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SYMMETRIC DIFFERENTIAL OPERATORS OF FRACTIONAL ORDER AND THEIR EXTENSIONS

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Abstract. This paper is devoted to the description of symmetric operators and the justification of Green's formula for a fractional analogue of the Sturm–Liouville operator of order 2α, where $\frac{1}{2} < \alpha < 1$.

1. Introduction

In the theory of differential equations, the description and study of self-adjoint problems is of particular significance. One of the methods for describing such problems is the theory of self-adjoint extensions, which is quite well developed, for example, in the monographs [1] and [2]. In the theory of extensions and restrictions, one of the main points is the justification of Green's formula. In this paper, Green's formula is derived and justified for a differential operator of fractional order. We also introduce the concept of fractional differentiation from generalised functions. For clarity, we present some classes of self-adjoint problems for a fractional analogue of the Sturm–Liouville operator. In view of the physical applications, the spectral properties of fractional operators are the subject of intensive investigation, especially in the applied papers [3], [4], [5], [6], and [7]. One of the first works to study the spectral properties of differential operators of fractional order was that of Dzhrbashyan [8]. After Dzhrbashyan's work, scientists began an active study of the properties of certain special functions generated by fractional equations. This work includes the papers [9], [10], [11], [12], [13], [14], [15], and [16]. Since, on the whole, fractional operators are not symmetric, only nonself-adjoint problems were considered in all of the above-mentioned papers, (see also [17], [18], and [19]). A symmetric differential operator of fractional order was first introduced in [20] by Klimek and Agrawal in a weight class of continuous functions. In this paper, an attempt is made to justify Green's formula for a differential equation of fractional order with a further description of the class of self-adjoint problems.

The ultimate goal of this paper is to describe the class of self-adjoint problems in a Hilbert space for a differential operator of fractional order. In fact, we find the symmetric Caputo–Riemann–Liouville operator of order 2α (where $\frac{1}{2} < \alpha < 1$), which is in some sense a fractional analogue of the Sturm–Liouville operator.

The problems considered in the paper can be applied in solving problems in the mathematical modeling of physical and mechanical processes. For example, such problems can arise in using the Fourier method to solve problems of subdiffusion, superdiffusion, anomalous diffusion, the fractional Laplacian, and others (see [21], [22], [23], and [24]).

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Moreover, the study of spectral problems for differential operators of fractional order is important for the enrichment and improvement of the theory of fractional calculi.

2. Operators of fractional differentiation and their properties

In this section, we define operators of fractional integro-differentiation; see [25], [26], and [27].

Definition 1. Given a function f, defined on the interval $[0,1]$, if the integrals

$$I_0^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in (0,1],$$

and

$$I_1^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s)\, ds, \quad t \in [0,1)$$

exist, then we call them the *left-* and *right-hand integral Riemann–Liouville operators of fractional order* $\alpha > 0$, respectively.

Definition 2. We define *left-* and *right-hand differential Riemann–Liouville operators of fractional orders* α $(0 < \alpha < 1)$ in the following manner:

$$D_0^\alpha[f](t) = \frac{d}{dt} I_0^{1-\alpha}[f](t) \quad \text{and} \quad D_1^\alpha[f](t) = -\frac{d}{dt} I_1^{1-\alpha}[f](t),$$

respectively.

Definition 3. For $0 < \alpha < 1$, we call the actions

$$\mathcal{D}_0^\alpha[f](t) = D_0^\alpha[f(t) - f(0)] \quad \text{and} \quad \mathcal{D}_1^\alpha[f](t) = D_1^\alpha[f(t) - f(1)],$$

respectively, the *left-* and *right-hand operators of differentiation of order* α *(*$0 < \alpha < 1$*) in the sense of Caputo.*

We remark that other types of operators of fractional differentiation and their basic properties are investigated in the monographs [25], [26], and [27]. In the following statements we give some properties of integral and integro-differential Riemann–Liouville operators and fractional Caputo operators which will be used extensively in the present work.

Property 1 (see [27, pp. 73, 76, 96])**.** *Let* $0 < \alpha < 1$*, and let* $f \in \mathrm{L}_1(0,1)$*,* $I_1^{1-\alpha} f, I_0^{1-\alpha} f \in AC[0,1]$*. Then for* $0 < x < 1$ *the following equalities hold:*

$$I_0^\alpha I_0^\beta f(x) = I_0^{\alpha+\beta} f(x), \quad I_1^\alpha I_1^\beta f(x) = I_1^{\alpha+\beta} f(x), \quad 0 < \beta < 1,$$

$$I_1^\alpha D_1^\alpha f(x) = f(x) - I_1^{1-\alpha} f(0) \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)}, \quad I_0^\alpha D_0^\alpha f(x) = f(x) - I_0^{1-\alpha} f(0) \frac{x^{\alpha-1}}{\Gamma(\alpha)}.$$

If $f \in AC[0,1]$*, then*

$$I_0^\alpha \mathcal{D}_0^\alpha f(x) = f(x) - f(0), \quad I_1^\alpha \mathcal{D}_1^\alpha f(x) = f(x) - f(1), \quad 0 < x < 1.$$

Property 2 (see [25, p. 87])**.** *Let* $\alpha, \beta > 0$*, and let* $0 \le \varepsilon \le 1$*. Then for the function*

$$f(x) = C \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (1-x-\varepsilon)_*^{\alpha+\beta-1} = \begin{cases} 0, & 1-x \le \varepsilon, \\ C \dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (1-x)^{\alpha+\beta-1}, & 1-x > \varepsilon, \end{cases}$$

where C is some constant, the following equality holds for all $0 < x < 1$:

$$I_1^\alpha f(x) = \begin{cases} 0, & 1 - x \le \varepsilon, \\ C(1-x)^{\beta-1}, & 1 - x > \varepsilon. \end{cases}$$

Property 3. *Let $0 < \alpha < 1$, and let $f \in \mathrm{L}_2(0,1)$. Then for any $\varepsilon \in (0,1)$ and a constant C, the function*

$$f(x) = C(1-x-\varepsilon)_*^{\alpha-1} = \begin{cases} 0, & 1 - x \le \varepsilon, \\ C(1-x)^{\alpha-1}, & 1 - x > \varepsilon, \end{cases}$$

satisfies the equation $D_1^\alpha f(x) = 0$, $0 < x < 1$.

Property 4. *Let $0 < \alpha < 1$, and let $f \in \mathrm{L}_2(0,1)$. Then for any $\varepsilon \in (0,1)$ and constant C, the function*

$$f(x) = C\theta(1-x-\varepsilon) = \begin{cases} 0, & 1 - x \le \varepsilon, \\ C, & 1 - x > \varepsilon, \end{cases}$$

where $\theta(x)$ is the Heaviside function, satisfies the equation $\mathcal{D}_0^\alpha f(x) = 0$, $0 < x < 1$.

Property 5. *Let $0 < \alpha < 1$, and let $f \in \mathrm{L}_2(0,1)$. Then for any $\varepsilon \in (0,1)$ and for arbitrary constants C_1 and C_2, the function*

$$f(x) = C_1(1-x-\varepsilon)_*^{\alpha-1} + C_2(1-x-\varepsilon)_*^{\alpha}$$

satisfies the equation $D_1^\alpha f(x) = C_2\theta(1-x-\varepsilon)$, $0 < x < 1$.

Property 6 (see [26, p. 34])**.** *Let $u, v \in \mathrm{L}_2(0,1)$, $0 < \alpha < 1$. Then a formula for fractional integration by parts is given by*

$$(I_1^\beta u(t), v(t)) = (u(t), I_0^\beta v(t)).$$

Here and in what follows, we denote by $(\cdot,\cdot)$ the scalar product in the Hilbert space $\mathrm{L}_2(0,1)$.

3. The main part

In what follows, let $\frac{1}{2} < \alpha < 1$. We consider the expression

$$Lu(x) := \mathcal{D}_0^\alpha[D_1^\alpha[u]](x), \quad 0 < x < 1. \tag{3.1}$$

Our goal is to investigate the spectral properties of operators generated by the differential equation of fractional order (3.1) in the space $\mathrm{L}_2(0,1)$. But first we define the operator in Hölder classes. We consider the spectral problem

$$Lu(x) = \lambda u(x),\ 0 < x < 1, \tag{3.2}$$

in the space $H_1^{2\alpha+o}([0,1]) := \{\varphi \in H^{2\alpha+o}([0,1])\colon \varphi(1) = 0, \ldots, \varphi^{(m)}(1) = 0\}$, where $m = [2\alpha + o]$, and $H^{2\alpha+o}([0,1])$ is the Hölder space with parameter $2\alpha + o$. Here o is a positive number that is sufficiently small so that $o < 1 - \alpha$. In fact, we are dealing with the following spaces:

$$\begin{aligned} H_1^{2\alpha+o}([0,1]) &:= \{\varphi \in H^{2\alpha+o}([0,1])\colon \varphi(1) = 0,\ \varphi'(1) = 0\}, \\ H_1^{\alpha+o}([0,1]) &:= \{\varphi \in H^{\alpha+o}([0,1])\colon \varphi(1) = 0\}, \\ H_1^{o}([0,1]) &:= \{\varphi \in H^{o}([0,1])\colon \varphi(1) = 0\}. \end{aligned}$$

It follows from Samko, Kilbas, and Marichev's book [25, Chapter 1, Theorem 3.2], that the integro-differential operator L is well defined in $H_1^{2\alpha+o}([0,1])$, which implies that the functionals

$$\begin{aligned} \xi_1^-(u) &:= I_1^{1-\alpha}[u](0), & \xi_2^-(u) &:= I_1^{1-\alpha}[u](1), \\ \xi_1^+(u) &:= D_1^\alpha[u](0), & \xi_2^+(u) &:= D_1^\alpha[u](1) \end{aligned}$$

are well defined for all $H_1^{2\alpha+o}([0,1])$. We let L_0 denote the operator generated by the fractional differential expression (3.1) with "boundary" conditions

$$\xi_2^-(u)=0 \quad \text{and} \quad \xi_1^+(u)=0. \tag{3.3}$$

Then, in view of the properties and definitions given in § 2 (see also [25, Chapter 1]), we represent the operator inverse to L_0 in the form

$$L_0^{-1}f(x)=I_1^\alpha I_0^\alpha f(x):=\int_0^1 K(x,s)f(s)\,ds, \quad 0<x<1,$$

for

$$f\in\tilde{H}_1^o([0,1]):=\Big\{v\in H_1^o([0,1]):\\ \int_0^1 v(s)(1-s)^{2\alpha}\,ds=0 \text{ and} \int_0^1 v(s)(1-s)^{2\alpha-1}\,ds=0\Big\},$$

as the operator $L_0^{-1}\colon \tilde{H}_1^o\to H_1^{2\alpha+o}$, with symmetric kernel $K(\cdot,\cdot)$ from $\mathrm{L}_2(0,1)\otimes\mathrm{L}_2(0,1)$. Since $S:=\operatorname{span}\{(1-x)^k,\,k\in\mathbb{N}\}\subset H_1^o([0,1])$, the sets S and

$$\tilde{S}:=\Big\{v\in S\colon \int_0^1 v(s)(1-s)^{2\alpha}ds=0 \text{ and } \int_0^1 v(s)(1-s)^{2\alpha-1}ds=0\Big\}$$

are equipotent, so we conclude that the closure of the space $\tilde{H}_1^o([0,1])$ with respect to the L_2-norm is $\mathrm{L}_2(0,1)$. Hence, L_0^{-1} has a continuous extension to a compact operator in $\mathrm{L}_2(0,1)$. Compactness, in turn, gives us the existence of a nonempty discrete spectrum with eigenfunctions that form an orthogonal basis of $\mathrm{L}_2(0,1)$.

We denote by λ_k, $k\in\mathbb{N}$, the eigenvalues of the spectral problem (3.2)–(3.3) ordered by increasing absolute value, and by u_k, $k\in\mathbb{N}$, the corresponding eigenfunctions, that is,

$$\mathcal{D}_0^\alpha[D_1^\alpha[u_k]](x)=\lambda_k u_k(x), \quad 0<x<1,$$
$$\xi_2^-(u_k)=0 \quad \text{and} \quad \xi_1^+(u_k)=0$$

for all $k\in\mathbb{N}$. In this way, the domain of definition of L_0,

$$\operatorname{Dom}(L_0):=\{u\in H^{2\alpha}([0,1]):\xi_2^-(u)=0,\;\; \xi_1^+(u)=0\},$$

is nonempty.

We now introduce a space of test functions $C_{L_0}^\infty([0,1])$ in the following manner:

$$C_{L_0}^\infty([0,1]):=\bigcap_{k=1}^\infty \operatorname{Dom}(L_0^k),$$

where $\operatorname{Dom}(L_0^k)$ is the domain of definition of the operator L_0^k. Here L_0^k is the k-fold iterated operator L_0 with domain of definition

$$\operatorname{Dom}(L_0^k):=\{L_0^{k-j-1}u\in\operatorname{Dom}(L_0), j=0,1,\dots,k-1\}$$

for $k\geq 2$. The space of test functions $C_{L_0}^\infty([0,1])$ is nonempty as a set, since the linear span of all eigenfunctions is contained in $C_{L_0}^\infty([0,1])$. For further properties of the space $C_{L_0}^\infty([0,1])$, see [29], [30], [31], where the properties of the test functions (introduced via basis functions) are well studied (see also [32], [33], where special cases are given). We denote the space dual to $C_{L_0}^\infty([0,1])$ by $\mathcal{D}'_{L_0}(0,1)$ (the space of continuous functionals defined on $C_{L_0}^\infty([0,1])$).

We now turn to the definition of fractional derivatives with respect to generalised functions. To begin with, we note that for $u,v\in C_{L_0}^\infty([0,1])$, the following equality holds:

$$(\mathcal{D}_0^\alpha[D_1^\alpha u],v)=(u,\mathcal{D}_0^\alpha[D_1^\alpha v]), \tag{3.4}$$

where both sides exist in the classical sense.

In fact, the equality (3.4) follows from direct calculation of $(\mathcal{D}_0^\alpha[D_1^\alpha u], v)$. By definition,

$$(\mathcal{D}_0^\alpha[D_1^\alpha u], v) = \int_0^1 \Big(\int_0^x (x-t)^{-\alpha} \frac{d}{dt} D_1^\alpha u(t)\, dt \Big) v(x)\, dx,$$

and, changing the order of integration, we obtain

$$\int_0^1 \Big(\int_0^x (x-t)^{-\alpha} \frac{d}{dt} D_1^\alpha u(t)\, dt \Big) v(x)\, dx = \int_0^1 \frac{d}{dt} D_1^\alpha u(t) \Big(\int_t^1 (x-t)^{-\alpha} v(x)\, dx \Big)\, dt. \tag{3.5}$$

Integrating by parts on the right-hand side of equation (3.5), we have

$$\begin{aligned}\int_0^1 \frac{d}{dt} D_1^\alpha u(t) \Big(\int_t^1 (x-t)^{-\alpha} v(x)\, dx \Big)\, dt &= D_1^\alpha u(t) I_1^{1-\alpha} v(t) \Big|_0^1 + (D_1^\alpha u, D_1^\alpha v) \\ &= D_1^\alpha u(t) I_1^{1-\alpha} v(t) \Big|_0^1 - I_1^{1-\alpha} u(t) D_1^\alpha v(t) \Big|_0^1 + \int_0^1 I_1^{1-\alpha} u(t) \frac{d}{dt} D_1^\alpha v(t)\, dt.\end{aligned}$$

Applying Property 6 to $(I_1^{1-\alpha}u, \frac{d}{dt}D_1^\alpha v)$, in view of the equivalent definition of the Caputo derivative (see [25, Chapter 1]) we obtain

$$\Big(I_1^{1-\alpha} u, \frac{d}{dt} D_1^\alpha v \Big) = (u, \mathcal{D}_0^\alpha[D_1^\alpha]v).$$

As a result, this gives us Green's formula

$$(\mathcal{D}_0^\alpha[D_1^\alpha u], v) = (u, \mathcal{D}_0^\alpha[D_1^\alpha]v) + \sum_{i=1}^{2} \big[\xi_i^-(u)\xi_i^+(v) - \xi_i^-(v)\xi_i^+(u) \big]. \tag{3.6}$$

Since $u, v \in C_{L_0}^\infty([0,1])$, (3.4) follows from (3.6).

We define an action of the operator L on a generalised function $u \in \mathcal{D}'_{L_0}(0,1)$. We set

$$(Lu, v) := (u, \mathcal{D}_0^\alpha[D_1^\alpha v]) \tag{3.7}$$

for all $v \in C_{L_0}^\infty([0,1])$. The value $(u, \mathcal{D}_0^\alpha[D_1^\alpha v])$ exists because it follows from $v \in C_{L_0}^\infty([0,1])$ that also $\mathcal{D}_0^\alpha[D_1^\alpha v] \in C_{L_0}^\infty([0,1])$. Thus, the action of the operator L in the space of generalised functions $\mathcal{D}'_{L_0}(0,1)$, introduced in formula (3.7), is well defined.

We now consider the expression

$$Lu(x) := \mathcal{D}_0^\alpha[D_1^\alpha[u]](x), \quad 0 < x < 1, \tag{3.8}$$

in $\mathrm{L}_2(0,1)$. In order for L to be properly defined in $\mathrm{L}_2(0,1)$, we introduce the space $\mathrm{W}_2^{2\alpha}(0,1)$ as the closure of $H_1^{2\alpha+o}([0,1])$ in the norm

$$\|u\|_{\mathrm{W}_2^{2\alpha}(0,1)} := \|u\|_{L_2(0,1)} + \|\mathcal{D}_0^\alpha D_1^\alpha u\|_{L_2(0,1)}.$$

In fact, $\mathrm{W}_2^{2\alpha}(0,1)$ with the norm we have introduced is a Banach space. Moreover, it is a Hilbert space with scalar product

$$(u,v)_{\mathrm{W}_2^{2\alpha}(0,1)} := (u,v) + (\mathcal{D}_0^\alpha D_1^\alpha u, \mathcal{D}_0^\alpha D_1^\alpha v).$$

We define L_m as the operator acting from $\mathrm{L}_2(0,1)$ into $\mathrm{L}_2(0,1)$ by the formula (3.8) with domain of definition

$$\mathrm{Dom}(L_m) = \big\{ u \in \mathrm{W}_2^{2\alpha}(0,1) \colon \xi_1^-(u) = \xi_2^-(u) = \xi_1^+(u) = \xi_2^+(u) = 0 \big\}.$$

We also introduce the operator $L_M \colon \mathrm{L}_2(0,1) \to \mathrm{L}_2(0,1)$ generated by the expression (3.8) with domain of definition $\mathrm{Dom}(L_M) := \{u \in \mathrm{W}_2^{2\alpha}(0,1)\}$.

Further, we introduce a class of 2×4 matrices, by means of which we define boundary forms for a differential equation of fractional order $\mathcal{D}_0^\alpha\,[D_1^\alpha[u]]$.

Definition 4. The matrix

$$\theta := \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}$$

is called an *S-matrix* if it can be written in one of the following forms:

$$\begin{pmatrix} 1 & 0 & r & c \\ 0 & 1 & -c & d \end{pmatrix}, \begin{pmatrix} d & 1 & 0 & r \\ c & 0 & 1 & d \end{pmatrix},$$

$$\begin{pmatrix} 1 & d & r & 0 \\ 0 & c & -d & 1 \end{pmatrix}, \begin{pmatrix} r & c & 1 & 0 \\ -c & d & 0 & 1 \end{pmatrix},$$

where $r, c, d \in \mathbb{R}$. We assume here that the matrices

$$\begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}, \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \gamma\theta_{21} & \gamma\theta_{22} & \gamma\theta_{23} & \gamma\theta_{24} \end{pmatrix} \text{ for } (\gamma \neq 0),$$

$$\begin{pmatrix} \theta_{11} \pm \theta_{21} & \theta_{12} \pm \theta_{22} & \theta_{13} \pm \theta_{23} & \theta_{14} \pm \theta_{24} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix} \text{ and } \begin{pmatrix} \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\ \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \end{pmatrix}$$

have one entry.

Theorem 1. *Let θ be an S-matrix. Then the operator L_θ generated by the expression*

$$\mathcal{D}_0^\alpha D_1^\alpha u(x) = f(x), \quad 0 < x < 1,$$

for $u \in \mathrm{W}_2^{2\alpha}(0,1)$ with "boundary" conditions

$$\theta_{11}\xi_1^-(u) + \theta_{12}\xi_2^-(u) + \theta_{13}\xi_1^+(u) + \theta_{14}\xi_2^+(u) = 0,$$
$$\theta_{21}\xi_1^-(u) + \theta_{22}\xi_2^-(u) + \theta_{23}\xi_1^+(u) + \theta_{24}\xi_2^+(u) = 0$$

is a self-adjoint extension of the operator L_m in the class $\mathrm{W}_2^{2\alpha}(0,1)$.

We note that, on the whole, the results of Theorem 1 are false for the case $\alpha < 1/2$.

4. Proof of Theorem 1

We give below some properties of the operators L_m and L_M.

Lemma 1. *For arbitrary $\varepsilon \in [0,1]$, any linear combination of the functions $(1-x-\varepsilon)_*^\alpha$ and $(1-x-\varepsilon)_*^{\alpha-1}$ is contained in the kernel of the operator L_M $(\operatorname{Ker} L_M)$.*

The proof of Lemma 1 follows from the assertions of Lemmas 2, 3, 4, and 5.

Lemma 2. *The equation $L_m u = g$ has a solution $u \in \operatorname{Dom}(L_m)$ if and only if there exists $f \in \mathrm{L}_2(0,1)$ such that $(f,v) = 0$ for arbitrary $v \in \operatorname{Ker} L_M$:*

$$\mathcal{R}(L_m) \oplus \operatorname{Ker} L_M = \mathrm{L}_2(0,1).$$

Proof. Let $f \in \mathcal{R}(L_m)$. Then there is a function $w \in \mathrm{L}_2(0,1)$ such that for any $v \in \operatorname{Ker} L_M$, we have the equality

$$(f,v) = (L_m w, v) = (w, L_M v) = 0.$$

We now fix a function $f \in \mathrm{L}_2(0,1)$ that satisfies the equality $(f,v) = 0$ for all $v \in \operatorname{Ker} L_M$. In view of the definition of L_M, there is a function $g \in \operatorname{Dom}(L_M)$ such that $L_M g = f$. It is easy to see that for arbitrary $v \in \operatorname{Ker} L_M$, we have

$$0 = (f,v) = (L_M g, v) = \sum_{i=1}^{2} [\xi_i^-(v)\xi_i^+(g) - \xi_i^-(g)\xi_i^+(v)]. \tag{4.1}$$

Further, it follows from Lemma 1 that the kernel of L_M consists of an infinite collection of linearly independent functions, and since v is arbitrary, it follows from the identity (4.1) that

$$\xi_i^-(g) = \xi_i^+(g) = 0, \quad i = 1, 2.$$

Consequently, $f \in \mathcal{R}(L_m)$. This completes the proof of Lemma 2. □

Corollary 1. *The set* $\mathrm{Dom}(L_m)$ *is dense in* $\mathrm{L}_2(0,1)$.

Proof. Let $g \in \mathrm{L}_2(0,1)$ be the orthogonal to the lineal of $\mathrm{Dom}(L_m)$. We find a function v that is an arbitrary solution of the equation $L_M v = g$. Then for any $u \in \mathrm{Dom}(L_m)$ we obtain

$$0 = (u, g) = (u, L_M v) = (L_m u, v).$$

In view of Lemma 2, we have $v \in \operatorname{Ker} L_M$. Consequently, $g = L_M v = 0$. The lemma is proved. □

4.1. **Proof of Theorem 1.** Since for any $u, v \in \mathrm{Dom}(L_m)$

$$(L_m u, v) = (u, L_m v),$$

by definition (see [28]) L_m is Hermitian. Also, in view of Corollary 1, L_m is symmetric. Thus, for L_θ to be a selfadjoint operator, it is sufficient that

$$\mathrm{Dom}(L_\theta) = \mathrm{Dom}(L_\theta^*). \tag{4.2}$$

As a result, the validity of Theorem 1 follows from simple calculations, taking formula (3.6) into account.

5. Some spectral properties of L_θ

Theorem 2. *Let θ have the form*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \theta_{21} & 0 & \theta_{23} & 0 \end{pmatrix}.$$

Then the operator L_θ is self-adjoint and the following assertions are valid for it:

(i) *The operator L_θ^{-1} is completely continuous in* $\mathrm{L}_2(0,1)$.

(ii) *The spectrum of the operator L_θ is real-valued and discrete, and the system of eigenfunctions forms a complete orthogonal system in* $\mathrm{L}_2(0,1)$.

Proof.

(i) If $\theta_{21} \neq 0$ and $\theta_{21} \neq \theta_{23}$, then we represent the inverse operator in the form

$$L_\theta^{-1} f(x) = \frac{\theta_{21}(1-x)^\alpha}{(\theta_{21} - \theta_{23})\Gamma(\alpha+1)} [I_0^{\alpha+1} f](1) + I_1^\alpha I_0^\alpha f(x).$$

For $\theta_{21} = 0$ we have

$$L_\theta^{-1} f(x) = I_1^\alpha I_0^\alpha f(x), \quad 0 < x < 1.$$

The compactness of the operator L_θ^{-1} in $\mathrm{L}_2(0,1)$ now follows.

(ii) In view of the complete continuity of L_θ^{-1}, we see that the spectrum is discrete, and the system of eigenfunctions forms a complete orthogonal system in $\mathrm{L}_2(0,1)$. Using [28], it then follows from the selfadjointness of L_θ that all the eigenvalues are real. □

Theorem 3. *Let θ have one of the following forms:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

$$\begin{pmatrix} \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Then for all $\rho \in \mathbb{R}$, the operator L_θ is positive in the space $\mathrm{L}_2(0,1)$.

Proof. To prove the theorem, it is sufficient to verify the inequality

$$(\mathcal{D}_0^\alpha[D_1^\alpha u], u) \geq 0.$$

So, we calculate

$$(\mathcal{D}_0^\alpha[D_1^\alpha u], u) = \int_0^1 \left(\int_0^x (x-t)^{-\alpha} \frac{d}{dt} D_1^\alpha u(t)\, dt \right) u(x)\, dx.$$

Changing the order of integration, we have

$$\int_0^1 \left(\int_0^x (x-t)^{-\alpha} \frac{d}{dt} D_1^\alpha u(t)\, dt \right) u(x)\, dx = \int_0^1 \frac{d}{dt} D_1^\alpha u(t) \left(\int_t^1 (x-t)^{-\alpha} u(x)\, dx \right) dt.$$

Integrating the right-hand side of the last integral by parts, we obtain

$$\int_0^1 \frac{d}{dt} D_1^\alpha u(t) \left(\int_t^1 (x-t)^{-\alpha} u(x)\, dx \right) dt = D_1^\alpha u(t) I_1^{1-\alpha} u(t) \Big|_0^1 + (D_1^\alpha u, D_1^\alpha u).$$

From any of the forms (5.1) and (5.2), we obtain the identity

$$D_1^\alpha u(t) I_1^{1-\alpha} u(t) \Big|_0^1 = 0,$$

which implies the validity of the theorem. □

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