weight $\prod_\alpha s_{a,\alpha}$, whence

$$\boldsymbol{\lambda} = \sum_{a,\alpha} \ln(z_a)\, \ln(s_{a,\alpha})\,. \tag{3.28}$$

A certain care is required in working with (3.26) because $\phi(\hbar T^{1/2})$ may contain non-equivariant non-invertible factors (since these singular terms involve neither equivariant nor Kähler variables, they are irrelevant from the point of view of difference equations). To avoid these complications, we define

$$\overline{\mathbf{V}} = \frac{\widehat{\mathsf{a}}(T)}{\theta(T^{1/2})}\,\mathbf{V} = \hbar^{-\frac{1}{4}\dim X}\,\frac{\mathbf{e}(z_\#)}{\varphi(qT^\vee)}\,\mathsf{Vertex}\,, \tag{3.29}$$

where $\theta(s)$ is the odd theta-function defined in (2.19). Note a slight difference with the odd theta-function $\vartheta(s) = s^{1/2}\theta(s)$ used in [5]. We extend (2.19) multiplicatively as before. Note that the division of the $\mathbf{V}$-function by the theta-function of the polarization $T^{1/2}$ is a part of the pole subtraction operator of [5]; see Section 6.3 there.

Substituting (3.29) in (3.25), we obtain

$$\begin{aligned}\chi(X, \overline{\mathbf{V}} \otimes \mathscr{F}) \\ = \frac{1}{|W|} \sum_{\{d_{a,\alpha}\}} z_\#^{\deg f} \int_{\gamma_\chi} \exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right) \frac{\mathscr{F}(s)\, d_{\mathrm{Haar}} s}{\theta(T^{1/2})}\, \frac{\varphi\left(q\mathscr{T}_0^{1/2}\right)}{\varphi\left(\hbar\mathscr{T}_0^{1/2}\right)}\,,\end{aligned} \tag{3.30}$$

where the computation on the prequotient and the extraction of the moving parts is understood.

3.4.7. Let

$$\mathbf{d} = \{d_{a,\alpha}\}$$

denote an effective splitting (3.19) and define

$$q^{\mathbf{d}} s = \{q^{d_{a,\alpha}} s_{a,\alpha}\}\,.$$

These are the weights of the bundles $\mathscr{V}_a$ over $0 \in C$. Clearly,

$$\frac{\varphi\left(q\mathscr{T}_0^{1/2}\right)}{\varphi\left(\hbar\mathscr{T}_0^{1/2}\right)} = \left.\frac{\varphi\left(qT^{1/2}\right)}{\varphi\left(\hbar T^{1/2}\right)}\right|_{s\mapsto q^{\mathbf{d}} s}\,.$$

Also, from (3.28), we have

$$z_\#^{\deg f} \exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right) = \left.\exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right)\right|_{s\mapsto q^{\mathbf{d}} s}\,.$$

Therefore, it is natural to change variables in the integral (3.30). So far, we made no assumptions on insertion $\mathscr{F}(s)$. In (3.30), it can be an arbitrary element of $K_{\mathsf{T}}(X)$ or, more generally, an arbitrary analytic function on the spectrum of the ring $K_{\mathsf{T}}(X) \otimes \mathbb{C}$. We now assume it has the same automorphy as $\theta(T^{1/2})$, that is, we assume

$$\frac{\mathscr{F}(s)}{\theta(T^{1/2})} \quad \text{is invariant under } s \mapsto q^{\mathbf{d}} s\,. \tag{3.31}$$

This means $\mathscr{F}$ is a section of a certain line bundle over the the scheme $\mathrm{Ell}_{\mathsf{T}}(X)$, the equivariant elliptic cohomology of X. In principle, this section is allowed to have singularities away from the integration cycle. With this assumption, a change of variables in (3.30) gives the following.

Proposition 2. *For any insertion $\mathscr{F}$ satisfying* (3.31), *we have*

$$\chi(X, \overline{\mathbf{V}} \otimes \mathscr{F}) = \frac{1}{|W|} \int_{\sum q^{\mathbf{d}} \cdot \gamma_\chi} \exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right) \frac{\mathscr{F}(s)\, d_{\mathrm{Haar}} s}{\phi(T^\vee_{\mathrm{moving}})}\,, \tag{3.32}$$

where the sum of residues is over all effective shifts of the cycle γ_χ.

Recall that the cycle γ_χ is, by construction, a sum of several cycles. For insertions $\mathscr{F}$ supported on its particular components (such as the classes of torus-fixed loci in X), the integration cycle will be correspondingly smaller.

3.4.8. As explained in Section 3.4.4, the integration contour in (3.32) may be extended to all q-shifts of γ_χ as long as $\mathscr{F}(s)$ is non-singular on γ_χ. Obviously, the integration $\int_{\sum q^{\mathbf{d}} \cdot \gamma_\chi} d_{\mathrm{Haar}} s$ where the sum is over the whole lattice of splittings, is invariant under q-shifts.

3.4.9. For a simplest example, let us examine the statement of Proposition 2 for

$$X = T^*\mathbb{P}^{n-1}\,.$$

We will follow the notation of Section 6.2 of [5] and of Section A.0.6 below. We take

$$\mathsf{T} = \mathbb{C}^\times_\hbar \times \mathsf{A},$$

where A is as in (A.8) and the first factor scales the cotangent directions with weight $\hbar^{-1}$. We denote the T-fixed points by

$$X^{\mathsf{T}} = \{p_1, \dots, p_n\}\,.$$

The elementary analysis of the quasimap spaces recalled in [5] shows

$$\chi(\mathbf{V} \otimes \mathscr{O}_{p_k}) = \frac{\hbar^{\frac{1}{4}\dim X}}{2\pi i} \int_{\gamma_k} \frac{ds}{s}\, e^{\dfrac{\ln z_\# \,\ln s}{\ln q}}\, \varphi((q-\hbar)\, T^{1/2})_{\mathrm{moving}}\,, \tag{3.33}$$

where

$$T^{1/2} = -\frac{1}{\hbar} + \sum_i \frac{1}{\hbar a_i s}\,, \tag{3.34}$$

and the contour γ_k enclosed the poles

$$x = \frac{q^d}{a_k}\,, \quad d = 0, 1, 2, \dots\,. \tag{3.35}$$

In (3.33) we restored the power of $\hbar$ that comes from $\mathscr{K}_X \cong \hbar^{\frac{1}{2}\dim X} \mathscr{O}_X$.

Tautologically,

$$\chi(\mathbf{V} \otimes \mathscr{O}_{p_k}) = \chi\left(\overline{\mathbf{V}} \otimes \frac{\mathscr{O}_{p_k}}{\hat{\mathsf{a}}(T)} \otimes \theta(T^{1/2})\right)\,.$$

We have

$$\left.\frac{\mathscr{O}_{p_k}}{\hat{\mathsf{a}}(T)}\right|_{p_i} = \hbar^{\frac{1}{4}\dim X} \delta_{ki}\,, \tag{3.36}$$

which means that this insertion serves as a delta-function restricting the residues to the sequence (3.35). Thus setting

$$\mathscr{F}(s) = \theta(T^{1/2}) \otimes \mathscr{F}'(s)$$

where $\mathscr{F}'(s)$ is q-periodic and non-singular at the points $\{a_i^{-1}\}$, we get from (3.33)

$$\chi\left(\overline{\mathbf{V}} \otimes \mathscr{F}(s)\right) = \int_\gamma \frac{ds}{2\pi i s}\, e^{\dfrac{\ln z_\# \, \ln s}{\ln q}} \, \mathscr{F}'(s)\, \varphi((q-\hbar)\, T^{1/2})_{\text{moving}}$$

$$= \int_\gamma \frac{ds}{2\pi i s}\, e^{\dfrac{\ln z_\# \, \ln s}{\ln q}} \, \frac{\mathscr{F}(s)}{\varphi(T^\vee)_{\text{moving}}}\,,$$

where $\gamma = \sum_{k=1}^n \gamma_k$. This is a specialization of the general formula (3.32).

3.4.10. Heuristically, it may be argued that (3.32) is an infinite-dimensional version of the formula (A.5), in which

$$\begin{aligned} \widetilde{X} \quad &\mapsto \quad \mathsf{QM}(\widetilde{X}) \\ G \quad &\mapsto \quad \text{gauge transformations}\,. \end{aligned}$$

Such or a similar viewpoint is implicit in many papers on supersymmetric gauge theories. Here, we don't try to turn this heuristic into precise mathematical statements. The argument given above is technically much simpler and sufficient for our needs.

3.5. **Vertex functions and $\mathcal{W}_{q,t}$ algebra correlators.** In this section, we prove the following.

Theorem 3.1. *The vertex function*

$$\chi(X, \mathbf{V} \otimes \mathscr{F}') = \frac{1}{|W|} \int_{\gamma_\chi} \exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right) \mathscr{F}'(s) \phi((q-\hbar)T^{1/2})\, d_{\text{Haar}} s, \tag{3.37}$$

is a $\mathcal{W}_{q,t}(\mathfrak{g})$ correlator

$$\langle \mu' | \prod_{a,i} V_a^\vee(a_{a,i}) \, \prod_a (Q_a^\vee)^{d_a} |\mu\rangle, \tag{3.38}$$

where a choice insertion $\mathscr{F}'$ corresponds to a choice of $\mathscr{F}'(s)$ contours of integration in the definition of screening charge operators, and $\mu = \frac{z}{\ln q}$.

Proof. In Section 2.2.10, we gave an explicit integral form of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra correlator in (3.38). We will now show that this exactly equals the integral in (3.37).

Consider the $\phi((q-\hbar)T^{1/2})$ in the integrand (3.37). Recall that for the Nakajima variety X, with quiver $\mathcal{Q}$

$$T^{1/2}X = \sum_a V_a \otimes W_a^* + \sum_{a,b} (I_{ab} - \delta_{ab})\, V_a \otimes V_b^*,$$

where I_{ab} is the adjacency matrix of the Dynkin diagram, and we have identified the vector space $T^{1/2}X$ with its character under the action of the torus $S \times T$. We can choose coordinates on S so that

$$V_a = \sum_\alpha x_{a,\alpha}\, \hbar^{a/2}, \qquad W_a = \sum_i a_{a,i}\, \hbar^{(a-1)/2},$$

where relative to conventions elsewhere in this section, $x_{a,\alpha} = s_{a,\alpha}^{-1}$, and the powers of $\hbar$ are a convenient choice of coordinates. (Hopefully, the reader will distinguish the subscript a labeling the node of $\mathcal{Q}$ and taking values from 1 to $\operatorname{rk}\mathfrak{g}$.) With this, the contributions to $\phi((q-\hbar)T^{1/2})$ are:

— From $\mathrm{Hom}(V_a, W_a)$, we get

$$\prod_{\alpha,i} \frac{\varphi(q\, x_{\alpha,a}/\hbar^{1/2} a_{i,a})}{\varphi(\hbar x_{\alpha,a}/\hbar^{1/2} a_{i,a})}. \tag{3.39}$$

This coincides with (2.24) if we recall that $v_a = \hbar^{1/2}$ and $t = q/\hbar$. Then

$$\prod_{\alpha,i} \langle S_a^\vee(x_{a,\alpha}) V_a^\vee(a_{a,i})\rangle.$$

— From for every pair of nodes adjacent nodes a, b with $I_{ab} = 1$, we get

$$\prod_{\alpha,\beta} \frac{\varphi(q\hbar^{a/2} x_{a,\alpha}/\hbar^{b/2} x_{b,\beta})}{\varphi(\hbar\, \hbar^{a/2} x_{a,\alpha}/\hbar^{b/2} x_{b,\beta})}$$

from the contributions of $\mathrm{Hom}(V_a, V_b)$ to $T^{1/2}X$. This coincides with

$$\prod_{\alpha,\beta} \langle S_a^\vee(x_{a,\alpha}) S_b^\vee(x_{b,\beta})\rangle$$

in (2.23).

— From $\mathrm{Hom}(V_a, V_a)$, we get

$$\prod_{\alpha\neq\beta} \frac{\varphi(\hbar\, x_{a,\alpha}/x_{a,\beta})}{\varphi(q x_{a,\alpha}/x_{a,\beta})}$$

up to an overall constant. Recall that (2.22)

$$\prod_{\alpha<\beta} \langle S_a^\vee(x_{a,\alpha}) S_a^\vee(x_{a,\beta})\rangle = \prod_{\alpha\neq\beta} \frac{\varphi(x_{a,\alpha}/x_{a,\beta})}{\varphi(t\, x_{a,\alpha}/x_{a,\beta})} \prod_{\alpha<\beta} \frac{\theta(t x_{a,\alpha}/x_{a,\beta})}{\theta(\, x_{a,\alpha}/x_{a,\beta})}.$$

Using $t = q/\hbar$, and the $\theta(x) = \theta(q/x) = \varphi(x)\varphi(q/x)$ property of theta-function, the above coincides with

$$\prod_{\alpha<\beta} \langle S_a^\vee(x_{a,\alpha}) S_a^\vee(x_{a,\beta})\rangle = \prod_{\alpha\neq\beta} \frac{\varphi(\hbar x_{a,\alpha}/x_{a,\beta})}{\varphi(q\, x_{a,\alpha}/x_{a,\beta})} \prod_{\alpha<\beta} \frac{\theta(q x_{a,\alpha}/x_{a,\beta})}{\theta(\hbar\, x_{a,\alpha}/x_{a,\beta})} \tag{3.40}$$

up to the ratio of theta-functions.

In summary, we showed that the contribution of $\phi((q-\hbar)T^{1/2})$ to (3.37) coincides with the contribution of $\Phi(x, a)$ to (3.38), up to the collection of theta-functions in (3.40). The exponential terms in (3.37) correspond to the exponential x^μ terms in (2.21), with identification

$$z_a = q^{(\mu, e_a)}.$$

From the perspective of the difference equations the effect of the ratio of the theta-functions in (3.40) is to produce a shift in the Kähler variables $z_a = q^{(\mu,e_a)}$ by a power of $\hbar^{1/2}$. These shifts are collected in (3.37) in replacing $z \to z_\#$. This proves the theorem. □

3.5.1. Explicitly, for $X = T^*\mathbb{P}^{n-1}$, the right hand side of (3.38) becomes a q-conformal block

$$\langle \mu' | V^\vee(a_1) \dots V^\vee(a_n)\, Q^\vee |\mu\rangle$$

of the $\mathcal{W}_{q,t}(A_1)$ algebra (the algebra is also known as the q-Virasoro algebra). The algebra has a single family of generators $e[k]$, $k \in \mathbb{Z}$, satisfying

$$[e[k], e[\ell]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})(q^{\frac{k}{2}} t^{-\frac{k}{2}} + q^{-\frac{k}{2}} t^{\frac{k}{2}})\delta_{k,-\ell},$$

with Fock representation π_μ defined as

$$e[k]|\mu\rangle = 0, \;\; \text{for } \; k > 0, \;\; \text{and } \; e[0]|\mu\rangle = (\mu, e)|\mu\rangle.$$

The screening charge operator is

$$Q^\vee = \int dx\ S^\vee(x) \ : \pi_0 \to \pi_{-e\beta}, \tag{3.41}$$

where

$$S^\vee(x) = [\dots]x^{-e[0]} : \exp\Big(\sum_{k\neq 0} \frac{e[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}} x^k\Big) : . \tag{3.42}$$

The $[\dots]$ stands for operators responsible for the shift of μ in (2.14).The magnetic degenerate vertex operator is

$$V^\vee(x) = [..]x^{w[0]} : \exp\Big(-\sum_{k\neq 0} \frac{w[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}\, x^k\Big) :,$$

where $w[k] = e[k]/(q^{\frac{k}{2}}t^{-\frac{k}{2}} + q^{-\frac{k}{2}}t^{\frac{k}{2}})$, and the dots stand for the operator responsible for shifting the Fock vacuum

$$V^\vee(x) \ : \pi_0 \to \pi_{w\beta}.$$

From the definitions, one computes

$$\langle \mu' | V^\vee(a_1) \dots V^\vee(a_n)\ Q^\vee|\mu\rangle = \int dx\ x^{-(\mu,e)-1}\Phi(x,a),$$

where

$$\Phi(x,a) = \prod_{j=1}^n \frac{\varphi(tx/a_j)}{\varphi(x/a_j)},$$

$t = q/\hbar$, and

$$\mu' = \mu + (nw - e)\beta.$$

By inspection, $\Phi(x,a) = \varphi((q-\hbar)T^{1/2})$, with $z_\# = q^{(\mu,e)}$.

3.5.2. While the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra has a nice conformal limit, the same is not true of quantum K-theory. While we can formally take the limit (2.25) of the generating functions, their contributions have no enumerative meaning. (There is a natural limit of the theory where degenerate counts in K-theory to cohomology, but this is not the limit we need here.)

4. Vertex functions and qKZ

4.1. **Degeneration formula.**

4.1.1. Recall that the domain C of the quasimaps (3.2) is a fixed, that is, parametrized curve. We can let it degenerate to a union C_0 of two rational curves, e.g., by taking a trivial family $C \times \mathbb{C}$ over $\mathbb{C}$ and blowing up a point $(c, 0) \in C \times \mathbb{C}$. We denote by ε the parameter of the degeneration and write $\mathbb{C}_\varepsilon$ to denote the base

$$\pi : \mathbf{C} = \mathrm{Bl}_{(c,0)} C \times \mathbb{C}_\varepsilon \to \mathbb{C}_\varepsilon$$

of the degenerating family.

Clearly,

$$C_\varepsilon = \pi^{-1}(\varepsilon) \cong C$$

canonically for $\varepsilon \neq 0$, while the special fiber of the new family is the union

$$C_0 = C_{0,1} \cup C_{0,2}\,, \quad C_{0,1} \cong C, \tag{4.1}$$

of two rational curves joined at the point $c \in C_{0,1}$. If $c = 0$, that is, if c is a fixed point of $\mathbb{C}_q^\times$ other than $\infty \in C$, then this degeneration is $\mathbb{C}_q^\times$-equivariant.

4.1.2. A key geometric question is to put a good central fiber into the family $\mathsf{QM}(C_\varepsilon \to X)$ over $\mathbb{C}_\varepsilon \setminus \{0\}$ and it is answered by a beautiful theory due principally to Jun Li; see [72–74] and also, for example, [93] for an introductory discussion. The central fiber, which we still denote $\mathsf{QM}(C_0 \to X)$, is the moduli space of quasimaps from a nodal curve C_0; however, an important geometric idea is hidden here in the definition of a quasimap from a nodal source curve.

To keep the obstruction theory perfect, quasimaps need to be non-singular at the nodes of the source curve. To satisfy this constraint and the usual properness requirements at the same time, one has to say what to do with a one-parameter family of quasimaps that develops a singularity at the node of a special fiber. It is treated by a version of the familiar semistable reduction process, in which the offending node is being blown up until the singularity at it goes away. In the process, the node becomes replaced by a chain of rational curves, considered up to an isomorphism fixing the points at which it is attached to the original nodal curve.

This motivates defining $\mathsf{QM}(C_0 \to X)$ as the moduli spaces of quasimaps of the form

$$\begin{tikzcd} C_0' \arrow[r, dashed, "f'"] \arrow[d, "g"] & X \\ C_0 & \end{tikzcd} \tag{4.2}$$

in which

— the map g collapses a chain of rational curves to the node of C_0,
— f' is non-singular at the nodes of C_0',
— the automorphism group of f' is finite.

Here the source of automorphisms is the group

$$\mathrm{Aut}(C_0', g) = \left(\mathbb{C}^\times\right)^{\#\text{ of new components}} .$$

Pictorially, the domain C_0' may be represented as in Figure 3. As usual, one of the uses of the finiteness of $\mathrm{Aut}(f')$ is to prevent unnecessary blowups in the course of the semistable reduction.

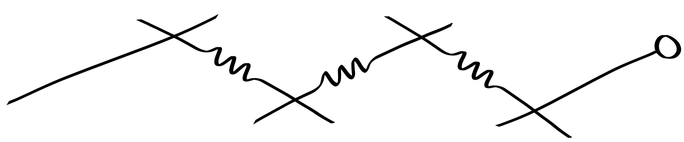

FIGURE 3. A semistable curve C_0' whose stabilization is the nodal curve C_0. Components with $\mathbb{C}^\times$ automorphisms are indicated by springs; they are often called accordions. The point $\infty \in C \cong \mathbb{C}_{0,1}$ at which the quasimaps are required to be non-singular is indicated by a circle.

4.1.3. The family

$$\mathsf{QM}(C_\varepsilon \to X) \to \mathbb{C}_\varepsilon \tag{4.3}$$

has a natural relative obstruction theory, given by the cohomology, that is, push-forward along π, of quiver sheaves in question. Its restrictions to fibers of (4.3) is the obstruction theory for the spaces $\mathsf{QM}(C_\varepsilon \to X)$ and hence the virtual structure sheaves and the symmetrized virtual structure sheaves of these spaces fit into a flat family over $\mathbb{C}_\varepsilon$. This means, one can count quasimaps from C in terms of quasimaps from C_0.

4.1.4. Quasimaps from C_0 can, in turn, be glued out of pieces that correspond to the pieces in the domain curve in Figure 3. Indeed, moduli of quasimaps from a fixed nodal curve C'_0 non-singular at the nodes are the product of moduli of quasimaps from the components over the evaluation maps at the nodes. Because the number of the accordions, that is, non-parametrized components of C'_0 is dynamical, the correct decomposition to take is the one depicted in Figure 4.

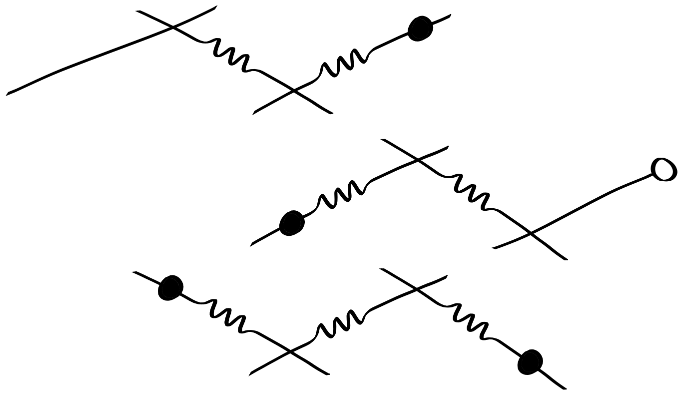

FIGURE 4. Three kinds of moduli spaces that appear as pieces in the degeneration formula. In the first line, $C_{0,2}$ is linked by a chain of accordions to a marked point (bold circle), at which quasimaps must be non-singular. In the second line, we have the same for $C_{0,1}$ together with the original evaluation point $\infty \in C \cong C_{0,1}$. In the third line, the domain is a chain of accordions joining two marked points.

4.1.5. The difference between the new marked points shown in bold and the original evaluation point $\infty \in C \cong C_{0,1}$ is the following. While evaluation at ∞ requires explicitly throwing out quasimaps with singularities there, singularities cannot get to the bold points by the nature of moduli spaces. Any time a singularity tries to get to the point $\bullet$, a new accordion opens (by semistable reduction), and the point $\bullet$ gets away.

These new kinds of quasimaps are called quasimaps *relative* a point $\bullet$ of the domain. The above informal discussion means formally that the evaluation maps

$$\begin{aligned} \mathrm{ev}_\bullet :\ & \mathsf{QM}(C_{0,1})_{\text{relative }\bullet} && \to X \\ \mathrm{ev}_{\bullet,\bullet} :\ & \mathsf{QM}(\text{accordions})_{\text{relative }\bullet,\bullet} && \to X \times X \end{aligned} \tag{4.4}$$

are *proper*. Using them, we can define

$$\mathsf{Cap} = \mathrm{ev}_{\bullet,*}\left(\widehat{\mathscr{O}}_{\mathrm{vir}}\, z^{\deg f}\right) \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X) \otimes \mathbb{Q}[[z]] \tag{4.5}$$

and

$$\mathsf{Glue} = \mathrm{ev}_{\bullet,\bullet,*}\left(\widehat{\mathscr{O}}_{\mathrm{vir}}\, z^{\deg f}\right) \in K_{\mathsf{T}}(X^2) \otimes \mathbb{Q}[[z]]\,, \tag{4.6}$$

where localization is not required in contrast to (3.7). Note that (4.6) does not depend on q since $\mathbb{C}_q^\times$ acts trivially on quasimaps from non-parametrized curves. The absence of denominators makes these tensors much simpler objects than the vertex, or than its analog

$$\mathsf{J} = \mathrm{ev}_{\bullet,\circ,*}\left(\widehat{\mathscr{O}}_{\mathrm{vir}}\, z^{\deg f}\right) \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X^2)_{\text{localized}} \otimes \mathbb{Q}[[z]]\,. \tag{4.7}$$

Here the quasimaps are from the domain shown in the middle line of Figure 4. The tensor (4.7) is called the *capping* operator in [93] and denoted by J there. Here we use the same notation. It would be nice to have a name for this operator which better reflects the role it plays in the correspondence studied in the the present paper.

4.1.6. The correspondences (4.6) and (4.7) act on $K_{\mathsf{T}\times\mathbb{C}_q^\times}(X)$ and the statement of the degeneration formula may be written as follows:

$$\mathsf{Vertex} = \mathsf{Cap}\,\mathsf{Glue}^{-1}\,\mathsf{J} \tag{4.8}$$

where we compose the operators in the order in which we draw the component of C_0'. From definitions,

$$\mathsf{Glue} = \mathscr{K}_X^{1/2} + O(z)$$

where $\mathscr{K}_X$ is the canonical bundle of X viewed as an operator of tensor multiplication, and so the inverse Glue^{-1} is well-defined as a formal series in z.

The discovery that the operator Glue^{-1} enters the degeneration formula was originally made by Givental in his study of K-theoretic analogs of Gromov-Witten counts; see [58, 70]. The adaptation of this idea to K-theory of quasimap moduli spaces is straightforward; see, e.g., [93] for the details.

4.2. **Difference equations.**

4.2.1. The geometric construction of the operator (4.7) makes it easy to show that it is a fundamental solution to a compatible system of difference equations in both Kähler and equivariant variables; see Section 8 of [93].

Here by a fundamental solution we mean an operator that conjugates a difference connection to a constant coefficient difference connection or to some other standard form. Concretely, for q-shifts of equivariant variables discussed in Section 8.2 of [93], that standard form is a difference equation solvable in φ-functions. This is the origin of φ-prefactors in (3.26).

4.2.2. An algebraic identification of these q-difference equations requires a development of geometric representation theory ideas in the present setting. In includes an identification

$$K_{\mathsf{T}}(X) = \begin{array}{l}\text{weight subspace in}\\ \text{a representation } F \text{ of } \mathscr{U}_\hbar(\widehat{\mathfrak{g}})\,,\end{array} \tag{4.9}$$

for a certain quantum group $\mathscr{U}_\hbar(\widehat{\mathfrak{g}})$. Such geometric realizations of quantum groups go back to the pioneering work of Nakajima [81] and have been studied by many researchers since. The particular point of view on (4.9) developed in [79] and further in [5, 93, 94] will be important in what follows. It gives, among other things, a natural collection of identifications

$$F \otimes \mathbb{Q}(\mathsf{T}) \cong \bigotimes_{a\in I} \bigotimes_{\alpha=1}^{\dim W_a} F_a(a_{a,\alpha}) \otimes \mathbb{Q}(\mathsf{T}) \tag{4.10}$$

indexed by all possible orderings of the coordinates of the maximal torus

$$\mathsf{A} = \{\operatorname{diag}(a_{a,\alpha})\} \subset \prod_a \operatorname{GL}(W_a) \subset \operatorname{Aut}(X,\omega)\,.$$

In (4.10), we have

$$
\begin{aligned}
a = \quad & \text{an element of the set } I \text{ of vertices of the quiver}\,, \\
F_a = \quad & \text{the corresponding fundamental representation of } \mathscr{U}_\hbar(\widehat{\mathfrak{g}}) \\
& \text{a.k.a. Kirillov-Reshetikhin module}\,, \\
a_{a,\alpha} = \quad & \text{equivariant parameter for } \mathrm{GL}(W_a) \text{ and} \\
& \text{an evaluation parameter for } F_a\,.
\end{aligned}
$$

The identification (4.9) is in integral K-theory, and so a certain integral form of both the quantum group and of its module appears in the right hand side. In (4.10) we tensor with the field $\mathbb{Q}(\mathsf{T})$ of rational functions of T, which corresponds to localization in T-equivariant K-theory. Correspondingly, R-matrices that intertwine the identifications (4.10) for different ordering of the evaluation points act in localized K-theory.

Geometrically, it is the tensor structure, that is, the maps (4.10) that are constructed first in the approach of [79]. They are a particular instance of certain very special maps of the form

$$K_{\mathsf{T}}(X^{\mathsf{A}}) \to K_{\mathsf{T}}(X)$$

called *stable envelopes*; see, e.g., Section 9 of [93] for an introduction. The structure of a module over a quantum group is then reconstructed from this tensor structure.

4.2.3. On the right hand side of (4.10) we have a canonical q-difference connection in the evaluation parameters $a_{a,\alpha}$, namely the quantum Knizhnik-Zamolodchikov connection of I. Frenkel and N. Reshetikhin [51]. It takes as a parameter an element

$$z \in \left(\mathbb{C}^\times\right)^I = e^{\mathfrak{h}}\,, \quad \mathfrak{h} \subset \mathfrak{g} \subset \widehat{\mathfrak{g}},$$

of the torus of group-like elements of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}})$. It corresponds to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$, the affinization of which is $\widehat{\mathfrak{g}}$. This torus is naturally identified with the Kähler torus from before.

A technical result of [93] identifies the geometric q-difference connection in variables $\{a_{a,\alpha}\}$ with the qKZ connection. See Section 10 in [93] and also [79] for a proof in the setting of equivariant cohomology. Thus

$$\mathsf{J} = \text{fundamental solution of qKZ}\,, \tag{4.11}$$

with the following very important detail that needs to be mentioned.

4.2.4. The difference connection in a and z solved by (4.7) is by construction flat. Moreover, it is regular in either a or z separately. However, it is *not* jointly regular in the variables a and z. This simple, but important new phenomenon for difference equations is discussed at length in [5]. It does not occur for differential equations by a deep theorem of Deligne. As a result, one cannot find a fundamental solution which will be holomorphic in both z and a in some asymptotic region of the torus of variables.

Recall that one usually looks for solutions of qKZ analytic in an asymptotic region of the schematic form

$$|a_{2,5}| \gg |a_{1,7}| \gg |a_{3,3}| \gg \dots\,,$$

that is, in a certain neighborhood of a fixed point in a toric compactification of A. We will call such solutions a-solutions. By the results of [51], q-deformed WZW conformal blocks are a-solutions of the qKZ equations.

Instead, (4.7) is a series in z, which means it is holomorphic is a neighborhood of a torus-fixed point of the Kähler moduli space. This Kähler moduli space is the toric variety constructed from the fan of ample cones of flops of X in $\mathrm{Pic}(X)$. We call such solutions z-solutions. A more precise version of (4.11) is thus the following.

Theorem 2 ([93]). *The operator* (4.7) *is the fundamental z-solution of qKZ equations in variables* $\{a_{a,\alpha}\}$.

4.2.5. Meromorphic solutions to a q-difference equation form a vector space over q-periodic meromorphic functions of dimension equal to the rank. Therefore, there exists a uniquely defined matrix transforming z-solutions to a-solutions. Taking into account the constant coefficient q-difference equations to which fundamental solutions conjugate the original equation, this matrix is best seen as a meromorphic section of a certain vector bundle on the elliptic curve

$$E = \mathbb{C}^\times / q^{\mathbb{Z}} .$$

It is called the *pole subtraction* matrix in [5], as it quite literally removes the poles in one set of variables at the expense of poles in another set of variables.

This matrix is linked to an elliptic analog of stable envelopes in [5]. Concretely, Theorem 4 in [5] shows that elliptic stable envelopes transform z-solutions of the equations satisfied by the vertex functions to the corresponding a-solutions.

4.2.6. Difference equations satisfied by the vertex functions follow from the following qualitative.

Proposition 3 ([94, 109]). *The cap* (4.5) *and the glue operator* (4.6) *are rational functions of all variables, including the Kähler variables.*

The statement about the glue operator follows from the results of [94] because the glue operator may be obtained as a $q \to \infty$ limit of operators of the Kähler q-difference connection; see Section 8.1 of [93]. The statement about cap is shown in [109]. In both cases, there is an explicit formula for these objects that makes rationality manifest.

Thus (4.8) gives an explicit gauge equivalence between the scalar difference equation of degree $\mathrm{rk} K(X)$ satisfied by the vertex functions and the quantum Knizhnik-Zamolodchikov equations.

The results of [94] identify the operators of the Kähler q-difference connection with the lattice in what can be called the dynamical quantum affine Weyl group of $\mathscr{U}_\hbar(\widehat{\mathfrak{g}})$. It coincides with the object studied by Etingof and Varchenko in [33] for quivers of finite-type and generalizes it to the case when $\widehat{\mathfrak{g}}$ is not generated by real root subspaces. From this perspective, the glue operator generalizes the longest element in the finite quantum dynamical Weyl group.

4.2.7. The cap with descendents mentioned above refers to the generalization of (4.5) constructed as follows:

$$\mathsf{Cap}(\lambda) = \mathrm{ev}_{\bullet,*}\left(\widehat{\mathscr{O}}_{\mathrm{vir}}\, z^{\deg f} \otimes \lambda(\mathscr{V}_i|_0)\right) \tag{4.12}$$

where

$$\lambda \in K_{\mathsf{T}\times \mathrm{GL}(V)}(\mathrm{pt})$$

is a tensor functor in the fibers of the tautological bundles $\mathscr{V}_i$ over a $\mathbb{C}^\times_q$-fixed point 0 in the domain of the quasimap. We can identify $K_{\mathsf{T}\times \mathrm{GL}(V)}(\mathrm{pt}) = K_{\mathsf{T}}(\mathfrak{R})$, where $\mathfrak{R}$ is the stack of quiver representations that contains X as the set of the stable points satisfying the moment map equations. Using the surjectivity of [80], we get

$$\begin{gathered} K_{\mathsf{T}\times\mathbb{C}^\times_q}(\mathfrak{R})[[z]] \xrightarrow{\mathsf{Cap}(\,\cdot\,)} K_{\mathsf{T}\times\mathbb{C}^\times_q}(X)[[z]] \to 0\,, \\ \lambda \mapsto \lambda\big|_X + O(z), \end{gathered} \tag{4.13}$$

and Smirnov gives an explicit rational function formula for this map [109].

The degeneration formula (4.8) remains unmodified, giving

$$\mathsf{Vertex}(\lambda) = \mathsf{Cap}(\lambda)\,\mathsf{Glue}^{-1}\,\mathsf{J}\,. \tag{4.14}$$

In particular, one can choose the descendent insertions so that they precisely cancel the glue matrix in (4.14), and this shows

$$\mathsf{J} \subset \{\text{Vertices with descendents}\}\,. \tag{4.15}$$

4.2.8. Let $\overline{\mathbf{V}}(\lambda)$ denote the vertex with descendents normalized as in (3.29). The descendents are expressed in terms of the Chern roots of the bundles $\mathscr{V}_i\big|_0$ which are precisely the integration variables in (3.32). Therefore, we have the following immediate generalization of Proposition 2.

Proposition 4. *For any insertion $\mathscr{F}$ satisfying* (3.31)*, we have*

$$\chi(X, \overline{\mathbf{V}}(\lambda) \otimes \mathscr{F}) = \frac{1}{|W|} \int_{\sum q^{\mathbf{d}} \cdot \gamma_\chi} \exp\left(\frac{\boldsymbol{\lambda}(z_\#, s)}{\ln q}\right) \frac{\mathscr{F}(s)\, \lambda(s)\, d_{\mathrm{Haar}} s}{\phi(T^\vee_{\mathrm{moving}})}\,, \tag{4.16}$$

where the sum of residues is over all effective shifts of the cycle γ_χ.

Smirnov's formula lets one construct collections $\{\lambda_k\}$ such that the matrix of the corresponding descendent vertices is the fundamental z-solution of qKZ. Theorem 4 of [5] applies equally well to both ordinary and descendent vertices, therefore, elliptic stable envelopes provide a connection matrix between this fundamental solution and the fundamental a-solutions. In particular, for quivers of finite-type, these a-solutions are the q-deformed WZW conformal blocks.

This can be summarized as follows.

Theorem 3. *There exists a linear map*

$$K_T(X) \ni \alpha \mapsto \lambda_\alpha \in K_T(\mathfrak{R}) \otimes \mathbb{Q}(z,q) \tag{4.17}$$

such that

$$\lambda_\alpha|_{X,z=0} = \alpha$$

and such that the corresponding vertex functions (4.16) *form a fundamental z-solution of qKZ. With the insertions of the elliptic stable envelopes, these become the fundamental a-solutions of qKZ, that is, a basis of the q-conformal blocks for $\mathscr{U}_\hbar(\widehat{\mathfrak{g}})$. The entry corresponding to the identity function $\lambda = 1$ is the corresponding $\mathcal{W}$-algebra q-conformal block.*

A remarkably simple formula for an equivalent version of (4.17) is obtained in [6].

5. $\mathfrak{g} = A_1$ example

To illustrate the results, it may be helpful to work out one example in its completeness. Take $\mathfrak{g} = sl_2$ with finite-dimensional representations ρ_i of highest weights w_i attached to points

$$x = a_i, \qquad i = 1, \dots, n,$$

of the Riemann surface.

5.1. **qKZ equation and its z-solutions.** The q-conformal block of $U_\hbar(\widehat{\mathfrak{g}})$ with this data is a chiral correlation function from (2.1)

$$\Psi(a_1, \dots, a_\ell, \dots a_n) \in (\bigotimes_i \rho_i)_{\lambda - \lambda'}, \tag{5.1}$$

and where $\lambda_{0,\infty}$ are the weights $|\lambda_{0,\infty}\rangle$, the highest weight vectors of Verma module representations which enter (2.1).

As [51] explained, (5.1) solves the qKZ equation of (2.4) where the $\mathcal{R}$ matrices take the following explicit form.

5.1.1. Let v_i be the highest weight vector of representation ρ_i (with weight w_i). Let f be the lowering operator of $\mathfrak{g} = sl_2$. The $\mathcal{R}$ matrix acts by

$$\mathcal{R}_{ij}(a) v_i \otimes v_j = v_i \otimes v_j, \tag{5.2}$$

$$\mathcal{R}_{ij}(a) f v_i \otimes v_j = \frac{a\hbar^{m_j} - \hbar^{m_i}}{a - \hbar^{m_i+m_j}} f v_i \otimes v_j + \frac{1 - \hbar^{2m_j}}{a - \hbar^{m_i+m_j}} v_i \otimes f v_j, \tag{5.3}$$

$$\mathcal{R}_{ij}(a) v_i \otimes f v_j = \frac{a(1 - \hbar^{2m_i})}{a - \hbar^{m_i+m_j}} f v_i \otimes v_j + \frac{a\hbar^{m_i} - \hbar^{m_j}}{a - \hbar^{m_i+m_j}} v_i \otimes f v_j, \tag{5.4}$$

where

$$m_i = (w_i, e)/2,$$

and e is the positive root of sl_2. Throughout, one should keep in mind the identifications in (2.5). Furthermore, $(\hbar^\mu)_\ell$ acts on the ℓth component of the tensor, corresponding to representation ρ_ℓ of highest weight vector v_ℓ of highest weight w_ℓ by

$$\hbar^\mu(v_\ell) = \hbar^{(\mu, w_\ell)} v_\ell, \qquad \hbar^\mu(f v_\ell) = \hbar^{(\mu, w_\ell - e)}.$$

The Weyl vector ρ, which also enters (2.4), is equal to half the sum of positive roots, $\rho = e/2$ in this case.

5.1.2. The solutions to the qKZ equation, in n-dimensional the subspace of weight

$$\lambda' - \lambda = w_1 + \ldots + w_n - e,$$

can be written out explicitly, as follows [77]. Let

$$\Psi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \varphi_i(a_1, \ldots, a_n) \, v_1 \otimes \ldots \otimes f v_i \otimes \ldots \otimes v_n. \tag{5.5}$$

Further, it is useful to define

$$\varphi_i(a_1, \ldots, a_n) = q^{(\beta_{i+1} + \ldots + \beta_n)/2} \, a_1^{\beta_1} \ldots a_n^{\beta_n} \, \mathcal{F}_i(q^{\beta_1/2} a_1, \ldots, q^{\beta_n/2} a_n) \tag{5.6}$$

where

$$q^{\beta_i} = \hbar^{(w_i, e)}, \qquad q^\eta = \hbar^{-(\lambda, e)}.$$

Then, [77] proves (5.5) is the solution of the qKZ equation for

$$\mathcal{F}_i(a) = \int_\gamma dx \, x^{\eta - 1} \, K_i(x, a) \times \prod_{j=1}^{n} \frac{\varphi(x/a_j)}{\varphi(q^{\beta_j} \, x/a_j)}, \tag{5.7}$$

where we defined

$$K_i(x, a) = \prod_{j=1}^{i-1} \frac{(1 - q^{\beta_j} \, x/a_j)}{(1 - x/a_j)} \times \frac{1}{1 - x/a_i},$$

and $\varphi(x) = \prod_{n=0}^{\infty} (1 - q^n x)$, as before. The equation (5.5) gives solutions to qKZ for any contour γ for which the integral[8] is invariant under $x \to qx$. One proves this by explicitly studying difference equations satisfied by (5.7) with respect to operators that take $a_i \to q a_i$.

[8]This implies that, from the perspective of difference equations, the integrand $I(x)$ in (5.7) is equivalent to $I(qx)$: $\int \frac{dx}{x} I(x) = \int \frac{dx}{x} I(qx)$.

5.1.3. Our main example corresponds to ρ_i which is the two-dimensional representation $\rho_i = \rho$ for all i. Its highest weight is the fundamental weight $w_i = w$ of $\mathfrak{g} = sl_2$, $(w, e) = 1$, so that $q^{\beta_i} = \hbar$. Up to redefinition of integration variable x, replacing it with $\hbar x$, we have

$$\mathcal{F}_i(a) = \int_\gamma dx\, x^{\eta-1} K_i(x,a) \times \prod_{j=1}^{n} \frac{\varphi(\hbar^{-1}x/a_j)}{\varphi(x/a_j)}, \tag{5.8}$$

where we defined

$$K_i(x,a) = \prod_{j=1}^{i-1} \frac{(1 - x/a_j)}{(1-\hbar^{-1}x/a_j)} \times \frac{1}{1-\hbar^{-1}x/a_i}.$$

The equations (5.6) and (5.8) provide a solution to the qKZ equation, for any choice of the contour C. The set of linearly independent solutions one gets by varying the contour C has a geometric and representation theoretic interpretation.

5.2. **Geometric interpretation in terms of $X = T^*\mathbb{P}^{n-1}$.** The geometric interpretation is in terms of counts of quasimaps to

$$X = T^*\mathbb{P}^{n-1}, \tag{5.9}$$

where z keeps track of the degree of the map. X is the Nakajima quiver variety (1.11) corresponding to a $\mathfrak{g} = A_1$ quiver $\mathcal{Q}$ a single node and a pair of vector spaces $V = \mathbb{C}$ and $W = \mathbb{C}^n$ associated to it, acted on by

$$G_{\mathcal{Q}} = \mathrm{GL}(1), \qquad G_W = \mathrm{GL}(n).$$

The dimension vectors of W and V are determined, respectively, by the highest weight of the module

$$\bigotimes_i \rho_i = \bigotimes^n \rho,$$

and the weight of its subspace in which (5.1) takes values, as explained in Section 1.3.

5.2.1. The vertex function of X, counting quasimaps from C to X, has an integral representation (3.33), as one recalls from Section 3.2:

$$\mathbf{V} = \int_\gamma dx\, x^{\eta-1}\Phi(x,a) \tag{5.10}$$

where, in terms of $t = q/\hbar$,

$$\Phi(x,a) = \prod_{j=1}^{n} \frac{\varphi(t\,x/a_j)}{\varphi(x/a_j)}.$$

Using that, it is easy to recognize that the solutions to the qKZ equation in (5.8) can be rewritten in terms of the geometric quantities of X:

$$\mathcal{F}_i = \int_\gamma dx\, x^{\eta-1}\mathrm{Stab}_i^K(x,a)\Phi(x,a), \tag{5.11}$$

where

$$\mathrm{Stab}_i(x,a) = \prod_{j=1}^{i-1}(1 - x/a_j) \times \prod_{j=i+1}^{n} (1-\hbar^{-1}x/a_i) \tag{5.12}$$

is a collection of classes in $K_T(X)$. The integrands in (5.11) and (5.8) are equal.

This shows that $\mathcal{F}$, the fundamental z-solution to the qKZ equation in (5.8), is the geometrically defined operator J in (4.11),

$$\mathcal{F} = \mathsf{J},$$

and that the geometric corresponding of (5.11) is in terms of vertex functions, counting quasimaps $\mathbb{C} \dashrightarrow X$, with descendant insertions at $0 \in X$ from (4.15). The basis of insertions that leads to the qKZ equation with $\mathcal{R}$ matrices in the standard form is a special one, as will be explained in [6]. The classes in (5.12) give the K-theoretic stable basis of X, defined in [93]. For a suitable choice of a chamber, slope, and polarization [93], (5.12) gives the basis element corresponding to a stable envelope of the ith T-fixed point in X.

5.2.2. We can also consider the stable envelope with slope s [93]

$$\mathrm{Stab}^K_{i,(s)}(x,a) \equiv x^s \,\mathrm{Stab}^K_i(x,a). \tag{5.13}$$

The role of the slope s is to change the weight λ in (2.1), and leads to a family of solutions to qKZ, differing by the choice of the highest weight vector $|\lambda\rangle$ in (2.1).

5.3. **q-Virasoro conformal blocks.** The vertex function in (5.10) as we saw in Section 3.2, coincides with a q-conformal block of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra for $\mathfrak{g} = A_1$; the algebra which is the q deformation of Virasoro algebra. Its q-conformal blocks are

$$\langle \mu' | V^\vee(a_1)\dots V^\vee(a_n)\, Q^\vee|\mu\rangle = \int_\gamma dx\, x^{-(\mu,e)-1}\Phi(x,a). \tag{5.14}$$

To completely define the q-conformal block in (5.14) we need to specify the contour γ. As in Section 3.5.1, we will define the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra blocks to be the components of the vertex function of $X = T^*\mathbb{P}^{n-1}$,

$$\mathbf{V} = \langle \mu' | V^\vee(a_1)\dots V^\vee(a_n)\, Q^\vee|\mu\rangle,$$

where the Kähler variable z equals $z = q^{(\mu,e)}$, up to unimportant shift. The component of the vertex function $\mathbf{V}_\ell$ where the point at infinity in C maps to the fixed point p_ℓ in X corresponds to

$$\mathbf{V}_\ell = \chi(X, \mathbf{V}\otimes \mathcal{O}_{p_\ell})$$

as in Section 3.4.7. The insertion of $\mathcal{O}_{p_\ell}$ amounts to picking the contour γ_ℓ which picks up the poles at

$$\gamma_\ell: \qquad x = q^{-n}\, a_\ell, \qquad n = 0,1,\dots. \tag{5.15}$$

Computing the integral by residues, we find

$$\mathbf{V}_\ell = (a_\ell)^\eta \,\frac{\varphi(t)}{\varphi(q)} \prod_{i\neq\ell} \frac{\varphi(t a_\ell/a_i)}{\varphi(a_\ell/a_i)}\, \mathbb{F}\left[\begin{matrix} \hbar a_1/a_\ell, & \hbar a_2/a_\ell, & \dots \\ q a_1/a_\ell, & q a_2/a_\ell, & \dots \end{matrix}\,\middle|\, z/\hbar^n\right] \tag{5.16}$$

and

$$\mathbb{F}\left[\begin{matrix} \hbar a_i/a_\ell \\ q a_i/a_\ell \end{matrix}\,\middle|\, z/\hbar^n\right] = \sum_{d\geq 0} (z/\hbar^n)^d \prod_i \frac{(\hbar a_i/a_\ell)_d}{(q a_i/a_\ell)_d} = \mathsf{Vertex}_\ell$$

is the q-hypergeometric function. It is also the Vertex function of X, in its canonical normalization. The function $\mathbf{V}$ differs from it by contributions of constant, zero degree maps (see (3.26)).

5.3.1. The $\mathbf{V}$ function (a vector) can be written as the covector W contracted with the operator $\mathcal{F} = \mathsf{J}$. Then

$$\sum_{i=1}^{n} W_i \, \mathcal{F}^i{}_\ell = \mathbf{V}_\ell \tag{5.17}$$

which is the content of (4.14), in the present example.

The coefficients W_i can be found as follows: The K-theoretic stable envelopes of fixed slope provide a basis of $K(X)$-theory of X, so in particular, the trivial insertion 1 at 0 in X written in the stable basis from (5.12) as:

$$1 = \sum_{i=1}^{n} W_i \, \mathrm{Stab}_i^K(x,a), \tag{5.18}$$

where W_i are the coefficients in (5.17). The stable basis is upper triangular, as $\mathrm{Stab}_i^K(x,a)$ vanishes at $x = a_j/\hbar$, for $i < j$. This lets us find W_i solving (5.18) recursively, solving for W_i in terms of $W_{i+1}, \ldots, W_n$.

5.4. **Elliptic stable envelope and z- and a-solutions.** $\mathcal{F}_i$'s generate a space of solutions of qKZ equation, by varying the contours γ. The solutions to qKZ obtained in (5.8) or (5.11) are not q-conformal blocks of $U_\hbar(\widehat{\mathfrak{g}})$ since they are not a-solutions of qKZ, which are solutions *jointly* analytic in a chamber of A-parameter space.[9] Instead, they are the z-solutions, analytic functions of the Kähler variable z. The map between the z solutions and the a-solutions is provided by elliptic stable envelopes of X.

5.4.1. Pick an a-chamber

$$\mathfrak{C}: \qquad |a_j| < |a_i| \qquad \text{for} \qquad j < i.$$

Starting with the vertex function $\mathbf{V} = \int dx \; x^{\eta-1} \; \Phi(x)$, we obtain a new vertex function $\mathbf{V}_{\mathfrak{C}}$, which solves the same set of difference equations as $\mathbf{V}$ and which is analytic in chamber $\mathfrak{C}$, as follows. We take

$$\mathbf{V}_{\mathfrak{C}} = \oint dx \, x^{\eta-1} \; \Phi(x) \; \mathfrak{P}_{\mathfrak{C}}(x) \tag{5.19}$$

with

$$\mathfrak{P}_{\mathfrak{C},\ell}(x,a) = U_{\mathfrak{C},\ell} \, \frac{\prod\limits_{i<\ell} \theta(a_i/x) \; \theta(z^{-1}\hbar^\ell \, a_\ell/x) \; \prod\limits_{\ell<i} \theta(\hbar \, a_i/x)}{\theta(z^{-1}\hbar^\ell) \, \prod_i \theta(\hbar \, a_i/x)} \mathbf{e}(z,x)^{-1}. \tag{5.20}$$

The contour of integration is spelled out below equation (5.25). $\mathbf{V}$ and $\mathbf{V}_{\mathfrak{C}}$ solve the same set of difference equations, since $\mathfrak{P}_{\mathfrak{C},\ell}(x)$ are pseudo-constants satisfying

$$\begin{aligned} \mathfrak{P}_{\mathfrak{C},\ell}(x, a_1, \ldots, a_i, \ldots, a_n; z) &= \mathfrak{P}_{\mathfrak{C},\ell}(qx, a_1, \ldots, a_i, \ldots, a_n; z) \\ &= \mathfrak{P}_{\mathfrak{C},\ell}(x, a_1, \ldots, qa_i, \ldots, a_n; z) \\ &= \mathfrak{P}_{\mathfrak{C},\ell}(x, a_1, \ldots, a_i, \ldots, a_n; qz). \end{aligned} \tag{5.21}$$

The function in (5.20) is, up to normalizations, the *elliptic stable envelope* of a fixed point p_ℓ in X in the chamber $\mathfrak{C}$

$$\mathrm{Stab}^{\mathrm{ell}}_{\mathfrak{C},\ell}(x,a) = \frac{\prod\limits_{i<\ell} \theta(a_i/x) \; \theta(z^{-1}\hbar^\ell \, a_\ell/x) \; \prod\limits_{\ell<i} \theta(\hbar \, a_i/x)}{\theta(z^{-1}\hbar^\ell)}$$

[9]For example, $\mathbf{V}\big|_{\ell=1}$ is analytic for $|a_1| < |a_2|, |a_3|, \ldots$, but $\mathbf{V}\big|_{\ell=2}$ is analytic for $|a_2| < |a_1|, |a_3|, \ldots$.

defined geometrically in [5]. The normalizations involve

$$\mathbf{e}(z,x)^{-1} = \exp \frac{\log(x)\log(z)}{\log(q)} \tag{5.22}$$

and

$$U_{\mathfrak{C},\ell} = \exp \frac{\Big(\log(a_\ell)\log(z^{-1}\hbar^\ell) - \sum_{i\leq \ell}\log(a_i)\log(\hbar)\Big)}{\log(q)}, \tag{5.23}$$

which help ensure that $\mathfrak{P}_{\mathfrak{C},\ell}(x)$ satisfies (5.21).

5.4.2. The contour of integration in (5.19) is defined to separate the poles of the integrand, located at

$$x = q^{-n}\, a_i, \qquad \ell \leq i\,, \quad n = 0,1,\dots, \tag{5.24}$$

$$x = q^{n}\, a_i\, \hbar\,, \qquad i \leq \ell\,, \quad n = 0,1,\dots\,. \tag{5.25}$$

For $|q|<1$, the poles in (5.24) accumulate to $x=\infty$ while the poles in (5.25) accumulate to $x=0$.

For $z<1$ we can deform the contour to enclose all poles of the form (5.24), to obtain

$$\mathbf{V}_{\mathfrak{C},\ell} = \sum_{\ell'} \mathbf{V}_{\ell'}\, \mathfrak{P}^{\ell'}_{\mathfrak{C},\ell} \tag{5.26}$$

where

$$\mathfrak{P}^{\ell'}_{\mathfrak{C},\ell} = \mathfrak{P}_{\mathfrak{C},\ell}(a_{\ell'}).$$

The linear change of basis in (5.26), determined by elliptic stable envelopes, is the *pole subtraction matrix* for chamber $\mathfrak{C}$. The name reflects the fact that $\mathbf{V}_{\mathfrak{C}}$ is pole-free in a neighborhood of $0_{\mathfrak{C}}$, the origin of the chamber $\mathfrak{C}$. Note the pole subtraction matrix is triangular,

$$\mathfrak{P}^{\ell'}_{\mathfrak{C},\ell} = 0, \qquad \ell' < \ell,$$

since the numerators in (5.21) eliminate poles from (5.24) for $i=\ell'<\ell$.

5.4.3. A contour integral becomes singular when the poles of the integrand, located on opposite sides of the contour, coalesce. By studying poles of the integrand in (5.19), it follows that $\mathbf{V}_{\mathfrak{C}}$ is pole-free in a neighborhood of $0_{\mathfrak{C}}$. This motivates the name we gave to the matrix $\mathfrak{P}_{\mathfrak{C}}$ in (5.26).

