NOTICES
OF THE
AMERICAN MATHEMATICAL SOCIETY

Edited by John W. Green and Gordon L. Walker

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MEETINGS

Calendar of Meetings

NOTE: This Calendar lists all of the meetings which have been approved by the Council up to the date at which this issue of the NOTICES was sent to press. The summer and annual meetings are joint meetings of the Mathematical Association of America and the American Mathematical Society. The meeting dates which fall rather far in the future are subject to change. This is particularly true of the meetings to which no numbers have yet been assigned.

<table>
<thead>
<tr>
<th>Meeting No.</th>
<th>Date of Meeting</th>
<th>Place</th>
<th>Deadline for Abstracts*</th>
</tr>
</thead>
<tbody>
<tr>
<td>609</td>
<td>February 29, 1964</td>
<td>New York, New York</td>
<td>Jan. 16</td>
</tr>
<tr>
<td>610</td>
<td>April 18, 1964</td>
<td>Reno, Nevada</td>
<td>Mar. 5</td>
</tr>
<tr>
<td>611</td>
<td>April 20-23, 1964</td>
<td>New York, New York</td>
<td>Mar. 5</td>
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<tr>
<td>612</td>
<td>April 24-25, 1964</td>
<td>Chicago, Illinois</td>
<td>Mar. 5</td>
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<tr>
<td>613</td>
<td>June 20, 1964</td>
<td>Pullman, Washingotn</td>
<td>May 7</td>
</tr>
<tr>
<td>614</td>
<td>August 24-28, 1964 (69th Summer Meeting)</td>
<td>Amherst, Massachusetts</td>
<td>July 3</td>
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<tr>
<td></td>
<td>January 25-29, 1965 (71st Annual Meeting)</td>
<td>Denver, Colorado</td>
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<tr>
<td></td>
<td>August 30-September 3, 1965 (70th Summer Meeting)</td>
<td>Ithaca, New York</td>
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<tr>
<td></td>
<td>August 1966 (71st Summer Meeting)</td>
<td>New Brunswick, New Jersey</td>
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<tr>
<td></td>
<td>August, 1967 (72nd Summer Meeting)</td>
<td>Toronto, Canada</td>
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</table>

* The abstracts of papers to be presented in person at the meetings must be received in the Headquarters Offices of the Society in Providence, Rhode Island, on or before these deadlines. The deadlines also apply to news items. The next two deadline dates for by title abstracts are February 27, and April 30, 1964.
Seventieth Annual Meeting
University of Miami, Coral Gables,
and
Miami, Florida
January 23-27, 1964

PROGRAM

The seventieth annual meeting of the American Mathematical Society will be held in Miami, Florida in conjunction with the annual meeting of the Mathematical Association of America. The Society will meet from Thursday, January 23, 1964 through Sunday, January 26, and the Mathematical Association of America will meet on Saturday through Monday, January 25-27. All sessions will be held on the campus of the University of Miami, except as noted below.

The thirty-seventh Josiah Willard Gibbs Lecture will be delivered by Professor Lars Onsager at 8:00 P.M. on Friday, January 24, in the Everglades Room of the Everglades Hotel. The title of Professor Onsager's talk is "Mathematical Problems of Cooperative Phenomena".

Professor Deane Montgomery of the Institute for Advanced Study will deliver the Presidential Address at 9:00 A.M. on Friday, January 24 in Room 110 University College. He will speak on "Compact Groups of Transformations".

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, hour addresses will be given by Professor Morton Brown of the University of Michigan and by Professor Heisuke Hironaka of Brandeis University. Professor Brown will speak at 9:00 A.M. Thursday, January 23 on "Topological Manifolds", and Professor Hironaka's talk, entitled "Singularities in Algebraic Varieties" will be given at 2:00 P.M. on Friday, January 24. Both of these addresses will be presented in Room 110 University College. A special feature of this meeting will be four sessions of selected twenty minute papers.

The first Special Session, devoted to Finite Groups, will be held at 10:15 A.M. on Thursday in Room 160 University College. This session was arranged by Walter Feit, and it consists of papers by Walter Feit, Daniel Gorenstein, Marshall Hall, Donald Livingstone, and M. Suzuki. The Special Session on Geometry will be held in Room 160 University College at 3:15 on Friday. Professor Herbert Busemann has arranged a program of papers by Louis Auslander, R. H. Bruck, Herbert Busemann, L. W. Green, and T. F. Branchoff. The Special Session on Homological Algebra will begin at 2:00 P.M. on Saturday in Room 160 University College. Professor D. A. Buchsbaum has arranged the program for this Session, consisting of papers by Maurice Auslander, Hyman Bass, D. A. Buchsbaum, Peter Freyd, and R. O. Swan.

The last Special Session on Partial Differential Equations will be at 2:00 P.M. on Sunday in Room 160 University College. Professor David Gilbarg has arranged a program of papers by D. G. Aronson, David Gilbarg, Peter Lax, Ralph Phillips, and Leon Ehrenpreis.

There will be sessions for contributed ten minute papers on Thursday at 10:15 A.M. and 3:45 P.M.; on Friday, at 10:15 A.M. and 3:15 P.M.; on Saturday at 2:00 P.M.; and on Sunday at 2:00 P.M.

Both the first and second awards of the Veblen Prize in Geometry and the Bocher Prize will be awarded at the Business Meeting which will be held at 1:30 P.M. on Thursday, January 23 in Room 110 University College.

The Society is reminded that by action of the Council the number of contributed papers will be limited to 200 on a "first come, first served" basis, and there will be no sessions for late papers.

The Annual Business Meeting will be held at 1:30 P.M. on Thursday, January
23 in Room 110 University College.
The Council of the Society will meet at 4:00 P.M. on Wednesday, January 22, at the Everglades Hotel.

REGISTRATION
Registration headquarters will be in the lobby of the 730 Building on the University of Miami campus. (This is not a building number. It is a girl's dormitory and derives its name from the number of girls residing there.) On Wednesday, January 22, a Registration Desk will be maintained on the Mezzanine of the Everglades Hotel from 2:00 P.M. to 8:00 P.M. The Registration Desk in the 730 Building will be open from Thursday through Sunday from 9:00 A.M. to 5:00 P.M. On Monday, the Registration Desk will be located in the lobby of the University College and will be open from 9:00 A.M. to 2:00 P.M. All members attending the meetings are requested to register upon arrival.

The registration fee will be $2.00 for each member of a participating organization (except students), $0.50 for the first non-member in a member's family. There will be no fee for other non-members in a member's family and for students. The fee for non-members not in any of the above categories is $5.00.

The Mathematical Sciences Employment Register will be located in the Westminster Chapel Fellowship Room, 5225 Ponce de Leon Boulevard, on the University of Miami campus, just across Miller Drive from the 720 Dormitory, and will function from 9:00 A.M. to 5:00 P.M. on Friday, Saturday, and Sunday.

EXHIBITS
Book and other exhibits will be located in the Recreation Room of the 730 Building.

ACCOMMODATIONS
Accommodations for the meeting will be handled by the Housing Bureau of the Convention Bureau of the City of Miami. The reservation form on the inside back cover of these NOTICES should be used in requesting accommodations. The Housing Bureau will make reservations as nearly as possible in accordance with the member's request at one of the hotels in the list below. All reservations will be confirmed by the Housing Bureau.

The University cafeterias will be open for breakfast, lunch and dinner during the meetings.

Special Gray Line shuttle buses will be provided to transport persons from the hotels in Miami to the University of Miami campus. The trip takes approximately 40 minutes on week days and 30 minutes on Saturdays and Sundays. Buses will be leaving from the boarding areas at the Everglades, Dupont Plaza and Patricia Hotels in Miami at 8:00 A.M. and 8:20 A.M. Thursday through Sunday, and on Monday at 7:50 A.M. and 8:10 A.M., returning daily, except Friday, with departures at 5:00 P.M. and 5:20 P.M. at the University of Miami; on Friday, the buses will leave at 5:40 P.M. and 6:10 P.M. from the University of Miami.

<table>
<thead>
<tr>
<th>Hotel</th>
<th>Singles</th>
<th>Doubles</th>
<th>Twins</th>
<th>Suites</th>
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<tr>
<td>Alcazar</td>
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<td>$9.00-$10.00</td>
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<td>Dupont Plaza</td>
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<tr>
<td>McAllister</td>
<td>9.00-$11.00</td>
<td>15.00-$16.00</td>
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<tr>
<td>Miami Colonial</td>
<td>9.00-$10.00</td>
<td>12.00-$14.00</td>
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<td>Ponce de Leon</td>
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<td>Patricia</td>
<td>8.50</td>
<td>10.50</td>
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<td>Triples 12.50</td>
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<tr>
<td>Robert Clay</td>
<td>7.00</td>
<td>9.00</td>
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<tr>
<td>Towers</td>
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<td>Dallas Park</td>
<td>12.00</td>
<td>12.00</td>
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<td>20.00</td>
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</table>
Miami. On Saturday and Sunday morning, the buses will make a stop directly at the Ponce de Leon Junior High School Auditorium for the Association sessions scheduled there. All other hotels on the list above are within a few minutes walk from the above three boarding areas. Most Miami Beach hotels are an additional ten or fifteen miles away from the University of Miami, and shuttle bus service is not available to Miami Beach.

ENTERTAINMENT AND RECREATION

There will be a President's tea and reception sponsored by the University of Miami on Friday from 4:30 P.M. to 6:00 P.M. in the Richter Library Lecture Hall.

A bus will leave the 720 Dormitory Lounge at 12:30 P.M. on Sunday for Flamingo (Everglades National Park), from where a sightseeing boat trip through the Everglades National Park will start at 3:00 P.M. The boat has been reserved for a two hour bird watchers' and photographers' trip. Space on this boat is limited to 30 passengers. The cost is $3.00 per passenger. Reservations will be made in the order in which requests are received and should be requested from Mrs. Georgia Del Franco, Department of Mathematics, University of Miami, Coral Gables, Florida 33146, as early as possible.

A tour to the International Design Center will leave the Everglades Hotel at 9:00 A.M. on Saturday. There will be a $1.00 transportation charge for each lady making the tour.

Gray Line Sightseeing Tours will have an agent at the registration area at the University of Miami from 1:00 P.M. to 5:00 P.M., Thursday through Sunday, and at the Everglades Hotel from 9:00 A.M. to 12:00 noon Thursday through Monday, to provide information on sightseeing and travel. The agent will be able to arrange, either individually or for groups, attractive visits, post meeting trips to the Caribbean, sightseeing, night club tours, fishing trips, and general transportation. For golfing, a taxi will take a group to the Miami Springs or Le Jeune courses, which are city owned, for about $0.40 per person. A day's notice will allow the agent to have reservations made for a group. Deep sea fishing is available from a selection of charter boats merely by walking through Bayfront Park directly across from the Biscayne Boulevard Hotels to Pier 5. Tennis courts at the University of Miami are available.

Further information may be obtained by contacting Mr. Joe Boise, Assistant Manager, City of Miami Convention Bureau, 320 N. E. Fifth Street, Miami, Florida.

LOCAL AND TRAVEL INFORMATION

Miami is on Eastern Standard Time. The expected diurnal temperature variation for January is between 55 and 75 degrees. Rain is unlikely at this time of the year.

There is regular airline service to Miami International Airport by the following airlines: Braniff, Delta, Eastern, National, Northeast, Pan American, Trans-World and United. If members plan a stop-over in Nassau before or after the meetings, it is advisable to make travel and hotel reservations early, as this is the peak season in Nassau.

Nassau can be reached from Miami by air by: Bahamas Airways, Cunard-Eagle, BOAC, and Pan American; by sea by: S. S. Bahama Star, and S. S. Florida. From St. Petersburg, Tampa, Fort Lauderdale and West Palm Beach, Mackey Airlines has several flights daily to Nassau. From New York to Nassau by air by: BOAC and PAA, 2 1/2 hours non-stop; by sea: M. V. Italia, weekly sailings. From Montreal and Toronto to Nassau by air: Trans-Canada Airlines. Pan American and BOAC issue seventeen-day round-trip tickets from Miami to Nassau, tourist class, at a 10 percent discount.

Some airlines offer San Francisco to Miami passengers a detour through New York for only $11.10 extra.

The following railroad lines connect with the Seaboard Airline Railroad and offer direct pullman service to Miami: Baltimore and Ohio, Chesapeake and Ohio, Louisiana and Nashville, New York Central and New Haven, Santa Fe, Seaboard, Southern Pacific and Texas Pacific.

Miami is served by Greyhound and Trailways bus lines.

MAIL AND TELEGRAMS

Correspondence for members attending the meetings should be addressed in care of the American Mathematical Society, University of Miami, Coral Gables,

**TIME TABLE**

*(Eastern Standard Time)*

<table>
<thead>
<tr>
<th>WEDNESDAY</th>
<th>American Mathematical Society</th>
<th>Mathematical Association of America</th>
</tr>
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<tbody>
<tr>
<td>January 22</td>
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</table>

2:00 P.M. - 8:00 P.M.  
REGISTRATION -- MEZZANINE -- EVERGLADES HOTEL

4:00 P.M.  
Council Meeting  
Banyan Room, Everglades Hotel

<table>
<thead>
<tr>
<th>THURSDAY</th>
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<tbody>
<tr>
<td>January 23</td>
<td>Society</td>
<td>Association</td>
</tr>
</tbody>
</table>

9:00 A.M. - 5:00 P.M.  
REGISTRATION  
Lobby of the 730 Building  
University of Miami

9:00 A.M.  
Invited Address:  
Morton Brown  
110 University College

10:15 A.M.  
Special Session on Finite Groups  
160 University College  
Session on Analysis I  
110 University College  
Session on Analysis II  
120 University College  
Session on Algebra  
130 University College  
Session on Topology  
140 University College

1:30 P.M.  
Business Meeting and Awarding of  
Bôcher Prize and two Veblen Prizes  
110 University College

2:30 P.M.  
Executive and Finance Committees  
226 Ashe

3:30 P.M.  
Session on Analysis  
110 University College  
Session on Logic and Foundations  
120 University College  
Session on Statistics and Probability  
140 University College  
Session on Algebra  
130 University College  
Session on Geometry  
160 University College
<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
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</table>
| 9:00 A.M. - 5:00 P.M. | **REGISTRATION**  
Lobby of the 730 Building  
University of Miami  
**EMPLOYMENT REGISTER**  
Westminster Chapel Fellowship Room  
5225 Ponce de Leon Boulevard  
University of Miami  
**EXHIBITS**  
730 Building |
| 9:00 A.M. | Presidential Address  
Deane Montgomery  
110 University College |
| 10:10 A.M. | Session on Analysis I  
110 University College |
| 10:15 A.M. | Session on Analysis II  
120 University College  
Session on Algebra  
130 University College  
Session on Applied Mathematics  
140 University College  
Session on Logic and Foundations  
160 University College |
| 2:00 P.M. | Invited Address  
H. Hironaka  
110 University College |
| 3:15 P.M. | Special Session on Geometry  
160 University College  
Session on Analysis  
110 University College  
Session on Applied Mathematics  
120 University College  
Session on Statistics and Probability  
130 University College  
Session on Topology  
140 University College |
| 4:30 P.M. - 6:00 P.M. | **RECEPTION**  
Richter Library Building  
Lecture Hall |
| 8:00 P.M. | Gibbs Lecture  
Everglades Room, Everglades Hotel  
Lars Onsager |

**SATURDAY**

<table>
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<tr>
<th>Time</th>
<th>Event</th>
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| 9:00 A.M. - 5:00 P.M. | **REGISTRATION**  
Lobby of the 730 Building  
University of Miami  
**EMPLOYMENT REGISTER**  
Westminster Chapel Fellowship Room  
5225 Ponce de Leon Boulevard  
University of Miami  
**EXHIBITS**  
730 Building |
TIME TABLE  
(Eastern Standard Time)  

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<tr>
<td><strong>SATURDAY</strong></td>
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<tr>
<td><strong>January 25</strong></td>
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</table>
| 9:00 A.M. - 10:00 A.M. | FIRST SESSION  
Ponce de Leon Junior High School Auditorium  
Goals for School Mathematics  
A. M. Gleason  
G. F. Carrier  
R. J. Wisner  
Panel Discussion  
Retiring Presidential Address  
A. W. Tucker  
Film Program (Color Animations)  
Ponce de Leon Junior High School Auditorium |
| 10:10 A.M. - 10:50 A.M. |                                                                                   |
| 11:00 A.M. - 11:50 A.M. |                                                                                   |
| 1:00 P.M. - 1:55 P.M. |                                                                                   |
| 2:00 P.M. | Special Session on Homological Algebra  
160 University College  
Session on Analysis I  
110 University College  
Session on Analysis II  
120 University College  
Session on Algebra  
130 University College  
Session on Applied Mathematics  
140 University College  
PSSC Physics Films  
Everglades Room, Everglades Hotel  
ETS Film: Thinking Machines  
Everglades Room, Everglades Hotel |
| 7:00 P.M. - 8:38 P.M. |                                                                                   |
| 8:39 P.M. - 8:57 P.M. |                                                                                   |
| **SUNDAY**    |                                                                                   |
| **January 26**|                                                                                   |
| 9:00 A.M. - 5:00 P.M. | REGISTRATION  
Lobby of the 730 Building  
EMPLOYMENT REGISTER  
Westminster Chapel Fellowship Room  
EXHIBITS  
730 Building  
SECOND SESSION  
Ponce de Leon Junior High School Auditorium  
Content of the First Course in Real Variables  
E. J. McShane  
Edwin Hewitt  
Walter Rudin  
Panel Discussion  
Business Meeting  
Distinguished Service Award  
Chauvenet Prize |
| 9:00 A.M. - 10:10 A.M. |                                                                                   |
| 10:20 A.M. - 10:50 A.M. |                                                                                   |
| 11:00 A.M. - 11:50 A.M. |                                                                                   |
| TIME TABLE  
| (Eastern Standard Time) |
|---|---|---|
| **SUNDAY**  
| January 26 | American Mathematical Society | Mathematical Association of American |
| 2:00 P.M. | Meeting, Council of the Conference Board of the Mathematical Sciences — Room 323 Everglades |
| 2:00 P.M. | Special Session on Partial Differential Equations  
160 University College |
| 2:00 P.M. | Session on Analysis  
110 University College |
| 2:00 P.M. | Session on Topology  
120 University College |
| 2:00 P.M. | Session on Applied Mathematics  
140 University College |
| 2:00 P.M. | Session on Algebra  
130 University College |
| 7:00 P.M. - 7:34 P.M. | Film: Theory of Limits (Part I)  
Everglades Room, Everglades Hotel |
| 7:40 P.M. - 8:40 P.M. | Film: The Kakeya Problem  
Everglades Room, Everglades Hotel |
| 8:45 P.M. | Film: Title to be announced  
Everglades Room, Everglades Hotel |
| **MONDAY**  
| January 27 | Society | Association |
| 9:00 A.M. - 2:00 P.M. | REGISTRATION  
Lobby of University College |
| 9:00 A.M. - 9:50 A.M. | The Commutator Equation and Some Allied Equations  
Einar Hille |
| 10:00 A.M. - 10:50 A.M. | Analytic Properties of the Solutions of Certain Functional Equations  
J. H. B. Kemperman |
| 11:00 A.M. - 11:50 A.M. | The Functional Equation of Associativity  
Berthold Schweizer |
| **THIRD SESSION**  
| 120 University College | Functional Equations |
| 2:00 P.M. - 2:50 P.M. | The Polyhedron Inequality  
Herbert Busemann |
| 3:00 P.M. - 3:50 P.M. | Some Problems of Elementary Euclidean Geometry  
H. G. Eggleston |
| 4:00 P.M. - 4:50 P.M. | Facial Structure of Convex Polytopes  
Victor Klee |
PROGRAM OF THE SESSIONS

The time limit for each contributed paper in the ordinary sessions is ten minutes. The papers are scheduled at 15 minute intervals so that listeners can circulate between different sessions. To maintain the schedule, the time limit will be strictly enforced.

THURSDAY, 9:00 A.M.

Invited Address, Room 110, University College

Topological Manifolds
Professor Morton Brown, University of Michigan

THURSDAY, 10:15 A.M.

Special Session on Finite Groups, Room 160, University College

10:15 - 10:35
Groups of exponent 4. Preliminary report
Professor Marshall Hall, Jr.* and Professor D. E. Knuth, California Institute of Technology (608-217)

10:45 - 11:05
The classification of finite groups with dihedral Sylow 2-subgroups
Professor Daniel Gorenstein*, Clark University and Professor J. H. Walter, University of Illinois (608-224)

11:15 - 11:35
On the nonexistence of certain generalized polygons
Professor Walter Feit* and Mr. Graham Higman, Cornell University (608-216)

11:45 - 12:05
A characterization of 3-dimensional unitary groups
Professor Michio Suzuki, University of Illinois (608-229)

12:15 - 12:35
Group rings of finite groups. II. Preliminary report
Mr. J. A. Cohn and Professor Donald Livingstone*, The University of Michigan (68-223)

THURSDAY, 10:15 A.M.

Session on Analysis, Room 110, University College

10:15 - 10:25
(1) Inequalities and the saturation classes of Bernstein polynomials
Professor G. G. Lorentz, Syracuse University (608-63)

10:30 - 10:40
(2) Integral inequalities for functions with nondecreasing increments
Professor H. D. Brunk, University of Missouri (608-80)

10:45 - 10:55
(3) Chain sequences and univalence
Professor T. L. Hayden*, University of Kentucky and Professor E. P. Merkes, University of Cincinnati (608-92)

11:00 - 11:10
(4) On extremal decompositions of a quadrilateral
Professor F. C. Huckemann, University of Giessen, Germany and University of Tennessee (608-96)

*For papers with more than one author, an asterisk follows the name of the author who plans to present the paper at the meeting.
11:15 - 11:25
(5) Continuity and location of zeros of linear combinations of polynomials
Professor Mishael Zedek, University of Maryland (608-101)

11:30 - 11:40
(6) Interpolation of operators for Lip spaces
Professor R. C. O'Neil, Rice University (608-131)

11:45 - 11:55
(7) A criterion for the comparability of means
Dr. Oved Shisha*, Aerospace Research Laboratories, Wright-Patterson Air Force Base and Professor G. T. Cargo, Syracuse University (608-132)

12:00 - 12:10
(8) On extreme Banach limits. Preliminary report
Dr. S. P. Lloyd, Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey (608-199)

THURSDAY, 10:15 A.M.

Session on Analysis, Room 120, University College
10:15 - 10:25
(9) Characterization of group algebras in terms of their translations. Preliminary report
Mr. F. P. Greenleaf, Yale University (608-58)

10:30 - 10:40
(10) States and ideals in Jordan operator algebras
Dr. D. M. Topping, University of Chicago (608-64)

10:45 - 10:55
(11) On restrictions of functions of the spaces $P^{a,p}$ and $B^{a,p}$
Professor Pawel Szeptycki, University of Kansas (608-78)

11:00 - 11:10
(12) A new group algebra for locally compact groups
Professor J. A. Ernest, University of Rochester (608-89)

11:15 - 11:25
(13) The bidual of a locally multiplicatively-convex algebra
Mr. S. L. Gulick, University of Pennsylvania (608-104)

11:30 - 11:40
(14) An operational calculus for meromorphic functions
Professor H. A. Gindler, University of Pittsburgh (608-124)

11:45 - 11:55
(15) The Banach algebra of multipliers on a functional Banach space
Professor Nachman Aronszajn, University of Kansas (608-142)

12:00 - 12:10
(16) Normed modules and almost periodicity
Professor J. W. Kitchen, Jr., Duke University (608-201)

THURSDAY, 10:15 A.M.

Session on Algebra, Room 130, University College
10:15 - 10:25
(17) Derivations on $p$-adic fields
Professor Joseph Neggers, Florida State University (608-35)

10:30 - 10:40
(18) Uniqueness of invariant Cartan subalgebras of solvable Lie algebras
Professor E. J. Taft, Rutgers, The State University (608-52)
10:45 - 10:55
(19) A theorem on automorphisms of classical Lie algebras
Professor D. A. Smith, Duke University (608-90)

11:00 - 11:10
(20) A torsion theory for Abelian categories
Professor S. E. Dickson, University of Nebraska (68-141)

11:15 - 11:25
(21) Perfect closure of a scheme
Professor M. J. Greenberg, University of California, Berkeley (608-149)

11:30 - 11:40
(22) First integrals of prime differential ideals
Mr. Shlomo Halfin, University of California, Berkeley (608-157)

11:45 - 11:55
(23) Autonomy in equational categories. Preliminary report
Mr. F. E. Linton, Wesleyan University (608-183)

12:00 - 12:10
(24) Rings of integer-valued functions and non-standard models. Preliminary report
Dr. N. L. Alling, Massachusetts Institute of Technology and Purdue University (608-193)

THURSDAY, 10:15 A.M.

Session on Topology, Room 140, University College

10:15 - 10:25
(25) On the faithfulness of homology functors
Dr. G. M. Kelly, Tulane University (608-26)
(Introduced by Dr. G. S. Young)

10:30 - 10:40
(26) Factorization of differentiable maps with small branch set dimension
Professor P. T. Church, Syracuse University (608-29)

10:45 - 10:55
(27) The rank of $S^3$
Dr. E. L. Lima, Columbia University (608-127)

11:00 - 11:10
(28) The product of polyhedral homotopy manifolds is a homotopy manifold
Professor P. M. Rice, University of Georgia (608-134)

11:15 - 11:25
(29) A note on counting isotropy subgroups
Professor L. N. Mann, Institute for Defense Analyses, Princeton, New Jersey and University of Virginia (608-169)

11:30 - 11:40
(30) A generalization of the EHP-sequence
Professor Tudor Ganea, University of Washington (608-175)

11:45 - 11:55
(31) Isotopies which remove closed sets of rank 1 critical points of $\infty$-smooth homeomorphisms of the 2-sphere
Professor R. M. Gillette, University of Oregon (608-178)

12:00 - 12:10
(32) On the genus of links
Mr. C. B. Schaufele, Florida State University (608-181)

12:15 - 12:25
(33) A Samelson product in $SO(2n)$
Professor A. T. Lundell, Purdue University (608-189)
THURSDAY, 1:30 P.M.

**Business Meeting, Room 110, University College**

The Bôcher Prize and the first two Veblen Prizes will be awarded at this Meeting

THURSDAY, 3:45 P.M.

**Session on Analysis, Room 110, University College**

3:45 - 3:55  
(34) Perturbations of linear operators and summability. Preliminary report  
Mr. I. D. Berg, Yale University (608-55)

4:00 - 4:10  
(35) A limitation of uniform cluster operators  
Dr. M. L. Weiss, Institute for Advanced Study (608-59)

4:15 - 4:25  
(36) A structure theorem of set-valued additive functions  
Miss D. R. Henney, University of Maryland (608-61)

4:30 - 4:40  
(37) Theorem on uniform approximation and its applications  
Professor Ammon Jakimovski, Tel-Aviv University, Tel-Aviv, Israel and Professor M. S. Ramanujan*, The University of Michigan (608-93)

4:45 - 4:55  
(38) A norm bound for the inverse shift operator  
Professor J. W. Moeller, Case Institute of Technology (608-95)

5:00 - 5:10  
(39) Stronger forms of inequalities of Kantorovich and Strang for operators in a Hilbert space  
Professor J. B. Diaz and Professor F. T. Metcalf*, University of Maryland (608-121)

THURSDAY, 3:45 P.M.

**Session on Logic and Foundations, Room 120, University College**

3:45 - 3:55  
(40) A maximal set which is not complete  
Professor G. E. Sacks, Cornell University (608-1)

4:00 - 4:10  
(41) The ordering theorem does not imply the axiom of choice. Preliminary report  
Mr. J. D. Halpern*, California Institute of Technology and Dr. Azriel Levy, Hebrew University (608-11)

4:15 - 4:25  
(42) Bounded ALGOL-like languages  
Dr. Seymour Ginsburg*, System Development Corporation, Santa Monica, California and Professor E. H. Spanier, University of California, Berkeley (608-16)

4:30 - 4:40  
(43) Structural criterion for recursive enumeration without repetition, I  
Professor M. B. Pour-El*, Institute for Advanced Study and Professor W. A. Howard, Pennsylvania State University (608-24)

4:45 - 4:55  
(44) A recursively enumerable splinter which is not a cylinder  
Mr. P. R. Young, Reed College (608-47)
THURSDAY, 3:45 P.M.

Session on Algebra, Room 130, University College
3:45 - 3:55
(45) On Baer semigroups
Mr. M. F. Janowitz, University of New Mexico (608-21)

4:00 - 4:10
(46) Decomposition of sets of group elements
Professor W. B. Laffer*, Western Washington State College and
Professor H. B. Mann, Ohio State University (608-54)

4:15 - 4:25
(47) On the extensions of lattice-ordered groups
Mr. J. R. Teller, Tulane University (608-79)

4:30 - 4:40
(48) The root problem in one-eighth groups
Professor Seymour Lipschutz, Polytechnic Institute of Brooklyn (608-98)

4:45 - 4:55
(49) A natural lattice-theoretic concept of convergence in abelian 1-groups
Dr. F. Papangelou, The Catholic University of America (608-136)

5:00 - 5:10
(50) An 0-topological lattice which cannot be regularly extended to a ∑-complete
0-topological one
Professor D. A. Kappos* and Dr. F. Papangelou, The Catholic University of America (608-137)

THURSDAY, 3:45 P.M.

Session on Statistics and Probability, Room 140, University College
3:45 - 3:55
(51) Some Smirnov type limit theorems of probability theory
Mr. Miklós Csörgő, Princeton University (608-62)
(Introduced by Dr. S. H. Gould)

4:00 - 4:10
(52) Fluctuation and periodicity
Professor Seymour Sherman, Wayne State University (608-67)

4:15 - 4:25
(53) A winning bet in Nevada Baccarat
Professor E. O. Thorp, New Mexico State University and Mr. W. E.
Walden*, Los Alamos Scientific Laboratory, Los Alamos, New Mexico
(608-103)

4:30 - 4:40
(54) Continuous transformations and stochastic differential equations
Dr. D. A. Woodward, Argonne National Laboratory, University of Chicago (608-150)

4:45 - 4:55
(55) Tests of hypotheses using quantiles
Mr. Isidore Eisenberger, Jet Propulsion Laboratory, Pasadena, California (608-204)

THURSDAY 3:45 P.M.

Session on Geometry, Room 160, University College
3:45 - 3:55
(56) The solution of a problem of Bonnice-Klee
Professor J. R. Reay, Western Washington State College (608-8)
4:00 - 4:10
(57) A property of two chords which divide a convex curve into four arcs of equal length
Professor H. G. Eggleston, Bedford College and University of Washington and Professor Aboulghassem Zirakzadeh*, University of Colorado (608-28)

4:15 - 4:25
(58) A geometric proof of Hadamard's inequality
Professor S. B. Townes, University of Hawaii (608-68)

4:30 - 4:40
(59) On piecewise linear separability
Dr. Edmund Eisenberg and Professor Eugene Wong*, University of California, Berkeley (608-151)

4:45 - 4:55
(60) Universal Laplacians
Professor A. W. Adler, Purdue University (608-191)

FRIDAY, 9:00 A.M.

Presidential Address, Room 110, University College
Compact Groups of Transformation
Professor Deane Montgomery, Institute for Advanced Study

FRIDAY, 10:15 A.M.

Session on Analysis, Room 110, University College
10:15 - 10:25
(61) The domain of univalence of certain classes of meromorphic functions
Mr. R. J. Distler, University of Kentucky (608-44)

10:30 - 10:40
(62) Zeros of exponential sums
Professor D. G. Dickson, University of Michigan (608-49)

10:45 - 10:55
(63) Sets of convergence of Dirichlet series
Professor D. R. Lick, New Mexico State University (608-56)

11:00 - 11:10
(64) Weakly reproducing differentials on open Riemann surfaces
Professor Albert Marden, University of Minnesota (608-74)

11:15 - 11:25
(65) Power series whose sections have zeros of large modulus
Professor J. D. Buckholtz, University of North Carolina (608-102)

11:30 - 11:40
(66) Holomorphic functions with gap power series
Mr. Alfred Gray*, University of California, Los Angeles and Professor S. M. Shah, University of Kansas (608-110)

11:45 - 11:55
(67) On entire functions and a conjecture of Erdös
Mr. Alfred Gray, University of California, Los Angeles and Professor S. M. Shah*, University of Kansas (608-111)

12:00 - 12:10
(68) Two new characterizations for plane quasiconformal mappings
Professor J. A. Kelingos, Duke University (608-180)
FRIDAY, 10:15 A.M.

Session on Analysis, Room 120, University College
10:15 - 10:25  
(69) Factor sets for doubly stochastic operators on a Hilbert space  
Professor R. D. Sinkhorn, University of Houston (608-39)

10:30 - 10:40  
(70) Strictly singular operators in locally convex spaces  
Dr. H. E. Lacey, Abilene Christian College (608-42)

10:45 - 10:55  
(71) A representation theorem for a continuous linear transformation on a space of continuous functions  
Professor D. H. Tucker, University of Utah (608-122)

11:00 - 11:10  
(72) Vector-valued summability methods on a linear normed space  
Mr. L. C. Kurtz* and Professor D. H. Tucker, University of Utah (608-123)

11:15 - 11:25  
(73) An open mapping theorem  
Dr. G. M. Koethe, University of Maryland (608-152)

11:30 - 11:40  
(74) The solution of systems of functional equations  
Professor R. C. Buck, University of Wisconsin (608-164)

11:45 - 11:55  
(75) $w^*$-bases of subspaces  
Mr. J. R. Retherford, Florida State University (608-194)

12:00 - 12:10  
(76) Orbits of $L^1$-functions under doubly stochastic transformations  
Dr. J. V. Ryff, Harvard University (608-195)

FRIDAY, 10:15 A.M.

Session on Algebra, Room 130, University College
10:15 - 10:25  
(77) The exponent set of primitive matrices  
Professor N. S. Mendelsohn* and Professor A. L. Dulmage, University of Manitoba (608-3)

10:30 - 10:40  
(78) Linear homogeneous equations over finite rings  
Dr. H. R. Stevens, Pennsylvania State University (608-14)

10:45 - 10:55  
(79) Generalized Stirling numbers defined by generalized powers  
Professor Gloria Olive, Anderson College (608-37)

11:00 - 11:10  
(80) Improvement of the error term in an asymptotic formula  
Dr. D. R. Hayes, University of Tennessee (608-200)

11:15 - 11:25  
(81) On the class of subdirect powers of a finite algebra  
Professor G. A. Gratzer, The Pennsylvania State University (608-46)

11:30 - 11:40  
(82) Fermat's last theorem is true for any exponent up to 25,000  
Professor J. L. Selfridge*, University of Washington and Mr. B. W. Pollack, University of California, Los Angeles (608-138)

11:45 - 11:55  
(83) Kronecker and other special products of matrices. Preliminary report  
Mrs. M. H. Fitzpatrick, Auburn University (608-158)
12:00 - 12:10
(84) On representing groups by permutations
Dr. J. T. Moore, University of Florida and Mr. S. S. Magliveras*, Florida Presbyterian College (608-147)

FRIDAY, 10:15 A.M.

Session on Applied Mathematics, Room 140, University College
10:15 - 10:25
(85) Some special results on the asymptotic behavior of the solution of the initial-boundary value problem of the two dimensional wave equation for large time
Mr. Y. M. Chen, Purdue University (608-32)

10:30 - 10:40
(86) Mixed problems for the wave equation in a time dependent domain, Preliminary report
Mr. E. D. Rogak, University of Michigan (608-83)

10:45 - 10:55
(87) Three new integral relations
Dr. W. W. Turner*, Western Michigan University and Dr. Alfred Leitner, Michigan State University (608-94)

11:00 - 11:10
(88) On the existence in the large of a solution to a nonlinear wave equation with mixed boundary conditions
Professor H. L. Johnson, Purdue University (608-107)

11:15 - 11:25
(89) Bohr's quantization rule, Preliminary report
Dr. P. B. Bailey, Sandia Corporation, Albuquerque, New Mexico (608-135)

11:30 - 11:40
(90) The Dirichlet and Neumann problems for the wave equation and slotted coaxial cylindrical boundary
Dr. Yoshio Hayashi, The Radiation Laboratory, The University of Michigan (608-144)

11:45 - 11:55
(91) A new class of self-adjoint differential operators of the pure wave type
Mr. J. E. Lagnese*, National Bureau of Standards, Washington, D. C. and Mr. K. L. Stellmacher, University of Maryland (608-174)

12:00 - 12:10
(92) Multiple scattering of elastic waves by cylindrical cavities
Professor N. R. Zitron, Purdue University (608-188)

FRIDAY, 10:15 A.M.

Session on Logic and Foundations, Room 160, University College
10:15 - 10:25
(93) A generalization of Axt's primitive recursive hierarchy. Preliminary report
Mr. R. J. Fabian, Case Institute of Technology (608-57)

10:30 - 10:40
(94) On a class of sums of regressive isols
Mr. F. J. Sansone, Case Institute of Technology (608-66)

10:45 - 10:55
(95) Implicative semi-lattices
Professor W. C. Nemitz, Southwestern at Memphis (608-69)

11:00 - 11:10
(96) Grzegorczyk's primitive recursive classes $\mathcal{C}_n$ and Ackermann's functions
Professor R. W. Ritchie, University of Washington (608-75)
11:15 - 11:25  
(97) Reduction of ordinal recursion  
Professor C. F. Kent, Case Institute of Technology (608-128)

11:30 - 11:40  
(98) An isomorphism theorem  
Mr. S. P. Franklin, University of Washington (608-185)

FRIDAY, 2:00 P.M.

Invited Address, Room 110, University College

Singularities in Algebraic Varieties  
Professor H. Hironaka, Brandeis University

FRIDAY, 3:15 P.M.

Special Session on Geometry, Room 160, University College

3:15 - 3:35  
Algebraic groups attached to complete compact locally affine spaces  
Professor L. Auslander, Purdue University and University of California, Berkeley (608-225)

3:40 - 4:00  
Geodesic inequalities  
Professor L. W. Green, University of Minnesota (608-226)

4:05 - 4:25  
Length preserving maps  
Professor Herbert Busemann, University of Southern California (608-228)

4:30 - 4:50  
The construction of translation planes from projective space  
Professor R. H. Bruck* and Professor R. C. Bose, University of North Carolina (608-221)

4:55 - 5:15  
Minimally imbedded two-dimensional polyhedral manifolds  
Professor T. F. Banchoff, University of California (608-30)

FRIDAY, 3:15 P.M.

Session on Analysis, Room 110, University College

3:15 - 3:25  
(99) A nonlinear integral operation  
Professor J. S. MacNerney, University of North Carolina (608-2)

3:30 - 3:40  
(100) Integral operators in ordinary differential equations  
Professor M. Z. v. Krzywoblocki, Michigan State University (608-19)

3:45 - 3:55  
(101) Partial differential equations and difference equations  
Dr. Leopold Flatto, Yeshiva University (608-48)

4:00 - 4:10  
(102) On existence of asymptotically almost periodic and almost periodic solutions of nonlinear operational-differential equations  
Professor Witold Bodganowicz, Georgetown University (608-53)

4:15 - 4:25  
(103) Parabolabolic differential equations and Lyapunov like functions  
Dr. V. Lakshmikantham, University of Alberta, (608-60)
4:30 - 4:40  
(104) Reflection laws of fourth order elliptic differential equations  
Professor R. D. Brown, University of Kansas (608-70)

4:45 - 4:55  
(105) Asymptotic solutions of systems of nonlinear difference equations, II  
Professor W. A. Harris, Jr. and Professor Yasutaka Sibuya*, University of Minnesota (608-91)

5:00 - 5:10  
(106) Multiplicative summability methods and the Stone-Cech compactification, II  
Professor Melvin Henriksen*, Purdue University and Professor J. R. Isbell, University of Washington (608-116)

FRIDAY, 3:15 P.M.,

Session on Applied Mathematics, Room 120, University College
3:15 - 3:25  
(107) The Peaceman-Rachford method for small mesh increments  
Mr. W. H. Guilinger, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania (608-13)

3:30 - 3:40  
(108) The quadrature of some exponential transforms of Bessel functions  
Professor Julius Kane, University of Rhode Island (608-17)

3:45 - 3:55  
(109) Infinite integrals containing Bessel functions  
Dr. W. C. Lindsey, Jet Propulsion Laboratory, University of California, Pasadena (608-25)  
(Introduced by Dr. E. C. Posner)

4:00 - 4:10  
(110) Alternating direction methods for operator equations  
Dr. R. B. Kellogg, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania (608-87)

4:15 - 4:25  
(111) Steady-state diffusion thru an impervious finite cylinder into a porous semi-infinite cylinder: an exact solution. Preliminary report  
Dr. R. B. Kelman, University of Maryland (608-115)

4:30 - 4:40  
(112) Remarks on the zeros of certain combinations of Bessel functions  
Dr. J. A. Cochran, Bell Telephone Laboratories, Incorporated, Whippany, New Jersey (608-143)

4:45 - 4:55  
(113) Zeroes of Bessel functions and eigenvalues of non-self-adjoint boundary value problems  
Dr. D. S. Cohen, Rensselaer Polytechnic Institute (608-162)

5:00 - 5:10  
(114) An extension of Ascoli's theorem and its applications to the theory of optimal control  
Professor S. S. L. Chang, State University of New York at Stony Brook (608-186)  
(Introduced by Dr. S. H. Gould)

FRIDAY, 3:15 P.M.

Session on Statistics and Probability, Room 130, University College
3:15 - 3:25  
(115) Mean random path across a square  
Dr. Maurice Horowitz, The Magnavox Company, Fort Wayne, Indiana (608-6)
3:30 - 3:40  
(116) Exact and limit distributions of a branching stochastic process  
Mr. H. H. Stratton, Jr., University of California and Professor H. G. Tucker*, University of California, Riverside and The Institute for Advanced Study (608-15)

3:45 - 3:55  
(117) Probability and the (C,r) sums of Fourier series  
Mr. W. E. Rosenkrantz, Dartmouth College (608-23)

4:00 - 4:10  
(118) Contractive projections on an L_1-space  
Professor R. G. Douglas, University of Michigan (608-27)

4:15 - 4:25  
(119) An uncertainty function arising in search theory  
Dr. E. C. Posner* and Dr. H. C. Rumsey, Jr., Jet Propulsion Laboratory, California Institute of Technology (608-43)

4:30 - 4:40  
(120) Nonsingular recurrent Markov processes have stationary measures  
Professor R. E. Isaac, Hunter College (608-45)

FRIDAY, 3:15 P.M.

Session on Topology, Room 140, University College

3:15 - 3:25  
(121) On topologically induced generalized proximity relations. II  
Mr. M. W. Lodato, The Mitre Corporation, Bedford, Massachusetts (608-41)

3:30 - 3:40  
(122) Homomorphisms of semirings of continuous functions. Preliminary report  
Professor K. D. Magill, Jr., State University of New York at Buffalo (608-72)

3:45 - 3:55  
(123) Some cardinality conditions for metrization of locally compact, normal Moore spaces  
Professor D. R. Traylor, University of Houston (608-97)

4:00 - 4:10  
(124) A short proof of Koch's theorem on the existence of arcs  
Professor L. E. Ward, Jr., University of Oregon (608-109)

4:15 - 4:25  
(125) Spaces without large projective subspaces  
Dr. J. R. Isbell, Institute for Advanced Study (608-129)

4:30 - 4:40  
(126) Approximation of maps of inverse limit spaces by induced maps  
Professor M. K. Fort, Jr., University of Georgia and Professor M. C. McCord*, University of Wisconsin (608-146)

4:45 - 4:55  
(127) Some theorems paralleling a theorem by Kuratowski  
Professor D. H. Staley, Ohio Wesleyan University (608-148)

5:00 - 5:10  
(128) The product theorem for minimal Hausdorff spaces and related results. Preliminary report  
Mr. C. T. Scarborough, Tulane University (608-172)  
(Introduced by Professor M. P. Berri)
FRIDAY, 8:00 P.M.

Gibbs Lecture, Everglades Room, Everglades Hotel
Mathematical Problems of Cooperative Phenomena
Professor Lars Onsager, Yale University

SATURDAY, 2:00 P.M.

Special Session on Homological Algebra, Room 160, University College

2:00 - 2:20
Cramer's rule, complexes and multiplicity
Professor D. A. Buchsbaum* and Professor D. S. Rim, Brandeis University (608-211)

2:30 - 2:50
Brauer groups of fields with discrete rank one valuations
Professor Maurice Auslander*, Brandeis University and Mr. Armund Brumer, Boston University (608-212)

3:00 - 3:20
Minimal resolutions for finite groups
Professor Richard Swan, University of Chicago (608-213)

3:30 - 3:50
Abstract theorems and concrete problems
Professor Peter Freyd, University of Pennsylvania (608-214)

4:00 - 4:20
Grothendieck determinants
Professor Hyman Bass, Columbia University (608-215)

SATURDAY, 2:00 P.M.

Session on Analysis, Room 110, University College

2:00 - 2:10
(129) Necessary and sufficient conditions for a matrix distribution to have a positive-real Laplace transform
Professor A. H. Zemanian, State University of New York at Stony Brook (608-10)

2:15 - 2:25
(130) Inversion for Hankel convolutions
Professor D. T. Haimo, Southern Illinois University (608-82)

2:30 - 2:40
(131) An integral theorem for analytic intrinsic functions on quaternions
Professor C. G. Cullen, University of Pittsburgh (608-113)

2:45 - 2:55
(132) Regularity of domains for semi-linear partial differential equations of parabolic type
Dr. Harus Murakami, University of Kansas (608-160)

3:00 - 3:10
(133) Theta functions and elliptic curves
Professor W. E. Jenner, University of North Carolina (608-161)

3:15 - 3:25
(134) On the absolute Norlund summability of a Fourier series
Mr. B. A. Johns, Jr., University of Kansas (608-163)

3:30 - 3:40
(135) Some properties of Jacobi polynomials
Professor A. F. Danese, State University of New York at Buffalo (608-167)
3:45 - 3:55  
(136) Bessel potentials on regular Riemannian manifolds  
Professor R. D. Adams* and Professor Nachman Aronszajn, University of Kansas (608-179)

4:00 - 4:10  
(137) Some formulas to represent functions by means of derivatives  
Professor K. T. Smith, New York University (608-205)

SATURDAY, 2:00 P.M.

Session on Analysis, Room 120, University College
2:00 - 2:10  
(138) A maximum property of Cauchy’s problem in n-dimensional space-time  
Mr. Duane Sather, University of Minnesota (608-76)  
(Introduced by Professor H. F. Weinberger)

2:15 - 2:25  
(139) A uniqueness theorem  
Professor R. F. DeMar, Miami University (608-118)

2:30 - 2:40  
(140) A continuous function whose Fourier series diverges everywhere  
Professor D. R. Lick and Professor N. S. Scarritt, Jr.*, New Mexico State University (608-154)

2:45 - 2:55  
(141) Operator representation theorems  
Professor E. O. Thorp and Mr. R. J. Whitley*, New Mexico State University (608-155)

3:00 - 3:10  
(142) Eigenfunction expansions in proper functional Hilbert spaces  
Mr. E. G. P. Gerlach, University of Kansas (608-159)

3:15 - 3:25  
(143) A representation theorem for linear operators  
Professor J. T. Darwin, Jr., University of Texas and Auburn University (608-168)

3:30 - 3:40  
(144) Completions  
Professor J. W. Brace and Mr. R. M. Nielsen*, University of Maryland (608-170)

3:45 - 3:55  
(145) Invariant subspaces of continuous functions  
Mr. Morisuke Hasumi*, University of California, Berkeley and Professor T. P. Srinivasan, Panjab University, Chandigarh, India (608-177)

4:00 - 4:10  
(146) General solution of nonlinear difference equations  
Professor W. A. Harris, Jr.*, and Professor Yasutaka Sibuya, University of Minnesota (608-166)

SATURDAY, 2:00 P.M.

Sessions on Algebra, Room 130, University College
2:00 - 2:10  
(147) On the indecomposability of torsion free groups  
Mr. J. W. Armstrong, University of Illinois (608-5)

2:15 - 2:25  
(148) On high subgroups. Preliminary report  
Mr. C. K. Megibben, Texas Technological College (603-20)
2:30 - 2:40
(149) Quasi-isomorphism of direct sums of cyclic groups
Professor R. A. Beaumont* and Professor R. S. Pierce, University of Washington (608-51)

2:45 - 2:55
(150) On Howarth endomorphisms
Professor R. T. J. Mahoney, Syracuse University (608-73)

3:00 - 3:10
(151) An algebraic application of the wreath product
Dr. W. Kuyk, Ottawa University (608-99)

3:15 - 3:25
(152) ZA groups satisfying an Engel condition
Mr. K. W. Weston, University of Notre Dame (608-105)

3:30 - 3:40
(153) Spectral sequences and Frobenius groups
Professor Ernst Snapper, Dartmouth College (608-139)

3:45 - 3:55
(154) Minimal pure subgroups in primary Abelian groups
Professor P. D. Hill*, Emory University and Professor C. K. Megibben, Texas Technological College (608-156)

4:00 - 4:10
(155) Homotopism problems treated with the aid of functional equations
Professor J. Aczel, University of Debrecen, Hungary and University of Florida (608-165)

SATURDAY, 2:00 P.M.

Session on Applied Mathematics, Room 140, University College
2:00 - 2:10
(156) The solutions of a second order differential equation containing a parameter on a region surrounding four turning points
Mr. G. A. Stengle, New York University (608-30)

2:15 - 2:25
(157) Quasi-one-dimensional magnetohydrodynamic flow with heat addition, II, Oblique field
Professor R. M. Gundersen, The University of Wisconsin-Milwaukee (608-40)

2:30 - 2:40
(158) Circle theorem for the biharmonic equation. Interior problem
Dr. Johann Martinek* and Dr. H. P. Thielman, United Electrodynamics, Incorporated, Alexandria, Virginia (608-100)

2:45 - 2:55
(159) On the nonexistence of limit-cycles of a system of differential equations of nonlinear oscillations

3:00 - 3:10
(160) A finite difference analog of the Neumann problem for Poisson's equation
Dr. J. H. Bramble and Professor B. E. Hubbard*, University of Maryland (608-133)

3:15 - 3:25
(161) On the numerical solution of the Dirichlet problem for \( \Delta u + ku = F \)
Professor J. H. Bramble, University of Maryland (608-140)
3:30 - 3:40

(162) Concerning the location of singularities of solutions to certain classes of
elliptic partial differential equations in four variables
Professor R. P. Gilbert, University of Maryland (608-187)

3:45 - 3:55

(163) Maximum principle
Dr. Rudolf Vyborny, Czechoslovak Academy and University of Maryland
(608-196)

(Introduced by Professor J. B. Diaz)

SUNDAY, 2:00 P.M.

Special Session on Partial Differential Equations, Room 160, University College

2:00 - 2:20

The exterior problem for nonlinear elliptic equations in n-variables
Professor David Gilbarg, Stanford University

2:30 - 2:50

Scattering theory
Professor P. D. Lax, New York University and Professor Ralph Phillips*, Stanford University (608-219 and 608-220)

3:00 - 3:20

Scattering theory
Professor P. D. Lax*, New York University and Professor Ralph Phillips, Stanford University (608-219 and 608-220)

3:30 - 3:50

Removable singularities for the equation of heat conduction
Professor D. G. Aronson, University of Minnesota (608-222)

4:00 - 4:20

The Cauchy problem for overdetermined systems
Professor Leon Ehrenpreis, New York University (608-227)

SUNDAY, 2:00 P.M.

Session on Analysis, Room 110, University College

2:00 - 2:10

(164) Summability of Laguerre series. Preliminary report
Professor G. G. Bilodeau, Boston College (608-31)

2:15 - 2:25

(165) On the compatibility of the uniform integral, II
Mr. R. T. Sandberg, University of Arizona (608-33)

2:30 - 2:40

(166) On families of functions characterized by kernels
Professor R. F. Jolly, University of California, Riverside (608-36)

2:45 - 2:55

(167) On recursively defined orthogonal polynomials
Professor T. S. Chihara, Seattle University (608-81)

3:00 - 3:10

(168) Summation methods on locally compact spaces
Professor Arne Persson, Lund University and University of Kansas
(608-88)

(Introduced by Dr. S. H. Gould)

3:15 - 3:25

(169) Summability C on the (k - 1)-dimensional hypersphere
Professor Aaron Siegel, State University of New York at Buffalo
(608-106)
3:00 - 3:10
(172) Minimal Gerschgorin sets
Professor R. S. Varga, Case Institute of Technology (608-12)

2:15 - 2:25
(173) Goddard problem with bounded thrust
Mr. Herman Munick, Grumman Aircraft Engineering Corporation, Bethpage, New York (608-22)

2:30 - 2:40
(174) New bound on (n,k) binary group codes
Dr. Gustave Solomon* and Mr. J. J. Stiffler, Jet Propulsion Laboratory, California Institute of Technology (608-34)

3:00 - 3:10
(175) Characteristic values of circulant matrices
Dr. M. M. Lotkin, General Electric Company, Philadelphia, Pennsylvania (608-38)

3:15 - 3:25
(176) On some problems in the optimal design of shields and reflectors in particle physics
Mr. Martin Tierney, Dr. P. E. Waltman* and Mr. G. M. Wing, Sandia Corporation, Albuquerque, New Mexico (608-120)

3:30 - 3:40
(177) Fuel-optimal controls
Professor A. A. Goldstein, University of Texas and Dr. T. I. Seidman*, Boeing Scientific Laboratory, Seattle, Washington (608-117)

3:45 - 3:55
(178) A boundary-value control problem with guidance theory applications
Professor R. W. Hunt* and Mr. Robert Silber, Southern Illinois University (608-130)

4:00 - 4:10
(180) Bauer fields on values of a matrix
Mr. N. E. Nirschl*, St. Norbert College and Mr. Hans Schneider, University of Wisconsin (608-176)

SUNDAY, 2:00 P.M.

Session on Algebra, Room 130, University College

2:00 - 2:10
(181) Middle nucleus = center in a simple Jordan ring
Professor Edwin Kleinfeld, Syracuse University (608-114)

2:15 - 2:25
(182) Double loops and ternary rings
Professor Peter Wilker, University of Berne, Switzerland and State University of New York at Buffalo (608-153)
2:30 - 2:40  
(183) Generalized quotient rings. Preliminary report  
Professor Fred Richman, New Mexico State University (608-171)

2:45 - 2:55  
(184) Almost-Gaussian domains  
Mr. G. L. Kerr, State University of Iowa (608-173)

3:00 - 3:10  
(185) Ancestral rings and left-idealizers  
Professor R. E. Peinado, State University of Iowa (608-182)

3:15 - 3:25  
(186) Kurosh radicals of rings with operators  
Professor N. J. Divinsky, University of British Columbia and  
Mr. A. Sulinski*, University of Warsaw, Poland (608-192)

3:30 - 3:40  
(187) Nonideal topologies on the ring of integers  
Professor L. A. Hinrichs, Duke University (608-202)

3:45 - 3:55  
(188) Rings of arithmetic functions  
Professor Leonard Carlitz, Duke University (608-203)

SUNDAY, 2:00 P.M.

Session on Topology, Room 120, University College
2:00 - 2:10  
(189) Some results on crumpled cubes  
Mr. L. L. Lininger, State University of Iowa (608-65)

2:15 - 2:25  
(190) Taming 2-complexes in high-dimensional manifolds  
Professor C. H. Edwards, Jr., University of Wisconsin (608-84)

2:30 - 2:40  
(191) The genus of the n-cube  
Mr. L. W. Beineke and Professor Frank Harary*, University of Michigan (608-85)

2:45 - 2:55  
(192) On the thickness of the complete graph  
Mr. L. W. Beineke* and Professor Frank Harary, University of Michigan (608-86)

3:00 - 3:10  
(193) Cellular subcomplexes of piecewise-linear manifolds  
Mr. R. E. Chandler, Florida State University and Duke University (608-125)

3:15 - 3:25  
(194) Pairs of 3-cells with intersecting boundaries in E^3  
Professor C. E. Burgess, University of Utah (608-126)

3:30 - 3:40  
(195) Extended topology: connected sets and Wallace separations  
Professor P. C. Hammer, University of Wisconsin (608-184)

3:45 - 3:55  
(196) Applications of a result of Capel and Strother  
Professor A. D. Wallace, University of Florida (608-197)

4:00 - 4:10  
(197) Neighborhood product and quotient spaces. Preliminary report  
Dr. Z. P. Mamuzić, University of Florida (608-198)  
(Introduced by Professor A. D. Wallace)
PRELIMINARY ANNOUNCEMENTS OF MEETINGS

Six Hundred Ninth Meeting
New York University, Washington Square Campus
New York, New York
February 29, 1964

The American Mathematical Society will hold its six hundred ninth meeting on February 29, 1964 at the Washington Square Campus of New York University.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Heisuke Hironaka of Brandeis University will give an address entitled "Formal, analytic, henselian and algebraic structures" at 2:00 P.M.

There will be sessions for contributed papers both morning and afternoon. Abstracts for contributed papers should be sent to the American Mathematical Society, 190 Hope Street, Providence, Rhode Island 02906 so as to arrive prior to the deadline of January 16, 1964.

The Washington Square Campus may be reached by public transportation as follows: Lexington Avenue (Interborough) Subway (IRT)--Local to Astor Place Station. Walk west on Astor Place to Broadway, then south on Broadway to Waverly Place, and west on Waverly Place to Washington Square.

Seventh Avenue (Interborough) Subway (IRT)--Local to Sheridan Square Station, Walk east on Waverly Place to Washington Square.

Broadway (Brooklyn - Manhattan) Subway (BMT)--Brighton local or Fourth Avenue local to Eighth Street Station, Walk south on Broadway to Waverly Place, then west on Waverly Place to Washington Square.

Sixth or Eighth Avenue (Independent) Subway (IND)--Express to West Fourth Street - Washington Square Station, Walk east on West Fourth Street or Waverly Place to Washington Square.

Fifth Avenue Bus--Busses numbered 3, and some numbered 5, to University Place. Walk south one block to Waverly Place.

Everett Pitcher
Associate Secretary
Bethlehem, Pennsylvania

Six Hundred Eleventh Meeting
Hotel New Yorker
New York, New York
April 20-23, 1964

The six hundred eleventh meeting of the American Mathematical Society will be held at the Hotel New Yorker in New York on April 20-23, 1964.

Contributed papers and invited addresses will be scheduled on Monday, April 20 and Tuesday, April 21.

The Association for Symbolic Logic

The Association for Symbolic Logic will meet at the same place on Tuesday, April 21. Their sessions will follow a session of the Society for contributed papers in Logic and Foundations.

Abstracts of papers contributed to the meeting of the Association for Symbolic Logic (but not abstracts in Logic and foundations contributed to the meeting of the Society) should be sent to Professor Raymond Smullyan, Yeshiva University, Amsterdam Avenue and 186th Street, New York 33, New York to arrive prior to his

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The Symposium in Applied Mathematics

There will be a Symposium on Applications of Nonlinear Partial Differential Equations in Mathematical Physics on the afternoon of Tuesday, April 21 and on April 22 and 23.

The subject was chosen by the Committee on Applied Mathematics, consisting of V. Bargmann, G. E. Forsythe, P. R. Garabedian, C. C. Lin, Chairman, Alfred Schild, and David Young.


The Organizing Committee, responsible for the planning of the program and the choice of speakers, consists of

Professor Robert Finn, Stanford University  
Professor C. B. Morrey, University of California at Berkeley  
Professor Walter Noll, Carnegie Institute of Technology  
Professor J. B. Serrin, University of Minnesota  
Professor A. H. Taub, University of Illinois

The Organizing Committee, responsible for the planning of the program and the choice of speakers, consists of

Professor Robert Finn, Stanford University  
Professor C. B. Morrey, University of California at Berkeley  
Professor Walter Noll, Carnegie Institute of Technology  
Professor J. B. Serrin, University of Minnesota  
Professor A. H. Taub, University of Illinois

The five sessions of the Symposium will be the following:

I. General non-linear theory  
II. Finite elasticity, compressible fluids  
III. Viscous fluids, magnetohydrodynamics  
IV. General relativity, quantum field theory  
V. Round table discussion

Deadlines for contributed papers

The deadline for receipt of abstracts of contributed papers for the sessions on Monday is March 5, 1964.

The Round Table Discussion at the conclusion of the Symposium will be devoted to extended comments on the invited addresses and to the presentation and discussion of selected short contributed papers. Abstracts intended for consideration for the Round Table Discussion should be submitted on the usual abstract blank before February 28 and should be marked For Symposium on Partial Differential Equations, Section (Number). A discussion leader will select those papers he wishes to use. The number on the abstract should be one of the numbers I, II, III, IV, namely that of the section to which it is relevant, so that it will come to the attention of the appropriate discussion leader. The selection of papers for the Round Table will be made on the basis of consistency of the program and relevance to the papers of the invited speakers. The selection will be limited by the available time. Papers not selected for the Round Table will be considered along with the other contributed papers for a special session on Partial Differential Equations on Tuesday and for sessions on Monday. The abstracts contributed for the Symposium will be considered for publication in the same manner as other abstracts.

All abstracts should be sent as usual to the American Mathematical Society, 190 Hope Street, Providence, Rhode Island 02906.

Everett Pitcher  
Associate Secretary  
Bethlehem, Pennsylvania
The eleventh American Mathematical Society Summer Institute will be held from July 6 to July 31, 1964. The topic of this Summer Institute is Algebraic Geometry. It will probably be held at the Whitney Estate, Woods Hole, Massachusetts; however, Cornell University, Ithaca, New York, is being considered as an alternate location. The Institute will be sponsored by the American Mathematical Society, and proposals for financial support have been sent to the National Science Foundation, the Air Force Office of Scientific Research, and the North Atlantic Treaty Organization.

The activities of the Institute will center about current work in Algebraic Geometry, with special emphasis on the following topics: theory of singularities; classification of surfaces and moduli; Grothendieck cohomology; zeta-functions and arithmetic of abelian varieties.

The Organizing Committee consists of Professor Oscar Zariski, Harvard University, Chairman; Professor W. L. Chow, Johns Hopkins University; Professor Maxwell Rosenlicht, University of California, Berkeley; Professor D. C. Spencer, Stanford University; and Professor John Tate, Harvard University.

In addition to the financial support that will be offered to invited participants, a limited amount of such support is expected to be available to interested persons; these should write to a Committee member and have letters of recommendation from their references sent to that Committee member.

Oscar Zariski,
Chairman
A conference on Manpower Problems in the Training of Mathematicians, sponsored by the Conference Board of the Mathematical Sciences, with support from the National Science Foundation, was held in Washington, D. C., on 16, 17 April 1963. The conference was a response to the report of the President's Science Advisory Committee on Graduate Training in Engineering, Mathematics and Physical Sciences, which was issued on 12 December 1962. The sense of the conference is contained in its recommendations listed below. These recommendations were drafted by panels of participants named by J. Barkley Rosser, who chaired the conference, and discussed by the conference as a whole.

1. The existing forms of support for graduate students--fellowships, research assistantships, teaching assistantships-- are valuable and should be continued. However, a training grant program is required which (a) provides a department of mathematics with the opportunity of attracting students in number and interests corresponding to its capabilities; (b) enables the department to expand its faculty and facilities to handle more students; (c) facilitates the establishment of new centers of excellence. Such a program is required to complete and balance the existing forms of graduate student support and to stabilize the work of departments of mathematics. The programs should engage the participation of mission-oriented agencies as well as the National Science Foundation.

Note on Recommendation 1.

The National Science Foundation has been developing plans for training grants for graduate students. After the end of the conference, the recommendations of the conference relative to training grants were communicated verbally to officials of the National Science Foundation. These officials summarized their current thinking on the subject in the following points:

1. Training grants are a new item in NSF budgets, appearing for the first time in the budget for 1 July 1963 to 30 June 1964. It is not known yet if any funds will be appropriated, so that all plans must necessarily be tentative.

2. For the year 1963-1964 the request for training grants included provision for first year graduate students only. It is hoped that in subsequent years funds will be available for grants for more advanced students.

3. The selection of first year students to receive training grants, and the determination of which students shall have training grants renewed for a subsequent year, shall be in the hands of the grantee university.

2. New centers of excellence should be developed at universities whose departments of mathematics already give the doctorate and which have demonstrated the desire and capability for such development. These universities should be selected primarily where there are untapped pools of prospective graduate students in the mathematical sciences.

3. The support of Summer Seminars for Graduate Student Research, presently undertaken by NSF, should be continued and strengthened. One or more Summer Institutes for Advanced Study, modeled on the Canadian Mathematical Congress, should be established for post-doctoral research.

4. Pre-doctoral fellows should normally be expected to do some teaching. They should receive compensation for such
work beyond their fellowship stipends.

5. Policies governing awards of post-doctoral research support by both private and government agencies should take account of the extent to which applicants have participated in the educational process, particularly in cases where the applicants have had recent support for two consecutive years.

6. Young mathematicians who have established their research productivity should be encouraged to engage in thesis direction by reducing their teaching loads. Federal support can be of real help in this respect.

Graduate students should be provided with effective information as to the availability of productive thesis advice in departments other than those of the best known centers. Such information should be gathered and distributed regularly, e.g. biennially, by a responsible mathematical organization.

A study of faculty resources for Ph.D. production on a national scale should be made so as to obtain a realistic estimate of possible output.

7. Suitable arrangements should be found by which universities encourage the use of the substantial number of qualified mathematicians in industry and government who can participate in the supervision of doctoral dissertations and graduate instruction.

8. Centers of mathematical excellence should be provided with funds to develop in the direction of advanced technology.

9. The policy of the Federal government which provides funds for laboratories and other research space should be broadened to provide for seminar rooms, office space, and libraries. Such facilities are essential for graduate student research in mathematics as laboratories are for the physical sciences and engineering. Attention is called to the report on Buildings and Facilities for the Mathematical Sciences by J. S. Frame, sponsored by CBMS, and to be published July, 1963, by the Columbia University Press.

10. The relationship between pure and applied mathematics should be examined in the light of the small number of doctorates being produced in statistics, applied mathematics, and computation, and the large demand for such mathematicians. It may be desirable, in the larger universities, to form separate departments in such fields and unite them with mathematics in a division of mathematical sciences.

11. Adequate funds should be provided for the maintenance of Mathematical Reviews and other abstracting and reviewing journals in view of their indispensability for graduate research training.

12. The universities should obtain support to cover the full proportion of the costs of computer facilities used in the graduate training required for engineering, mathematics, and physical sciences.

13. Departments of mathematics should maintain a continuing review of their courses, taking note of recommendations of such bodies as the Committee on the Undergraduate Program in Mathematics, so that students entering graduate study in engineering, mathematics, and physical sciences shall be properly prepared to seek advanced degrees.

14. Departments of mathematics using graduate students as teaching assistants should take responsibility for raising the level of their teaching performance.

A few copies of the conference report are available gratis at CBMS, Mills Building, 17th Street and Pennsylvania Avenue, N. W., Washington 6, D. C.

Leon W. Cohen
Executive Secretary CBMS,
ACTIVITIES OF OTHER ASSOCIATIONS

THE MATHEMATICAL ASSOCIATION OF AMERICA
Miami and Coral Gables, Florida
January 25-27, 1964

The forty-seventh Annual Meeting of the Mathematical Association of America will be held at the University of Miami, Coral Gables, Florida, from Saturday to Monday, January 25-27, 1964.

FIRST SESSION, SATURDAY: 9:00 A.M.
Ponce de Leon Junior High School Auditorium

GOALS FOR SCHOOL MATHEMATICS
(Report of the Cambridge Conference)

9:00 - 10:00 The Earliest Grades
   Professor A. M. Gleason, Harvard University
Viewpoints of a User of Mathematics
   Professor G. F. Carrier, Harvard University
Implications for Teacher Education
   Professor R. J. Wisner, New Mexico State University

10:00 - 10:10 INTERMISSION

10:10 - 10:50 Panel Discussion, Critique, Rebuttal, Discussion from the Floor

11:00 - 11:50 Retiring Presidential Address:
   PIVOTAL METHODS IN LINEAR ALGEBRA
   Professor A. W. Tucker, Princeton University

SECOND SESSION, SUNDAY: 9:00 A.M.
Ponce de Leon Junior High School Auditorium

CONTENT OF THE FIRST COURSE IN REAL VARIABLES

9:00 - 10:10 Professor E. J. McShane, University of Virginia
   Professor Edwin Hewitt, University of Washington
   Professor Walter Rudin, University of Wisconsin

10:10 - 10:20 INTERMISSION

10:20 - 10:50 Panel Discussion, Critique, Rebuttal, Discussion from the Floor

11:00 - 11:50 Business Meeting of the Association; the Association's Third Award for Distinguished Service to Mathematics, and the Award of the 1964 Chauvenet Prize

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THIRD SESSION, MONDAY: 9:00 A.M.
Room 120, University College Building

FUNCTIONAL EQUATIONS

9:00 - 9:50  The Commutator Equation and Some Allied Equations
   Professor Einar Hille, Yale University

10:00 - 10:50  Analytic Properties of the Solutions of Certain Functional Equations
   Professor J. H. B. Kemperman, The University of Rochester

11:00 - 11:50  The Functional Equation of Associativity
   Professor Berthold Schweizer, The University of Arizona

FOURTH SESSION, MONDAY: 2:00 P.M.
Room 120, University College Building

CONVEXITY

2:00 - 2:50  The Polyhedron Inequality
   Professor Herbert Busemann, University of Southern California

3:00 - 3:50  Some Problems of Elementary Euclidean Geometry
   Professor H. G. Eggleston, University of London and University
   of Washington

4:00 - 4:50  Facial Structure of Convex Polytopes
   Professor V. L. Klee, Jr., University of Washington

FILM PROGRAM
Ponce de Leon Junior High School Auditorium

Saturday,
1:00 - 1:55 P.M.  Color Animations by Bruce Cornwell: "Seven Bridges of Königsberg," "Possibly so, Pythagoras," "How Do We Count?," "Big Numbers, Little Numbers," "Sets, Crows, and Infinity"

Everglades Room, Everglades Hotel, Miami

Saturday,
7:00 - 7:11 P.M.  "Four Line Conics" (color, silent), by Fletcher
7:12 - 7:45 P.M.  "Straight Line Kinematics" (black and white), a PSSC physics film
7:55 - 8:21 P.M.  "Frames of Reference" (black and white), a PSSC physics film
8:22 - 8:38 P.M.  "Vector Kinematics" (black and white), a PSSC physics film
8:39 - 8:57 P.M.  "Thinking Machines" (color), an ETS film

Sunday,
7:00 - 7:34 P.M.  "Theory of Limits," Part I (black and white), by Professor E. J. McShane
7:40 - 8:40 P.M.  "The Kakeya Problem" (color and with animation), by Professor A. S. Besicovitch
8:45  P.M.  A short film to be announced at the meeting.
SIAM's 1964 Spring Meeting will include sessions in memory of Theodore von Kármán with the theme "Applied Mathematics and Mechanics". A special committee under the chairmanship of Professor W. R. Sears, Cornell University, Ithaca, New York, is planning these sessions. There will be invited papers by distinguished mathematicians and scientists, as well as sessions for contributed papers which are consistent with the central theme. Abstracts for contributed papers should be sent to Professor Sears, deadline February 1.

The von Kármán sessions will comprise two days of the four-day meeting. A banquet scheduled for Wednesday night, May 13, will feature an address in harmony with the von Kármán theme.

In addition to the von Kármán Days, there will be sessions featuring invited papers on current important areas of applied mathematics and the customary sessions of contributed papers.

The broad capabilities of TRW Space Technology Laboratories range from research and development through hardware production, to launch supervision, tracking and evaluation of flight data. Today, more than 6,500 engineers, scientists and support personnel are working on major NASA and Air Force Programs. STL is building OGO and Pioneer spacecraft for NASA, and Program 823 spacecraft for the Air Force. STL is also developing special engines for Apollo's Lunar Excursion Module and other spacecraft, and continuing Systems Management for the Air Force's ATLAS, TITAN and MINUTEMAN Programs.

These many activities create these immediate openings at the Computation and Data Reduction Center and the Systems Analysis and Design Laboratories of STL's new advanced Space Technology Center in Redondo Beach, near Los Angeles International Airport.

GENERAL SCIENTIFIC PROGRAMMERS • TEST EVALUATION PROGRAMMERS • MANAGEMENT SYSTEMS PROGRAMMERS • COMPUTATIONAL SYSTEMS PROGRAMMERS • SYSTEMS TEST ANALYSTS • SYSTEMS ANALYSTS • TRAJECTORY ANALYSTS • GUIDANCE ANALYSTS • MASS PROPERTIES ANALYSTS • ASTRODYNAMICISTS • PRELIMINARY DESIGN WEIGHT ENGINEERS •

For interview in your area in the near future, act NOW! Forward your resume to STL Professional Placement, Department 0-1, One Space Park, Redondo Beach, California. TRW is an equal opportunity employer.
The following is a summary of the sessions so far arranged for 1964. The dates include arrival and departure days.

**Jan. 3-6** Research sessions of the Frankfurt seminar  
Leader: R. Baer, Frankfurt a.M.

**Feb. 12-18** Seminar: Group theory and elementary particles  
Leaders: P. Beckmann, Mainz; H. G. Tillmann, Mainz

**Feb. 29** Research session  
Leader: P. L. Butzer, Aachen

**Mar. 8** Mathematical methods in celestial mechanics and astronautics and related questions of numerical analysis  
Leader: E. Stiefel, Zurich

**Mar. 15-21** Mathematical logic  
Leaders: H. Hermes, Münster; H. A. Schmidt, Marburg

**Apr. 14-18** Foundations of geometry  
Leaders: F. Bachmann, Kiel; E. Sperner, Hamburg

**Apr. 26-**  
**May 2** Research group  
Leader: K. Krickeberg, Heidelberg

**May 18-23** The geometry of groups and the groups of geometry, with especial attention to finite structures  
Leader: R. Baer, Frankfurt a.M.

**June 7-13** Numerical methods for problems in linear algebra  
Leader: F. L. Bauer, München

**June 14-20** Methods of functional analysis applied to numerical mathematics  
Leader: L. Collatz, Hamburg

**June 21-27** Differential geometry in the large  
Leaders: S. S. Chern, Berkeley; H. Hopf, Zurich  
In charge of preliminary work: M. Barner, Freiburg; W. Klingenberg, Mainz

**Aug. 2-8** Mathematical statistics and the theory of probability  
Leader: J. Pfanzagl, Köln

**Aug. 9-16** Meeting of the international committee for improvement of the teaching of mathematics  
Guest session: F. Denk, Erlangen; G. Papy, Brussel

**Aug. 17-22** Reserved for an international session of guest mathematicians.

**Aug. 26-**  
**Sept. 6** Topology  
Leaders: A. Dold, Heidelberg; D. Puppe, Saarbrucken; H. Schibert, Kiel

**Sept. 6-12** Algebraic number theory  
Leaders: H. Hasse, Hamburg; P. Roquette, Tubingen
Sept. 20-26  History of problems in mathematics  
      Leader:  J. E. Hofmann, Ichenhausen

Sept. 27-Oct. 3  Research group  
      Leader:  K. Krickeberg, Heidelberg

Oct. 4-10  Geometry  
      Leader:  K. H. Weise, Kiel

Oct. 11-17  Curriculum in advanced mathematics for secondary school teachers  
      Leaders:  H. Kneser, Tübingen; W. Vogel, Tübingen

Oct. 25-31  Pedagogy (cooperation of school, university and administration)  
      Leaders:  M. Barner, Freiburg; K. Fladt, Freiburg  
      (Subject to be announced later)

Oct. 18-24  Guest session  
      Colloquium on musicology  
      Leader:  H. H. Eggebrecth, Freiburg

M. Barner  
Director of the Institute  
78 Freiburg i. Br.  
Hebelstrasse 29

A NEW POLICY OF THE PACIFIC JOURNAL OF MATHEMATICS

The Board of Governors of the Pacific Journal of Mathematics has adopted the policy that beginning January 1, 1964, all manuscripts submitted to the journal for publication must be accompanied by an author's résumé. The author's résumé will be submitted to the referee at the same time as the manuscript. The referee of a manuscript may suggest changes in or offer criticisms of an author's résumé in order that it be more meaningful and appropriate. It is expected by the editors that the necessary changes in the résumé will be made by the author before publication.

The change in policy adopted by the Pacific Journal of Mathematics will unquestionably cause some inconvenience during an initial period. The editors of the journal can only suggest what an author's résumé should contain because the policy is new to most mathematical journals. In a sense, the editors believe that the résumé should represent an objective compromise between Mathematical Reviews and the abstracts now appearing in the Notices of the American Mathematical Society of papers to be given at a meeting. The length of the résumé should be sufficient to ensure an adequate description of the important results and their relationship to the mathematical literature.

Lowell J. Paige, Managing Editor  
Pacific Journal of Mathematics

AFOSR DIVISION OF MATHEMATICS

The MATHEMATICS DIVISION of the AIR FORCE OFFICE OF SCIENTIFIC RESEARCH has announced a redirection of its continuing research program, to emphasize the areas of analysis, functional analysis, probability and statistics. The Division also wishes to announce the continuation of its special research sabbatical and book-writing programs, for the coming fiscal year (FY 65). The latter programs are not restricted in terms of mathematical areas of interest.

The redirection of the continuing program is being affected because of the changing "national research-support picture," i.e., the total amount of funds now available nationally for the support of research in mathematics, the amount available to the Mathematics Division, AFOSR, and considered judgments to reinforce the
best interests of the latter. Proposals in appropriate areas may continue to be submitted, preferably for a two- or three-year period.

Support for a research sabbatical may be requested for a nine or twelve month period, in this instance to begin 1 October 1964 or 1965. Partial support by the proposer's University is preferred, but not essential. Proposals for such support should follow the outlines of the conventional research proposal, including a statement of the research problems and methods of attack, a biographical sketch, the proposer's bibliography, and a budget. The project officer for this program is Major Joseph P. Martino.

The goal of the book-writing program is to bridge the gap between professional-level and user-level mathematics, in areas of current and potential Air Force interest. Proposals for books which come closest to this goal will be given preferential treatment. Support may be for a nine or twelve month period, beginning 1 October 1964 or 1965. Again, they should follow the outlines of the conventional research proposal as appropriate. They should include a table of contents, sample chapters, and reference to the unique features of the book. The project officer is Captain John F. Gander.

Proposals and inquiries should be directed to the Mathematics Division, Air Force Office of Scientific Research, Washington, D. C. 20333. Proposals under either of the special programs should be directed to the attention of the appropriate project officer. Decisions under the special programs should be forthcoming by Spring 1964.

**A NEW PH.D. PROGRAM**

The University of Wisconsin-Milwaukee has just been authorized to begin a Ph.D. program in Mathematics starting September, 1964. Although thesis work may be done in a number of subjects, the program will emphasize classical analysis and applied mathematics. A number of major appointments are now being made along these lines.

**TAX STATUS OF GRANTS**

The Internal Revenue Service announced in a technical information release on October 16 that it will dispose of pending income tax cases, involving recipients of scholarships and fellowship grants, which are substantially identical in their facts to two cases decided in Federal tax courts in accordance with the decisions in those cases.

In the two cases, the courts held that the stipends which the students received through their universities' research programs were primarily to further their education and training and therefore were excludable from gross incomes as scholarships or fellowship grants under section 117 of the Internal Revenue Code. In both cases, the taxpayers were candidates for the Ph.D. degree and the research in connection with which the stipends were paid was accepted by the universities in satisfaction of their degree requirements. Equivalent research was required of all candidates for the degree at each university.

The Internal Revenue Service is now in the process of revising the regulations. Representatives of higher education, through the auspices of the American Council on Education, will meet with Treasury officials on October 30 to discuss and clarify more specific guidelines for determining when a scholarship or fellowship exists.

(Reprinted from Higher Education and National Affairs, Volume XII(October 25, 1963) published by the American Council on Education.)

**UNPUBLISHED MATHEMATICAL TABLES**

The editorial office of Mathematics of Computation maintains a repository of Unpublished Mathematical Tables (UMT). When a table is deposited in the UMT repository a brief summary of its contents is published in the section, Reviews and Descriptions of Tables and Books. Readers may request copies of the tables from the editor, which are made available at a nominal cost.
Editor, The NOTICES

Getting good advanced training for the doctorate for enough budding young Mathematicians is a tough problem. There is clearly a need (due to the rapid growth of Science and Technology) and an opportunity (many more potentially able and interested undergraduates). But, it is not clear how many Ph.D's can be trained well. The recent "Conference on Manpower Problems in the Training of Mathematicians" struggled with this problem. Nobody could expect a pat solution at the fag end of a two day conference: A glance at the recommendations of this conference (This issue of NOTICES, page 32) confirms this expectation.

For example, it is by no means clear that a "training grant program" (see recommendation 1) is the best way of supporting more good graduate students; perhaps more cooperative fellowships with their yearly renewal is a better device. Directing Ph.D. theses is hard individual work. Maybe there is a commodity called "productive thesis advice", but then it is often the case that advice profitable for one student is disastrous for another. Thus it is hard to imagine (recommendation 6) how any organization is going to collect "effective information as to the availability of productive thesis advice". That young Mathematicians should start directing theses is good. Should they start when they have established their "research productivity" and are pushed by administrators bearing reduced teaching loads (recommendation 6) or should they start when they have established mastery of some field in which they are anxious to encourage interest?

More doctorates in Applied Mathematics or in Computation would be excellent (recommendation 10), but it will probably take more than part time supervision of dissertations (recommendation 7). Distant direction of part time dissertations is notoriously a dismal process.

Many important considerations simply do not appear among the 14 recommendations of the conference. Most important is quality: The American Mathematical community has in recent years developed a number of young men of the highest quality, How does one continue this accomplishment in the face of much larger numbers of graduate students? This point should take some hard-headed recommendations.

The conference was stimulated in part by the "Gilliland report" from PSAC (The President's Scientific Advisory Committee). But there are substantial reservations about the tenor of this report: The panel which prepared it, with Mathematics represented but sparsely, proposed drastic increases in the annual number of Ph.D.'s in Mathematics and this with little attention to the problem of maintaining quality. One might regret that the Manpower conference did not in its recommendations take some note of these reservations. Is the word passed down from PSAC to be sacrosanct?

One conclusion is clear: A brief conference held in the bustling atmosphere of Washington cannot settle problems of University policy which call for some considerable measure of wisdom and restraint.

Saunders Mac Lane

Editor, The NOTICES

As a reviewer for Mathematical Reviews, may I enter a plea in behalf of its present mode of operation? (This is in reply to the letter contained in the Secretary's Report, published in the NOTICES, Vol. 10, p. 548.) I think the method of abstracting by an original review, written by an experienced worker in the field, is one of the few significant advances in the methods of scientific publication during the modern era.

The reviews published in Mathematical Reviews make vivid reading, and are far more useful than authors' abstracts. An outsider who abstracts an article will have more of a sense of pro-
portion about it than the author, and understand better what is important and what is unimportant to a reader who comes to it without previous acquaintance. Furthermore, the reviews stimulate the field, I think, by allowing a forum for opinions (expressed in some of the more outspoken reviews) on the merits of new developments.

The monetary compensation I receive as a reviewer (namely, a small discount on subscription price) does not pay me for my time, but I consider my efforts to be in discharge of a civic duty to my field, and I am proud to contribute to this useful activity. I think the multitude of other reviewers must feel the same way.

Given the relatively immense amounts spent on publishing original work the world over, and in view of the "knowledge explosion" these publications reveal to be taking place in the mathematical sciences, isn't a publication which makes all this work readable and comprehensible to scientists a necessity rather than a luxury? I hope the funds for its continuation will be found.

Armand Siegel
Professor of Physics
Boston University

MEMORANDA TO MEMBERS

PANEL OF CONSULTANTS ON GRADUATE INSTRUCTION

At its meeting in January, 1963, the Council of the Society authorized the preparation of a list of persons who have had considerable experience with the establishment and administration of Ph.D. programs in mathematics and who would be willing to serve as consultants to institutions instituting such programs. This list was prepared by a committee of C. W. Curtis, Pasquale Porcelli, L. H. Loomis, H. F. Bohnenblust, M. L. Curtis, and Kirk Fort, Chairman, and is now available by writing to the Secretary, John W. Green, University of California, Los Angeles, California.

Those appearing in this list have expressed willingness to consult with mathematicians and administrators of institutions wishing to establish Ph.D. programs in mathematics. Arrangements for such consultation should be made with the individual mathematicians concerned.

John W. Green

THE NATIONAL REGISTER OF SCIENTIFIC AND TECHNICAL PERSONNEL

The Mathematical and Statistical Sciences Section of the Register will maintain a desk during the Annual Meeting at the University of Miami, Coral Gables, Florida on January 24, 25, and 26, 1964. The National Register Desk will be located in the Westminster Chapel Fellowship Room, 5225 Ponce de Leon Boulevard, Coral Gables, Florida. The attendants will be pleased to assist with registrations and to supply information. The National Register as a whole is a responsibility of the National Science Foundation. The Mathematical and Statistical Sciences Section is operated by the American Mathematical Society with the cooperation of the Association for Computing Machinery, the Association for Symbolic Logic, the Biometric Society, the Econometric Society, the Industrial Mathematical Society, the Institute of Mathematical Statistics, the Mathematical Association of America, the Operations Research Society of America, the Society for Industrial and Applied Mathematics, the Society of Actuaries, and the American Statistical Association.

CHAIRMEN

The Annual List of Chairmen of Departments has been compiled. Copies may be obtained free of charge by writing to the Headquarters Offices, 190 Hope Street, Providence, Rhode Island 02906.
NEW AMS PUBLICATIONS

SELECTED TRANSLATIONS SERIES II

Volume 33

444 pages; List Price $5.80; 25% discount to members.


Volume 34

416 pages; List Price $5.40; 25% discount to members.


MATHEMATICAL SURVEYS

Volume 8

DISCONTINUOUS GROUPS AND AUTOMORPHIC FUNCTIONS
by Joseph Lehner

432 pages; List Price $12.60; 25% discount to members.

Much has been written on the theory of discontinuous groups and automorphic functions since 1880, when the subject received its first formulation. The purpose of this book is to bring together in one place both the classical and modern aspects of the theory, and to present them clearly and in a modern language and notation. The emphasis in this book is on the fundamental parts of the subject.

The book is directed to three classes of readers: graduate students approaching the subject for the first time, mature mathematicians who wish to gain some knowledge and understanding of automorphic function theory, and experts.

Volume 10

AN INTRODUCTION TO THE ANALYTIC THEORY OF NUMBERS
by Raymond Ayoub

396 pages; List Price $10.20; 25% discount to members.

There exist relatively few books, especially in English, devoted to the analytic theory of numbers and virtually none suitable for use in an introductory course or suitable for a first reading. This is not to imply that there are no excellent books devoted to some of the ideas and theorems of number theory. Mention must certainly be made of the pioneering and monumental work of Landau and in more recent years of the excellent books of Estermann, Ingham, Prachar, Vinogradoff and others. For the most part, however, these works are aimed at the specialist rather than at the general reader.

The book is divided into five chapters: I. Dirichlet's theorem on primes in an arithmetic progression; II. Distribution of primes; III. The theory of partitions; IV. Waring's problem; V. Dirichlet L-functions and class number of quadratic fields.
SUMMER EMPLOYMENT OPPORTUNITIES

The Mathematical Sciences Employment Register, 190 Hope Street, Providence, Rhode Island 02906, has compiled a list of opportunities for summer employment for mathematicians and college mathematics students. Members who are interested in summer employment and would like to obtain a copy of this list should write to the Mathematical Sciences Employment Register. There is no charge for this list.

Copies will be available at the Annual Meeting in the Westminster Chapel Fellowship Room, 5225 Ponce de Leon Boulevard, Coral Gables, Florida.

THE EMPLOYMENT REGISTER

The following item is repeated from the November 1963 issue of the NOTICES and gives a more detailed time schedule.

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the University of Miami, Coral Gables, Florida, in the Westminster Chapel Fellowship Room on January 24, 25, and 26, 1964. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registration, either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $3.00 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence, Rhode Island 02906, for application forms or for position description forms. These forms must be completed and returned to Providence not later than December 15, 1963, in order to be included in the listings at the Annual Meeting in Miami. Position description forms which arrive after this closing date, but before January 5, will be included in the register at the meeting for a late registration fee of $3.00. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

LONDON MATHEMATICAL SOCIETY

The Council of the London Mathematical Society has decided to increase the size of the JOURNAL, and to increase the price to non-members for both the JOURNAL and the PROCEEDINGS. At the same time, dues of ordinary members and of reciprocity members under the agreement with the American Mathematical Society are being increased so all members will continue to receive both the JOURNAL and PROCEEDINGS as a privilege of membership.

Under this arrangement the reciprocity membership dues will be raised from $6.00 to $8.80 per year, beginning in 1964.

The London Mathematical Society is one of the sponsors of a new periodical, The Journal of Applied Probability, which starts publication in 1964. In future members of the London Mathematical Society may elect to receive the Journal of Applied Probability instead of either the Journal or the Proceedings of the London Mathematical Society, or they may receive it in addition to the present periodicals on payment to the London Mathematical Society of an additional $4.20 per year.

Applications for membership should be sent to Professor S. J. Taylor, Secretary, London Mathematical Society, Westfield College, London N. W. 3, England.
Mr. JOSE BARROS-NETO of the Massachusetts Institute of Technology has been appointed to a visiting assistant professorship at Brandeis University.

Mr. A. T. BEYER of the Laboratory for Electronics, Incorporated has accepted a position as Member of the Technical Staff with Data Dynamics, Incorporated, Monterey, California.

Mr. H. H. BROWN of the Goddard Space Flight Center has accepted a position as Mathematician with the Research and Advanced Development Division, Systems Analysis Section, of the AVCO Corporation.

Professor H. D. BRUNK of the University of California, Riverside has been appointed Professor and Chairman of Statistics at the University of Missouri.

Mr. D. H. CARLSON of the University of Wisconsin has been appointed to an assistant professorship at Oregon State University.

Assistant Professor HAROLD DAVIS of the University of California, Los Angeles has accepted a position as Senior Scientist at Hughes Aircraft Company, Culver City, California.

Mr. R. F. DEMAR has returned to Miami University as an Associate Professor after a year's leave at the National Bureau of Standards in Washington, D. C.

Dr. A. J. FABENS of the Australian National University has been appointed to an assistant professorship at the University of Rhode Island.

Assistant Professor A. N. FELDZAMEN of the University of Wisconsin will be on leave during the academic year 1963-1964 to serve as the Executive Director of the Committee on Educational Media of the Mathematical Association of America, San Francisco, California.

Dr. D. L. HANSON of Sandia Corporation, has been appointed Associate Professor of Statistics at the University of Missouri.

Dr. HARISH-CHANDRA of Columbia University has been appointed to a professorship at the Institute for Advanced Study.

Dr. ISIDORE HELLER of Stanford University has been appointed to a professorship at The Catholic University of America.

Mr. J. R. JOHNSON of Appalachian State Teachers College has been appointed an Associate Professor of Electrical Engineering at Louisiana State University.

Mr. R. J. JOLLY of the University of Texas has been appointed to an assistant professorship at the University of California, Riverside.

Mr. M. P. JONES of the University of Virginia has been appointed to an assistant professorship at Southwestern at Memphis, Memphis, Tennessee.

Dr. TATSUJI KAMAYASHI of Brown University has been appointed a Lecturer at Indiana University.

Dr. D. A. KAPPOS of the University of Athens, Athens, Greece, has been appointed to a visiting professorship in the Statistical Laboratory at The Catholic University of America.

Miss P. A. KNOOP of the Lewis Research Center has accepted a position as Research Mathematician with the Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio.

Dr. C. F. KOSSACK, University and Scientific Representative of International Business Machines Corporation, has accepted a position as Director of the Laboratory of Computer Sciences at the Graduate Research Center of the Southwest, Dallas, Texas.

Dr. YEHIEL LEHRER-LLAMED of Israel Atomic Energy Commission will spend the academic year 1963-1964 as a Visiting Associate Professor at McGill University, Montreal, Canada.

Dr. SHEN LIN of Ohio State University has accepted a position as a member of the Technical Staff with the Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey.

Mr. N. Y. LUTHER of the State University of Iowa has received a National Science Foundation Postdoctoral Fellowship at the University of California, Berkeley.
Mr. J. C. MCMILLAN of Edcliff Instruments, Incorporated has been appointed an Assistant Professor of Electrical Engineering at the California State Polytechnic College, Pomona, California.

Mr. AMRAM MEIR of the Hebrew University, Jerusalem has been appointed to an assistant professorship at the University of Alberta, Calgary, Alberta, Canada.

Dr. R. E. MESSICK of the California Institute of Technology has been appointed to an assistant professorship at the Case Institute of Technology.

Mr. R. E. MOSHER of Brandeis University has been appointed to an assistant professorship at Northwestern University.

Mr. J. H. NICHOLSON of the University of Texas has been appointed to an assistant professorship at Clemson College.

Mr. MARIO PETRICH of International Business Machines Corporation has been appointed to an assistant professorship at Lehigh University.

Mr. I. H. POMPER of New York University has accepted a position as Member of the Research Staff at the Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights.

Mr. G. S. RINEHART of Columbia University has been appointed to an assistant professorship at Cornell University.

Mr. W. H. RUCKLE of Florida State University has been appointed to an assistant professorship at Lehigh University.

Professor ALBERTO SAEZ of the Universidad de Los Andes, Merida, Venezuela has been appointed to a professorship at the Universidad Central de Venezuela, Caracas, Venezuela.

Associate Professor ICHIROS TAKE of the University of Tokyo has been appointed to a professorship at the University of Chicago.

Associate Professor S. C. SAXENA of Atlanta University has been appointed to an associate professorship at Northern Illinois University.

Associate Professor DANA SCOTT of the University of California, Berkeley has been appointed to an associate professorship at Stanford University.

Associate Professor MORRIS SKIBINSKY of Purdue University has accepted a position as Mathematician with the Brookhaven National Laboratory.

Assistant Professor W. E. SMITH on leave from the University of Colorado has been appointed Visiting Assistant Professor of Biostatistics, School of Public Health at the University of California, Los Angeles, California.

Professor ERNST SNAPPER of Indiana University has been appointed to a professorship at Dartmouth College.

Professor D. C. SPENCER of Princeton University has been appointed to a professorship at Stanford University.

Mr. P. F. G. STANEK of the Operations Evaluation Group has been appointed to an assistant professorship at the University of Southern California.

Professor J. M. STARK of Lemer College has been appointed a National Science Foundation Science Faculty Fellow at Stanford University for the academic year 1963-1964.

Professor M. H. STONE of the University of Chicago has accepted the first Ramanujam Visiting Professorship of MATSCIENCE, Institute of Mathematical Sciences, Madras, India. The visiting professorship was instituted in January 1963 to honor the memory of the great Indian Mathematician.

Associate Professor H. G. TUCKER on a sabbatical leave from the University of California, Riverside has been appointed a Member of the Institute for Advanced Study for the academic year 1963-1964.

Associate Professor J. H. WALTER has been appointed an Associate Member of the Center for Advanced Study at the University of Illinois.

Mr. JU-KWEI WANG of the University of California, Berkeley has been appointed a Visiting Lecturer at Yale University.

Mr. K. W. WESTON of the University of Wisconsin has been appointed to an assistant professorship at the University of Notre Dame.

Professor M. C. WICHT of North Georgia College, active duty in the U. S. Naval Reserve as Commander, has been appointed to a professorship at the U. S. Naval Postgraduate School, Monterey, California.

The following promotions are announced:
M. W. AL-DHAHIR, University of Baghdad, Baghdad, Iraq, to a professorship.
K. W. ANDERSON, Harpur College, to an associate professorship.
SISTER MARION BEITER, Rosary Hill College, to a professorship.
G. A. CRAFT, Harpur College, to an associate professorship.
J. H. H. CHALK, University of Toronto, Toronto, Ontario, Canada to a professorship.
MAKOTO ITOH, North Carolina State University, Raleigh, to a professorship.
K. R. LUCAS, U. S. Naval Postgraduate School, to an associate professorship.
ROBERT MCGEE, Rosary Hill College, to an assistant professorship.
R. W. RECTOR of the Aerospace Corporation, Los Angeles has been promoted to Assistant Director of the Computation and Data Processing Center.
C. K. TSAO, Wayne State University, to a professorship.

The following appointments to instructorships are announced:

University of Chicago: D. M. TOPPING; Louisiana State University in New Orleans: D. E. PENNEY; Loyola College: J. A. HIGGINS; Queensborough Community College: B. R. BERNSTEIN; Smith College: R. M. O'CONNOR.

Deaths:

Lieutenant Commander J. H. BILLINGS of the Portsmouth Naval Shipyard died on April 10, 1963 at the age of 35.
Dr. MAREK FISZ of New York University died on November 4, 1963 at the age of 53.
Reverend L. J. HEIDER of Marquette University died on August 5, 1963 at the age of 49.
Professor H. A. MEYER of the University of Florida died on September 1, 1963 at the age of 58. He was a member of the Society for 33 years.

ERRATA

Following is a correction of an announcement in the October issue of the NOTICES.

Dr. ROSCOE WOODS of the State University of Iowa, died on June 20, 1963 at the age of 73.

NEWS ITEMS AND ANNOUNCEMENTS

GRADUATE INSTRUCTION IN NUMERICAL ANALYSIS AT THE UNIVERSITY OF WISCONSIN

Graduate instruction, now offered in the Numerical Analysis Department of the University of Wisconsin, is designed to provide the student with broad knowledge of numerical methods of applied mathematics and of topics selected from research interests of the staff. A student may earn a Master of Arts or Master of Science degree in Numerical Analysis or jointly with Mathematics upon completion of a specified course of study.

The Doctor of Philosophy degree in Numerical Analysis is granted only upon the basis of convincing evidence of a high degree of competency in Numerical Analysis and a suitable related field, not upon any program of credits or course work.

Students may apply for research assistantships or assistantships in computing. Independent research into new and challenging fields of numerical analysis is encouraged.
During the interval from September 25, 1963 through November 19, 1963 the papers listed below were accepted by the American Mathematical Society for presentation by title. After each title on this program is an identifying number. The abstracts of the papers will be found following the same number in the section on Abstracts of Contributed Papers in this issue of these NOTICES.

(1) The geometry of triangles in the Minkowski relativistic plane
   Mr. R. V. Anderson, Illinois Institute of Technology (64T-58)
   (Introduced by Professor John DeCicco)

(2) On complementary regressive sets
   Professor K. I. Appel and Dr. T. G. McLaughlin, University of Illinois
   (64T-45)

(3) A convergence theorem. Preliminary report
   Professor W. D. L. Appling, North Texas State University (64T-19)

(4) Refinement-unbounded interval functions and absolute continuity. Preliminary report
   Dr. W. D. L. Appling, North Texas State University (64T-62)

(5) Recursive functions and regressive isols. Preliminary report
   Mr. Joseph Barback, University of Pittsburgh (64T-71)

(6) On functions that commute with full functions
   Professor G. E. Baxter and Professor J. T. Joichi, University of Minnesota (64T-56)

(7) Circular regions covered by Schlicht functions
   Professor S. D. Bernardi, New York University (64T-8)

(8) Measures on the computation speed of partial recursive functions
   Mr. Manuel Blum, Massachusetts Institute of Technology (64T-7)
   (Introduced by Professor Michael Arbib)

(9) Functions whose best rational Tchebycheff approximations are polynomials
   Mr. B. W. Boehm, University of California, Los Angeles (64T-59)

(10) Convergence of best rational Tchebycheff approximations
    Mr. B. W. Boehm, University of California, Los Angeles (64T-60)

(11) The degree of convergence of best rational Tchebycheff approximations
    Mr. B. W. Boehm, University of California, Los Angeles (64T-61)

(12) Existence of best rational Tchebycheff approximations
    Mr. B. W. Boehm, University of California, Los Angeles (64T-73)

(13) On a functional-integral equation
    Professor Witold Bogdanowicz, Georgetown University, and Mr. Tad Krauze, American Airlines (64T-63)

(14) Existence and uniqueness of asymptotically almost periodic and almost periodic solutions of nonlinear operational-differential equations
    Professor Witold Bogdanowicz, Georgetown University (64T-64)

(15) On a nonlinear hyperbolic system of differential equations
    Professor Witold Bogdanowicz, Georgetown University (64T-65)

(16) Existence of periodic solutions of nonlinear operational-differential equations
    Professor Witold Bogdanowicz, Georgetown University (64T-66)

(17) Dual integral equations, Bessel kernels
    Professor R. G. Buschman, State University of New York at Buffalo (64T-38)

(18) On the representation of the weight operator corresponding to certain ordinary differential operators of even order. Preliminary report
    Professor J. B. Butler, Jr. Portland State College (64T-87)

(19) Recurrences for the Bernoulli and Euler polynomials, II
    Professor Leonard Carlitz, Duke University (64T-75)

(20) Generalized Dedekind sums
    Professor Leonard Carlitz, Duke
(21) An extension of the congruences of Bauer and Lubelski
Professor Leonard Carlitz, Duke University (64T-77)

(22) The distribution of irreducible polynomials in several indeterminates
Professor Leonard Carlitz, Duke University (64T-76)

(23) Existence theorems for Tchebycheff approximation, I
Professor E. W. Cheney, University of California, Los Angeles and Professor A. A. Goldstein, University of Texas (64T-46)

(24) Characterization theorems for Tchebycheff approximation
Professor E. W. Cheney, University of California, Los Angeles and Professor A. A. Goldstein, University of Texas (64T-48)

(25) Remark on a set of integers
Professor Eckford Cohen, University of Tennessee (64T-52)

(26) A generalization of the Dirichlet product of arithmetical functions
Dr. K. J. Davis, University of Tennessee (64T-44)

(27) Physical systems in a bicentral positional field of force in the Euclidean plane
Professor John DeCicco and Mr. P. M. Krajekiewicz, Illinois Institute of Technology (64T-35)

(28) Theory of isoclines in a complex inner product space
Professor John DeCicco and Mr. J. A. Synowiec, Illinois Institute of Technology (64T-40)

(29) The initial rates of departure in a directional field of force
Professor John DeCicco, and Mr. S. T. Ramchandran, Illinois Institute of Technology (64T-51)

(30) Idempotents in the group algebra of a compact group
Professor R. G. Douglas, University of Michigan (64T-50)

(31) On extreme points and subspace density. Preliminary report
Professor R. G. Douglas, University of Michigan (64T-55)

(32) Decision problems of extensions of second order theory of successor
Dr. C. C. Elgot, IBM Research Center, Yorktown Heights, New York and Professor M. O. Rabin, Hebrew University, Jerusalem, Israel (64T-20)

(33) On the first order theory of generalized successor
Dr. C. C. Elgot, IBM Research Center, Yorktown Heights, New York and Professor M. O. Rabin, Hebrew University, Jerusalem, Israel (64T-21)

(34) An inequality on doubly stochastic matrices
Dr. C. J. Everett, Los Alamos Scientific Laboratories, University of California, Los Alamos, (64T-29)

(35) Discretization methods for ordinary differential equations with retarded argument, I. Preliminary report
Mr. M. A. Feldstein, University of California, Los Angeles (64T-24)

(36) A product theorem for compact 2-manifolds
Mr. R. L. Finney, Princeton University (64T-18)

(37) On relatively prime automorphism groups
Mr. George Glauberman, University of Wisconsin (64T-10)

(38) Correspondence of characters in relatively prime automorphism groups
Mr. George Glauberman, University of Wisconsin (64T-11)

(39) A new unified field and particle theory
Dr. Andre Gleyzal, U. S. Naval Ordnance Laboratory, White Oak, Maryland (64T-30)

(40) Representation for Hankel convolutions
Professor D. T. Haimo, Southern Illinois University (64T-49)

(41) Boundary behaviour of temperatures. I. Preliminary report
Mr. J. R. Hattemer, Washington University (64T-88)

(42) Boundary behaviour of temperatures. II. Preliminary report
Mr. J. R. Hattemer, Washington University (64T-89)

(43) Averages of continuous functions on countable spaces
Professor Melvin Henriksen and Professor J. R. Isbell, The Institute for Advanced Study (64T-41)

(44) $B(\mathcal{F})$-spaces and the closed graph theorem. II
Professor Taqdir Husain, Univer-
(45) Remarks on Post normal systems
Miss A. H. Ihrig, University of Illinois (64T-9)

(46) Extremal quasi doubly stochastic matrices. Preliminary report
Mr. R. E. Jaffa, University of California, Berkeley (64T-82)

(47) Properties of Zassenhaus refinement-groups and lattices
Dr. Ludvik Janos, George Washington University (64T-57)
(Introduced by Professor David Nelson)

(48) All symmetric operators have a weakly-continuous functional calculus on (BV)
Professor G. L. Krabbe, Purdue University (64T-26)

(49) Some elementary properties of analytic polygenic functions
Mr. P. M. Krajkiewicz, Illinois Institute of Technology (64T-33)

(50) Existence theorems for Tchebycheff approximation. II
Mr. B. R. Kripke and Mr. R. T. Rockafellar, University of Texas (64T-47)

(51) Identity and uniqueness theorems for automorphic functions
Professor P. A. Lappan, Jr., Lehigh University (64T-39)

(52) A condition for regularity in local rings
Mr. G. L. Levin, University of Chicago (64T-17)

(53) On uniformly continuous maps between Banach spaces
Professor Joram Lindenstrauss, Yale University (64T-31)

(54) On operators which attain their norm
Professor Joram Lindenstrauss, Yale University (64T-32)

(55) The union of a crumpled cube and a 3-cell is topologically S
Mr. L. L. Lininger, State University of Iowa (64T-5)

(56) Commutativity and orthogonality. Preliminary report
Mr. F. E. J. Linton, Wesleyan University (64T-72)

(57) A theorem on indecomposable sets
Mr. T. G. McLaughlin, University of Illinois (64T-34)

(58) An algorithm for the solution of a word problem
Professor N. S. Mendelsohn, University of Manitoba (64T-84)

(59) WITHDRAWN.

(60) On ring properties of injective hulls
Mrs. B. L. Ososky, Rutgers University (64T-86)

(61) Ratio limit theorems for functionals on a Markov chain. Preliminary report
Professor S. C. Port, Rand Corporation, Santa Monica, California (64T-67)

(62) Structural criterion for recursive enumeration without repetition. II
Professor M. B. Pour-El, Institute for Advanced Study and Professor W. A. Howard, Pennsylvania State University (64T-13)

(63) Sufficient conditions that a group not act expansively
Mr. William Reddy, Syracuse University (64T-2)

(64) Upper equicontinuity in measure
Mr. William Reddy, Syracuse University (64T-3)

(65) Remark on expansive autohomeomorphisms
Mr. William Reddy, Syracuse University (64T-4)

(66) On the number of those integers below a positive bound and in a prime residue class which can be represented as the sum of two squares
Professor G. J. Rieger, University of Munich, Munich, West Germany (64T-14)

(67) On the number of integers of the form \( p_1^n + p_2^m \) and below a positive bound
Professor G. J. Rieger, University of Munich, Munich, West Germany (64T-83)

(68) Algebras with all commutators in the center. Preliminary report
Professor D. J. Rodabaugh, Vanderbilt University (64T-25)

(69) Le groupe d'anti-autotopie
Professor Albert Sade, Pertuis, Vaucluse, France (64T-79)

(70) On Poincaré duality
Professor Hans Samelson, Stanford University (64T-85)

(71) A recurrence criterion for random walk
Professor F. L. Spitzer, Cornell University (64T-1)
(72) Products of nearly compact spaces
Professor A. H. Stone, University of Rochester (64T-16)

(73) An extended theory of the Cauchy-Riemann equations
Mr. Michael Sullivan, Illinois Institute of Technology (64T-74)

(74) On a stronger version of Wallis' formula
Mr. V. R. R. Uppuluri, Oak Ridge National Laboratory, Oak Ridge, Tennessee (64T-90)

(75) Padé approximants as limits of rational functions of best approximation
Professor J. L. Walsh, Harvard University (64T-22)

(76) A theorem on definitions of the Zermelo-Neumann ordinals
Professor Hao Wang, Harvard University (64T-36)

(77) Theorems on Brewer and Jacobsthal sums
Professor A. L. Whiteman, University of Southern California, Los Angeles (64T-42)

(78) Abstract characterization of an algebra of multiplace functions. II. Preliminary report
Mr. H. I. Whitlock, Illinois Institute of Technology (64T-27)
(Introduced by Professor Haim Rein-gold)

(79) Abstract characterization of an algebra of multiplace functions. III
Mr. H. I. Whitlock, Illinois Institute of Technology (64T-28)

(80) Eigenvalue distribution of certain analytic difference kernels. Preliminary report
Professor Harold Widom, Cornell University (64T-54)

(81) Translation invariant function algebras on compact groups
Professor J. A. Wolf, University of California, Berkeley (64T-80)

(82) On the classification of hermitian symmetric spaces
Professor J. A. Wolf, University of California, Berkeley (64T-81)

(83) A theorem on manifolds of the same homotopy type. Preliminary report
Mr. Y. F. Wong, Cornell University (64T-53)

(84) On the existence of additive generators for a t-norm. Preliminary report

Dr. J. Z. Yao and Mr. C. H. Ling, Illinois Institute of Technology (64T-6)

(85) An interpolation theorem
Professor Guido Stampacchia, University of Pisa and New York University (64T-23)

NEWS ITEMS

BENJAMIN PIERCE INSTRUCTORSHIP

Applications for the Benjamin Pierce Instructorship at Harvard University are invited. The teaching commitment is six hours and the salary is $7,000 for the academic year. There is also the possibility of summer income through teaching or research contracts. Appointments are annual but carry the presumption of two renewals. Additional information and application forms are available from the Department of Mathematics, Harvard University, 2 Divinity Avenue, Cambridge, Massachusetts, 02138.
ABSTRACTS OF CONTRIBUTED PAPERS

The October Meeting in Brooklyn, New York
October 26, 1963


Given a simple n-person game, it is always possible to find a solution by taking the set of all imputations in which a minimal winning coalition splits the profits in all different ways, while excluding the remaining players. Sometimes it is possible to assign fixed positive amounts to some of the discriminated players and still maintain a solution, providing these amounts are not too large. The problem is to determine when these amounts are permissible. It is found that the sets of permissible amounts correspond to convex sets; the inequalities defining these convex sets are determined by the values of certain sets of two-person games whose game matrices are incidence matrices. Applications of this theory are made to special cases which include symmetric games and simple games. (Received August 23, 1963.)


Let R be a complex commutative Banach ring with identity e and an involution *. Let M be the maximal ideal space, and the Gelfand representation, let P denote all elements of the form xx*. Theorem 1. A is a topological *-isomorphism onto C(M) iff there exists a positive K such that \|\exp ih\| \leq K for all Hermitian h. Theorem 2. \( \Lambda \) is an isometric *-isomorphism onto C(M) iff \( \|\exp ih\| = 1 \) for all Hermitian h. Corollary 1. Let N be a neighborhood of e, suppose that G maps N \cap P continuously into the non-negative reals so that if xx* is in N, \( \|x\| \|x^*\| \leq G(xx^*) \). Then \( \Lambda \) is a topological *-isomorphism of R onto C(M). Furthermore, if G(1) = 1, \( \Lambda \) is isometric. Corollary 1 generalizes well-known results of Gelfand and Naïmark and of Arens (Rickart, Banach algebras, p. 190). The main device in the proof of Theorem 2 is the proof by induction on N that \( \Lambda \) is norm-preserving on elements with spectral radius 1 whose spectra avoid the intersection of some ray with the outside of the disc with center 0 and radius 1 - (1/2N). (Received September 13, 1963.)

In this paper, the convergence of the iterations of a generalized method of steepest descent is established for a class of nonlinear gradient mappings [E. Rothe, Gradient mappings, Bull. Amer. Math. Soc. 59 (1953), 5-19] whose Fréchet differential is a weak Legendre form which is positive in a certain subset of the Hilbert space. The proof is first given for the case when the conditions on the nonlinear operator are imposed globally. It is then shown that if \( f(x) \) is a functional whose gradient is the nonlinear operator under consideration, and if the level surfaces of \( f(x) \) near a solution \( x = \tilde{x} \) can be approximated by hyper-ellipsoids, in some abstract sense, then there is a neighborhood of \( \tilde{x} \) such that if the initial approximation lies in this neighborhood, all the iterates will lie in the same neighborhood and will converge to \( \tilde{x} \). Extensions are made for operators which are not necessarily gradient mappings. These extensions are based on decompositions of the nonlinear operator, on symmetrizableability of the Frechet derivative of the nonlinear operator by an invertible bounded linear transformation, and on general integrability conditions. (Received October 2, 1963.)

605-26. EDWIN DUDA, University of Miami, Coral Gables 46, Florida. A locally compact separable metric space is almost invariant under a closed mapping.

Let \( f \) be a closed mapping (continuous transformation) of a topological space \( X \) onto \( Y \). It is known that normality is an invariant (G. T. Whyburn, Duke Math. J. 17 (1950), 69-74) and that paracompactness is an invariant (E. Michael, Proc. Amer. Math. Soc. 8 (1957)). From the first reference above we obtain the result that if \( X \) is perfectly separable, \( f \) is closed and \( Y \) is weakly separable then \( Y \) is a separable metric space. Let \( f \) be a closed mapping of a locally compact separable metric space \( X \) onto a topological space \( Y \). Say a set \( S \) is a scattered set if every subset of \( S \) is closed. Our results show that \( Y \) minus a scattered set \( S \), which has at most a countable number of points, is also a locally compact separable metric space. (Received October 3, 1963.)

The November Meeting in Pasadena, California
November 21-23, 1963

606-32. BASIL GORDON, University of California, