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A MODIFICATION OF THE BLOOM–GRAHAM THEOREM: THE INTRODUCTION OF WEIGHTS IN THE COMPLEX TANGENT SPACE

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1. Introduction

Polynomial models are widely used in CR-geometry. Their main application is connected with the study of automorphisms of germs of CR-manifolds, and here an essential role is played by the extension of a function by homogeneous and quasihomogeneous components (see [5–8]). Via this approach, the coordinates of the complex tangent and the transversal directions are distinguished by means of weights. But there are situations when it is necessary to introduce differences within the complex tangent by means of weights—such a case is considered in Example 3 (see Section 3).

In [1] a characteristic of the germ of a real submanifold of a complex space was introduced; this is now called the Bloom–Graham germ type. There is also a distinction between germs of finite and infinite type. The significance of this characteristic is based on the fact that if the type is finite, then it has two equivalent definitions: coordinateless, in the form of the Lie algebra of tangent vector fields, and coordinate, in the form of local equations of the germ. We give more precise definitions.

Let $\mathcal{L}^1$ be the distribution of complex tangents to a generating CR-manifold in a neighbourhood of a point p. We can associate a system of distributions $\mathcal{L}^j$ with it, defined inductively, namely $\mathcal{L}^{j+1} = [\mathcal{L}^j, \mathcal{L}^1] + \mathcal{L}^j$, where $[\mathcal{L}^j, \mathcal{L}^1]$ are the distributions generated by the brackets of the fields from $\mathcal{L}^j$ and $\mathcal{L}^1$ and conjugate to them. We set n_1 equal to the number of the first jump of dimension of $\mathcal{L}^j$, namely to the first number $j > 1$, so that $q_1 = \dim \mathcal{L}^j - \dim \mathcal{L}^1 > 0$. In the same way, we encode the place (n_2) of the second jump of dimension of $\mathcal{L}^j$ and this jump itself (q_2). And so on. Let s be the number of jumps. Then we obtain a nondecreasing sequence of natural numbers $2 \le m_1 \le \cdots \le m_k \le \infty$.

Definition 1. The sequence

$$(m_1, \dots, m_k) = (\underbrace{n_1, \dots, n_1}_{q_1}, \underbrace{n_2, \dots, n_2}_{q_2}, \dots, \underbrace{n_s, \dots, n_s}_{k-q_1-\cdots-q_{k-1}})$$

is called the *geometric type*.

If all $m_i \le \infty$, then we say that the germ is of *finite type*.

We turn to the definition of coordinate type.

Suppose that a germ M_p generating a CR-manifold whose CR-dimension is equal to $n-k$ and with codimension equal to k is given at a point $p \in \mathbb{C}^n$. We introduce a coordinate system $(z_1, \dots, z_{n-k}, w_1, \dots, w_k)$ in a neighbourhood of p, and assign a

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weight to all variables: the weight of each z_j is equal to 1, and that of w_i is equal to m_i, $2 \le m_1 \le \dots \le m_k$. Suppose that the equations of the germ can be written in the form

$$r_i = 2\,\mathrm{Re}\; w_i + p_i(z, \bar{z}, w, \bar{w}) + o(m_i) = 0, \quad i = 1, \dots, k, \tag{1.1}$$

where p_i is a homogeneous polynomial of weight m_i, and $o(m_i)$ is a function whose formal Taylor series consists of terms of higher weight.

Definition 2. The resulting sequence $(m_1, \dots, m_k)$ is called the *coordinate type.*

Theorem 1 (Bloom–Graham, 1977, [1])**.** *Let M_p be the germ of a generating analytic manifold of finite geometric type $(m_1, \dots, m_k)$. Then there exists a coordinate system in which it is given by equations of the form* (1.1)*, that is, it has the same coordinate type.*

It is clear that the coordinates $z = (z_1, \dots, z_{n-k})$ are the coordinates of a complex tangent to M at the point p, and $w = (w_1, \dots, w_k)$ are the coordinates which complete z to the whole space.

In this paper we give a generalisation of the Bloom–Graham type: weighted type. The modification, connected with the choice of weights in the complex tangent, is the subject both of a definition of the same kind as the Bloom–Graham type itself and to an assertion analogous to the Bloom–Graham theorem. Thus, this weighted type can turn out to be finite for one choice of weights, but infinite for another (see Example 3). The essence of this generalisation lies in the fact that arbitrary natural weights are assigned to different coordinates of the complex tangent $(z_1, \dots, z_{n-k})$. This gives the weighted type more flexibility and convenience in applications [5,8]. Further (see Theorem 2), we prove an analogue of the Bloom–Graham theorem on the relationship between the coordinate and coordinateless definitions of this weighted type. The given theorem demonstrates the connection between geometric data which can be obtained by calculating the commutators of holomorphic tangent vector fields (all the necessary information about these commutators is contained in the geometric type of the surface) and the representability of the surface by equations of special form. The main difference is that in our considerations we use the weights of commutators of vector fields, rather than their lengths. In the notation and the structure of the article, we follow [1], with the exception of the term "holomorphic vector field", which is now understood differently. Namely, those fields that are called holomorphic vector fields in [1] are now called fields of type (1,0) or complex vector fields.

2. Basic definitions

Let M be a smooth real submanifold of an open subset U in $\mathbb{C}^n$, $p \in M$. A *complex vector field* on U is a smooth vector field F for which $F(p) \in T^{1,0}(\mathbb{C}^n, p)$, where $T^{1,0}(\mathbb{C}^n, p)$ is a holomorphic subspace of the complexified tangent space $\mathbb{C}T(\mathbb{C}^n, p) = T^{1,0}(\mathbb{C}^n, p) \oplus T^{0,1}(\mathbb{C}^n, p)$ at the point p. A complex vector field tangent to M is an F such that $F(p) \in \mathbb{C}T(M, p)\ \forall p \in M$, and $F(p) \in T^{1,0}(\mathbb{C}^n, p)\ \forall p \in U$.

Let F_i, $i = 1, \dots, k$, be a collection of complex vector fields. We say that an expression of the form $F = [F_{i_\mu}, [F_{i_{\mu-1}}, \dots [F_{i_2}, F_{i_1}] \dots]]$, where $1 \le i_j \le k$, is a commutator of length μ.

Let $\mathcal{L}^\mu$ be the module of vector fields (over $C^\infty(U)$, $U \in \mathbb{C}^n$ an open subset) generated by complex tangent vector fields, their conjugates and commutators of such fields of length at most μ.

We now describe the modification of the geometric definition of type.

The *weights* for local coordinates $(z_1, \dots, z_{n-k}, w_1, \dots, w_k)$ at a point p are defined as follows: z_i has weight λ_i, where $\lambda = (\lambda_1, \dots, \lambda_{n-k})$ is an *arbitrary* collection of natural numbers, and w_i has weight l_i, where $l = (l_1, \dots, l_k)$, $2 \le l_1 \le \dots \le l_k \le$

$+\infty$, is a collection of natural numbers whose values we define later. Such a coordinate system is called a *weighted coordinate system*. The differential operators $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial w_j}$ have the corresponding negative weights $-\lambda_i, -l_j$.

We write $wt_P(\phi) = \gamma$ if in the formal Taylor series of the function ϕ there is a monomial of weight γ, but no monomials of smaller weight. A differential monomial is an expression of the form

$$B\, z^\alpha \bar{z}^\beta w^\gamma \bar{w}^\delta \frac{\partial^a}{\partial z^a} \frac{\partial^b}{\partial \bar{z}^b} \frac{\partial^c}{\partial w^c} \frac{\partial^d}{\partial \bar{w}^d}, \tag{2.1}$$

which is an operator of partial differentiation with respect to a collection of coordinates with coefficients equal to the product of a constant and the degree of the coordinate (here B is a nonzero constant, and $\alpha = (\alpha_1, \dots, \alpha_{n-k})$, $\beta = (\beta_1, \dots, \beta_{n-k})$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\delta = (\delta_1, \dots, \delta_k)$, $a = (a_1, \dots, a_{n-k})$, $b = (b_1, \dots, b_{n-k})$, $c = (c_1, \dots, c_k)$, $d = (d_1, \dots, d_k)$ are multi-indices). The weight of the differential monomial (2.1) is set equal to

$$\sum_{i=1}^{n-k} (\alpha_i + \beta_i - a_i - b_i)\lambda_i + \sum_{i=1}^{k} (\gamma_i + \delta_i - c_i - d_i) l_i.$$

The weight of a differential operator that is the sum of differential monomials is the minimum of the weights of these monomials. If the weights of all monomials of a differential operator Q are the same and are equal to γ, then we say that Q is of *homogeneous weight* γ.

We say that a manifold is *homogeneous* with respect to a given choice of weights if, in a system $(z, w; \lambda, l)$ of local coordinates with weights, it is given by a function of the form

$$r_i = 2\,\mathrm{Re}\; w_i + p_i(z, \bar{z}, w_1, \bar{w}_1, \dots, w_i, \bar{w}_i),$$

where p_i is of homogeneous weight m_i.

Surfaces that are not homogeneous can be regarded as perturbations of homogeneous surfaces by terms of larger weight, since the homogeneous part of smaller weight plays the main role in the proof. It is a model with respect to the class of perturbed surfaces (see [8]).

We choose the weights of the variables $z = (z_1, \dots, z_{n-k})$ to be $\lambda = (\lambda_1, \dots, \lambda_{n-k})$. Then $\mathcal{L}^\mu$ breaks down into the weights of the components $\mathcal{L}^\mu_m = \mathcal{L}^\mu_m(\lambda)$, which are combined in the direct sum $\mathcal{L}_m = \bigoplus_\mu \mathcal{L}^\mu_m$.

Definition 3. We define the *weighted geometric type* at a point as the collection of those μ that correspond to an increase in the dimension of $\mathcal{L}_m(p)$, that is, those n_i for which $\dim_{\mathbb{C}}(\mathcal{L}_{m_i}(p)/\mathcal{L}_{m_i-1}(p)) = q_i \geq 1$, and moreover for an increase in dimension of at least 1, the corresponding m_i is repeated q_i times.

Thus, we obtain the collection

$$(m_1, \dots, m_k) = (\underbrace{n_1, \dots, n_1}_{q_1}, \underbrace{n_2, \dots, n_2}_{q_2}, \dots, \underbrace{n_s, \dots, n_s}_{k-q_1-\dots-q_{k-1}}),$$

where s is the number of jumps.

Suppose that a germ M_p of a generating manifold whose CR-dimension is equal to $n-k$ and codimension is equal to k is given at a point p. We introduce a coordinate system $(z_1, \dots, z_{n-k}, w_1, \dots, w_k)$ in a neighbourhood of p, and assign weights to all variables: the weight of all the z_j is equal to 1, and the weight of w_i is equal to m_i, $2 \leq m_1 \leq \dots \leq m_k$. Suppose that the equations of the germ can be written in the form

$$r_i = 2\,\mathrm{Re}\; w_i + p_i(z, \bar{z}, w, \bar{w}) + o(m_i) = 0, \quad i = 1, \dots, k, \tag{2.2}$$

where p_i is a homogeneous polynomial of weight m_i, and $o(m_i)$ is a function whose formal Taylor series consists of terms of higher weight.

Definition 4. The resulting sequence $(m_1, \dots, m_k)$ is called the *weighted coordinate type.*

We introduce a *scalar product* in the space of algebraic polynomials of homogeneous weight in the variables $z, \bar{z}, w, \bar{w}$ (that is, those polynomials that are equal to a sum of monomials of the same weight) in the following manner:

$$(p, q) = p\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}\right) \bar{q}(z, \bar{z}, w, \bar{w}), \tag{2.3}$$

that is, we replace the variables $z, \bar{z}, w, \bar{w}$ in $p(z, \bar{z}, w, \bar{w})$ by the corresponding differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}$ and apply the resulting differential polynomial to $\bar{q}(z, \bar{z}, w, \bar{w})$.

In the space of differential operators of homogeneous weight with constant coefficients, we put

$$(P, Q) = P\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}\right) \bar{Q}(z, \bar{z}, w, \bar{w}), \tag{2.4}$$

that is, we replace the differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}$ in $\bar{Q}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}\right)$ by the corresponding variables $z, \bar{z}, w, \bar{w}$ and apply the differential operator $P\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}\right)$ to the resulting polynomial $\bar{Q}(z, \bar{z}, w, \bar{w})$.

Let $\mathcal{A}$ be the set of all collections $(l_1, \dots, l_k)$, $l_1 \le \dots \le l_k$, where $2 \le l_i \le +\infty$. We introduce a partial order on $\mathcal{A}$ in the following manner: if $l = (l_1, \dots, l_k)$, $l' = (l'_1, \dots, l'_k)$, then $l \le l'$, if $l_i \le l'_i$ for all i.

3. Examples

We now look at some examples of types of surfaces.

Example 1. We consider an example of a surface of infinite type. Suppose that in the space $\mathbb{C}^6$ with coordinates $(z_1, z_2, z_3, w_1 = u_1 + iv_1, w_2 = u_2 + iv_2, w_3 = u_3 + iv_3)$, a surface of CR-dimension 3 and codimension 3 is given by the equations

$$r_1 = 2\,u_1 + |z_1|^2 = 0,$$
$$r_2 = 2\,u_2 + |z_1|^4 + |z_2|^2 + |z_3|^2 = 0,$$
$$r_3 = 2\,u_3 = 0.$$

We choose the weights of the coordinates $(z_1, z_2, z_3; w_1, w_2, w_3)$ to be equal to $(\lambda, l) = (1, 2, 2; 2, 4, \infty)$ so that the equations take a homogeneous form as in (2.2). Then the weighted geometric type at the origin is $(2, 4, \infty)$. Thus, the surface under consideration is a surface of infinite type.

Example 2. Suppose that in the space $\mathbb{C}^4$ with coordinates (z_1, z_2, w_1, w_2), a surface of CR-dimension 2 and codimension 2 is given by the equations

$$r_1 = 2\,u_1 + |z_1|^4 = 0,$$
$$r_2 = 2\,u_2 + |z_2|^2 = 0;$$

the collection of weights of the coordinates $(z_1, z_2; w_1, w_2)$ is equal to $(\lambda, l) = (1, 2; 4, 4)$. Then the weighted geometric type at the origin is equal to $(4, 4)$.

Example 3 (see also [4]). In $\mathbb{C}^3$ with coordinates (z_1, z_2, w) we consider the surface of CR-dimension 2 and codimension 1 given by the equation

$$r = 2\,u + |z_1|^2 + |z_2|^2(\bar{z}_1 + z_1) = 0.$$

This example in $\mathbb{C}^3$ is interesting because when we choose equal weights $(\lambda, l) = (1, 1; \infty)$ for the variables the type assumes an infinite value, but for an unequal choice of weights $(\lambda, l) = (2, 1; 4)$ the type is finite.

4. Formulation of the theorem

Theorem 2. *Let M_p be the germ of a generating analytic manifold of finite weighted geometric type $(m_1, \dots, m_k)$. Then there exists a system of weighted coordinates in which it is given by equations of the form* (2.2)*, that is, it has the same weighted coordinate type.*

5. Proof in the case of a homogeneous surface

We note first of all that if M is given by a system of equations reduced to canonical form (see below), then, as will be seen from the proof of the theorem, in this case the weighted coordinate type determines a weighted geometric type equal to the coordinate type.

Without loss of generality, we assume that the point p is the origin. Let $r_1, \dots, r_k$ be real-valued smooth functions, and let M be a homogeneous manifold, that is, in a system $(z, w; \lambda, l)$ of local coordinates with weights the defining functions have the form

$$r_i = 2\,\mathrm{Re}\; w_i + p_i(z, \bar{z}, w_1, \bar{w}_1, \dots, w_i, \bar{w}_i), \quad i = 1, \dots, k,$$

where p_i is of homogeneous weight m_i.

We consider the differential operators:

$$T_s^0 = \frac{\partial}{\partial z_s}, \quad s = 1, \dots, n-k,$$
$$T_s^i = T_s^{i-1} - T_s^{i-1}(r_i)\frac{\partial}{\partial w_i}, \quad i = 1, \dots, k.$$

Lemma.

1) *The T_s^i are tangents to the surface $r_1 = \dots = r_i = 0$ and have homogeneous weight $-\lambda_s$.*

2) *The commutator of the vector fields T_s^i, $\bar{T}_s^i$, $s = 1, \dots, k$, of length μ for fixed i have homogeneous weights $-\sum_{j=1}^{\mu} \lambda_{s_j}$, where s_j corresponds to the occurrence of $T_{s_j}^i$ or $\bar{T}_{s_j}^i$ in the commutator.*

Proof. The proof is by induction on i and is completely analogous to that given in [1] for the case when the variables z have equal weights. □

Considering the weight C of the commutator from the lemma, we remark that its coefficients A_j, B_j for $\frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_j}$, where $j \le i$, vanish at the origin if C is less than m_j, and are identically zero if C is greater than m_j. This follows from the fact that the weight of the coefficients is nonnegative. Thus, the origin has type $(n_1, \dots, n_k)$, where $n_i = m_i$, if there exists a commutator of weight m_i with nonzero coefficient for $\frac{\partial}{\partial w_j}$, and $n_i = +\infty$ otherwise.

Let X_s, Y_s, $s = 1, \dots, n-k$, be two groups of noncommuting variables. We consider a nested commutator of length μ of these variables of the form $[\dots, [X_s, Y_s] \dots]$, that is, the bracket nested inside all the rest contains variables from X_s and from Y_s. Such a commutator is represented in the form of a sum of noncommutative polynomials of the form $C = C_1 + C_2$, where C_1 is a sum of monomials of the form $A_1 \cdots A_\mu$, in which A_μ is a variable from X_s (the remaining A_j can be variables from both groups), and C_2 is a sum of monomials of the form $B_1 \cdots B_\mu$, in which B_μ is a variable from Y_s (the remaining B_j can be variables from both groups). The notation $C(T^i, \bar{T}^i)$ means that

T_s^i, $\bar{T}_s^i$, are substituted into the commutator instead of the variables X_s, Y_s, respectively. The following relation holds for commutators of the given type ([1]):

$$C\left(T^i,\bar{T}^i\right)=C\left(T^{i-1},\bar{T}^{i-1}\right)-C_1\left(T^{i-1},\bar{T}^{i-1}\right)(r_i)\frac{\partial}{\partial w_i}-C_2\left(T^{i-1},\bar{T}^{i-1}\right)(r_i)\frac{\partial}{\partial \bar{w}_i}.$$

If the length is $C\geq 2$, then repeated application of this relation gives

$$C\left(T^i,\bar{T}^i\right)=-\sum_{j=1}^{i}C_1\left(T^{j-1},\bar{T}^{j-1}\right)(r_j)\,\frac{\partial}{\partial w_j}-\sum_{j=1}^{i}C_2\left(T^{j-1},\bar{T}^{j-1}\right)(r_j)\,\frac{\partial}{\partial \bar{w}_j}.$$

Let $(w_j=u_j+iv_j)$ π_i be the operator dual to p_i, that is, $\pi_j=p_j\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar{z}_j},\frac{\partial}{\partial w_j},\frac{\partial}{\partial \bar{w}_j}\right)$ ($\pi_j=0$ if $p_j=0$). By obvious algebraic changes of the coordinates and defining functions that are carried over verbatim from the case considered in [1], when the variables have equal unit weight, the system of equations defining M can be reduced to the form

$$r_i=2u_i+p_i(z,\bar{z},v_1,\dots,v_{i-1}),$$

where the p_i satisfy the following conditions:

(1) the p_i are of homogeneous weights m_i or identically zero;
(2) there are no holomorphic or antiholomorphic terms;
(3) p_i depends on $z,\bar{z},v_1,\dots,v_{i-1}$ and not on $v_i,\dots,v_k$;
(4) there are no terms of the form cv^az^α or $\bar{c}v^az^\alpha$, where $v^a=v_1^{a_1}\cdots v_{i-1}^{a_{i-1}}$, $c\in\mathbb{C}$;
(5) $\left(\frac{\partial}{\partial v_{i-1}}\right)^{a_{i-1}}\left(\frac{\partial}{\partial v_{i-2}}\right)^{a_{i-2}}\cdots\left(\frac{\partial}{\partial v_j}\right)^{a_j}\pi_j(p_i)(0)=0$ $\forall j<i$ and nonnegative a^{i-1}, $\dots,a^j$.

The proof is by induction. Since homogeneity is preserved for the substitutions under consideration, the proof for equal weights also remains in force for arbitary weights.

Basis of induction: the functions p_1 depends only on $z,\bar{z}$, and the pluriharmonic terms can be removed by adding a holomorphic polynomial to w_1.

Induction step: we fix i and suppose that $p_1,\dots,p_{i-1}$ have the required form. The pluriharmonic terms can be removed by a holomorphic change of coordinates. In terms containing u_j, $\quad j<i$, we replace u_j by $-\frac{1}{2}p_j$. Further, we note that $cv^az^\alpha+\bar{c}v^a\bar{z}^\alpha=2\operatorname{Re}(bw^az^\alpha)+$(terms containing variables u) for a suitable choice of $b\in\mathbb{C}$ (here $c\in\mathbb{C}$). The term bw^az^α can be removed by adding it to the variable w_i. We replace the variables u_j by $-\frac{1}{2}p_j$, and by the induction hypothesis new terms of the forms v^az^α or $v^a\bar{z}^\alpha$ do not appear.

We now consider condition (5). Suppose that it is not satisfied. We first consider the case $(a_{i-1},a_{i-2},\dots,a_j)\neq 0$. We order the terms v^a lexicographically with respect to the exponents $(a_{i-1},a_{i-2},\dots,a_j)$. We show that we can add a term of the form $ev_{i-1}^{a_{i-1}}\cdots v_j^{a_j}p_j$, $e\in\mathbb{R}$, to p_i so that condition (5) is satisfied.

We choose $b\in\mathbb{C}$ such that $b(\sqrt{-1})^{a_{i-1}+\cdots+a_j}=-1$.

Then $\frac{2}{a_j+1}\operatorname{Re}\left(bw_{i-1}^{a_{i-1}}\cdots w_{j+1}^{a_{j+1}}w_j^{a_j+1}\right)=-2v_{i-1}^{a_{i-1}}\cdots v_j^{a_j}u_j+$(terms that contain either the factor u_k, $k\neq j$, or at least three factors u_k, $k=j,\dots,i-1$). We make the change of variables $w_i\longrightarrow w_i-\frac{be}{a_j+1}w_{i-1}^{a_{i-1}}\cdots w_{j+1}^{a_{j+1}}w_j^{a_j+1}$, $p_i\longrightarrow p_i-2ev_{i-1}^{a_{i-1}}\cdots v_j^{a_j}u_j$, substituting, as above, $\frac{1}{2}p_k$ for u_k. Now we need to take the remaining terms into account: first, it is clear that terms of the form $cv^az^\alpha+\bar{c}v^a\bar{z}^\alpha$ do not appear; second, we can obtain terms that change the value of the expression $\left(\frac{\partial}{\partial v_{i-1}}\right)^{b_{i-1}}\left(\frac{\partial}{\partial v_{i-2}}\right)^{b_{i-2}}\cdots\left(\frac{\partial}{\partial v_l}\right)^{b_l}\pi_l(p_i)(0)$ for some exponents $(b_{i-1},\dots,b_l)$. But of the remaining terms it is clear that $(b_{i-1},\dots,b_l)<(a_{i-1},a_{i-2},\dots,a_j)$. Therefore, repeating the described procedure, we obtain (5) in a finite number of steps.

It remains to consider the case $(a_{i-1}, a_{i-2}, \dots, a_j) = 0$. In this case, the required equality can be obtained by applying the Gram–Schmidt orthogonalisation method with respect to the scalar product (2.3).

Thus, we have obtained the required form. Such a form will be called canonical.

6. Decomposition of the space of differential operators of homogeneous weight with constant coefficients into a direct sum

We fix $i, 1 \le i \le k$. Let $p_1, \dots, p_i$ be functions of homogeneous weight and canonical form, with $p_1, \dots, p_{i-1}$ not identically zero. Let $\mathcal{Q}_\mu$ be the space of operators with constant coefficients of weight $-\mu$, depending on $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_{i-1}}$. We denote by $\mathcal{Q}^j_\mu$ the operators depending only on $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_{j-1}}$.

We define the subspace $\mathcal{G}^j_\mu$ in $\mathcal{Q}^j_\mu$ as the subspace consisting of those operators Q that contain mixed derivatives with respect to z and $\bar{z}$, and that satisfy the condition $Q(v_{j-1}^{a_{j-1}} v_{j-2}^{a_{j-2}} \cdots v_l^{a_l} p_l)(0) = 0$.

We define the subspaces $\mathcal{H}^j_\mu$ in $\mathcal{Q}^j_\mu$ inductively. We suppose that $\mathcal{H}^1_\mu$ is the zero subspace of $\mathcal{Q}^j_\mu$, and for $j > 1$ define $\mathcal{H}^j_\mu$ as the linear span of operators of the form $\left(\frac{\partial}{\partial v_{j-1}}\right)^{a_{j-1}} \left(\frac{\partial}{\partial v_{j-2}}\right)^{a_{j-2}} \cdots \left(\frac{\partial}{\partial v_l}\right)^{a_l} \pi_l$, where π_l is the operator dual to p_l, $1 \leqq l \leqq j-1$, and all degrees are nonnegative.

We define the subspace $\mathcal{Z}^j_\mu$ in $\mathcal{Q}^j_\mu$ as the subspace of those operators that do not contain simultaneous differentiation with respect to z and $\bar{z}$.

We note that replacing z_s by $z_s^{\lambda_s}$, where λ_s is the weight of the corresponding variable z_s, and replacing the weight λ_s by unity, preserves the homogeneity of the equations and of the differential operators. We apply the theorem on the decomposition of $\mathcal{Q}^j_\mu$ into an orthogonal direct sum of operators to the system of equations obtained. Selecting the subspace $\tilde{\mathcal{Q}}^j_\mu$ in $\mathcal{Q}^j_\mu$, corresponding to the original operator, and the subspaces $\tilde{\mathcal{G}}^j_\mu$, $\tilde{\mathcal{H}}^j_\mu$ and $\tilde{\mathcal{Z}}^j_\mu$ in $\tilde{\mathcal{Q}}^j_\mu$, defined in a similar way, and returning to the original variables and weights, we obtain an analogous decomposition:

$$\tilde{\mathcal{Q}}^j_\mu = \tilde{\mathcal{G}}^j_\mu \oplus \tilde{\mathcal{H}}^j_\mu \oplus \tilde{\mathcal{Z}}^j_\mu.$$

7. Calculating the type in the homogeneous case and the completion of the proof

Now, relying on the equalities obtained in the preceding sections, which are analogues of the auxiliary assertions which were used to determine the type in the homogeneous case, repeating the same arguments, we obtain that for $j = 1, \dots, i$ there exists a commutator $F^j \in \mathcal{L}^{m_j}$ such that

$$F^j(T^k, \bar{T}^k) = \frac{\partial}{\partial v_j} + \sum_{l=j+1}^{k} \left(A^j_l \frac{\partial}{\partial w_l} + B^j_l \frac{\partial}{\partial \bar{w}_l} \right).$$

It follows that zero is a point of type $(m_1, \dots, m_{i-1}, m_i, \dots)$ if $p_i \neq 0$ identically, and type $(m_1, \dots, m_{i-1}, \infty, \dots)$ otherwise.

To pass to the general case, in the equations we single out the part with homogeneous lowest weight. Acting in the same way as in the case when the variables z have equal weights, we find that the terms with the highest weight in the equations do not affect the terms having weight of the highest absolute value in the tangent fields, which means that the type of a point is determined by the homogeneous part we singled out, and so the conclusion of the theorem follows.

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References

[1] T. Bloom and I. Graham, *On 'type' conditions for generic real submanifolds of* $\mathbb{C}^n$, Invent. Math. **40**(3) (1977), 217–243. MR0589930

[2] M. S. Baouendi, P. Ebenfelt and L. P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999. MR1668103

[3] V. Beloshapka, *Automorphisms of degenerate hypersurfaces in* $\mathbb{C}^2$ *and a dimension conjecture*, Russian J. Math. Phys. **4**(3) (1996), 393–396. MR1443181

[4] A. E. Ershova, *Automorphisms of 2-nondegenerate hypersurfaces in* $\mathbb{C}^3$, Matem. Zametki **69**(2) (2001), 214–222; MR1830221; English transl.: Math. Notes **69**(2) (2001), 188–195.

[5] M. Kolar, F. Meylan and D. Zaitsev, *Chern–Moser operators and polynomial models in CR geometry*, Adv. Math. **263** (2014), 321–356. MR3239141

[6] D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. Math. (2) **120**(3) (1984), 529–586. MR769163

[7] D. Catlin, *Subelliptic estimates for the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. Math. (2) **126**(1) (1987), 131–191. MR898054

[8] V. K. Beloshapka, *A universal model of a real submanifold*, Matem. Zametki **75**(4) (2004), 507–522. MR2068283; English transl.: Math. Notes **75**(4) (2004), 475–488.

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