The contour of the integration in (5.19) per definition separates the poles in (5.24) accumulate to $x=\infty$ while the poles in (5.25) accumulate to $x=0$. This means that the contour integral in (5.19) has singularities at

$$\frac{a_i}{a_j} = q^n \hbar\,,$$

with $j\leq \ell \leq i$ and $n\geq 0$. This is the complement of the chamber $\mathfrak{C}$ in which $|a_j/a_i|<1$ for $j<i$.

5.4.4. More generally, replacing $\mathcal{F}_i$ in (5.19) with

$$(\mathcal{F}^{\mathfrak{C}})_{i\ell} = \oint dx\; x^{\eta-1}\, \mathrm{Stab}^K_i(x)\; \Phi(x)\; \mathfrak{P}_{\mathfrak{C},\ell}(x) \tag{5.27}$$

for each fixed ℓ, we get a solution of qKZ of the form (5.5)–(5.6) which is analytic in chamber $\mathfrak{C}$. This is a q-conformal block of $U_\hbar(\widehat{{}^L\mathfrak{g}})$. Both the K-theoretic and the elliptic stable envelopes enter (5.27), but their roles are different. The K-theoretic stable envelope produces vector-valued solutions of $U_\hbar(\widehat{\mathfrak{g}})$ qKZ from scalar q-conformal blocks of $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra. To get analytic solutions of qKZ in chamber $\mathfrak{C}$ requires knowing the elliptic stable envelope, which enters the definition of $\mathfrak{P}_{\mathfrak{C},\ell}(x,a)$.

5.5. $X = T^*\mathbb{P}^1$ **example.** Let's make this fully explicit for $n = 2$, when $X = T^*\mathbb{P}^1$. The vertex functions associated to the two fixed points in X, corresponding to the north and the south poles of the $\mathbb{P}^1$ are:

$$\begin{aligned} \mathbf{V}_1 = \ & a_1^{\eta} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(ta_1/a_2)}{\varphi(a_1/a_2)} \mathbb{F}\left[\begin{matrix} \hbar & \hbar\frac{a_2}{a_1} \\ q & q\frac{a_2}{a_1}\end{matrix} \,\middle|\, tz'\right], \\ & = a_1^{\eta\#} \frac{\theta(ta_1/a_2)}{\theta(a_1/a_2)} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(\hbar z')}{\varphi(z')} \mathbb{F}\left[\begin{matrix} t & tz' \\ q & qz'\end{matrix} \,\middle|\, \hbar\frac{a_2}{a_1}\right], \\ \mathbf{V}_2 = \ & a_2^{\eta} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(ta_2/a_1)}{\varphi(a_2/a_1)} \mathbb{F}\left[\begin{matrix} \hbar & \hbar\frac{a_1}{a_2} \\ q & q\frac{a_1}{a_2}\end{matrix} \,\middle|\, tz'\right] \\ & = a_2^{\eta\#} \frac{\theta(ta_2/a_1)}{\theta(a_2/a_1)} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(\hbar z')}{\varphi(z')} \mathbb{F}\left[\begin{matrix} \hbar & \hbar z' \\ q & qz'\end{matrix} \,\middle|\, \hbar\frac{a_2}{a_1}\right], \end{aligned}$$

where we defined $z' = tz$. The right hand side of the equations follows using standard identities for q-hypergeometric functions. Clearly the vertex function $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2)$ has no nice analyticity properties as functions of a's, but they are analytic for $|z| < 1$.

5.5.1. The elliptic stable envelopes provide a change of basis to solutions which are (quasi)-analytic in a's. In the chamber $\mathfrak{C}$ where $|a_1| < |a_2|$, we have

$$\mathfrak{P}_{\mathfrak{C}} = U_{\mathfrak{C}} \begin{pmatrix} \frac{1}{\theta(\hbar)} & 0 \\ \frac{\theta(\hbar a_1/za_2)}{\theta(\hbar a_1/a_2)\theta(\hbar/z)} & \frac{\theta(a_1/a_2)}{\theta(\hbar a_1/a_2)\theta(\hbar)} \end{pmatrix} \mathbf{e}^{-1} \tag{5.28}$$

where the $\ell\ell'$ entry of the matrix in (5.28) corresponds to $\mathfrak{P}_{\mathfrak{C},\ell}^{\ell'}$ in (5.26). The matrices $\mathbf{e}$, $U_{\mathfrak{C}}$ are both diagonal, with eigenvalues, and can be read off from (5.23), (5.22).

Explicitly, using various q-hypergeometric function identities, we find:

$$\begin{aligned} \mathbf{V}_{\mathfrak{C},1} &= \frac{1}{\theta(t)} \, a_1^{\eta\#} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(\hbar z')}{\varphi(z')} \mathbb{F}\left[\begin{matrix} t & t/z' \\ q & q/z'\end{matrix} \,\middle|\, \hbar\frac{a_1}{a_2}\right], \\ \mathbf{V}_{\mathfrak{C},2} &= \frac{1}{\theta(t)} \, a_2^{\eta\#} \, \frac{\varphi(t)}{\varphi(q)} \frac{\varphi(\hbar/z')}{\varphi(1/z')} \mathbb{F}\left[\begin{matrix} t & tz' \\ q & qz'\end{matrix} \,\middle|\, \hbar\frac{a_1}{a_2}\right]. \end{aligned}$$

The vertex function in the stable basis $\mathbf{V}_{\mathfrak{C}} = (\mathbf{V}_{\mathfrak{C},1}, \mathbf{V}_{\mathfrak{C},2})$ is now clearly analytic in the chamber $\mathfrak{C}$, corresponding to $|a_1| < |a_2|$. The map to conformal blocks can be read off from the elliptic stable envelope, and goes as follows:

$$\begin{aligned} \mathbf{V}_{\mathfrak{C},1} \quad &\longrightarrow \quad H^{\rho_1}_{\lambda_0,\lambda_0-w} \otimes H^{\rho_2}_{\lambda_0-w,\lambda_0}, \\ \mathbf{V}_{\mathfrak{C},2} \quad &\longrightarrow \quad H^{\rho_1}_{\lambda_0,\lambda_0+w} \otimes H^{\rho_2}_{\lambda_0+w,\lambda_0}. \end{aligned} \tag{5.29}$$

5.6. **Conformal limit.** In the conformal limit, $q \to 1$ limit, one gets the familiar expressions for the integral solutions of the KZ equation, and the corresponding Virasoro conformal blocks. We will review these in detail in Section 6, for now simply note that the Virasoro block, given by (5.14) still has the same form, with

$$\langle \mu' | V^\vee(a_1) \dots V^\vee(a_n) \, Q^\vee | \mu \rangle = \int_\gamma dx \; x^{-(\mu,e)-1} \Phi(x,a) \tag{5.30}$$

with

$$\Phi(x,a) \quad \longrightarrow \quad \prod_{j=1}^{n} (1 - x/a_i)^{-\beta}. \tag{5.31}$$

Now consider the limit of the z-solutions of qKZ equations, as obtained from the J operator of X, geometrically. becomes

$$\mathcal{F}_i = \int_\gamma dx\, x^{-(\mu,e)-1}\, \mathrm{Stab}_i^K(x,a)\, \Phi(x,a), \tag{5.32}$$

which leads to the [32] integral form of solutions of the KZ equation, which we review in Section 6. Namely, the limit of stable envelopes is

$$\mathrm{Stab}_i^K(x,a) \qquad \longrightarrow \qquad \prod_{\substack{j\neq i\\ j=1}}^{n} (1 - x/a_j)$$

and taking (5.31) and (5.32) together, we get

$$\mathcal{F}_i(a) = \int_\gamma dx\, x^{(\mu,e)-1} \frac{1}{1-x/a_i} \;\times\; \prod_{j=1}^{n} (1-x/a_j)^{-\beta+1}. \tag{5.33}$$

Recalling (1.2),

$$\beta - 1 = \theta = 1/^L(k + h^\vee),$$

we recognize in (5.32) the integral form of solutions of the KZ equation in the weight $n-1$ subspace.

5.6.1. Recall that $\mathfrak{P}_{\mathfrak{C},\ell}(x,a)$ is a pseudo-constant with respect to q-shifts of all the variables; see (5.21). Thus, when q goes to 1 it becomes a constant, depending only on

$$q' = e^{-2\pi i \frac{\ln \hbar}{\ln q}} = e^{\frac{2\pi i}{^L(k+h^\vee)}},$$

but not on any continuous variables. It follows

$$\mathfrak{P}_{\mathfrak{C},\ell}{}^{\ell'} \to (q')^{\#_{\mathfrak{C},\ell,\ell'}}$$

where $\#_{\mathfrak{C},\ell,\ell'}$ is a number depending only on ℓ and $\mathfrak{C}$.

6. Isomorphism of conformal blocks and the geometric Langlands correspondence

In the previous sections, we have established an isomorphism of q-deformed conformal blocks of the deformed $\mathcal{W}$-algebra associated to a simple Lie algebra $\mathfrak{g}$ and the quantum affine algebra associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$. It is natural to ask whether the appearance of dual Lie algebras here is in some ways related to the geometric Langlands correspondence and its one-parameter deformation known as the quantum geometric Langlands.

The way the Langlands dual Lie algebra manifests here is all the more striking because deformed $\mathcal{W}$-algebras do not exhibit the duality known to exist in the conformal limit $q \to 1$. Indeed, recall that in the conformal case, we have an isomorphism between the $\mathcal{W}$-algebras associated to $\mathfrak{g}$ and ${}^L\mathfrak{g}$, after a change of the parameter [36] (it is recalled in Theorem 6.1 below). But after the q-deformation, no such isomorphism is available. In other words, there is no longer an isomorphism between the deformed $\mathcal{W}$-algebras associated to $\mathfrak{g}$ and ${}^L\mathfrak{g}$ (unless of course ${}^L\mathfrak{g} = \mathfrak{g}$). This brings the difference between the above two algebras, $\mathcal{W}_{q,t}(\mathfrak{g})$ and $U_\hbar(\widehat{{}^L\mathfrak{g}})$, into a sharper focus.

We stress that it is quite common in representation theory that introducing an additional parameter enables one to see a particular phenomenon more clearly, and notice aspects of it that hitherto could be more easily missed or ignored. Often, this results in revisiting the original phenomenon (before the deformation) and adjusting one's point of view.

For instance, consider the Harish-Chandra isomorphism $c : Z(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Fun}(\mathfrak{h}^*)^W$ between the center of the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ and the algebra of Weyl-invariant polynomial functions on the dual space to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. There is a strange aspect of this formula that is easy to ignore: c maps $Z(\mathfrak{g})$ to $\mathrm{Fun}(\mathfrak{h}^*)^W$ rather than $\mathrm{Fun}(\mathfrak{h})^W$. But actually this is significant: the Harish-Chandra isomorphism already contains a germ of the Langlands duality. The point is that we have a canonical isomorphism between $\mathfrak{h}^*$ and the Cartan subalgebra ${}^L\mathfrak{h}$ of ${}^L\mathfrak{g}$. It is more insightful to express this isomorphism as $Z(\mathfrak{g}) \simeq \mathrm{Fun}({}^L\mathfrak{h})^W$, but it is difficult to convince oneself that this is how we should view it if we remain squarely within the finite-dimensional context because $\mathfrak{h}$ and ${}^L\mathfrak{h}$ are so close to each other (they are canonically isomorphic up to an overall scalar).

However, this phenomenon becomes much more clear after affinization (which we can think of as introducing an additional parameter into the picture). Indeed, the affine analogue of the Harish-Chandra isomorphism is the isomorphism of [36, 43] between the center $Z_{\mathrm{crit}}(\widehat{\mathfrak{g}})$ of the enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level and the classical $\mathcal{W}$-algebra $\mathcal{W}_\infty({}^L\mathfrak{g})$, viewed as a subalgebra of the algebra $\mathrm{Fun}(\mathrm{Conn}_{{}^L\mathfrak{h}})$ of functions on the space of connections on a certain ${}^LH$-bundle on the punctured disc. As explained in [43], the algebra $\mathrm{Fun}(\mathrm{Conn}_{{}^L\mathfrak{h}})$ can be viewed as an affine analogue of $\mathrm{Fun}({}^L\mathfrak{h})^W$. Furthermore, $\mathcal{W}_\infty({}^L\mathfrak{g})$, and hence $Z_{\mathrm{crit}}(\widehat{\mathfrak{g}})$, can be described as the subalgebra of $\mathrm{Fun}(\mathrm{Conn}_{{}^L\mathfrak{h}})$ consisting of elements invariant under the classical limits of the screening operators (which can be viewed here as affine analogues of the simple reflections from W). The essential point is that, unlike in the finite-dimensional case, $\mathcal{W}_\infty({}^L\mathfrak{g})$ and $\mathcal{W}_\infty(\mathfrak{g})$ are no longer isomorphic to each other (as Poisson algebras) if ${}^L\mathfrak{g} \neq \mathfrak{g}$. Therefore the phenomenon of Langlands duality becomes much more transparent, and in retrospect, it forces us to look at the original Harish-Chandra isomorphism in a new light.

This is what we hope our results on the q-deformed conformal blocks can bring us as well: a sharper manifestation of certain phenomena that would be difficult to see or appreciate in the context of the undeformed quantum geometric Langlands, at $q = 1$ (the same way as the appearance of ${}^L\mathfrak{h}$ in the Harish-Chandra isomorphism would be difficult to appreciate). Understanding such phenomena for $q \neq 1$ could then shine a new light on what was considered as well-known or well-understood in quantum geometric Langlands.

While we do not claim that we fully understand it yet, we consider the canonical isomorphism of q-deformed conformal blocks conjectured in this paper (and proved in the simply-laced case) as a significant phenomenon in the framework of a conjectural q-deformed quantum Langlands correspondence. We believe that it deserves further study. As far as we know, this is the first attempt to make a precise statement about q-deformed quantum Langlands correspondence (even though its existence had been anticipated; see, e.g., the end of [42]). We hope that more information will come to light in the future that will enable one to formulate the q-deformed quantum Langlands more precisely.

In this section, we give a brief overview of some aspects of the geometric Langlands correspondence and its quantum deformation, and then explain in what sense our isomorphism of q-deformed conformal blocks could be seen as a manifestation of a q-deformation of the quantum geometric Langlands correspondence.

6.1. **Overview.** As we mentioned in the introduction, the geometric Langlands correspondence is usually understood today as a conjectural equivalence between certain categories of sheaves on two moduli stacks related to a smooth projective algebraic curve $\mathcal{C}$ and a pair of connected Langlands dual complex reductive Lie groups G and ${}^LG$. One is the derived category of $\mathcal{D}$-modules on the moduli stack $\mathrm{Bun}_{{}^LG}$ of ${}^LG$-bundles on $\mathcal{C}$ and the other is a certain modification of the derived category of $\mathcal{O}$-modules on the moduli stack Loc_G of flat G-bundles on $\mathcal{C}$ (see [12] for a precise formulation; in the abelian case this equivalence is a version of the Fourier–Mukai transform that has been proved in [71, 103]).[10]

In [65], Kapustin and Witten have connected this equivalence to the homological mirror symmetry of sigma models with the Hitchin moduli spaces of G and ${}^LG$ as target manifolds and to the S-duality of maximally supersymmetric 4d gauge theories with the gauge groups being the compact forms of G and ${}^LG$.

This equivalence is expected to satisfy various properties; in particular, the compatibility with certain functors acting on the two categories: the Hecke functors on the ${}^LG$ side and the "Wilson functors" on the G side (they are connected to the 't Hooft and Wilson line operators of the 4d gauge theory [65]).

In [17], Beilinson and Drinfeld constructed an important part of the geometric Langlands correspondence in which on the G-side one takes the subcategory of $\mathcal{O}$-modules supported on a substack of G-opers in Loc_G. In the case that G is a simple Lie group of adjoint-type (i.e., with the trivial center), to which we restrict ourselves in this subsection, Op_G is an affine space that is isomorphic to the space of all flat connections on a specific G-bundle on $\mathcal{C}$ [17].

In their construction, Beilinson and Drinfeld used the description of the center of the vertex algebra of $\widehat{{}^L\mathfrak{g}}$ at the critical level given in [36]. Namely, it was proved in [36] (see also [43] for a survey) that the center of the completed enveloping algebra of $\widehat{{}^L\mathfrak{g}}$ at the critical level ${}^Lk = -{}^Lh^\vee$ is isomorphic (as a Poisson algebra) to the classical $\mathcal{W}$-algebra associated to $\mathfrak{g}$. The latter is, by definition, the algebra of functions on the space of G-opers on the punctured disc, and the Poisson structure on it is obtained via the (classical) Drinfeld–Sokolov reduction. Equivalently, the center of the vertex (or chiral) algebra of $\widehat{{}^L\mathfrak{g}}$ at the critical level is isomorphic to the commutative (Poisson) vertex algebra $\mathcal{W}_\infty(\mathfrak{g})$.

This isomorphism enabled Beilinson and Drinfeld to construct a family of critically twisted $\mathcal{D}$-modules on $\mathrm{Bun}_{{}^LG}$ parametrized by those conformal blocks of $\mathcal{W}_\infty(\mathfrak{g})$ on X that are algebra homomorphisms. Furthermore, the Beilinson–Drinfeld construction can be placed in the framework of 2d CFT, even though the Kac–Moody chiral algebra at the critical level is quite unusual (it is missing the stress tensor $T(z)$ because the quadratic Sugawara current becomes central in this case). See Section 9 of [44] for more details.

6.2. **Global quantum Langlands correspondence.** As soon as it became clear that there is a link between the Beilinson–Drinfeld construction of the geometric Langlands correspondence and 2d CFT at the critical level, a natural question arose: is it possible to deform the geometric Langlands correspondence away from the critical level? The first conjectural formulation was proposed by Beilinson and Drinfeld themselves (see [110]): the global quantum geometric Langlands correspondence should be an equivalence of suitably modified derived categories of twisted $\mathcal{D}$-modules on Bun_G and $\mathrm{Bun}_{{}^LG}$, provided that the corresponding twist parameters, which can be identified with the levels ${}^Lk$ and k, satisfy the relation (6.4) below. There is a precise sense in which the k-twisted $\mathcal{D}$-modules can be identified with $\mathcal{O}$-modules on Loc_G in the limit $k \to \infty$ (see, e.g., Section 6.3 of [44]), and it is in this sense that the $k \to \infty$ limit of this equivalence is expected to yield the categorical Langlands correspondence of the previous subsection (that is, for ${}^Lk = -{}^Lh^\vee$). A closely related equivalence (of certain categories of A-branes) was also suggested in the framework of the 4d gauge theory picture in [64, 65].

On the other hand, it is natural to try to develop "quantum geometric Langlands" within the framework of 2d CFT, as a deformation of the Beilinson–Drinfeld construction at the critical level.

[10] Note that our notation for G and ${}^LG$ is opposite to the standard one.

One immediate complication for doing this is that while the chiral algebra $V_{-{}^Lh^\vee}(\widehat{{}^L\mathfrak{g}})$ of $\widehat{{}^L\mathfrak{g}}$ of level $-{}^Lh^\vee$ deforms to the chiral algebra $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ of $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$, only the part of the center of $V_{-{}^Lh^\vee}(\widehat{{}^L\mathfrak{g}})$ generated by the quadratic Sugawara operators can be deformed. The center itself cannot be deformed inside $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ if $\mathfrak{g} \neq sl_2$. Luckily, there is another definition of the center that can be deformed: namely, the definition via the quantum Drinfeld–Sokolov reduction.

The quantum Drinfeld–Sokolov reduction [36] (see [19, 35] for earlier works and [46, Ch. 15] for a survey) is defined by introducing a BRST complex which is the tensor product of $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ and the free fermion vertex algebra built on the Clifford algebra generated by ${}^L\mathfrak{n}(\!(z)\!) \oplus {}^L\mathfrak{n}^*(\!(z)\!)dz$. Mathematically, it is the complex of Feigin's semi-infinite cohomology of the Lie algebra ${}^L\mathfrak{n}(\!(z)\!)$ with coefficients in $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ tensored with a non-degenerate (Whittaker-like) character. It turns out that this cohomology is non-zero only in cohomological degree 0, and the cohomological degree 0 part is a vertex algebra called the (quantum) $\mathcal{W}$-algebra associated to ${}^L\mathfrak{g}$ and level ${}^Lk$ (see [36, 46]). (This is one of two known definitions of this $\mathcal{W}$-algebra; the other definition, as the intersection of kernels of the screening operators, is equivalent to it, as explained in [36, 46].)

The notation used for this algebra in [36, 46] is $\mathcal{W}_{{}^Lk}({}^L\mathfrak{g})$, but here we will use the notation $\mathcal{W}_{{}^L\beta}({}^L\mathfrak{g})$, where ${}^L\beta = m \cdot {}^L(k + h^\vee)$ (m being the lacing number of ${}^L\mathfrak{g}$ and $\mathfrak{g}$). In particular, in our notation $\mathcal{W}_\infty({}^L\mathfrak{g})$ is the classical $\mathcal{W}$-algebra associated to ${}^L\mathfrak{g}$ (viewed as a commutative vertex Poisson algebra).

It turns out that if ${}^Lk = -{}^Lh^\vee$, then the corresponding $\mathcal{W}$-algebra $\mathcal{W}_0({}^L\mathfrak{g})$ also becomes commutative and is in fact isomorphic to the center of $V_{-{}^Lh^\vee}(\widehat{{}^L\mathfrak{g}})$. More precisely, the natural embedding of the center (placed in cohomological degree 0) into the above BRST complex induces an isomorphism of the cohomologies. This is proved in [36] (see also [46, Ch. 15]). Thus, we obtain an alternative description of the center at the critical level as the (commutative) vertex algebra $\mathcal{W}_0({}^L\mathfrak{g})$. This description makes it clear how to deform this vertex algebra: we simply take $\mathcal{W}_{{}^L\beta}({}^L\mathfrak{g})$.

Now recall the isomorphism of [36] between the center of $V_{-{}^Lh^\vee}(\widehat{{}^L\mathfrak{g}})$ and the classical $\mathcal{W}$-algebra associated to $\mathfrak{g}$. In our current notation, it takes the form

$$\mathcal{W}_0({}^L\mathfrak{g}) \simeq \mathcal{W}_\infty(\mathfrak{g}). \tag{6.1}$$

It turns out that this isomorphism has a one-parameter deformation [36] as follows:

Theorem 6.1. *For arbitrary complex parameters β and ${}^L\beta$ satisfying the relation*

$$\beta = \frac{m}{{}^L\beta}, \tag{6.2}$$

there is an isomorphism of vertex algebra

$$\mathcal{W}_{{}^L\beta}({}^L\mathfrak{g}) \simeq \mathcal{W}_\beta(\mathfrak{g}), \tag{6.3}$$

whose limit as $\beta \to \infty$ is the isomorphism (6.1).

Proof. In [36, Proposition 5], the isomorphism (6.3) was proved for generic values of β and ${}^L\beta$ satisfying (6.2). Note that relation (6.2) is equivalent to

$$m(k + h^\vee) = \frac{1}{{}^L(k + h^\vee)}. \tag{6.4}$$

Furthermore, in [36] the precise definition of the limit $\beta \to \infty$ was given so that (6.3) becomes (6.1) in this limit.

The isomorphism for arbitrary β and ${}^L\beta$ satisfying (6.2) follows easily from the results of [37]. Namely, according to Theorem 4.6.9 of [37],[11] both $\mathcal{W}_\beta(\mathfrak{g})$ and $\mathcal{W}_{{}^L\beta}({}^L\mathfrak{g})$, with the parameters $\beta, {}^L\beta$ satisfying (6.2), can be embedded as vertex subalgebras of the Heisenberg vertex algebra associated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, so that their graded characters are independent of β. Since the two images coincide for generic β, they coincide for all β. $\square$

In the next subsection, as a small aside, we express Theorem 6.1 in a slightly more satisfying way, as an isomorphism over the ring of Laurent polynomials $\mathbb{C}[\beta^{\pm 1}] = \mathbb{C}[({}^L\beta)^{\pm 1}]$.

6.3. **Oneness.** One possible point of view on the isomorphism of Theorem 6.1 is that there are two families of $\mathcal{W}$-algebras: for $\mathfrak{g}$ and for ${}^L\mathfrak{g}$, and there is an isomorphism between them if we reverse the parameter: $\beta \mapsto m/\beta$. However, a more fruitful point of view might be that there is only one $\mathcal{W}$-algebra, but we can look at it from two different points of view: as being associated to $\mathfrak{g}$ or to ${}^L\mathfrak{g}$. Accordingly, this quantum $\mathcal{W}$-algebra has two classical limits corresponding to these two points of view. In other words, there is one quantum $\mathcal{W}$-algebra, but it can be perceived as the quantization of two different vertex Poisson algebras.

This can be made more precise by exhibiting this "unified" $\mathcal{W}$-algebra as a free $\mathbb{C}[\beta^{\pm 1}]$-module which contains inside a $\mathbb{C}[\beta^{-1}]$-lattice and a $\mathbb{C}[\beta]$-lattice, the former "hailing" from $\mathfrak{g}$ and the latter from ${}^L\mathfrak{g}$ (in other words, β^{-1} is the quantization parameter from the point of view of $\mathfrak{g}$, and β is the quantization parameter from the point of view of ${}^L\mathfrak{g}$).

According to Theorem 4.6.9 of [37], $\mathcal{W}_\beta(\mathfrak{g})$ is freely generated by $\ell = \mathrm{rank}(\mathfrak{g})$ generators $W_1, \ldots, W_\ell(z)$ such that the degree (or conformal dimension) of W_i is $d_i + 1$, where d_i is the ith exponent of $\mathfrak{g}$. For non-zero β, the first of these generators, W_1, generates the Virasoro algebra, and each of the remaining generators $W_i, i = 2, \ldots, \ell$, can be chosen so that it is a highest weight vector of this Virasoro algebra.

The Heisenberg vertex algebra is, as a vector space, the Fock representation π_0 of the Heisenberg Lie algebra with the generators $b^i_n, i = 1, \ldots, \ell; n \in \mathbb{Z}$, which we normalize by the requirement that they satisfy the relations

$$[b^i_n, b^j_m] = \beta^{-1}(\alpha_i, \alpha_j) n \delta_{n,-m}.$$

Consider π_0 as a free $\mathbb{C}[\beta^{-1}]$-module with the basis of monomials in $b^i_n, i = 1, \ldots, \ell; n < 0$, applied to the vacuum vector. This is a vertex algebra over $\mathbb{C}[\beta^{-1}]$. It follows from the proof of Theorem 4.6.9 of [37] that each W_i can be normalized in such a way that $W_i = W^{(0)} + \beta^{-1}(\ldots)$, where $W^{(0)}_i$ is a polynomial in $b^i_{-1}, i = 1, \ldots, \ell$, invariant under the action of the Weyl group. Furthermore, $W^{(0)}_i, i = 1, \ldots, \ell$, is a set of generators of the ring of Weyl group invariant polynomials in b^i_{-1} (in fact, using the conformal dimension $\mathbb{Z}$-grading on π_0, we find also that $W^{(0)}_i$ is the symbol of W_i with respect to the standard PBW filtration on π_0; note also that the numbers $d_i + 1$ are precisely the degrees of the generators of the ring of Weyl group invariant polynomials).

According to Theorem 4.6.9 of [37], the lexicographically ordered monomials in the creation operators corresponding to the $W_i, i = 1, \ldots, \ell$, applied to the vacuum vector in π_0, span a free $\mathbb{C}[\beta^{-1}]$-submodule and a vertex subalgebra of π_0. This is $\mathcal{W}_\beta(\mathfrak{g})$, viewed as a vertex algebra over $\mathbb{C}[\beta^{-1}]$.

On the other hand, let ${}^LW_i = \beta^{(d_i+1)} W_i$. Then, applying the same argument, but from the point of view of ${}^L\mathfrak{g}$ and $({}^L\beta)^{-1} = \beta/m$ and using the ${}^LW_i$'s, we construct a free

[11] To avoid confusion, we note that the parameter β we use here is equal to m/β_{FF}^2, where β_{FF} is the parameter denoted by β in [37].

$\mathbb{C}[\beta]$-submodule of $\pi_0 \bigotimes_{\mathbb{C}[\beta^{-1}]} \mathbb{C}[\beta^{\pm 1}]$. This is $\mathcal{W}_{m/\beta}({}^L\mathfrak{g})$, viewed as a vertex algebra over $\mathbb{C}[\beta]$.

Finally, tensoring both of these vertex algebras with $\mathbb{C}[\beta^{\pm 1}]$, we obtain the promised "unified" $\mathcal{W}$-algebra (of $\mathfrak{g}$ *and* ${}^L\mathfrak{g}$), which contains $\mathcal{W}_\beta(\mathfrak{g})$ and $\mathcal{W}_{m/\beta}({}^L\mathfrak{g})$ as a $\mathbb{C}[\beta^{-1}]$- and a $\mathbb{C}[\beta]$-lattice, respectively. The two classical limits, $\mathcal{W}_\infty(\mathfrak{g})$ and $\mathcal{W}_\infty({}^L\mathfrak{g})$, are defined using these two lattices (as quotients by the maximal ideal in $\mathbb{C}[\beta^{-1}]$ and $\mathbb{C}[\beta]$, respectively). They are commutative vertex Poisson algebras.

6.4. **Conformal blocks and quantum geometric Langlands.** These results offer a particular interpretation of the quantum geometric Langlands correspondence in the language of 2d CFT. Recall from [44] (see also the discussion at the end of Subsection 6.1) that the fibers of the D-modules $\mathrm{Bun}_{{}^LG}$ constructed by Beilinson and Drinfeld [17] can be identified with the duals of the spaces of conformal blocks of $V_{-{}^Lh^\vee}(\widehat{{}^L\mathfrak{g}})$, which can in turn be described in terms of certain conformal blocks of the (commutative) classical $\mathcal{W}$-algebra $\mathcal{W}_\infty(\mathfrak{g})$ (namely, those conformal blocks that are algebra homomorphisms $\mathcal{W}_\infty(\mathfrak{g}) \to \mathbb{C}$).

Motivated by this observation, we propose that one of the manifestations of quantum Langlands correspondence is an *isomorphism between conformal blocks of representations from certain categories of representations of the two vertex* (*or chiral*) *algebras: the affine Kac–Moody vertex algebra* $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ *and the* $\mathcal{W}$*-algebra* $\mathcal{W}_\beta(\mathfrak{g})$, provided that β and ${}^Lk$ are generic or rational with $\beta < 0$ and satisfy the relation equivalent to (6.2) and (6.4):

$$\beta = \frac{1}{{}^L(k + h^\vee)}. \tag{6.5}$$

We stress that this is not the only manifestation of the quantum Langlands correspondence, but it is one that fits well with the isomorphism of q-deformed conformal blocks established in this paper (see Subsection 6.6 for a brief discussion of the links with other approaches).

Here, and in what follows, "generic" means a complex number that is not rational. However, we expect that most of our results and conjectures below also hold for those rational ${}^Lk$ that are less than $-{}^Lh^\vee$ (which is equivalent to $\beta < 0$ under the relation (6.5)). We will refer to such ${}^Lk$ as negative rational.

Let us define precisely the two categories of representations mentioned above. Representations of the vertex algebra $V_{{}^Lk}(\widehat{{}^L\mathfrak{g}})$ are the same as representations of the affine Lie algebra $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$ satisfying a finiteness condition: every vector is annihilated by the Lie subalgebra $z^N \cdot {}^L\mathfrak{g}[[z]]$ for sufficiently large N. We denote the category of such representations by $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$-mod. Let $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$-mod^0 be the category of those representations of $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$ on which the action of the Lie subalgebra ${}^L\mathfrak{g}[[z]]$ can be exponentiated to the corresponding Lie group ${}^LG[[z]]$. This is the same as a full subcategory of the usual category $\mathcal{O}$ of $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$ whose objects are the representations whose restriction to the constant Lie subalgebra ${}^L\mathfrak{g}$ decomposes as a direct sum of finite-dimensional representations.

We will assume that ${}^Lk$ is generic or negative rational. Then simple objects of this category are labeled by dominant integral weights $\lambda \in {}^LP^+$ of ${}^L\mathfrak{g}$. The simple object $L_{\lambda,{}^Lk}$ corresponding to $\lambda \in {}^LP^+$ is the unique irreducible quotient of the Weyl module over $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$ induced from the irreducible representation of ${}^L\mathfrak{g}$ with highest weight λ (note that for any dominant integral weight, $L_{\lambda,{}^Lk}$ is the Weyl module itself if ${}^Lk$ is not a rational number). The category $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$-mod^0 can be defined as the full subcategory of $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$-mod whose objects are representations with irreducible subquotients of this form. Imposing an additional property that representations have finitely many irreducible subquotients (i.e., have finite composition series), we obtain the category extensively studied by Kazhdan and Lusztig, who in particular defined the structure of a braided tensor category on it; see [66]. This will be our category of representations of the vertex algebra $V_{^Lk}(\widehat{^L\mathfrak{g}})$.

Note that if ^Lk is generic in the above sense, then the category $\widehat{^L\mathfrak{g}}_{^Lk}$-mod^0 is a semi-simple abelian category that is equivalent to the category of finite-dimensional representations of $^L\mathfrak{g}$.

Next, we define a subcategory of the category of representations of the $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$. Recall that the quantum Drinfeld–Sokolov reduction yields a functor [36, 48] from the category $\widehat{^L\mathfrak{g}}_{^Lk}$-mod to an analogous category of modules over $\mathcal{W}_{^L\beta}(^L\mathfrak{g})$, which is isomorphic $\mathcal{W}_\beta(\mathfrak{g})$ if β satisfies (6.5) [36]. We will henceforth denote the latter category by $\mathcal{W}_\beta(\mathfrak{g})$-mod and the Drinfeld–Sokolov reduction functor by $H^{^L\mathfrak{g}}_{\mathrm{DS}}$. This functor sends a $\widehat{^L\mathfrak{g}}_{^Lk}$-module M to the semi-infinite cohomology of $^L\mathfrak{n}(\!(z)\!)$ with coefficients in M tensored with the same non-degenerate character that was used to define $\mathcal{W}_{^L\beta}(^L\mathfrak{g})$. We denote it by $H^{^L\mathfrak{g}}_{\mathrm{DS}}(M)$.

It follows from the results of Arakawa [9] that for generic k the functor $H^{^L\mathfrak{g}}_{\mathrm{DS}}$ is exact on $\widehat{^L\mathfrak{g}}_{^Lk}$-mod^0 (see also [99]).

Let now $\mathcal{W}_\beta(\mathfrak{g})$-mod^0 be the full subcategory of $\mathcal{W}_\beta(\mathfrak{g})$-mod whose objects have finite composition series with irreducible subquotients being the modules $H^{^L\mathfrak{g}}_{\mathrm{DS}}(L_{\lambda,{^Lk}}), \lambda \in {^LP^+}$. These modules are irreducible and non-zero, according to [10]. This will be our category on the $\mathcal{W}$-algebra side.

There is an alternative way to describe the simple objects of $\mathcal{W}_\beta(\mathfrak{g})$-mod^0, in terms of the quantum Drinfeld–Sokolov reduction of $\widehat{\mathfrak{g}}_k$ rather than $\widehat{^L\mathfrak{g}}_{^Lk}$. Namely, instead of applying the quantum Drinfeld–Sokolov reduction functor $H^{^L\mathfrak{g}}_{\mathrm{DS}}$ to $L_{\lambda,{^Lk}}, \lambda \in {^LP^+}$, we apply the quantum Drinfeld–Sokolov reduction $H^{\mathfrak{g},\lambda}_{\mathrm{DS}}$, with the standard character twisted by the element $\lambda(z) \in H(\!(z)\!)$, to the vacuum module $L_{0,k}$ over $\widehat{\mathfrak{g}}_k$. Using the methods of [36], one can show[12] that for generic k we have

$$H^{\mathfrak{g},\lambda}_{\mathrm{DS}}(L_{0,k}) \simeq H^{^L\mathfrak{g}}_{\mathrm{DS}}(L_{\lambda,{^Lk}}), \tag{6.6}$$

Namely, using the free field realization of the $\mathcal{W}$-algebras along the lines of [36], we can construct the modules on the two sides of (6.6) as zeroth cohomologies of a finite BGG-type resolution and use it to see that their characters coincide. We expect (6.6) to be true for negative rational ^Lk as well.

From the point of view of $\mathcal{W}_\beta(\mathfrak{g})$, these simple modules correspond to the "magnetic" vertex operators, while from the point of view of $\mathcal{W}_{^L\beta}(^L\mathfrak{g})$ they correspond to the "electric" vertex operators (the duality of $\mathcal{W}$-algebras exchanges electric and magnetic vertex operators). We find it convenient to view quantum Drinfeld–Sokolov reduction $H^{^L\mathfrak{g}}_{\mathrm{DS}}$ as a functor from the category $\widehat{^L\mathfrak{g}}_{^Lk}$-mod^0 to the category $\mathcal{W}_\beta(\mathfrak{g})$-mod^0, which sends irreducible modules to irreducible "magnetic" modules (the reason for this will become clear in Subsection 6.5).

We conjecture that this functor is in fact an equivalence of braided tensor categories. This agrees with the fact known in 2d CFT that magnetic vertex operators for the $\mathcal{W}$-algebra and vertex operators corresponding to the Weyl modules over $\widehat{^L\mathfrak{g}}_{^Lk}$ braid as representations of the quantum group $U_{q'}(^Lg)$, where $q' = e^{2\pi i/^L(k+h^\vee)}$.

[12]Note added in proof: This has been proved in T. Arakawa and E. Frenkel, *Quantum Langlands duality of representations of $\mathcal{W}$-algebras.* arXiv:1807.01536.

The fusion tensor product of modules over a given vertex algebra has been defined by Huang and Lepowsky (see [62] and the references therein). Although $\mathcal{W}_\beta(\mathfrak{g})$ does not satisfy the conditions of [62], various results from 2d CFT suggest that the fusion tensor product endows $\mathcal{W}_\beta(\mathfrak{g})$-mod^0 with the structure of a braided tensor category. This leads us to the following.

Conjecture 6.2. *The category* $\mathcal{W}_\beta(\mathfrak{g})-\mathrm{mod}^0$ *is a braided tensor category* (*with respect to the fusion tensor product*) *which is equivalent to the category* $\widehat{^L\mathfrak{g}}_{^Lk}-\mathrm{mod}^0$ *as a braided tensor category if* β *and* ^Lk *satisfy equation* (6.5) *with* β *generic or negative rational.*

A more precise statement is that this should be an equivalence of chiral categories (see [100] for the definition).

We view Conjecture 6.2 as a purely algebraic manifestation of the quantum Langlands correspondence. (A closely related Gaitsgory–Lurie conjecture discussed in Subsection 6.6 below has the algebraically defined category $\widehat{^L\mathfrak{g}}_{^Lk}$-mod^0 on one side and a Whittaker category, which is defined in geometric terms, on the other side; here the role of the Whittaker category is played by an algebraically defined category $\mathcal{W}_\beta(\mathfrak{g})$-mod^0 as well.) It is a local statement, but it implies non-trivial global statements: namely, isomorphisms of the spaces of conformal blocks of representations from the categories $\widehat{^L\mathfrak{g}}_{^Lk}$-mod^0 and $\mathcal{W}_\beta(\mathfrak{g})$-mod^0.

Recall that for any vertex algebra V and a collection of V-modules $M_1, \dots, M_n$ attached to points $p_1, \dots, p_n$ of a smooth projective algebraic curve $\mathcal{C}$, one can define the vector space of conformal blocks $C_V(\mathcal{C}, (p_i), (M_i))$ (see [46, Section 10.1]). In the case of the vertex algebra $V = V_k(\widehat{\mathfrak{g}})$, this is the standard definition of the space of conformal blocks for $\widehat{\mathfrak{g}}$ (see, e.g., [44]).

Suppose that $\mathcal{C} = \mathbb{CP}^1$. If $M_1, \dots, M_n$ are objects of a braided tensor category with respect to a fusion tensor product $\otimes$, then the space $C_V(\mathbb{CP}^1, (p_i), (M_i))$ can be expressed as a Hom of this category:

$$C_V(\mathbb{CP}^1, (p_i), (M_i)) \simeq \mathrm{Hom}_V(M_1 \otimes \dots \otimes M_{n-1}, M_n^\vee).$$

Therefore Conjecture 6.2 implies (at least, for $\mathcal{C} = \mathbb{CP}^1$, and this should be true for all $\mathcal{C}$ if Conjecture 6.2 is true at the level of chiral categories) the following.

Conjecture 6.3. *There are isomorphisms of the spaces of conformal blocks*

$$C_{V_{^Lk}(\widehat{^L\mathfrak{g}})}(\mathcal{C}, (p_i), (L_{\lambda_i, {^Lk}})) \simeq C_{\mathcal{W}_\beta(\mathfrak{g})}(\mathcal{C}, (p_i), (H_{\mathrm{DS}}(L_{\lambda_i, {^Lk}}))) \tag{6.7}$$

provided that the parameters satisfy the conditions of Conjecture 6.2.

In the case of $\mathcal{C} = \mathbb{CP}^1$, then for generic ^Lk the isomorphisms (6.7) can indeed be constructed using the integral representation of the spaces of conformal blocks, as we discuss in Subsections 6.7 and 6.8. This gives us a concrete way to prove Conjectures 6.3, and 6.2, and more general Conjectures 6.5 and 6.4 below.

6.5. **A q-deformation.** At this point, it is natural to ask to what extent it is necessary to invoke the dual Lie algebra in the above conjectures. Indeed, using the duality of $\mathcal{W}$-algebras [36] (see Theorem 6.1), we can replace $\mathcal{W}_\beta(\mathfrak{g})$ by $\mathcal{W}_{^L\beta}(^L\mathfrak{g})$ with $^L\beta = m/\beta$ in Conjectures 6.2, 6.3. So, at first glance it may appear that the above results and conjectures can be accounted for by the Drinfeld–Sokolov reduction alone, and that there is no need to invoke the Langlands dual Lie algebra (in the same way, one would tend to dismiss the appearance of $^L\mathfrak{h}$ in the Harish-Chandra homomorphism, as we explained at the beginning of this section).

However, there are two reasons why Langlands duality is relevant here. First, as we already explained at the beginning of this section, *the isomorphism of conformal blocks*

$\mathcal{W}_\beta(\mathfrak{g})$ *and* $\widehat{{}^L\mathfrak{g}}$ *can be* q*-deformed*, and after the q-deformation the appearance of $\mathfrak{g}$ can't be written off because there is no longer an isomorphism between the deformed $\mathcal{W}$-algebras associated to $\mathfrak{g}$ and ${}^L\mathfrak{g}$ (if ${}^L\mathfrak{g} \neq \mathfrak{g}$). It is really $\mathcal{W}_{q,t}(\mathfrak{g})$ that appears in our isomorphism of q-deformed conformal blocks, and not $\mathcal{W}_{t,q}({}^L\mathfrak{g})$ (unless $\mathfrak{g} = {}^L\mathfrak{g}$). Furthermore, our isomorphism involves the deformations of the magnetic vertex operators over the $\mathcal{W}$-algebra of $\mathfrak{g}$, and these are no longer equal to the deformations of the electric vertex operators of the $\mathcal{W}$-algebra of ${}^L\mathfrak{g}$. Thus, the appearance of the Langlands dual Lie algebras becomes more meaningful after the q-deformation (similarly to how the appearance of the Langlands dual Lie algebra in the Harish-Chandra isomorphism becomes more meaningful after affinization). This suggests that it is fruitful to view the isomorphism between conformal blocks at $q = 1$, and the corresponding equivalences of categories, in the light of Langlands duality as well.

The second reason is that actually Conjectures 6.2 and 6.3 are special cases of more general conjectures corresponding to the generalized dualities $T^N S$ of the group $\mathrm{PSL}_2(\mathbb{Z})$ familiar from 4d gauge theory (the standard Langlands duality corresponds to S, i.e., $N = 0$). But to apply this duality we must first apply the duality S, exchanging $\mathfrak{g}$ and ${}^L\mathfrak{g}$, and then apply T^N (which preserves $\mathfrak{g}$ and ${}^L\mathfrak{g}$). Therefore, if we wish to look at the dualities $T^N S$ with $N \neq 0$, then using the dual Lie algebra is necessary already at $q = 1$.

In fact, and this is a crucial point, the isomorphism of conformal blocks obtained from our canonical isomorphism of q-deformed conformal blocks in the limit $q \to 1$ corresponds not to the standard relation (6.5) but to the relation

$$\beta = \frac{1}{{}^L(k + h^\vee)} + m, \tag{6.8}$$

where m is the lacing number of $\mathfrak{g}$. Indeed, this is the relation we obtain when we take the limit $q \to 1$ in the relation (1.6) between the parameters of the algebras $\mathcal{W}_{t,q}({}^L\mathfrak{g})$ and $U_\hbar(\widehat{{}^L\mathfrak{g}})$ using equation (1.5).

Formula (6.8) differs from formula (6.5) in the shift of β by m. This shift corresponds to applying, in addition to the standard Langlands duality S, the quantum Langlands duality T. Let us recall how the dualities T and S act on the parameters of 4d gauge theory.

The duality S exchanges the gauge groups G and ${}^LG$ (and hence the corresponding Lie algebras) and acts on the 4d gauge theory coupling constant τ as

$$S : \tau \mapsto -1/m\tau.$$

The duality T preserves the gauge group and acts on τ as

$$T : \tau \mapsto \tau + 1$$

(it is well-defined if G is simply-connected, which we will now assume to be the case; in general, only certain powers of T are well-defined). These two dualities generate a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ (see [65]).

The connection to our parameters is as follows:

$$\tau = \beta/m, \qquad {}^L\tau = -{}^L(k + h^\vee).$$

Hence, formula (6.8) is equivalent to

$$\tau = -1/m{}^L\tau + 1 = TS({}^L\tau). \tag{6.9}$$

In order to interpret the relation between conformal blocks corresponding to β and ${}^Lk$ related via formula (6.8), we need a generalization of Conjectures 6.2 and 6.3 in which we replace the relation (6.5) corresponding to the duality S with (6.8) corresponding to TS.

We will consider an even more general relation corresponding to the duality $T^N S$:

$$\beta = \frac{1}{{}^L(k+h^\vee)} + Nm, \qquad N \in \mathbb{Z}, \tag{6.10}$$

and the following conjectures.

Conjecture 6.4. *The categories* $\mathcal{W}_\beta(\mathfrak{g})-\mathrm{mod}^0$ *and* $\widehat{{}^L\mathfrak{g}}_{{}^Lk}-\mathrm{mod}^0$ *are equivalent as braided tensor categories* (*or chiral categories*) *if* β *and* ${}^Lk$ *satisfy equation* (6.10) *with* β *generic or negative rational.*

Conjecture 6.5. *There are isomorphisms* (6.7) *of the spaces of conformal blocks provided that the parameters satisfy the conditions of Conjecture* 6.4.

Now, the isomorphism (6.7) with β and ${}^Lk$ related by formula (6.8) is precisely the $q \to 1$ limit of the canonical isomorphism of q-deformed conformal blocks which we have conjectured in this paper and established in the simply-laced case. It is in this sense that we can view our isomorphism as a manifestation of a q-deformation of the quantum geometric Langlands.

6.6. **Connection with the Gaitsgory–Lurie conjecture.** Conjecture 6.2 is related to a conjecture of Gaitsgory and Lurie (proved by Gaitsgory in [55] for generic parameters; see also [56]) stating an equivalence of two braided tensor categories (or chiral categories in the terminology used in [55, 56, 100]). In our notation, one of them is the above category $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$-mod^0 (which is denoted by $\mathrm{KL}^{\check{c}}_{\check{G}}$ in [56]). The other is the "Whittaker category" denoted by $\mathrm{Whit}^c(\mathrm{Gr}_G)$ in [56].

Combining Conjecture 6.2 with the theorem of [55] (the Gaitsgory–Lurie conjecture for generic c), we obtain the following.

Conjecture 6.6. *The categories* $\mathrm{Whit}^c(\mathrm{Gr}_G)$ *and* $\mathcal{W}_c(\mathfrak{g})$-mod^0 *are equivalent as braided tensor* (*or chiral*) *categories for generic and negative rational* c.

We note that both categories have simple objects labeled by $\lambda \in {}^LP^+$, and they should correspond to each other under this equivalence. There is also a natural functor from $\mathrm{Whit}^c(\mathrm{Gr}_G)$ to $\mathcal{W}_c(\mathfrak{g})$-mod. Indeed, according to the definition given in [56], $\mathrm{Whit}^c(\mathrm{Gr}_G)$ is $\mathrm{Whit}(\mathcal{D}_k(\mathrm{Gr}_G)$-mod), the category of $(\mathfrak{n}(\!(z)\!), \chi)$-equivariant objects in the category of twisted $\mathcal{D}$-modules on the affine Grassmannian Gr_G [56] (here χ is the "Whittaker functional" used in the quantum Drinfeld-Sokolov reduction, and the twisting parameter should be, in our notation, the level k such that $c = m(k + h^\vee)$). The functor of global sections on Gr_G then yields a functor from the latter category to $\mathrm{Whit}(\widehat{\mathfrak{g}}_k$-mod), which is equivalent to the category $\mathcal{W}_c(\mathfrak{g})$-mod according to the results of [99].

Conjecture 4.5 of [56] links the statement of the Gaitsgory–Lurie conjecture to the global quantum Langlands correspondence discussed in Section 6.2 above. Therefore, Conjecture 6.6 provides a link between our Conjecture 6.2 and the global quantum Langlands correspondence.

6.7. **Integral representation of conformal blocks.** Conjecture 6.3 in genus zero can be tested using the integral formulas for the conformal blocks of affine Kac–Moody algebras obtained by Schechtman and Varchenko [105] (as solutions of the KZ equations). These formulas can also be obtained using the free field (Wakimoto) realization of $\widehat{{}^L\mathfrak{g}}$; see [14, 32, 39]. In this section we compare these formulas to the integral formulas for conformal blocks of $\mathcal{W}$-algebras. In gives us a concrete interpretation of the limit of our isomorphism of q-deformed conformal blocks as $q \to 1$.

Our notation for the conformal blocks will be similar to the notation we used for the q-deformed conformal blocks. Namely, we have a vertex operator $\Phi_{{}^L\rho_i}(a_i)$ corresponding to a finite-dimensional representation ${}^L\rho_i$ of ${}^L\mathfrak{g}$ of dominant integral highest weight $\lambda_i \in {}^LP^+$ inserted at the point $a_i \in \mathbb{CP}^1$, for $i = 1, \ldots, n$. Then conformal blocks may be viewed as (multivalued) functions of the a_i with values in a weight space in the tensor product $\bigotimes_i {}^L\rho_i$. This weight is given by the same formula as (1.8) (here we use a slightly different notation; in particular, we denote the simple roots of ${}^L\mathfrak{g}$ by α_i):

$$\gamma = \sum_{i=1}^{n} \lambda_i - \sum_{j=1}^{r} \alpha_{i_j}. \tag{6.11}$$

In the integral representation, these functions are written as integrals, over a suitable integration cycle Γ (discussed below), in the space

$$(\mathbb{C}\backslash\{a_1, \ldots, a_n\})^r \backslash \operatorname{diag}$$

with coordinates $x_1, \ldots, x_r$, of a function that is a product of two factors:

(1) The first factor is the multivalued function, denoted by $\mathcal{I}(x, a)$, which is the product of factors of three types:

$$(a_i - a_j)^{(\lambda_i,\lambda_j)/{}^L(k+h^\vee)},$$

$$(a_i - x_j)^{-(\lambda_i,\alpha_{i_j})/{}^L(k+h^\vee)}, \qquad (x_j - x_p)^{(\alpha_{i_j},\alpha_{i_p})/{}^L(k+h^\vee)} \tag{6.12}$$

(here we use the inner product normalized as in Section 2.1).

(2) The second factor is a rational function $|x_1^{i_1} \ldots x_r^{i_r}\rangle$ in the a_i and x_j with values in $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$; it can be realized as a conformal block of the bosonic $\beta\gamma$-system involved in the free field realization of $\widehat{{}^L\mathfrak{g}}$ (see Theorem 4 of [39] as well as [14]).

The product of the factors appearing in equation (6.12) defines a rank one local system $\mathcal{L}$ on $(\mathbb{C}\backslash\{a_1, \ldots, a_n\})^r \backslash \operatorname{diag}$. For the integral to be well-defined, the integration cycle Γ should be viewed as an element of the rth homology group of

$$(\mathbb{C}\backslash\{a_1, \ldots, a_n\})^r \backslash \operatorname{diag}$$

with coefficients in the dual local system $\mathcal{L}^*$.

It is known that for generic ${}^Lk$ the resulting integrals

$$\int_\Gamma \mathcal{I}(x, a)\, |x_1^{i_1} \ldots x_r^{i_r}\rangle\, dx_1 \ldots dx_r \tag{6.13}$$

(with varying Γ) span the subspace of highest weight vectors of the weight space $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$ with respect to the diagonal action of ${}^L\mathfrak{g}$. This may seem puzzling because if we only had vertex operators $\Phi_{{}^L\rho_i}(a_i), i = 1, \ldots, n$, in our set-up, then the space of conformal blocks would have been isomorphic to the subspace of ${}^L\mathfrak{g}$-invariant vectors in $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$. The explanation is that we have "cheated" a bit because to make this calculation work we actually need to insert a vertex operator at the point $\infty \in \mathbb{CP}^1$ with the lowest weight $-\gamma$ (this is explained in [39]). When we take this into account, the corresponding space of ${}^L\mathfrak{g}$-invariant vectors gets identified with the space of highest weight vectors in $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$. (Note that the measure in (6.13) is different from the rest of the paper because in this section x is a coordinate on the complex plane rather than a cylinder. To connect the formulas in this section to the formulas elsewhere, one should use the change of variables $x_{\text{cyl}} = e^{Rx_{\text{plane}}}$ and take R to zero.)

In fact, it follows from the results of Varchenko [121] that for generic ${}^Lk$ the above homology space can be identified with the space of highest weight vectors in $\left(\bigotimes_i {}^L\rho_i^{q'}\right)_\gamma$ where $\rho_i^{q'}$ is the representation with the same highest weight λ_i but over the quantum group $U_{q'}({}^L\mathfrak{g})$ with $q' = e^{2\pi i/{}^L(k+h^\vee)}$. As explained in [121], these integral formulas may therefore be thought of as providing a non-degenerate pairing between these spaces of highest weight vectors, one for the Lie algebra ${}^L\mathfrak{g}$ and one for the quantum group $U_{q'}({}^L\mathfrak{g})$. (As shown in [121], the fact that these are solutions of the KZ equations can be used to derive the Kohno–Drinfeld theorem identifying the R-matrices of $U_{q'}({}^L\mathfrak{g})$ with the "half-monodromies" of solutions of the KZ equations corresponding to exchanging the points a_i and a_j. See also [106] and the closely related work by Bezrukavnikov, Finkelberg, and Schechtman [20].)

It is possible to modify the construction slightly to obtain the entire weight space $\left(\bigotimes_i {}^L\rho_i^{q'}\right)_\gamma$ (rather than its subspace of highest weight vectors). For that, we also insert a vertex operator at the point $0 \in \mathbb{CP}^1$ as well as a vertex operator at the point ∞ (we assume that $a_i \neq 0$ for all $i = 1, \dots, n$). If the highest weight of the former is λ and the lowest weight of the latter is $-\lambda'$ so that $\lambda - \lambda' = \gamma$, then we can identify our conformal blocks with the matrix elements

$$\langle\lambda'| \prod_{i=1}^{n} \Phi_{{}^L\rho_i}(a_i)\, |\lambda\rangle \tag{6.14}$$

as in formula (1.7) (note that here we switch the points 0 and ∞ compared to Section 2.1; our λ, λ' are therefore $\lambda_\infty, \lambda_0$ of formula (1.7)).

The advantage is that we now get solutions that span the entire weight space $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$. Indeed, if λ is chosen to be generic, then the space of highest weight vectors in the tensor product of the Verma module with the highest weight λ and $\bigotimes_i {}^L\rho_i$ can be identified with $\bigotimes_i {}^L\rho_i$. The disadvantage, however, is that we have to modify formula (6.13) by inserting an additional factor, which is a product of powers of the x_j – this factor comes from the "interaction" of the vertex operator at the point 0 and the screening operators. The resulting formula for the conformal blocks reads

$$\int_\Gamma \prod_{j=1}^{r} x_j^{-(\lambda,\alpha_{i_j})/{}^L(k+h^\vee)}\, \mathcal{I}(x,a)\, |x_1^{i_1} \dots x_r^{i_r}\rangle\, dx_1 \dots dx_r. \tag{6.15}$$

Accordingly, Γ is now a cycle in the rth homology of

$$(\mathbb{C}\backslash\{0, a_1, \dots, a_n\})^r \backslash \operatorname{diag}$$

with coefficients in the dual local system of the rank one local system obtained by modifying $\mathcal{L}$ to include the monodromies around 0 specified by the extra factor in (6.15). This homology space is, according the results of [121], isomorphic to the weight space $\left(\bigotimes_i {}^L\rho_i^{q'}\right)_\gamma$.

Now let us discuss conformal blocks of the $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$, where β is related to ${}^Lk$ by formula (6.5). We now insert at the points a_i vertex operators of $\mathcal{W}_\beta(\mathfrak{g})$ corresponding to the representations $H_{\mathrm{DS}}(L_{\lambda_i, {}^Lk})$. In the free field realization of $\mathcal{W}_\beta(\mathfrak{g})$, these vertex operators are given by the standard bosonic vertex operators. However, in the same way as in the Kac–Moody case, we can insert integrals of the screening currents which commute with the $\mathcal{W}$-algebra.

As in the case of the deformed $\mathcal{W}$-algebra, there are two sets of screening currents: the "electric" and "magnetic" ones (see Section 8.6 of [44]). They are the conformal limits of the screening currents $S_a(x)$ and $S_a^\vee$ corresponding to the roots and coroots of $\mathfrak{g}$, respectively (see Section 2.2). However, since we only consider the insertions of the vertex operators corresponding to the representations $H_{\mathrm{DS}}(L_{\lambda_i, {}^Lk})$, where each λ_i is a dominant integral coweight of $\mathfrak{g}$ (equivalently, weight of ${}^L\mathfrak{g}$), only the magnetic screening currents corresponding to the coroots of $\mathfrak{g}$ (equivalently, roots of ${}^L\mathfrak{g}$) appear in the formulas for conformal blocks.

The resulting formula for the conformal blocks, which are the $q \to 1$ limits of the deformed blocks given by (1.9), is

$$\int_\Gamma \mathcal{I}(x,a) \ \ dx_1 \dots dx_r \tag{6.16}$$

if we do not include a vertex operator at the point 0, and

$$\int_\Gamma \prod_{j=1}^r x_j^{-(\mu,\alpha_{i_j})} \ \mathcal{I}(x,a) \, dx_1 \dots dx_r \tag{6.17}$$

if we do. We include the $\mathcal{W}$-algebra vertex operator at 0 with momentum μ. It is natural to use the definition of the momentum corresponding to $\mathfrak{g}$ rather than ${}^L\mathfrak{g}$; for this reason μ does not get rescaled by β in (6.17). The powers of the x_j are the same in (6.17) and (6.15), if we let $\mu = \lambda/{}^L(k+h^\vee)$.

The difference between formulas (6.13) and (6.15) on one side, and (6.16) and (6.17) is that the former take values in $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$ whereas the latter are scalar-valued functions. But what matters is that they are parametrized by the integration cycles Γ which belong to the same homology space. In both case (with or without a vertex operator at 0) the spaces of conformal blocks for $\widehat{{}^L\mathfrak{g}}$ and for $\mathcal{W}_\beta(\mathfrak{g})$ are therefore identified with the *same* homology space.

This enables us to identify the two spaces of conformal blocks, in effect proving Conjecture 6.3 for $\mathcal{C}$ of genus zero and generic values of β and ${}^Lk$ satisfying the relation (6.5).

However, it would be desirable to identify the integral formulas more directly. By that we mean finding a linear functional (covector) $\langle W|$ on $\left(\bigotimes_i {}^L\rho_i\right)_\gamma$ so that pairing it with the Kac–Moody conformal block (6.15) we would get the conformal block (6.17) of the $\mathcal{W}$-algebra. Morally, $\langle W|$ should be a "Whittaker-like" functional (which makes sense since the $\mathcal{W}$-algebra is obtained from the affine Kac–Moody algebra via the quantum Drinfeld-Sokolov reduction that uses a Whittaker functional).

However, by inspecting formulas (6.15) and (6.17) we can see that such a covector $\langle W|$ does *not* exist. Indeed, for the formulas to match, we need to have

$$\langle W|x_1^{i_1} \dots x_r^{i_r}\rangle = 1,$$

where $|x_1^{i_1} \dots x_r^{i_r}\rangle$ is the vector appearing in formula (6.15), but $\langle W|$ should not depend on the integration variables x_j. Explicit formula for $|x_1^{i_1} \dots x_r^{i_r}\rangle$ (see, e.g., Theorem 4 of [39]) shows that it goes to 0 if we take all of the x_j to ∞. Therefore the covector $\langle W|$ satisfying the above formula does not exist.

The results of this paper show nonetheless that such a covector does exist for the generic q-deformation of conformal blocks subject to the relation (1.6) (and it is indeed something like a Whittaker functional as it represents the identity in the equivariant K-theory of the corresponding quiver variety). However, the $q \to 1$ limit of this relation is not the standard relation (6.5) but rather the relation (6.8) in which β is shifted by m. We have conjectured in Conjecture 6.5 that there is an isomorphism of conformal blocks in this case, and even more general case of relation (6.10), in which β is shifted by Nm.

Let us discuss this shift in the framework of the above integral formulas. Recall that the inner product $(\cdot,\cdot)$ on the dual space to the Cartan subalgebra of ${}^L\mathfrak{g}$ is normalized in such a way that the long roots have square norm 2, and so the short roots have square norm $2/m$ (here m denotes the lacing number of $\mathfrak{g}$ and ${}^L\mathfrak{g}$, as before). Given that all the λ_i are dominant integral weights of ${}^L\mathfrak{g}$, we see that the rank one local system defined by the multivalued function $\mathcal{I}$ appearing in the above integral formulas does not change if we shift β by an integer multiple of m. Therefore we find that the corresponding homology groups remain the same, in agreement with Conjecture 6.5.

However, the case $N = 1$ (relation (6.8)) turns out to be special. In this case, we obtain a direct identification of the integral formulas for the conformal blocks using a covector $\langle W|$.

In the next subsection, we will give some explicit examples of this covector.

6.8. **Explicit identification of conformal blocks.** Let us discuss a concrete example of the identification of conformal blocks of $\widehat{{}^L\mathfrak{g}}$ and $\mathcal{W}_\beta(\mathfrak{g})$, with the parameters satisfying the relation (6.8), in the case of ${}^L\mathfrak{g} = sl_2$. To simplify our notation, we will denote ${}^Lk$ by k in this subsection.

First, suppose there are two points on the complex plane, a_1 and a_2, and we insert at each of them the vertex operator of $\widehat{sl_2}$ corresponding to the two-dimensional representation $\mathbb{C}^2$, in which we choose a basis $\{v, fv\}$, with v a highest weight vector and f the standard generator of sl_2. We also put a vertex operator at ∞, but for now we will not put a vertex operator at the point 0. We will choose the one at ∞ to be of lowest weight 0, so that the resulting conformal blocks take values in the subspace of weight 0 in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This subspace is two-dimensional, with a basis $\{v \otimes fv, fv \otimes v\}$.

Since we are not putting anything at 0, the space of conformal blocks is one-dimensional, and can be identified with the space of highest weight vectors of $\mathbb{C}^2 \otimes \mathbb{C}^2$ – but viewed as a representation of $U_{q'}(sl_2)$ (with $q' = e^{2\pi i/(k+2)}$) rather than sl_2.

As we discussed in the previous subsection, this is a general phenomenon: $\widehat{{}^L\mathfrak{g}}$-conformal blocks *take values* in the subspace of highest weight vectors in the tensor product of representations of ${}^L\mathfrak{g}$, but the space of conformal blocks itself is isomorphic to the space of the *cycles of integration* that can be identified [121] with the same subspace in the tensor product of the finite-dimensional representations of the same highest weights, but taken over the corresponding quantum group $U_{q'}({}^L\mathfrak{g})$.

In the case at hand, the integral solution (6.13) of the KZ equations is given by the formula

$$(a_1 - a_2)^{\theta/2} \int_\Gamma \left(\frac{fv \otimes v}{x - a_1} + \frac{v \otimes fv}{x - a_2} \right) (x - a_1)^{-\theta} (x - a_2)^{-\theta} \, dx, \tag{6.18}$$

where

$$\theta = \frac{1}{k+2}.$$

There are two things to note:

(1) This solution takes values in the subspace of the weight 0 subspace, spanned by the vector

$$fv \otimes v - v \otimes vf,$$

which is precisely the subspace of highest weight vectors of weight 0 in $\mathbb{C}^2 \otimes \mathbb{C}^2$, as expected (see the discussion in the previous subsection). This follows from the formula

$$\int_\Gamma \left(\frac{1}{x - a_1} + \frac{1}{x - a_2} \right) (x - a_1)^{-\theta} (x - a_2)^{-\theta} \, dx$$
$$= -\frac{1}{\theta} \int_\Gamma d\left((x - a_1)^{-\theta} (x - a_2)^{-\theta} \right) = 0.$$

(2) For generic k, the first twisted homology of $(\mathbb{C} \backslash \{a_1, a_2\})^2 \backslash \operatorname{diag}$ with coefficients in the rank one local system appearing in formula (6.18) is one-dimensional. There is a unique (up to a scalar) cycle of integration Γ, which generates this homology group.

Indeed, note that in this case the monodromies around a_1 and a_2 are the same: $e^{-2\pi i \theta}$. The cycle Γ can be chosen as follows: starting at some point z and going counterclockwise around a_1, coming back to z and then going clockwise around a_2 and returning to z. When we apply the differential of the standard twisted homology complex to this contour, the first of the two contours gives the point z multiplied by $(1 - e^{-2\pi i\theta})$, and the second one gives minus the same expression, so they cancel each other. As explained in [121] in the general case, the action of the differential can be identified with the action of the generator e of $U_{q'}(sl_2)$. In this case, it is the action on the weight 0 subspace of $\mathbb{C}^2_{q'} \otimes \mathbb{C}^2_{q'}$, where $\mathbb{C}^2_{q'}$ is the two-dimensional irreducible representation of $U_{q'}(sl_2)$ (and in general, with the action of a sum of the generators e_i of the quantum group, acting from the given weight space to the weight spaces corresponding to the shift of the weight by α_i). This is why one can identify the homology group with the space of highest weight vectors.

Now, let's see whether we can get a conformal block for the Virasoro algebra $\mathcal{W}_\beta(\mathfrak{g})$ by pairing the above solution with a covector $\langle W|$. Set

$$\langle W| = (a_1 - a_2)^{-1/2}((fv \otimes v)^* - (v \otimes fv)^*).$$

Applying this functional to the conformal block (6.18) and using formula

$$\frac{1}{x - a_1} - \frac{1}{x - a_2} = \frac{a_1 - a_2}{(x - a_1)(x - a_2)},$$

we get

$$(a_1 - a_2)^{1/2(\theta+1)} \int_\Gamma (x - a_1)^{-(\theta+1)}(x - a_2)^{-(\theta+1)}\, dx,$$

which is a conformal block of the Virasoro algebra with the parameter

$$\beta = \theta + 1 = \frac{1}{k+2} + 1.$$

Here we recognize the shift of β by 1, as in formula (6.8).

Let us now insert a vertex operator at the point 0 with generic (non-integral) highest weight λ while inserting a vertex operator with lowest weight $-\lambda$ at ∞. Then we again obtain conformal blocks with values in the weight 0 subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$, but now the highest weight condition is dropped. As the result, the formula for the conformal block becomes

$$(a_1 - a_2)^{\theta/2} \int_\Gamma x^{-\lambda\theta} \left(\frac{v \otimes fv}{x - a_2} + \frac{fv \otimes v}{x - a_1} \right) (x - a_1)^{-\theta}(x - a_2)^{-\theta}\, dx.$$

As in the general formula (6.15), there is an extra factor $x^{-\lambda\theta}$.

The cycle Γ is now in the first homology group of $(\mathbb{C}\backslash\{0, a_1, a_2\})^2 \backslash \mathrm{diag}$ which is two-dimensional and can be identified with the weight 0 subspace of $\mathbb{C}^2_{q'} \otimes \mathbb{C}^2_{q'}$. The corresponding integrals span the two-dimensional weight 0 subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

When we take the pairing with $\langle W|$, we obtain

$$(a_1 - a_2)^{(\theta+1)/2} \int_\Gamma x^{-\lambda\theta}(x - a_1)^{-(\theta+1)}(x - a_2)^{-(\theta+1)}\, dx. \tag{6.19}$$

This as a conformal block of the Virasoro algebra with $\beta = \theta + 1$ and momentum $\mu = \lambda\theta$ at the point 0.

Let us generalize the above example to the case of n points $a_1, \ldots, a_n$ with the insertion of the vertex operators corresponding to the two-dimensional representation $\mathbb{C}^2$. We will focus on the case of weight $2n - 2$ subspace, which corresponds to the case of a single screening operator.

The analogue of formula (6.18) is

$$\prod_{i<j} (a_i - a_j)^{\theta/2} \int_\Gamma \sum_{i=1}^n \frac{v \otimes \ldots \otimes \underset{i}{fv} \otimes \ldots \otimes v}{x - a_i} \prod_{i=1}^n (x - a_i)^{-\theta}\, dx. \tag{6.20}$$

The cycle Γ is an element of the first twisted homology group of

$$(\mathbb{C}\backslash\{a_1,\dots,a_n\})^n\backslash\,\mathrm{diag},$$

which is $(n-1)$-dimensional in this case (and can be identified with the space of highest weight vectors in the weight $2n-2$ subspace of $(\mathbb{C}^2_{q'})^{\otimes n}$).

The corresponding covector $\langle W|$ is given by the formula

$$\langle W| = \prod_{i<j}(a_i-a_j)^{1/2}\sum_{i=1}^n \frac{(v\otimes\ldots\otimes \underset{i}{fv}\otimes\ldots\otimes v)^*}{\prod_{j\neq i}(a_i-a_j)}.$$

Taking the pairing of $\langle W|$ and the Kac–Moody conformal block (6.20) and using the formula

$$\sum_{i=1}^n \frac{1}{(w-a_i)\prod_{j\neq i}(a_i-a_j)} = \frac{1}{\prod_{i=1}^n(w-a_i)},$$

we obtain the Virasoro conformal block

$$\prod_{i<j}(a_i-a_j)^{\beta/2}\int_\Gamma \prod_{i=1}^n (x-a_i)^{-\beta}\,dx, \tag{6.21}$$

where $\beta=\theta+1$.

For general highest weights $\lambda_1,\dots,\lambda_n$ and multiple screening operators, the explicit formula for the covector $\langle W|$ becomes increasingly complicated. However, our general results about the identification of the deformed conformal blocks guarantee that such a covector always exists and pairing it with a conformal block for $\widehat{{}^L\mathfrak{g}}$ of level ${}^Lk$, we obtain the corresponding $\mathcal{W}_\beta(\mathfrak{g})$-conformal block provided that the parameters are related by formula (6.8). This yields an explicit identification stated in Conjecture 6.3.

Remark 6.7. In the case $\mathfrak{g}=sl_2$, the Kac–Moody conformal blocks have been connected to the Virasoro conformal blocks, if the parameters are related by the formula $\beta = 1/(k+2)$, by two different changes of variables. In both cases, each representation of sl_2 (the ${}^L\rho_{\lambda_i}$ in the notation of the previous subsection) is realized in the space of polynomials in one variable x_i, viewed as the coordinate on the big cell of the flag manifold of SL_2. In the first approach, these variables x_i are identified with the positions a_i of the vertex operators [52, 96]. In the second approach, the change of variables is obtained by deforming Sklyanin's separation of variables in the SL_2 Gaudin model [47, 111, 118]. In this case, the Fourier dual variables to the x_i appearing on the Kac–Moody side are converted, on the Virasoro side, into positions of additional degenerate fields of type $\Phi_{1,2}$. It is unknown at present how to generalize these changes of variables to the case of arbitrary affine Kac–Moody algebras.

In contrast, here we do not introduce any additional degrees of freedom. Rather, as a consequence of our general results on the identification of the *q-deformed* conformal blocks, we obtain that there exists a covector $\langle W|$ on the tensor product $\bigotimes_i {}^L\rho_{\lambda_i}$ of finite-dimensional representations of ${}^L\mathfrak{g}$, such that when we couple it with the corresponding $\widehat{{}^L\mathfrak{g}}$ Kac–Moody blocks at level ${}^Lk$, we obtain conformal blocks of the $\mathcal{W}$-algebra $\mathcal{W}_\beta(\mathfrak{g})$, if $\beta = 1/{}^L(k+h^\vee)+m$. This provides an explicit identification of the two types of conformal blocks.

7. Quivers from string theory

7.1. **3d quiver gauge theory.** The quiver $\mathcal{Q}$ from Section 1.3 labels a gauge theory in three dimensions with $\mathcal{N}=4$ supersymmetry. The ranks of vector spaces V_a, W_a attached to the ath node of the quiver $\mathcal{Q}$ are the ranks of gauge $G_{\mathcal{Q}}$ and global symmetry groups G_W:

$$G_{\mathcal{Q}} = \prod_a U(d_a), \qquad G_W = \prod_b U(m_a). \tag{7.1}$$

The arrows of the quiver encode the representation in which the matter fields transform. For every pair a, b of nodes connected by a link of the Dynkin diagram we get a hypermultiplet transforming in bifundamental representation $(d_a, \overline{d_b})$ of under $U(d_a) \times U(d_b)$. There are also m_a hypermultiplets in fundamental representation d_a of the $U(d_a)$ gauge group.

7.1.1. The Nakajima quiver variety X is the Higgs branch of the gauge theory. The Kähler parameters of X encode Fayet-Illiopolous (FI) terms in the gauge theory. The equivariant parameters are the real masses, induced by weakly gauging G_W symmetry. Both the FI terms and the real masses get complexified once we compactify the gauge theory on S^1, as we will shortly do.[13] The $\mathbb{C}^\times_\hbar$ action that scales the symplectic form on X comes from a $U(1)$ subgroup of $SU(2)_H \times SU(2)_V$ R-symmetry group.

7.2. **Quiver gauge theory from IIB string.** The quiver gauge theory with quiver $\mathcal{Q}$ arises on D3 branes in IIB string theory compactified on

$$Y \times M_6.$$

Here, Y is an ADE surface, a resolution of $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ singularity, where $\Gamma_{\mathfrak{g}}$ is a discrete group of $SU(2)$ related to $\mathfrak{g}$ by McKay correspondence; $M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$ is the six-manifold in (1.17). The Riemann surface $\mathcal{C}$ is the same one we used to define the q-conformal blocks in Section 1.2.

7.2.1. The ranks of the vector spaces V_a, W_a are determined by the homology classes of 2-cycles in Y that the D3 branes wrap.

Recall the relation of geometry of Y to representation theory of $\mathfrak{g}$: The vanishing cycles of the ADE singularity are topologically S^2's which intersect according to the Dynkin diagram of $\mathfrak{g}$. Denote the vanishing cycles by S_a; their homology classes are the positive simple roots of $\mathfrak{g}$, $e_a = [S_a]$. They span $H_2(Y, \mathbb{Z})$, which can be identified with the root lattice of $\mathfrak{g}$ (with the norm coming from the intersection form on Y). The weight lattice of $\mathfrak{g}$ is the same as the relative homology group $H_2(Y, \partial Y; \mathbb{Z})$. The latter is spanned by a collection of non-compact cycles S^*_a whose homology classes are the fundamental weights, $w_a = [S^*_a]$. (A cycle in the class of S^*_a is the fiber of the cotangent bundle at a generic point on S_a.)

To get the quiver $\mathcal{Q}$, we take a collection of non-compact D3 branes in class $[S^*] \in H_2(Y, \partial Y; \mathbb{Z})$, where

$$[S^*] = \sum_a m_a [S^*_a], \qquad [S] = \sum_a d_a [S_a], \tag{7.2}$$

together with a collection of compact D3 branes in the class $[S] \in H_2(Y, \mathbb{Z})$. In addition to their support in Y, the D3 branes are distributed at a collection of points on $\mathcal{C}$, and on the complex plane in M_6, associated with q as the equivariant parameter.

[13] There are additional parameters needed to define the theory, such as the gauge couplings, which are not relevant for us, as they do not affect the partition function.

7.2.2. The D3 branes on the compact cycles in homology class $[S]$ in (7.2) support $G_{\mathcal{Q}}$ gauge fields in (7.1). The hypermultiplets in $(d_a, \overline{d}_b)$ arise from (zero-modes of) strings at the intersections of cycles in classes $[S_a]$ and $[S_b]$, for $a \neq b$. The intersection number $\#(S_a, S_b) = I_{ab}$ is identified with the incidence matrix I_{ab}. Strings at the intersections between S_a and S^*_a cycles give rise to hypermultiplets in $(d_a, \overline{m}_a)$ representation of $U(d_a) \times U(m_a)$. The flavor symmetry G_W in (7.1) is the gauge group of non-compact D3 branes on $[S^*]$ in (7.2); due to non-compactness, the corresponding gauge fields are frozen.

7.2.3. D3 branes give rise to a 3d gauge theory on $S^1_{R'} \times \mathbb{C}$. The circle $S^1_{R'}$ is not geometric in IIB. It arises due to a stringy effect.

The D3 branes are located at points on $\mathcal{C}$, which is a cylinder $\mathcal{C} = \mathbb{R} \times S^1_R$, with a circle of radius R. Due to strings which wind around the S^1_R, there are many infinitely many particles in the theory on $\mathbb{C}$. They are labeled by the winding modes on S^1_R, which are in turn equivalent to momentum modes on another circle $S^1_{R'}$, with radius $R' = 1/(m_s^2 R)$.

The three-dimensional nature of the theory can be made manifest by T-duality. The duality relates IIB on S^1_R with IIA on $S^1_{R'}$, and D3 branes in IIB at points on S^1_R with D4 branes in IIA wrapping the $S^1_{R'}$; these theories are the same. The winding on S^1_R corresponds to momentum on the $S^1_{R'}$.

7.2.4. The positions of the non-compact D3 branes on $\mathcal{C}$ are the A-equivariant parameters and the complexified real masses: a D3 brane supported at a point $x = a_i$ on $\mathcal{C}$ leads to an equivariant parameter with the same name. This is also an insertion point of a vertex operator in (1.7) and (1.9). The positions of compact D3 branes on $\mathcal{C}$ are dynamical parameters; they are the insertion points of screening charge operators in (1.9). The Kähler moduli of X are identified with the Kähler moduli of Y, as both correspond to FI parameters in the 3d gauge theory. They determine the weights λ in (1.7) and (1.9).

7.3. **Little string theory from IIB string.** The 10d IIB string on $Y \times M_6$ has many more degrees of freedom than we presently need. There is a smaller theory, which captures the physics relevant for us. It is a 6d string theory, "the little string theory" with $(2,0)$ supersymmetry on M_6.

7.3.1. The $\mathfrak{g}$-type little string theory with $(2,0)$ supersymmetry is defined as the limit of IIB string theory on $Y \times M_6$. The limit corresponds to taking the string coupling g_s to zero, keeping fixed the characteristic mass m_s of the IIB string, m_s and the moduli of the 6d $(2,0)$ theory. (The moduli come from periods of five 2-forms in IIB string compactified on $Y \times M_6$, coming from the triplet of self-dual 2-forms of Y and the two B-fields of IIB string, with appropriate normalizations.)

7.3.2. The theory one is left with is a string theory on M_6: it contains strings whose tension is m_s^2, which are inherited from IIB strings. One reflection of the fact one gets a string theory, and not a point particle theory, is that the little string theory has a T-duality symmetry. T-duality relates the $\mathfrak{g}$-type $(2,0)$ little string, compactified on a circle of radius R, with the $\mathfrak{g}$-type $(1,1)$ little string theory on a circle of radius $R' = 1/(m_s^2 R)$. (The latter is obtained from IIA string on Y, in an analogous g_s to zero limit.) The two string theories are equivalent.

7.3.3. The D3 branes of IIB string on Y give rise to codimension four defects of the little string theory on M_6. The theory on the defect D3 branes is the quiver gauge theory with quiver $\mathcal{Q}$. The limit which reduces the 10d IIB string to the 6d little string on M_6 does not affect the gauge theory on D3 branes at all. The triplet of FI parameters of the 3d gauge theory, for example, is given by R times the moduli of the little string, coming from the triplet of self-dual 3-forms on Y. The gauge couplings are R times the modulus originating from the NS B-field. Here, R is the radius of the S^1_R in $\mathcal{C}$. All these remain finite in the limit, since we are keeping both R and the moduli of $(2,0)$ little string fixed as we take g_s to zero. (See [2] for more details.)

7.4. **Non-simply-laced case.** To get non-simply-laced theories, we make use of the fact that every non-simply-laced Lie algebra $\mathfrak{g}$ arises as a subalgebra of a simply-laced Lie algebra $\mathfrak{g}_0$, invariant under the outer automorphism group H of $\mathfrak{g}_0$. Outer automorphisms of $\mathfrak{g}_0$ correspond to automorphisms in its Dynkin diagram.

7.4.1. We start with IIB string on Y_0 an ADE singularity corresponding to $\mathfrak{g}_0$. We take Y_0 to be fibered over M_6 in such a way[14] that, as we go around the origin of the complex plane in M_6 that supports the D3 branes, Y_0 comes back to itself only up to the action of a generator $h \in H$. The action of h on Y_0 is by permuting the 2-cycles classes in $H_2(Y_0, \mathbb{Z})$ in a way compatible with the action of h on the root lattice of $\mathfrak{g}_0$, and the identification of the latter with $H_2(Y_0, \mathbb{Z})$ (to our knowledge, this string construction was first used in [18]).

7.4.2. The automorphism groups H are all abelian, $H = \mathbb{Z}_m$, generated by a single element $h \in H$, with $h^m = 1$. The roots of $\mathfrak{g}$ are the combinations of roots of $\mathfrak{g}_0$ which are invariant under H. This way, from $(\mathfrak{g}_0, H)$ one gets $\mathfrak{g}$ with:

$$
\begin{aligned}
(A_{2n-1}, Z_2) &\to C_n,\\
(D_{n+1}, Z_2) &\to B_n,\\
(D_4, Z_3) &\to G_2,\\
(E_6, Z_2) &\to F_4.
\end{aligned}
\tag{7.3}
$$

The root lattice of $\mathfrak{g}$ is obtained from the root lattice of $\mathfrak{g}_0$ as follows. A simple positive root of $\mathfrak{g}$ is a sum over the simple positive roots of $\mathfrak{g}_0$ which are in a single orbit of H, normalized by the length of the orbit. The short roots of $\mathfrak{g}$ come from the simple roots in $\mathfrak{g}_0$ which lie in orbits of H of length m. The long roots of $\mathfrak{g}$ are the simple roots of $\mathfrak{g}_0$ invariant under H. The length of the root is defined by (e_a, e_a), where $(,)$ comes from the inner product on the root lattice of $\mathfrak{g}_0$. Since all the roots of $\mathfrak{g}_0$ have length 2, the length of a short simple root of $\mathfrak{g}$ is $2/m$, and the length of a long root is 2. The coroots of $\mathfrak{g}$ are related to the roots of $\mathfrak{g}$ in the usual way $e_a^\vee = 2e_a/(e_a, e_a)$. It is easy to show that the result is the Cartan matrix of $\mathfrak{g}$: $C_{ab} = (e_a^\vee, e_b)$.

7.4.3. The action of H on Y_0 translates into the action on D3 branes supported on the 2-cycles in (7.2), and on the quiver $\mathcal{Q}_0$ that describes them. The D3 brane configurations that are allowed in the fibered geometry are in one-to-one correspondence with the configurations of 2-cycles on Y_0 which are invariant under the H action: The D3 branes we are considering are supported on 2-cycles in Y_0 times the complex plane $\mathbb{C} \in M_6$ where the twist is; any H-invariant configuration in Y_0 gives rise to a configuration on the fibered product which comes back to itself up to the h-twist acting simultaneously on M_6 and on Y_0.

[14]In [67], the twist is around the S^1 in $\mathcal{C}$ instead, or more precisely, around its T-dual circle.

7.4.4. For H to leave the quiver $\mathcal{Q}_0$ invariant, the ranks of vector spaces (V_a, W_a) associated to the nodes of the Dynkin diagram of $\mathfrak{g}_0$ which lie in a single orbit of H have to be the same. From this it follows that the non-compact D3 branes, corresponding to W_a's, are labeled by fundamental weights of ${}^L\mathfrak{g}$, the Lie algebra Langlands dual to $\mathfrak{g}$. Similarly, the compact D3 branes, corresponding to V_a's, are labeled with the simple roots of ${}^L\mathfrak{g}$.

To see this, one recalls that the simple roots and the fundamental weights of ${}^L\mathfrak{g}$ coincide with the simple coroots and fundamental coweights of $\mathfrak{g}$, respectively. The latter are, in turn, simply the sums of the fundamental coweights and the simple coroots of $\mathfrak{g}_0$ lying in a single orbit of H. These are exactly the data labeling the H-invariant quivers $\mathcal{Q}_0$. (For the former statement, one merely needs to recall the relation of the root lattices of $\mathfrak{g}$ and $\mathfrak{g}_0$, and the definitions of the coroots and coweights. The coweight lattice is the lattice dual to the root lattice.)

7.4.5. The fields of the quiver gauge theory on the D3 branes are a subset of those of the original $\mathcal{Q}_0$ theory which are compatible with folding by H. Let z be the complex coordinate on the $\mathbb{C}$-plane that supports the D3 branes, and $\phi(z)$ a field of the $\mathcal{Q}_0$ quiver gauge theory. The fields must obey

$$\phi(e^{2\pi i}z) = h \cdot \phi(z), \tag{7.4}$$

where $h \cdot \phi$ denotes the image of ϕ under the h action on the quiver. The latter action is trivial for fields that only involve the long roots, corresponding to nodes of the Dynkin diagram of $\mathfrak{g}_0$ which are invariant under H. For fields ϕ that involve the short roots, coming from fields which transform in orbits of H of length m, the H action organizes $\phi(z), h\cdot\phi(z), \ldots, h^{m-1}\cdot\phi(z)$ into a single field (equal to their sum), which is single-valued only on the m-fold cover of the $\mathbb{C}$-plane. If w is a coordinate on the cover, $z = w^m$, fields coming from orbits of H of length m have integer mode expansion in terms of $w = z^{1/m}$, but fractional mode expansion in terms of z.

7.4.6. Langlands duality exchanges $\mathfrak{g}$ and ${}^L\mathfrak{g}$, and roots and coroots, while transposing the Cartan matrix. Since some define the Cartan matrix to be the transpose of ours, it is easy to mix-up $\mathfrak{g}$ and ${}^L\mathfrak{g}$. An unambiguous way to distinguish them is by the lengths of their roots. While the norm of the inner product $(,)$ is a matter of convention, the ratio of the lengths of the roots is not. For example, B_n has one short root, and $(n-1)$ long ones, while C_n has $(n-1)$ short roots, and one long one. (B_n and C_n are exchanged under Langlands duality, while F_4 and G_2 map to themselves.)

7.5. **Conformal limit.** The $(2,0)$ little string is a string theory, containing strings whose characteristic size is $1/m_s$. It becomes a point particle theory, the conformal field theory in 6d with $(2,0)$ supersymmetry in the limit where one sends m_s infinity,

$$m_s \to \infty.$$

We will call this theory theory $\mathcal{X}$, for short. In the conformal limit, we want to keep the moduli of the $(2,0)$ theory fixed, since they become the moduli of theory $\mathcal{X}$. We also want to keep fixed the Riemann surface it is compactified on, and the positions $x = a_i$ of D-branes on it.

In the conformal limit, the gauge theory description of the defects is lost. The inverse gauge coupling of the defect 3d quiver theory is given by the modulus of the $(2,0)$ theory (which has dimensions of mass square) times $1/m_s^4$. Thus, in the m_s to infinity limit, the gauge coupling becomes infinite. This means that there is no sense in which we can describe the theory on the defects as a gauge theory.

8. Vertex functions from physics

The vertex function of Nakajima quiver variety has two closely related, but distinct physics interpretations. Most directly, they are partition functions of 3d quiver gauge theory from Section 7 with quiver $\mathcal{Q}$, computed on $\mathbb{C}\times S^1_{R'}$. The gauge theory interpretation lets one make direct contact with vertex functions both in their defining formulation, in terms of counting quasimaps $\mathbb{C} \dashrightarrow X$, and in the integral form of Section 3.

The more far-reaching interpretation, however, is that they are also the partition functions of $\mathfrak{g}$-type $(2,0)$ little string theory on M_6, with codimension four defects, where the quiver $\mathcal{Q}$ captures data of the defect. This explains why vertex functions have implications for Langlands duality. We will return to this in Section 9.

The two interpretations are related: the partition function of little string theory we need, turns out to *equal* the partition function of the theory of its the defects. We will define the relevant partition functions, explain the mechanism between the equality of the bulk and the defect partition functions, and show how results of Section 3. emerge from the 3d gauge theory perspective.

8.1. **Little string partition function.** The partition function of the $\mathfrak{g}$-type $(2,0)$ little string on M_6 is most easily defined in the T-dual language, using T-duality with respect to the circle in $\mathcal{C}$.

The dual of the $(2,0)$ string theory on M_6 is the $(1,1)$ little string on

$$M_6' = \mathcal{C}' \times \mathbb{C} \times \mathbb{C},$$

where $\mathcal{C}' = S^1_{R'} \times \mathbb{R}$. The $(1,1)$ string theory is, at low energies, a 6d gauge theory with maximal supersymmetry, and gauge group based on the Lie algebra $\mathfrak{g}$. Its partition function on M_6' is a supersymmetric index

$$\mathsf{Index} = \mathrm{Tr}(-1)^F \mathbf{g}. \tag{8.1}$$

The trace is the trace going around the $S^1_{R'}$; F is the fermion number so $(-1)^F$ counts bosons and fermions with signs. The insertion of $\mathfrak{g}$ in the trace has the effect of turning M_6' into a twisted product: as we go around the $S^1_{R'}$, we rotate the two complex planes $\mathbb{C}\times\mathbb{C}$ by q and t^{-1}, respectively. This is known as the Ω-background, defined by Nekrasov and studied, e.g., in [75, 88, 92] and in many other papers.

8.1.1. Explicitly, $\mathbf{g}$ is the product of generators

$$\mathbf{g} = q^{S-S_H} \times t^{S_H - S_V}. \tag{8.2}$$

We denoted by S the generator of the rotation of the $\mathbb{C}$-plane in M_6' which is rotated by q. S_V generates the action that rotates the second $\mathbb{C}$-plane by t^{-1}. S_H is the generator of the $U(1)$ subgroup of $SU(2)$ R-symmetry group of the 6d theory. The R-symmetry twist is needed for the partition function to preserve supersymmetry.

8.2. **Localization to defects.** In the absence of defects, the partition function in (8.1) is trivial. In the presence of defects, it equals the partition function of the theory on the defects. One simply ends up computing (8.1), restricted to the modes on the defect.

8.2.1. Without any defects on M_6', the insertion of $\mathbf{g}$ in (8.1) ends up commuting with four of the sixteen supercharges of the 6d theory. This is too many for the index to receive non-trivial contributions: the supersymmetries end up relating bosons and fermions in pairs and their contributions to the index cancel out. To get a non-trivial partition function one must reduce the supersymmetries by a half. We will achieve this by adding defect D-branes.[15]

[15] The relation of bulk and defect perspective is described in more detail in [2–4, 7].

With defects present, supersymmetry is broken, but only near the defects [97]. Away from the defects, local physics is that of the $(1,1)$ little string, compactified on a circle, with all of its supersymmetries intact. This leads to localization: the only non-trivial contributions to the partition function can come from modes supported on the defects. Computing the trace restricted to such modes is the same as computing the partition function of the theory on the defect. (The notion of localization used here is in its essence the same mechanism as in the more familiar applications of the term. The defect is fixed by a linear combination of the supersymmetries in the bulk. See [123] for more explanation.)

8.2.2. The defects we will use are the D3 brane configurations in Section 7. The quiver $\mathcal{Q}$ which encodes the data of the defects, as in previous section, also encodes the 3d quiver gauge theory on the defects. T-duality maps D3 branes at points on $\mathcal{C}$ in M_6 to D4 branes winding around the $S^1_{R'}$ in $\mathcal{C}' = \mathbb{R} \times S^1_{R'}$, and at the same points in the radial direction. The position of D3 branes on S^1_R becomes the holonomies of the D4 brane gauge fields around $S^1_{R'}$. T-duality makes it manifest that the gauge theory on these D branes is a 3d theory on $S^1_{R'} \times C$, where C is identified with the complex plane in M_6' supporting the defect; this is the copy of $\mathbb{C}$ which is rotated by q.

(In addition to D3 brane defects, there are other kinds of defects which lead to the same localization effect. Adding D5 brane defects at points in $\mathcal{C}$ and filling $\mathbb{C} \times \mathbb{C}$, for example, will lead to Langlands correspondence with ramifications.)

8.3. **Defect partition function.** The index (8.1), computed in the 3d quiver gauge theory on the defect, becomes the supertrace over the Hilbert space of the theory on $\mathbb{C}$. The trace is around the $S^1_{R'}$ as before. The identification of $\mathbb{C}$ with the complex plane in M_6' supporting the defect, determines the action of all the generators of $\mathbf{g}$ in the 3d gauge theory.

8.3.1. From the 3d gauge theory perspective, the interpretation of various factors in $\mathbf{g}$ is as follows. Let $\hbar = q/t$. Then, (8.2) becomes

$$\mathbf{g} = q^S \times \hbar^{-S_H} \times t^{-S_V}.$$

S generates rotation of C, the copy of $\mathbb{C}$ that supports the defect. This is a geometric action from both the bulk and the defect perspective. S_V acts as a rotation of a complex plane transverse to the defect. It becomes an R symmetry generator in the gauge theory. It corresponds to the $U(1)$ subgroup of $SU(2)_V$ R-symmetry that acts on scalars in vector multiplets. (A complex scalar in the vector multiplet is the position of the D branes on $\mathbb{C}_t^{-1}$ plane.) S_H generates the $U(1)$ subgroup of $SU(2)_H$ R-symmetry group acting on hypermultiplet scalars; it generates an $R-$symmetry both in the bulk and on the defect.

There are factors in $\mathbf{g}$ we have refrained from writing out explicitly, to keep the formulas simpler. The remaining part of parameters come from global $U(1)$ symmetries of the 3d $\mathcal{N} = 4$ gauge theory. They enter $\mathbf{g}$ as the (complexified) holonomies of the corresponding gauge fields around the $S^1_{R'}$. They are associated with the

$$\mathsf{T} \times \mathsf{A}^\vee \times \mathbb{C}^\times_q, \qquad \mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_\hbar$$

symmetry of the theory. The symmetries in T are associated to real mass parameters; $\mathsf{A}^\vee$ are associated to the real FI parameters. (The parameters in A preserve $\mathcal{N} = 4$ supersymmetry, those in T but not in A break it to $\mathcal{N} = 2$.)

8.4. **Index for non-simply-laced $\mathfrak{g}$.** In non-simply-laced cases, there is an H-twist around the complex plane that supports the defect. The trace in (8.1) is the trace over states invariant under H (they correspond to fields obeying (7.4)). The generator S of rotations of the plane supporting the defects now has eigenvalues that are integer, and half integer, multiples of $1/m$, where m is the order of H. This is because some of the modes come back to themselves only upon going around the circle m or $2m$ times; see Section 7.4.5. We prefer that only integer and half integer powers of q appear in the partition function; to achieve this we will replace q by q^m, and define $\mathbf{g}$ in (8.2) as:

$$\mathbf{g} = q^{mS} \times \hbar^{-S_H} \times t^{-S_V}, \tag{8.3}$$

where now

$$\hbar = q^m/t \tag{8.4}$$

for the index to preserve supersymmetry. (The action of S, S_H, and S_V on the supersymmetry generators is independent of global identifications we make, so it is not sensitive to folding by H.) This is the string origin of the identification of parameters in (1.6).

8.5. **Vertex functions from 3d gauge theory.** The index in (8.1), computed in the 3d $\mathcal{N}=4$ gauge theory on $C \times S^1$ based on the quiver $\mathcal{Q}$, is the vertex function $\mathbf{V}$ of X from (3.26). Then

$$\mathsf{Index} = \mathrm{Tr}(-1)^F \mathbf{g} = \mathbf{V}. \tag{8.5}$$

The Index is not a function – it is a vector instead, because it is defined in the 3d gauge theory on $S^1 \times C$, and thus depends on the choice of the vacuum of the gauge theory at infinity in $C = \mathbb{C}$. We will show momentarily that the vector space it takes values in can be identified with $K_{\mathsf{T}}(X)$.

For a non-simply-laced Lie algebra $\mathfrak{g}$, the meaning of the Index is different. It is the vertex function $\mathbf{V}^H$ of X_0, restricted to H-invariant modes.

The relation between the partition function of 3d gauge theory on $\mathbb{C} \times S^1$ and vertex functions of quantum K-theory of its Higgs branch are well known [84, 89]. The integral representation of vertex functions, which we proved in Section 3 are also known in the physics literature; see for example [15]. We will briefly review the physics perspective on these.

8.5.1. The 6d little string Hilbert space effectively localizes to the Hilbert space of the 3d gauge theory on $C = \mathbb{C}$, but even that is much larger than the space of configurations that end up contributing to (8.1). The index receives contributions only from configurations that are annihilated by the pair of supersymmetry generators $\overline{Q}$, $\overline{Q}^\dagger$, which anti-commute with $(-1)^F \mathbf{g}$; all others come in pairs related by actions of these generators, and cancel out from the index. The field configurations which preserve the supersymmetries are "quasimaps" from C to X. The quasimaps are simply the solutions to vortex equations on C [84, 124]. In the adiabatic approximation, the supersymmetric path integral of the 3d theory on $R \times C$ (with R viewed as time direction) localizes to the supersymmetric quantum mechanics on the moduli space $\mathcal{M} = \mathsf{QM}_{\mathrm{nonsing}}(X)$ of quasimaps to X; see [84, 89]. The quasimaps are non-singular at infinity of C: this corresponds to working with boundary conditions which require the gauge field strength to vanish there. In addition, finite energy configurations require one to restrict the matter fields to approach a vacuum at infinity. In a theory deformed by masses, i.e., working equivariantly with respect to T, the latter corresponds to a fixed point of T-action on X.

In supersymmetric quantum mechanics with a pair of supercharges, the partition function $\mathrm{Tr}(-1)^F$ computes the index of the Dirac operator on $\mathcal{M}$. In the present case, the supersymmetric quantum mechanics have twice as many supersymmetries: there are in fact two more supercharges $Q, Q^\dagger$ that annihilate the solutions in $\mathcal{M}$, they just fail to commute with $\mathbf{g}$ for generic $\hbar$. The supercharges Q, $Q^\dagger$ and $\overline{Q}$, $\overline{Q}^\dagger$ are identified with Dolbeault operators $\partial, \partial^\dagger, \overline{\partial}, \overline{\partial}^\dagger$ acting on differential forms on $\mathcal{M}$. The index of $\not{D} = \overline{\partial} + \overline{\partial}^\dagger$ operator on $\mathcal{M}$ is the holomorphic Euler characteristic of the symmetrized virtual structure sheaf $\hat{\mathcal{O}}_{\mathrm{vir}}$ of $\mathcal{M}$ in (3.8); see [93] and also [89]. The $\hat{\mathcal{O}}_{\mathrm{vir}}$ bundle is the cohomology of the complex generated by the broken supersymmetries $Q \sim \partial$ and $Q^\dagger \sim \partial^\dagger$ acting on differential forms on $\mathcal{M}$, obtained by quantizing the collective coordinates of fermions. The Kähler variables of X come from the (complexified) real FI parameters in the 3d gauge theory; they lead to grading of quasimap moduli space by the degree.

In practice, we like to think about indices as functions of their parameters, so we want to extract a particular component of the vector $\mathbf{V}$. This corresponds to picking a specific vacuum state at infinity. The vacua lie on the T-fixed locus in X; if fixed points $p \in X_{\mathsf{T}}$ are isolated, it suffices to restrict $\mathcal{M}$ to the moduli space of maps $\mathcal{M}_p$ approaching $p \in X_{\mathsf{T}}$ at infinity. In that case, $K_{\mathsf{T}}(X)$ is spanned by classes of fixed points $\mathcal{O}_p$. A class in $K_T(X)$ labels the choice of a vacuum state even in more general situations. (More naturally, the supersymmetric vacua are ground states of effective supersymmetric quantum mechanics which arise in studying the 3d gauge theory on $R \times T^2$, with T^2 of complex structure parameter q, with equivariant/mass deformations turned on corresponding to parameters in T. In this setting, the ground states should be labeled by elements of $\mathrm{Ell}_{\mathsf{T}}(X)$, the equivariant elliptic cohomology of X. For Nakajima varieties, the ranks of $\mathrm{Ell}_{\mathsf{T}}(X)$ and $K_{\mathsf{T}}(X)$ turn out to be the same, so we will use the latter to label the vacua.)

8.5.2. The second way to compute (8.1), which leads to integral formulas, is simpler in many respects.

Since a q^S factor in $\mathfrak{g}$ regularizes the non-compactness of C, one can treat the 3d gauge theory on $S^1 \times C$ as (gauged) supersymmetric quantum mechanics on the S^1, with discrete spectrum. The computation becomes an elementary exercise in quantum mechanics (see [84] for more detail): enumerating the fields in the 3d theory, decomposing each field into modes on C of fixed momentum, and evaluating their contribution to the trace. For non-simply-laced Lie algebras one includes in the trace only the H-invariant configurations, obeying (7.4).

It is easiest to start by treating $G_{\mathcal{Q}}$ as a global symmetry; gauging it corresponds to projecting to $G_{\mathcal{Q}}$ invariant states, which one can do in the end. In addition, it is helpful to abelianize the theory, breaking the gauge group $G_{\mathcal{Q}}$ to its maximal abelian subgroup. Then, at the outset, the partition function depends on equivariant parameters associated with a maximal torus of $G_{\mathcal{Q}}$. These we denoted by x's elsewhere (and by s in the appendix and in Section 3) since they come from positions of compact D3 branes on $\mathcal{C}$. They are also (part of) the Coulomb branch moduli of the 3d gauge theory, so this computes the partition function from the Coulomb-branch perspective. The index in (8.1) depends on Kähler moduli of X via the classical FI terms in the Lagrangian. In the end, since $G_{\mathcal{Q}}$ is gauged one integrates over the x's. The contour is chosen to project to states which are neutral. This means integrating over

$$\int_{|x|=1} \ldots \, d_{\mathrm{Haar}} x \tag{8.6}$$

as in Appendix A and Section 3, where $d_{\mathrm{Haar}} x = \prod_{a,\alpha} dx_{a,\alpha}/x_{a,\alpha}$ and the contour is chosen to pick out contributions independent of x's. Depending on the values of FI parameters, one gets to deform the contour, picking up the residues in the process. This is the GIT quotient from Section 3.

The contribution to (8.1) of vector multiplets from the ath node of the Dynkin diagram of $\mathbf{g}$, is

$$\prod_{\alpha\neq\alpha'} \frac{\varphi_{q_a}(x_{\alpha,a}/x_{\alpha',a})}{\varphi_{q_a}(t\,x_{\alpha,a}/x_{\alpha',a})} \prod_{\alpha<\alpha'} \frac{\theta_{q_a}(tx_{\alpha,a}/x_{\alpha',a})}{\theta_{q_a}(\,x_{\alpha,a}/x_{\alpha',a})}; \tag{8.7}$$

see [84] for derivation. The one new aspect is the dependence, in non-simply-laced cases, on whether "a" labels a short, or a long root. Recall that a node corresponding to a short root of $\mathfrak{g}$ collects contributions m nodes of $\mathfrak{g}_0$ which are in a single orbit of H. The corresponding field configurations come back to themselves only after going around the origin of the $\mathbb{C}$-plane m times. By contrast, a node corresponding to a long root of $\mathfrak{g}$ comes from a node of $\mathfrak{g}_0$ which comes back to itself going around once. Since q keeps track of the minimum momentum on the disc, so that only (half-)integer powers of q enter the partition function, then for "a" a short root $q_a = q$, and for a long root, $q_a = q^m$. This coincides with the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra contributions from screening currents associated to a single node in (2.22). Similarly, hypermultiplets connecting a pair of distinct nodes a, b in the Dynkin diagram of $\mathfrak{g}$ contribute:

$$\prod_{\alpha,\beta} \frac{\varphi_{q_{ab}}(tv_{ab}\,x_{\alpha,a}/x_{\beta,b})}{\varphi_{q_{ab}}(v_{ab}\,x_{\alpha,a}/x_{\beta,b})} \tag{8.8}$$

where $v_{ab} = \sqrt{q_{ab}/t}$. If either of the nodes a, b is short, $q_{ab} = q$ since then the fields that contribute are single-valued only on the m-fold cover of the disc. If both of the nodes are long, then $q_{ab} = q^m$. This coincides with the two-point functions of screening currents associated to the distinct pair of nodes a, b, in (2.23). Finally, for each node of the $\mathfrak{g}$ Dynkin diagram, the charged fields in fundamental representation contribute

$$\prod_{i,\alpha} \frac{\varphi_{q_a}(tv_a\,a_{i,a}/x_{\alpha,a})}{\varphi_{q_a}(v_a\,a_{i,a}/x_{\alpha,a})} \tag{8.9}$$

where i runs from 1 to $\mathrm{rk}(W_a)$, and $q_a = q$ for short roots, and $q_a = q^m$ for the long roots, and $v_a = \sqrt{q_a/t}$. This coincides with the two-point function, from (2.24), of screening currents and vertex operators associated to this node.

We have yet to pick a specific vacuum at infinity. In simple cases, this can be done by changing the contour of integration (to replace the contour in (8.6) by an inequivalent one, that approaches thimble integrals in $q \to 1$ limit). This is not the most convenient way to do that, since in general cases construction of such contours becomes difficult. Instead, it is better to keep the contour of integration fixed to (8.6) and instead realize the choice of vacua as additional insertions in the integral. They arise as follows.

We treat $C = \mathbb{C}$ as a finite disc (since nothing in the computation depends on the area of C), with boundary. To reproduce the vertex function $\mathbf{V}$, we need to impose Dirichlet boundary conditions on the gauge fields, and place the conditions on the matter fields to localize them to a component of X_T at the boundary. Instead of imposing boundary conditions by hand, we couple the 3d theory to a 2d theory at the boundary, and integrate over all the fields with no restrictions (for examples, see [24]). Due to couplings in (8.1) the boundary theory has only $(0, 2)$ supersymmetry. The contribution of the elliptic genus of the boundary theory to the partition function leads to an additional insertion of (3.31)

$$\mathscr{F}(x)/\theta(T^{1/2}) = \mathscr{F}'(x),$$

in the integral; see (3.32). The condition, from (3.31), that $\mathscr{F}'(x)$ is invariant under $x \mapsto q^{\mathbf{d}}x$ says that the boundary theory has no gauge anomalies. While there are many different theories that can be coupled consistently (any anomaly free $(0, 2)$ theory would do), there is a finite-dimensional space of distinct non-trivial contributions they could give rise to, parametrized by classes in $\mathrm{Ell}_{\mathrm{T}}(X)$.

8.5.3. The vertex functions $\mathbf{V}$ lead solutions of qKZ which are holomorphic in z, per construction. We get a second basis of solutions to the same equation, which we denoted $\mathbf{V}_{\mathfrak{C}}$, which are holomorphic in a chamber $\mathfrak{C}$ of mass/equivariant parameter space, corresponding the choice of ordering of defects on $\mathcal{C}$. The vertex functions $\mathbf{V}_{\mathfrak{C}}$ and $\mathbf{V}$ solve the same set of difference equations in equivariant (and Kähler variables) since they originate from the same 3d gauge theory on $C \times S^1$. Correspondingly, the matrix $\mathfrak{P}_{\mathfrak{C}}$ relating them

$$\mathbf{V}_{\mathfrak{C}} = \mathbf{V}\,\mathfrak{P}_{\mathfrak{C}} \tag{8.10}$$

is a matrix of pseudo-constants. Theorem 4 of [5], gives the matrix elements of $\mathfrak{P}_{\mathfrak{C}}$ in terms of elliptic stable envelopes of X.

The change of basis in (8.10) corresponds to imposing different conditions on the fields of the 3d theory at the $\partial(C \times S^1) = T^2$ boundary. We can in principle impose boundary conditions leading to $\mathbf{V}_{\mathfrak{C}}$ in the same way as we did for $\mathbf{V}$, by coupling the 3d theory to a 2d theory on the boundary. This time the coupling, among other things, has an effect of imposing Neumann boundary conditions on the gauge fields. Having picked the chamber $\mathfrak{C}$, the stable basis leading to a-solutions of qKZ in this chamber is unique [5]. Here, we will only sketch some salient features of its construction.

To obtain a component of the covector $\mathbf{V}_{\mathfrak{C}}$, one starts by picking a component of the A-fixed point set $X_A \subset X$. The boundary conditions on matter fields parametrizing directions transverse to the fixed locus are either Neumann or Dirichlet boundary conditions depending on whether they correspond to attracting or repelling directions; this depends on $\mathfrak{C}$. The rest of the boundary theory is determined by cancellation of gauge anomalies. More precisely, the choices left to make are parametrized by equivariant elliptic cohomology classes of the corresponding fixed point locus. The elliptic genus of the boundary theory leads to a contribution to the integral which now takes the form

$$\mathscr{F}(x)/\theta(T^{1/2}) \to \mathrm{Stab}^{\mathrm{ell}}_{\mathfrak{C}}(x,z)\,\mathbf{e}(z)^{-1}/\theta(T^{1/2})\,. \tag{8.11}$$

(See Sec. 6.3. of [5] for a more precise statement.) Here $\mathrm{Stab}^{\mathrm{ell}}$ are elements of the elliptic stable basis, which assign, to every class in $\mathrm{Ell}_{\mathsf{T}}(X_{\mathsf{A}})$ a class in $\mathrm{Ell}_{\mathsf{T}}(X)$,

$$\mathrm{Stab}^{\mathrm{ell}}_{\mathfrak{C}}(X): \qquad \mathrm{Ell}_{\mathsf{T}}(X_{\mathsf{A}}) \longrightarrow \mathrm{Ell}_{\mathsf{T}}(X).$$

Per definition, the right hand side is invariant under $x \mapsto q^{\mathbf{d}}x$: the automorphy of the elliptic genus of the boundary theory cancels the bulk contribution coming from $\mathbf{e}(z) = \exp\left(\frac{\boldsymbol{\lambda}(z,x)}{\ln q}\right)$. This reflects the contribution of boundary degrees of freedom to the anomaly which cancels the anomaly the bulk theory has, in the presence of T^2 boundary. Since the right hand side is constant under $x \mapsto q^{\mathbf{d}}x$ in computing the integral by residues,

$$\mathfrak{P}_{\mathfrak{C}}(x) = \mathrm{Stab}^{\mathrm{ell}}_{\mathfrak{C}}(x)\,\mathbf{e}(z,x)^{-1}/\theta(T^{1/2}) \tag{8.12}$$

acts like a matrix of constants, so $\mathbf{V}_{\mathfrak{C}}$ is related to $\mathbf{V}$ by a linear operator $\mathfrak{P}_{\mathfrak{C}}$ in (8.10), obtained by evaluating (8.12) on classical vacua.

The matrix $\mathfrak{P}_{\mathfrak{C}}$ in (8.10) itself has a gauge theory interpretation. The partition function of the 3d gauge theory on $I \times T^2$ with Neumann-type boundary conditions that lead to a-solutions are imposed on one end of I, and those for Dirichlet-type z-solutions on the other. The supersymmetric partition function does not depend on the size of the interval, and shrinking it to zero, one gets an effective 2d gauge theory on T^2 with $(0,2)$ supersymmetry. The entries of the matrix $\mathfrak{P}_{\mathfrak{C}}$ are elliptic genera of the resulting theories.

8.5.4. Vertex functions with descendants correspond to placing line operators at $0 \subset \mathbb{C}$, winding around the S^1. The line operators one needs can be constructed geometrically as well; see [6], in terms of the K-theoretic stable basis Stab^K. The later is a $q \to 0$ limit of the elliptic stable basis. One can make use of this fact to obtain its gauge theory construction: first cut open a neighborhood of $0 \subset \mathbb{C}$, and impose the boundary conditions corresponding to elliptic stable basis. Then, shrinking the boundary back to a point has the same effect as taking q to zero. The elliptic genus of the boundary theory becomes a line operator insertion – this is the supersymmetric partition function on S^1 of the resulting quantum mechanics problem. Inserting the line operator, in the integral (8.6) takes $\mathbf{V}$ and $\mathbf{V}_{\mathfrak{C}}$ to fundamental z- and a-solutions of qKZ.

8.6. **Conformal limit.** The variables $q, t, \hbar$ are related to the parameters of the Ω-background as

$$q = \exp(R'\epsilon_q), \qquad t = \exp(R'\epsilon_t), \qquad \hbar = \exp(R'\epsilon_\hbar). \tag{8.13}$$

In the conformal limit, point particle limit, we send $m_s \to \infty$ and we keep ϵ's fixed, since they are part of the definition of the background the $(2,0)$ theory is compactified on. For the same reason, we keep the Riemann surface $\mathcal{C}$ fixed. This means the radius R of the circle in $\mathcal{C}$ must stay fixed, and hence the T-dual radius

$$R' = 1/(m_s^2 R) \to 0$$

goes to zero in the limit. Since R' goes to zero, with ϵ's fixed, we recover (2.6) and (2.25).

The positions of the points on the Riemann surface are fixed as well, but the z's have to scale to 1 to keep the moduli of the $(2,0)$ theory fixed in the limit. Namely,

$$z = \exp(R'\zeta) = q^\mu \tag{8.14}$$

where ζ is the 3d FI parameter complexified by the holonomy of the corresponding background gauge field around the S^1. It follows from its string theory origin that $\mathrm{Re}(\zeta_a)$ is R times the modulus of the $(2,0)$ theory, and both of these we need to fix in the limit. This implies that $\mathrm{Re}(R'\zeta_a)$ goes to zero in the conformal limit, and hence z goes to 1. The rate at which z goes to one is fixed, however, so μ defined by (8.14) remains fixed.

9. Langlands correspondence from little strings

It has been known for a long time that geometric Langlands correspondence should be a consequence of S-duality of the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory [65, 125, 129]. While some aspects of S-duality can be understood within the gauge theory, and many more from theory $\mathcal{X}$, to derive S-duality one needs string theory. This was shown in [119], and reviewed recently in [11].

In this section we will recall the derivation of S-duality from little string theory, as well as the expected relation between S-duality of the $\mathcal{N} = 4$ theory and the geometric Langlands. The fact that one is able to derive S-duality from little string theory offers an explanation why one can derive (quantum) geometric Langlands from it.

9.1. ***S*-duality of 4d Yang-Mills theory.** S-duality relates Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry and gauge groups based on Lie algebras $\mathfrak{g}$ and ${}^L\mathfrak{g}$,

$$S: \qquad ({}^L\mathfrak{g}, {}^L\tau) \qquad \longleftrightarrow \qquad (\mathfrak{g}, \tau). \tag{9.1}$$

The gauge coupling parameters are related by

$$m\,\tau\,{}^L\tau = -1, \tag{9.2}$$

where τ is given by $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$ in terms of the Yang-Mills coupling constant g_{YM} and the θ angle. The theory with Lie algebra $\mathfrak{g}$ has in addition a symmetry T corresponding to the 2π shift of the theta angle[16]

$$T: \qquad (\mathfrak{g}, \tau) \qquad \to \qquad (\mathfrak{g}, \tau + 1),$$

which maps the theory to itself for any $\mathfrak{g}$. The action of the S and T on the particles of the theory is always non-trivial.

9.2. **Derivation of S-duality from little string theory.** Start with IIB string on

$$(Y_0 \times S^1_q \times S^1_t)/H \times M_4, \tag{9.3}$$

where Y_0 is an ADE surface corresponding to a $\mathfrak{g}_0$ Lie algebra, as in Section 7, and S^1_q, S^1_t are a pair of circles (the subscripts are there to distinguish them). The H-twist acts by folding the Dynkin diagram of $\mathfrak{g}_0$, in going once around S^1_q. Nothing in what follows depends on the value of the string coupling, so we can take g_s to zero to get the $\mathfrak{g}_0$ little string theory on $S^1_q \times S^1_t \times M_4$, with the H-twist.

We would like to understand which 4d theory we get when we send to zero the characteristic size of the string $1/m_s$ and the area of the two torus $T^2 = S^1_q \times S^1_t$. The resulting theory can be derived using T-duality symmetry of string theory [119].

T-duality on the S^1_t circle leads to the description based on $\mathcal{N} = 4$ SYM theory with Lie algebra $\mathfrak{g}$. The description based on ${}^L\mathfrak{g}$ follows from T-duality on the S^1_q circle, instead. The more weakly coupled description comes from T-duality on the smaller of the two circles.

9.2.1. T-duality on the S^1_t circle relates IIB string on (9.3) to IIA string on

$$(Y_0 \times S^1_q \times S^1_{t'})/H \times M_4. \tag{9.4}$$

The two string theories are equivalent once we exchange the momentum and the winding modes around the S^1_t. Y_0 is the same ADE surface as in (9.3), and the H-twist is on S^1_q. If R_q and R_t are the radii of S^1_q and S^1_t circles, respectively, then $S^1_{t'}$ is a circle of radius $R_{t'} = 1/(R_t m_s^2)$. At the singularity in IIA theory we get the $(1,1)$ little string theory, as described in Section 7. This theory has a 6d gauge symmetry, with gauge group based on $\mathfrak{g}_0$ Lie algebra; its gauge coupling parameter is m_s^2 [107]. Presently, the $(1,1)$ little string is compactified on the two-torus $(S^1_q \times S^1_{t'})/H$ times M_4, where the H-twist around S^1_q permutes the fields of the gauge theory according to the action of H on the Dynkin diagram of $\mathfrak{g}_0$. The theta angle originates from the NS B-field on the two-torus; its periodicity results in the symmetry T we had before.

Starting with the gauge theory in six dimensions with $(1,1)$ supersymmetry, in the limit we take the area of the two-torus to zero, and m_s to infinity, we get a 4d gauge theory on M_4 with $\mathcal{N} = 4$ supersymmetry, gauge group based on $\mathfrak{g}$, and coupling $\tau = i m_s^2 R_q R_{t'} = i R_q/R_t$. This follows by restricting the fields of the 6d gauge theory to constant (zero momentum) modes around the T^2; these are the only excitations whose energy remains finite as the size of the torus goes to zero.

9.2.2. It was shown in [119] that T-duality on S^1_q circle relates IIB string on (9.3) to IIA string on

$$({}^LY_0 \times S^1_{q'} \times S^1_t)/{}^LH \times M_4, \tag{9.5}$$

with twist by ${}^LH$, going around $S^1_{q'}$ circle once. Here ${}^LY_0$ is the ADE singularity based on a simply-laced Lie algebra ${}^L\mathfrak{g}_0$. The Lie algebra ${}^L\mathfrak{g}_0$ has outer automorphism group

[16] This assumes the normalization of the invariant metric on $\mathfrak{g}$ we have chosen—the one in which the short coroots of $\mathfrak{g}$ have length squared equal to 2; see [11, 29].

${}^L H$, such that by projecting to its ${}^L H$ invariant part we get ${}^L\mathfrak{g}$, the Lie algebra which is the Langlands dual of $\mathfrak{g}$. The radius of $S^1_{q'}$ is $R_{q'} = 1/(m\, R_q\, m_s^2)$. The factor of m comes about due to the H-twist on the original circle in IIB: the momentum on S^1_q is quantized in units of $1/(mR_q)$ since all modes come back to themselves only after going m times around S^1_q. Hence, the strings wound on the T-dual circle $S^1_{q'}$ must have masses quantized in units of $m_s^2\, R_{q'} = 1/(mR_q)$.

The $(1,1)$ little string theory one gets by decoupling the modes far from the singularity in IIA theory now has the low energy description as a 6d maximally supersymmetric gauge theory based on the ${}^L\mathfrak{g}_0$ Lie algebra. The 4d theory on M_4, which we get in the limit of the area as the two-torus goes to zero, has $\mathcal{N}=4$ supersymmetry, gauge group based on ${}^L\mathfrak{g}$, and coupling ${}^L\tau = i m_s^2 R_t R_{q'} = iR_t/(mR_q)$. In particular, τ and ${}^L\tau$ are related by (9.2).

9.2.3. In principle, in addition to the Lie algebra, one should specify the global form of the gauge group on each side in (9.1). This corresponds to specifying the allowed representations of electrically charged fields, a character sublattice of the weight lattice of ${}^L\mathfrak{g}$; its dual lattice is the character lattice of $\mathfrak{g}$; see [60] for review. In this paper, we will allow for the most general choice of electric charges for ${}^L\mathfrak{g}$, choosing the character and weight lattices to coincide. This implicitly sets ${}^L G$ to be the simply connected group with Lie algebra ${}^L\mathfrak{g}$. The dual group G then is of adjoint-type, as its weight lattice is equal to its root lattice.

9.3. **Gauge theory partition function from little string.** We showed that partition functions of $(2,0)$ 6d theory on $M_6^\times$ and M_6 compute the conformal blocks of $\widehat{{}^L\mathfrak{g}}_{{}^L k}$ and $\mathcal{W}_\beta(\mathfrak{g})$ (in the conformal limit of our results). The relation of $(2,0)$ 6d theory to the pair of $\mathcal{N}=4$ gauge theories with gauge groups based on ${}^L\mathfrak{g}$ and $\mathfrak{g}$ then implies that conformal blocks are the partition functions of these gauge theories, in the background induced from their six dimensional origin.

This leads to an explicit relation between the S-duality of $\mathcal{N}=4$ gauge theories in four dimensions and the conformal field theory approach to geometric Langlands, which we reviewed in Section 6. We will describe some essential aspects (see also Sec. 8 of [54]), leaving a more detailed analysis for future work.

9.3.1. Consider $\mathfrak{g}_0$-type $(2,0)$ little string theory compactified on a six-manifold

$$M_6^\times = \mathcal{C} \times (\mathbb{C}\times\mathbb{C}^\times)/H,$$

with an H-twist around $\mathbb{C}^\times$. $M_6^\times$ differs from M_6 in (1.17) by having the origin of one of the complex planes deleted. This is merely a convenient choice made for ease of discussion. Working with $M_6^\times$ leads to partition functions which compute vector-valued q-conformal blocks, instead of scalar ones we get from M_6. The converse is that closing up the puncture, and thereby replacing $M_6^\times$ with M_6, corresponds to contraction of the vector-valued partition function with the Whittaker-type vector in (1.10).

The six-manifold $M_6^\times$ is a $T^2 = S^1_t \times S^1_q$ fibration

$$T^2 \;\to\; M_6^\times \;\to\; M_4 = \mathcal{C}\times B. \tag{9.6}$$

As we go once around the S^1_q circle, viewed as the circle fiber of $\mathbb{C}^\times$, we twist by H; the fiber of $\mathbb{C}$ is S^1_t. T-duality of little string theory on the S^1_t or on the S^1_q circle fiber leads to two distinct descriptions of the 4d theory on M_4, as we reviewed in Section 9.2: the first leads to the $\mathcal{N}=4$ SYM theory based on gauge group $\mathfrak{g}$, the second based on ${}^L\mathfrak{g}$.

The base M_4 is a manifold with a boundary, since $B = \mathbb{R}\times\mathbb{R}^+$. The boundary conditions [54, 65] for the two 4d gauge theories we need are defined by recalling their origin from the 6d $(2,0)$ theory on $M_6^\times$, which is a six-manifold without boundaries [127].

9.3.2. We studied the supersymmetric partition function of the $(2,0)$ little string theory on $M_6^\times$ in Section 8. In the conformal limit, the partition function we defined in Section 8 becomes the partition function of the $(2,0)$ 6d conformal field theory on $\mathcal{C}$ times the 4d Ω-background on $\mathbb{C}\times\mathbb{C}^\times$, or on $\mathbb{C}\times\mathbb{C}$ if we replace $M_6^\times$ by M_6.

It was argued in [92] that placing the $(2,0)$ 6d CFT theory on $\mathcal{C}$ times a 4d Ω-background leads to partition functions of the two S-dual $\mathcal{N}=4$ theories on M_4 with topological twist of geometric Langlands-type, studied in [65].

Further, [92] explained how to relate the parameters of Ω-background to the effective coupling constant of the gauge theory. One uses the fact that, asymptotically and locally, far away from the fixed points of rotations by $\mathbb{C}_t^\times$ and $\mathbb{C}_q^\times$, M_6 is a flat manifold. There, all effects of topological twisting go away and the Ω-background parameters of the twisted theory are identified[17] with the inverse radii of the two S^1's in $T^2=S^1_q\times S^1_t$ in the undeformed gauge theory: $\epsilon_t=2\pi/R_t$ and $\epsilon_q=2\pi/(imR_q)$.

Putting our results together with those of [92], we find the following.

9.3.3. The chiral conformal block of $\mathcal{W}_\beta(\mathfrak{g})$, corresponds to the partition function of 4d SYM theory with gauge group based on $\mathfrak{g}$ and coupling (see Section 9.2.1)

$$\tau=\epsilon_t/m\epsilon_q=\beta/m, \tag{9.7}$$

which one gets from $(2,0)$ theory on M_6. One wants to work with M_6 rather than $M_6^\times$ here since the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra blocks from (1.9) are naturally scalar.

This relation follows from AGT correspondence [8], and was used in [92]. It also follows from our results (and from [2]) by taking the conformal limit.

9.3.4. The chiral conformal blocks of $\widehat{{}^L\mathfrak{g}}_{{}^Lk}$ correspond to YM theory with gauge group ${}^L\mathfrak{g}$ and coupling parameter ${}^L\tau$ given by (see Section 9.2.2)

$${}^L\tau=\epsilon_q/\epsilon_\hbar=-{}^L(k+h). \tag{9.8}$$

It follows when we place the theory on $M_6^\times$.

This relation to WZW models was predicted in [95] nearly 20 years ago. The $(2,0)$ conformal field theory on any three-manifold M_3 times $\mathbb{C}\times S^1_q$ is expected [95] to compute the partition function of Chern-Simons theory based on ${}^L\mathfrak{g}$ Lie algebra, on M_3. In the present case, this applies with $M_3=\mathcal{C}\times\mathbb{R}$. For non-simply-laced ${}^L\mathfrak{g}$, one starts with the $\mathfrak{g}_0$ type $(2,0)$ theory, and introduces the twist by H [127], just as we did.

In the construction of [95], the level ${}^L(k+h^\vee)$ of Chern-Simons theory is determined by the parameter q' arising geometrically from the Ω-background on $\mathbb{C}\times S^1_q$. To get Chern-Simons theory at level ${}^Lk$ from $(2,0)$ theory on $M_3\times\mathbb{C}\times S^1_q$, we rotate $\mathbb{C}$ by $q'=\exp(\frac{2\pi i}{{}^L(k+h^\vee)})$ as we go around S^1_q, and accompany the rotation with an R-symmetry twist.

We can apply this here, with one subtle point. Namely, we need $\epsilon_\hbar$ not $-\epsilon_t$ to be the Ω-background that rotates the complex plane (the two are related by $\epsilon_\hbar=\epsilon_q-\epsilon_t$). This is related to the fact that since $\mathbb{C}^\times$ is a cylinder, the topological twist on it is trivial. We only need the R-symmetry twist with parameter $\epsilon_\hbar$, from Section 8, which gets compensated by a twist of the $\mathbb{C}$-plane in $M_6^\times$ by $\epsilon_\hbar$. Altogether, we find $q'=\exp(-mR_q\cdot\epsilon_\hbar)$, and since $imR_q=2\pi/\epsilon_q$, (9.8) follows.

[17] Our conventions for ϵ's are set in Section 8 and in (8.6) and they differ from those in [92] by factors of 2π.

9.3.5. Since $m\epsilon_q = \epsilon_t + \epsilon_\hbar$, we have that the two theories are related by

$$\tau - 1 = -1/(m^L\tau),$$

as in (1.3). This is the action of S-duality, together with the shift of the θ angle in the $\mathfrak{g}$-theory.

9.4. **Little string defects and line operators in gauge theory.** In the $(2,0)$ theory on $M_6^\times$, we have defects supported on $\mathbb{C}_q^\times$, labeled by collection of weights of ${}^L\mathfrak{g}$. These defects, which originate as D3 branes of IIB wrapping 2-cycles of Y_0, are self-dual strings of the $(2,0)$ theory. (The self-dual strings are strings present both in the $(2,0)$ little string theory, and in theory $\mathcal{X}$. They are distinct from fundamental strings of little string theory, which are not present in theory $\mathcal{X}$.)

Reducing the 6d $(2,0)$ theory on T^2 to $\mathcal{N}=4$ theory on $M_4 = \mathcal{C}\times\mathbb{R}_q\times\mathbb{R}_t^+$, the self-dual strings supported on $S^1\subset T^2$ become particles on M_4. We have that particles are supported at a collection of points on $\mathcal{C}$, with coordinates $\{a_i\}$, and charges which are labeled by weights of ${}^L\mathfrak{g}$. They are located at the tip of $\mathbb{R}_t^+$, and their world lines are along the "time" direction $\mathbb{R}_q$. Presence of such particles affects the partition function of the 4d theory by insertion of line operators. Which line operator we get depends on the $\mathcal{N}=4$ gauge theory description one uses.

9.4.1. In the $\mathcal{N}=4$ theory description based on ${}^L\mathfrak{g}$, the self-dual string of the 6d theory supported on $\mathbb{C}_q^\times$ becomes the Wilson line operator. Namely, when we view the T^2 compactification of theory $\mathcal{X}$ as a two step reduction, reducing on $S_q^1\in\mathbb{C}_q^\times$ first, the self-dual string defects become particles already in five dimensions, electrically charged under ${}^L\mathfrak{g}$-valued gauge field. Reducing further on S_q^1, we get the Wilson lines of $\mathcal{N}=4$ theory on M_4. This is as expected from our description, in Subsection 2.1, of the conformal blocks we study. Namely, the Wilson line operators of the Yang-Mills theory become Wilson line operators of $\widehat{{}^L\mathfrak{g}}$ Chern-Simons theory and the corresponding WZW model.

9.4.2. In the $\mathcal{N}=4$ theory based on $\mathfrak{g}$, the same strings give rise to 't Hooft line operators. Compactifying theory $\mathcal{X}$ on S_t^1, we get a 5d gauge theory with strings; the strings are charged under the magnetic dual of the $\mathfrak{g}$-valued 5d gauge field, a 2 form. After further compactifying on S_q^1 they become magnetically charged particles; their world lines introduce t'Hooft line operators in M_4. The 't Hooft line operators are labeled by coweights of $\mathfrak{g}$ and hence by weights of ${}^L\mathfrak{g}$. The effect the line operator in the $\mathfrak{g}$ gauge theory is described by how they affect the boundary conditions at the tip of $\mathbb{R}_t^+$. In the limit when the gauge theory become classical, [54] argued that the boundary conditions one gets are described in terms of $\mathfrak{g}$ opers. This agrees with what we find, since opers describe the classical limit of $\mathcal{W}_\beta(\mathfrak{g})$ algebra conformal blocks of our paper.

Appendix A. Integral formulas in K-theory of GIT quotients

Let a reductive group G act on a variety $\widetilde{X}$ and let $\mathscr{L}$ be a very ample G-linearized line bundle on $\widetilde{X}$. The GIT quotient

$$X = \widetilde{X}/\!\!/\!\!/\!\!/_{\mathscr{L}}G = \operatorname{Proj}\left(\bigoplus_n H^0(\widetilde{X},\mathscr{L}^{\otimes n})^G\right) \tag{A.1}$$

is the categorical quotient of the set of semistable points

$$\widetilde{X}_{\mathrm{ss}} = \{x\in\widetilde{X}, \exists s\in H^0(\widetilde{X},\mathscr{L}^{\otimes n})^G, s(x)\neq 0\}$$

by the action of G. We denote by

$$\pi_{\mathrm{GIT}}:\widetilde{X}_{\mathrm{ss}}\to X$$

the canonical affine morphism.

A.0.1. Let $\widetilde{\mathscr{F}}$ be a G-equivariant coherent sheaf on $\widetilde{X}$. It induces a coherent sheaf $\mathscr{F}$ on X by

$$\Gamma(U, \mathscr{F}) = \Gamma(\pi_{\mathrm{GIT}}^{-1}(U), \widetilde{\mathscr{F}})^G .$$

In particular, $\mathscr{L}$ itself induces the canonical line bundle $\mathscr{O}(1)$ on X.

Our interest is in the computation of $\chi(X, \mathscr{F})$ in terms involving the prequotient $\widetilde{X}$. This is the value of the quasipolynomial $\chi(X, \mathscr{F}(m))$ at $m = 0$, where $\mathscr{F}(m) = \mathscr{F} \otimes \mathscr{O}(m)$. By definition, a quasipolynomial in m is an element of a ring of the form

$$\mathbb{Q}[m, a_1^{\pm m}, a_2^{\pm m}, \dots] ,$$

where the parameters a_i may be roots of unity or weights of a group of automorphisms of X.

A.0.2. There is the following basic lemma.

Lemma 3. *For $m \gg 0$,*

$$\chi(X, \mathscr{F}(m)) = \chi(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G . \tag{A.2}$$

A more general formula, valid without the $m \gg 0$ assumption, follows from the results of Teleman [117]; see also [130] and [61].

Proof. Since $\mathscr{L}$ is ample, we have

$$\chi(X, \mathscr{F}(m)) = \Gamma(X, \mathscr{F}(m)) = \Gamma(\widetilde{X}_{\mathrm{ss}}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G$$

for $m \gg 0$. Therefore, it suffices to see that the natural restriction map

$$\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G \to \Gamma(\widetilde{X}_{\mathrm{ss}}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G \tag{A.3}$$

is an isomorphism for $m \gg 0$. The spaces in the source and the target in (A.3) form a module over the graded algebra in (A.1). The sheaf $\widetilde{\mathscr{F}}$ is coherent and the line bundle $\mathscr{L}$ is ample, hence for sufficiently large r and d there is a map

$$\left(\mathscr{L}^{-d}\right)^{\oplus r} \to \widetilde{\mathscr{F}}$$

inducing a surjection

$$\Gamma(X, \mathscr{L}^{m-d})^{\oplus r} \to \Gamma(X, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m) \to 0$$

for $m \gg 0$. Because G is reductive, we get a surjectivity for G-invariant sections and so the modules in question are finitely generated. Therefore,

$$\bigoplus_{d \le D} \Gamma(\widetilde{X}, \mathscr{L}^{m-d})^G \otimes \Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^d)^G \to \Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G \to 0$$

where D is the maximal degree of a generator, and similarly for $\widetilde{X}_{\mathrm{ss}}$ in place of X. Since all sections in $\Gamma(\widetilde{X}_{\mathrm{ss}} \otimes \mathscr{L}^m)^G$ extend by zero to $\widetilde{X}$, the isomorphism in (A.3) follows. $\square$

A.0.3. From now on we assume there is a torus

$$\mathsf{T} \subset \operatorname{Aut}_G(\widetilde{X})$$

acting on $\mathscr{L}$ and $\mathscr{F}$ that contracts $\widetilde{X}$ to a proper G-invariant set as $t \to 0_{\mathsf{T}}$, where 0_{T} is a point in a toric compactification of T. This is the case, for example, when $\widetilde{X}$ is a linear representation of G, or the zero locus of a moment map in a linear symplectic representation of G. The additional T-grading makes the trace

$$\operatorname{tr}_{\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)}(g,t) \in \mathbb{Q}(G \times \mathsf{T}) \tag{A.4}$$

converge for $|t| \gg 1$ to a rational function. Here $|t| \gg 1$ means that t^{-1} lies in a certain neighborhood of 0_{T} and in that region the poles of (A.4) are disjoint from any fixed maximal compact subgroup $G_{\text{compact}} \subset G$. Therefore

$$\begin{aligned} \operatorname{tr}_{\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)^G} t &= \int_{G_{\text{compact}}} \operatorname{tr}_{\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)}(g,t)\, d_{\text{Haar}} g \\ &= \frac{1}{|W|} \int_{|s|=1} \Delta_{\text{Weyl}}(s) \operatorname{tr}_{\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)}(s,t)\, d_{\text{Haar}} s \end{aligned} \tag{A.5}$$

for $|t| \gg 1$, where

$$\begin{aligned} W & \quad \text{is the Weyl group of } G, \\ \{|s|=1\} \subset G_{\text{compact}} & \quad \text{is a maximal torus,} \\ \Delta_{\text{Weyl}}(s) & \quad \text{is the Weyl denominator}\,, \end{aligned} \tag{A.6}$$

and the Haar measures are normalized to have total mass 1.

A.0.4. We denote by $\mathsf{S} \subset G$ the complexification of the torus in (A.6). By localization,

$$\operatorname{tr}_{\Gamma(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)}(s,t) = \sum_k \frac{p_k(s,t)}{\prod(1 - w_{k,i}^{-1})}\, \nu_k^m \tag{A.7}$$

where the sum in (A.7) is over the components $\widetilde{F}_k$ of the fixed locus

$$\widetilde{X}^{\mathsf{S} \times \mathsf{T}} = \bigsqcup \widetilde{F}_k\,,$$

p_k are certain Laurent polynomials in s and t, the characters ν_k are the $\mathsf{S} \times \mathsf{T}$-weights of $\mathscr{L}$ restricted to the components of the fixed locus, and $w_{k,i}$ are the weights in the denominators of the localization formula (i.e., normal weights to the fixed locus in some ambient smooth equivariant embedding of $\widetilde{X}$).

A.0.5. The integral in (A.5) may be computed by residues as follows. By linearity, it suffices to deal with each term in (A.7) separately. If $\nu_k\big|_{\mathsf{S}}$ is a trivial character, then

$$\int_{|s|=1} \dots \nu_k^m \, d_{\text{Haar}} s = \nu_k^m \int_{|s|=1} \dots d_{\text{Haar}} s\,,$$

which is tautologically a quasipolynomial in m.

If $\nu_k\big|_{\mathsf{S}}$ is a non-trivial character, then we deform the integration contour $\{|s| = 1\}$ to the region $|\nu_k| \ll 1$ in S while picking up residues in the process. These residues are integrals of the same form over translates of codimension one subtori in S, and so we can deal with them inductively.

The resulting quasipolynomial in m computes the quasipolynomial $\chi(X, \mathscr{F}(m))$.

A.0.6. For the most basic example, we can take $G = \mathrm{GL}(1)$ acting with weight one on $\mathbb{C}^n$ and $\mathscr{L} = \mathscr{O}_{\mathbb{C}^n}(k)$, where the twist is by the kth power of the defining representation. Then

$$\Gamma(\mathscr{L}^m)^G = \text{polynomials of degree } km\,,$$

and so

$$X = \begin{cases} \mathbb{P}^{n-1}, & k > 0\,, \\ \mathrm{pt}, & k = 0\,, \\ \varnothing\,, & k < 0\,, \end{cases}$$

or, taking polarization into account, X is the kth Veronese embedding of $\mathbb{P}^{n-1}$ for $k > 0$. We can take $\mathsf{T} = \mathsf{A}$, where

$$\mathsf{A} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \right\} \subset \mathrm{GL}(n)\,, \tag{A.8}$$

which gives normal weights $w_i = sa_i$, $i = 1, \ldots, n$ at the unique fixed point $0 \in \mathbb{C}^n$. Therefore, we get the integral

$$\chi(X, \mathscr{O}(m)) = \frac{1}{2\pi i} \int_{|s|=1} \frac{s^{km}}{\prod(1 - a_i^{-1} s^{-1})} \frac{ds}{s}\,, \quad |a_i| > 1\,,$$

which may be computed by deforming the contour to $|s| = \varepsilon^{\pm 1}$, depending on the sign of k.

A.0.7. Of importance to us will be the special case when $\mathscr{L}$ is twisted by a large power of a non-trivial G-character χ. In this case, we can start the analysis of (A.5) with deforming the contour into the region $|\chi| \ll 1$.

We denote by

$$\mathsf{S}^\circ = \left\{ s \,\middle|\, \forall w_{k,i}, w_{k,i} \neq 1 \right\} \subset \mathsf{S}$$

the regular locus of the integrand. The homology groups of S° have been studied in detail; see [28]. In particular, non-canonically,

$$H_*(\mathsf{S}^\circ, \mathbb{C}) = \bigoplus_{\mathsf{S}'} H_*(\mathsf{S}', \mathbb{C}) \otimes H_*(N_{\mathsf{S}/\mathsf{S}'} \setminus \{\text{hyperplanes}\}, \mathbb{C}) \tag{A.9}$$

where S' ranges over the components of all possible intersections of $\{w_{k,i} = 1\}$, and hyperplanes in the (trivial) normal bundle $N_{\mathsf{S}/\mathsf{S}'}$ to S' are cut out by the differentials of the characters $w_{k,i}$ trivial on S'. Since all homology groups vanish above the middle dimension, we have

$$H_{\mathrm{mid}}(\mathsf{S}^\circ, \mathbb{C}) = \bigoplus_{\mathsf{S}'} H_{\mathrm{mid}}(\mathsf{S}', \mathbb{C}) \otimes H_{\mathrm{mid}}(N_{\mathsf{S}/\mathsf{S}'} \setminus \{\text{hyperplanes}\}, \mathbb{C})\,. \tag{A.10}$$

For the computation of the integral, we are interested in homology relative to the subset $|\chi| \ll 1$, and for those we conclude

$$H_{\mathrm{mid}}(\mathsf{S}^\circ, \{|\chi| \ll 1\}, \mathbb{C}) = \bigoplus_{\mathsf{S}', \chi\big|_{\mathsf{S}'} = \mathrm{const}} \text{same as in (A.10)}\,, \tag{A.11}$$

which parallels the computation by residues discussed in Section A.0.5. We set

$$\gamma_\chi = \text{image of } \{|s| = 1\} \text{ in LHS of (A.11)}\,. \tag{A.12}$$

As a middle-dimensional cycle, it is represented by products of a maximal compact torus in S' with a middle-dimensional cycle in a certain hyperplane arrangement. We conclude the following.

Proposition 5. *If $\mathscr{L}$ is twisted by a sufficiently large power of a character χ, then*

$$\chi(X, \mathscr{F}(m)) = \frac{1}{|W|} \int_{\gamma_\chi} \Delta_{\mathrm{Weyl}}(s) \operatorname{tr}_{\chi(\widetilde{X}, \widetilde{\mathscr{F}} \otimes \mathscr{L}^m)}(s, t)\, d_{\mathrm{Haar}} s \tag{A.13}$$

for $m \gg 0$, and for all m if $\dim \mathsf{S}' = 0$ *for all* S' *in* (A.11).

If $\dim \mathsf{S}' > 0$ for a certain S' in (A.11), then the corresponding integral needs to be treated as in Section A.0.5 to pick the right quasipolynomial in m.

An important special case when one can be sure that $\dim \mathsf{S}' = 0$ for all S' in (A.0.5) is the case of Nakajima quiver varieties. More generally, we have the following simple lemma.

Lemma 4. *Suppose $G = \prod \mathrm{GL}(V_i)$ and*

$$\{w_k\} = \text{weights of } V_i,\ V_i^*, \text{ and } V_i \otimes V_j^*\,,$$

where $i, j = 1, \dots, n$. Then a generic character χ of G is non-trivial on every component S' of $\{w_{k_1} = \cdots = w_{k_l} = 1\}$ of positive dimension.

Proof. This is equivalent to the differential $d\chi$ being in the span of $dw_{k_1}, \dots, dw_{k_l}$ if and only if this span is the whole space. The generic character is not in the span of weights of $V_i \otimes V_j^*$ and so at least one fundamental or the dual fundamental weight has to appear among w_{k_i}. We can then argue modulo this weight and induct on $\sum \dim V_i$. □

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References

[1] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, and C. Vafa, *Topological strings and integrable hierarchies*, Comm. Math. Phys. **261** (2006), no. 2, 451–516. arXiv:hep-th/0312085. MR2191887

[2] M. Aganagic and N. Haouzi, *ADE little string theory on a Riemann surface (and triality).* arXiv:1506.04183 [hep-th].

[3] M. Aganagic, N. Haouzi, C. Kozcaz, and S. Shakirov, *Gauge/Liouville triality.* arXiv:1309.1687 [hep-th].

[4] M. Aganagic, N. Haouzi, and S. Shakirov, *A_n-triality.* arXiv:1403.3657 [hep-th].

[5] M. Aganagic and A. Okounkov, *Elliptic stable envelope.* arXiv:1604.00423 [math.AG].

[6] M. Aganagic and A. Okounkov, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J. **17** (2017), no. 4, 565–600. MR3734654

[7] M. Aganagic and S. Shakirov, *Gauge/vortex duality and AGT.* arXiv:1412.7132 [hep-th].

[8] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010), no. 2, 167–197. arXiv:0906.3219 [hep-th].

[9] T. Arakawa, *Representation theory of superconformal algebras and the Kac–Roan–Wakimoto conjecture*, Duke Math. J. **130** (2005), no. 3, 435–478. MR2184567

[10] T. Arakawa, *Representation theory of $\mathscr{W}$-algebras*, Invent. Math. **169** (2007), no. 2, 219–320. MR2318558

[11] P. C. Argyres, A. Kapustin, and N. Seiberg, *On S-duality for non-simply-laced gauge groups*, J. High Energy Phys. **6** (2006), 043. arXiv:hep-th/0603048. MR2233812

[12] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. (N. S.) **21** (2015), no. 1, 1–199. MR3300415

[13] H. Awata, D. Kubo, S. Odake, and J. Shiraishi, *Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials*, Comm. Math. Phys. **179** (1996), no. 2, 401–416. MR1400746

[14] H. Awata, A. Tsuchiya, and Y. Yamada, *Integral formulas for the WZNW correlation functions*, Nuclear Phys. B **365** (1991), no. 3, 680–696. MR1136712

[15] C. Beem, T. Dimofte, and S. Pasquetti, *Holomorphic blocks in three dimensions*, J. High Energy Phys. **12** (2014), 177–296. arXiv:1211.1986 [hep-th].

[16] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88. MR1437495

[17] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves,* preprint available at `www.math.uchicago.edu/~arinkin/langlands`.

[18] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, Nuclear Phys. B **481** (1996), no. 1–2, 215–252. MR1423982

[19] M. Bershadsky and H. Ooguri, *Hidden $SL(n)$ symmetry in conformal field theory*, Comm. Math. Phys. **126** (1989), no. 1, 49–83. MR1027913

[20] R. Bezrukavnikov, M. Finkelberg, and V. Schechtman, *Factorizable sheaves and quantum groups*, Lect. Notes in Math, vol. 1691, Springer-Verlag, Berlin, 1998. MR1641131

[21] A. Braverman, M. Finkelberg, and H. Nakajima, *Instanton moduli spaces and $\mathcal{W}$-algebras.* arXiv: 1406.2381 [math.QA]. MR3592485

[22] A. Braverman, M. Finkelberg, and H. Nakajima, *Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian* (with appendices by A. Braverman, M. Finkelberg, J. Kamnitzer, R. Kodera, H. Nakajima, B. Webster, and A. Weekes). arXiv:1604.03625 [math.RT].

[23] A. Braverman, M. Finkelberg, and H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II.* arXiv:1601.03586 [math.RT]. MR3565863

[24] M. Bullimore, T. Dimofte, D. Gaiotto, and J. Hilburn, *Boundaries, mirror symmetry, and symplectic duality in 3d $\mathcal{N} = 4$ Gauge theory.* arXiv:1603.08382 [hep-th]. MR3578533

[25] S. Cecotti and C. Vafa, *Topological–anti-topological fusion*, Nuclear Phys. B **367** (1991), no. 2, 359–461. S. Cecotti, D. Gaiotto, and C. Vafa, *tt^* geometry in 3 and 4 dimensions*, J. High Energy Phys. **5** (2014). S. Cecotti, A. Neitzke, and C. Vafa, *Twistorial topological strings and a tt^* geometry for $\mathcal{N} = 2$ theories in 4d.* arXiv:1412.4793 [hep-th]. MR1139739

[26] I. Ciocan-Fontanine and M. Kapranov, *Virtual fundamental classes via dg-manifolds*, Geom. Topol. **13** (2009), no. 3, 1779–1804. MR2496057

[27] I. Ciocan-Fontanine, B. Kim, and D. Maulik, *Stable quasimaps to GIT quotients*, J. Geom. Phys. **75** (2014), 17–47. MR3126932

[28] C. De Concini and C. Procesi, *On the geometry of toric arrangements*, Transform. Groups **10** (2005), no. 3–4, 387–422. MR2183118

[29] N. Dorey, C. Fraser, T. J. Hollowood, and M. A. C. Kneipp, *S-duality in $N = 4$ supersymmetric gauge theories with arbitrary gauge group*, Phys. Lett. B **383** (1996), no. 4, 422-428. MR1411828

[30] M. R. Douglas and G. W. Moore, *D-branes, quivers, and ALE instantons.* arXiv:hep-th/9603167.

[31] V. Drinfeld, *Hopf algebras and the quantum Yang–Baxter equation*, Dokl. Akad. Nauk SSSR **32** (1985), 254–258. MR802128

[32] P. I. Etingof, I. B. Frenkel, and A. A. Kirillov Jr., *Lectures on representation theory and Knizhnik–Zamolodchikov equations*, Math. Surveys and Monographs, vol. 58, AMS, Providence, RI, 1985. MR1629472

[33] P. I. Etingof and A. Varchenko, *Dynamical Weyl groups and applications*, Adv. Math. **167** (2002), no. 1, 74–127. MR1901247

[34] B. Fantechi and L. Göttsche, *Riemann–Roch theorems and elliptic genus for virtually smooth schemes*, Geom. Topol. **14** (2010), no. 1, 83–115. MR2578301

[35] B. Feigin and E. Frenkel, *Quantization of the Drinfel'd–Sokolov reduction*, Phys. Lett. B **246** (1990), no. 1–2, 75–81. MR1071340

[36] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gel'fand–Dikiĭ algebras*, Int. J. Mod. Phys. **A7** (1992), no. suppl. 1A, 197–215. Reprinted in *Infinite analysis (Kyoto, 1991)*, Adv. Ser. Math. Phys., vol. 16, World Sci. Publ., River Edge, NJ, 1992. MR1187549

[37] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lect. Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 349–418. arXiv:hep-th/9310022. MR1397275

[38] B. Feigin and E. Frenkel, *Quantum $\mathcal{W}$-algebras and elliptic algebras*, Comm. Math. Phys. **178** (1996), no. 3, 653–678. arXiv:q-alg/9508009. MR1395209

[39] B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. **166** (1994), no. 1, 27–62. MR1309540

[40] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves.* arXiv:hep-th/9407154. MR1404026

[41] E. Frenkel, *$\mathcal{W}$-algebras and Langlands–Drinfel'd correspondence*, New symmetry principles in quantum field theory (Cargèse, 1991), NATO Adv. Sci. Inst. Ser. B Phys., vol. 295, Plenum Press, New York, 1992, pp. 433–447. MR1204467

[42] E. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, XIth International Congress of Mathematical Physics (Paris, 1994), Int. Press, Cambridge, MA, 1995, pp. 606–642. arXiv:q-alg/9506003. MR1370720

[43] E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005), no. 2, 297–404. arXiv:math/0210029. MR2146349

[44] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, Frontiers in number theory, physics, and geometry, II, Springer, Berlin, 2007, pp. 387–533. arXiv:hep-th/0512172. MR2290768

[45] E. Frenkel, *Gauge theory and Langlands duality*, Astérisque **332** (2010), 369–403. Séminaire Bourbaki. Volume 2008/2009. Exp. 1010. arXiv:0906.2747. MR2648685

[46] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, 2nd ed., Mathematical Surveys and Monographs, vol. 88, AMS, Providence, RI, 2004. MR2082709

[47] E. Frenkel, S. Gukov, and J. Teschner, *Surface operators and separation of variables*, J. High Energy Phys. **1** (2016), 179. arXiv:1506.07508 [hep-th]. MR3471401

[48] E. Frenkel, V. Kac, and M. Wakimoto, *Characters and fusion rules for W-algebras via quantized Drinfel'd–Sokolov reduction*, Comm. Math. Phys. **147** (1992), no. 2, 295–328. MR1174415

[49] E. Frenkel and N. Reshetikhin, *Towards deformed chiral algebras*, Proceedings of the Quantum Group Symposium at the XXIth International Colloquium on Group Theoretical Methods in Physics (Goslar, 1996) (1997), 27–42. arXiv:q-alg/9706023.

[50] E. Frenkel and N. Reshetikhin, *Deformations of $\mathscr{W}$-algebras associated to simple Lie algebras*, Comm. Math. Phys. **197** (1998), no. 1, 1–32. arXiv:q-alg/9708006. MR1646483

[51] I. B. Frenkel and N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60. MR1163666

[52] P. Furlan, A. Ch. Ganchev, R. Paunov, and V. B. Petkova, *Solutions of the Knizhnik–Zamolodchikov equation with rational isospins and the reduction to the minimal models*, Nuclear Phys. B **394** (1993), no. 3, 665–706. MR1215177

[53] D. Gaiotto and P. Koroteev, *On three dimensional quiver gauge theories and integrability*, J. High Energy Phys. **5** (2013), 126. arXiv:1304.0779 [hep-th]. MR3080535

[54] D. Gaiotto and E. Witten, *Knot invariants from four-dimensional gauge theory*, Adv. Theor. Math. Phys. **16** (2012), no. 3, 935–1086. arXiv:1106.4789 [hep-th]. MR3024278

[55] D. Gaitsgory, *Twisted Whittaker model and factorizable sheaves*, Selecta Math. (N. S.) **13** (2008), no. 4, 617–659. arXiv:0705.4571. MR2403306

[56] D. Gaitsgory, *Quantum Langlands correspondence*. arXiv:1601.05279.

[57] V. Ginzburg, *Lectures on Nakajima's quiver varieties*, Geometric methods in representation theory. I, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 145–219. MR3202703

[58] A. Givental, *On the WDVV equation in quantum K-theory*, Michigan Math. J. **48** (2000), 295–304. Dedicated to William Fulton on the occasion of his 60th birthday. MR1786492

[59] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518. MR1666787

[60] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*, Current developments in mathematics, 2006 (2008), 35–180. MR2459305

[61] D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912. MR3327537

[62] Y.-Z. Huang and J. Lepowsky, *Tensor categories and the mathematics of rational and logarithmic conformal field theory*, J. Phys. A **46** (2013), no. 49, 494009. arXiv:1304.7556 [hep-th]. MR3146015

[63] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69. MR797001

[64] A. Kapustin, *A Note on Quantum Geometric Langlands Duality, Gauge Theory, and Quantization of the Moduli Space of Flat Connections*. arXiv:0811.3264 [hep-th]. MR2596636

[65] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Comm. Number Theory Phys. **1** (2007), no. 1, 1–236. arXiv:hep-th/0604151. MR2306566

[66] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, J. Amer. Math. Soc. **6** (1993), no. 4, 905–947. MR1186962

[67] T. Kimura and V. Pestun, *Quiver W-algebras*. arXiv:1512.08533 [hep-th].

[68] V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess–Zumino model in two-dimensions*, Nuclear Phys. B **247** (1984), no. 1, 83–103. MR853258

[69] K. Kuroki and A. Nakayashiki, *Free field approach to solutions of the quantum Knizhnik–Zamolodchikov equations*, SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), Paper 049. MR2425643

[70] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. **121** (2004), no. 3, 389–424. MR2040281

[71] G. Laumon, *Transformation de Fourier généralisée.* Preprint alg-geom/9603004. MR908218

[72] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578. MR1882667

[73] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293. MR1938113

[74] J. Li and B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Comm. Anal. Geom. **23** (2015), no. 4, 841–921. MR3385781

[75] A. S. Losev, A. Marshakov, and N. A. Nekrasov, *Small instantons, little strings and free fermions*, From Fields to Strings, Vol. 1, World Sci. Publ., Singapore, 2005, pp. 581–621. arXiv:hep-th/0302191. MR2330857

[76] A. Losev, G. W. Moore, and S. L. Shatashvili, *M & m's*, Nuclear Phys. B **522** (1998), no. 1–2, 105–124. arXiv:hep-th/9707250.

[77] A. Matsuo, *Jackson integrals of Jordan–Pochhammer type and quantum Knizhnik–Zamolodchikov equations*, Comm. Math. Phys. **151** (1993), no. 2, 263–273. MR1204771

[78] A. Matsuo, *Quantum algebra structure of certain Jackson integrals*, Comm. Math. Phys. **157** (1993), no. 3, 479–498. MR1243708

[79] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology.* arXiv:1211.1287, to appear in Astérisque.

[80] K. McGerty and T. Nevins, *Kirwan surjectivity for quiver varieties.* arXiv:1610.08121. MR3773791

[81] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238. MR1808477

[82] H. Nakajima, *Towards a mathematical definition of Coulomb branches of* 3*-dimensional* $\mathcal{N} = 4$ *gauge theories, I.* arXiv:1503.03676 [math-ph]. MR3565863

[83] N. Nekrasov, *On the BPS/CFT correspondence.* Seminar at the University of Amsterdam, Feb 2004.

[84] N. A. Nekrasov, *Instanton partition functions and M-theory*, Jpn. J. Math. **4** (2009), no. 1, 63–93. MR2491283

[85] N. Nekrasov, *BPS/CFT correspondence: non-perturbative Dyson–Schwinger equations and qq-characters*, J. High Energy Phys. **3** (2016), 181. arXiv:1512.05388 [hep-th]. MR3751819

[86] N. Nekrasov, *BPS/CFT correspondence II: instantons at crossroads, moduli and compactness theorem*, Adv. Theor. Math. Phys. **21** (2017), no. 2, 503–583. arXiv:1608.07272 [hep-th]. MR3678005

[87] N. Nekrasov, *BPS/CFT correspondence III: gauge origami partition function and qq-characters*, Comm. Math. Phys. **358** (2018), no. 3, 863–894. arXiv:1701.00189 [hep-th]. MR3778345

[88] N. Nekrasov and A. Okounkov, *Seiberg–Witten theory and random partitions*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 525–596. arXiv:hep-th/0306238. MR2181816

[89] N. Nekrasov and A. Okounkov, *Membranes and sheaves*, Algebr. Geom. **3** (2016), no. 3, 320–369. arXiv:1404.2323 [math.AG]. MR3504535

[90] N. Nekrasov and V. Pestun, *Seiberg–Witten geometry of four dimensional* $N = 2$ *quiver gauge theories.* arXiv:1211.2240 [hep-th].

[91] N. Nekrasov, A. Rosly, and S. Shatashvili, *Darboux coordinates, Yang-Yang functional, and gauge theory*, Nucl. Phys. B. Proc. Suppl. **216** (2011), 69–93. arXiv:1103.3919 [hep-th]. MR2851597

[92] N. Nekrasov and E. Witten, *The omega deformation, branes, integrability, and Liouville theory*, J. High Energy Phys. **1009** (2010), 092. arXiv:1002.0888 [hep-th]. MR2776942

[93] A. Okounkov, *Lectures on K-theoretic computations in enumerative geometry*, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, AMS, Providence, RI, 2017, pp. 251–380. MR3752463

[94] A. Okounkov and A. Smirnov, *Quantum difference equations for Nakajima varieties.* arXiv:1602.09007.

[95] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nuclear Phys. B **577** (2000), no. 3, 419–438. arXiv:hep-th/9912123.

[96] J. L. Petersen, J. Rasmussen, and M. Yu, *Conformal blocks for admissible representations in* SL(2) *current algebra*, Nuclear Phys. B **457** (1995), no. 1–2, 309–342. MR1366393

[97] J. Polchinski, *Tasi lectures on D-branes.* arXiv:hep-th/9611050.

[98] A. Polishchuk and M. Rothstein, *Fourier transform for D-algebras*, Duke Math. J. **109** (2001), no. 1, 123–146. MR1844207

[99] S. Raskin, *$\mathcal{W}$-algebras and Whittaker categories.* arXiv:1611.04937.

[100] S. Raskin, *$\mathcal{W}$-algebras and Whittaker categories*. available at `http://math.mit.edu/~sraskin/chiralcats.pdf`.

[101] N. Reshetikhin, *Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links, I and II*, LOMI Preprints, E-4-87 and E-17-87, USSR Academy of Sciences.

[102] N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik–Zamolodchikov system*, Lett. Math. Phys. **26** (1992), no. 3, 153–165. MR1199739

[103] M. Rothstein, *Connections on the total Picard sheaf and the KP hierarchy*, Acta Appl. Math. **42** (1996), no. 3, 297–308. MR1376873

[104] V. Schechtman, *Dualité de Langlands quantique*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 1, 129–158. MR3204734

[105] V. V. Schechtman and A. N. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), no. 1, 139–194. MR1123378

[106] V. V. Schechtman and A. N. Varchenko, *Quantum groups and homology of local systems*, Algebraic geometry and analytic geometry (Tokyo, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 182–197. MR1260946

[107] N. Seiberg, *New theories in six-dimensions and matrix description of M-theory on T^5 and T^5/Z_2*, Phys. Lett. B **408** (1997), no. 1–4, 98–104. arXiv:hep-th/9705221. MR1483580

[108] J. Shiraishi, H. Kubo, H. Awata, and S. Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996), no. 1, 33–51. MR1401054

[109] A. Smirnov, *Rationality of capped descendent vertex in K-theory*. arXiv:1612.01048, and in preparation.

[110] A. Stoyanovsky, *On quantization of the geometric Langlands correspondence I*. arXiv:math/9911108.

[111] A. Stoyanovsky, *A Relation between the Knizhnik–Zamolodchikov and Belavin–Polyakov–Zamolodchikov systems of partial differential equations*. arXiv:math-ph/0012013v3. MR2521319

[112] A. Stoyanovsky, *Quantum Langlands duality and conformal field theory*. arXiv:math/0610974.

[113] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, Invent. Math. **128** (1997), no. 3, 501–588. MR1452432

[114] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups*, Astérisque **246** (1997). MR1646561

[115] V. Tarasov and A. Varchenko, *Combinatorial formulae for nested Bethe vectors*, SIGMA Symmetry Integrability Geom. Methods Appl. **9** (2013), Paper 048. MR3116184

[116] V. Tarasov and A. Varchenko, *Jackson integral representations for solutions of the Knizhnik–Zamolodchikov quantum equation*, Algebra i Analiz **6** (1994), no. 2. 90–137; St. Petersburg Math. J. **6** (1995), no. 2, 275–313. MR1290820

[117] C. Teleman, *The quantization conjecture revisited*, Ann. of Math. (2) **152** (2000), no. 1, 1–43. MR1792291

[118] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I*, Adv. Theor. Math. Phys. **15** (2011), no. 2, 471–564. MR2924236

[119] C. Vafa, *Geometric origin of Montonen–Olive duality*, Adv. Theor. Math. Phys. **1** (1997), no. 1, 158–166. arXiv:hep-th/9707131. MR1489300

[120] A. Varchenko, *Quantized Knizhnik–Zamolodchikov equations, quantum Yang–Baxter equation, and difference equations for q-hypergeometric functions*, Comm. Math. Phys. **162** (1994), no. 3, 499–528. MR1277474

[121] A. Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, World Sci. Publ., River Edge, NJ, 1995. MR1384760

[122] E. Witten, *Quantum field theory, Grassmannians, and algebraic curves*, Comm. Math. Phys. **113** (1988), no. 4, 529–600. MR923632

[123] E. Witten, *Mirror manifolds and topological field theory*, Essays on mirror symmetry I, 1992, pp. 121–160. arXiv:hep-th/9112056. MR1191422

[124] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nuclear Phys. B **403** (1993), no. 1–2, 159–222. arXiv:hep-th/9301042.

[125] E. Witten, *Geometric Langlands from six dimensions*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, AMS, Providence, RI, 2010, pp. 281–310. arXiv:0905.2720 [hep-th]. MR2648898

[126] E. Witten, *Geometric Langlands duality and the equations of Nahm and Bogomolny*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), no. 4, 857–895. arXiv:0905.4795 [hep-th]. MR2672073

[127] E. Witten, *Fivebranes and knots*, Quantum Topol. **3** (2012), no. 1, 1–137. arXiv:1101.3216 [hep-th]. MR2852941

[128] E. Witten, *A new look at the path integral of quantum mechanics*, Surveys in differential geometry. Volume XV. Perspectives in mathematics and physics, Surv. Differ. Geom., vol. 15, Int. Press, Somerville, MA, 2011, pp. 345–419. arXiv:1009.6032 [hep-th]. MR2817248

[129] E. Witten, *More on gauge theory and geometric Langlands*, Adv. Math. **327** (2018), 624–707. arXiv:1506.04293 [hep-th]. MR3762001

[130] C. Woodward, *Moment maps and geometric invariant theory.* arXiv:0912.1132.

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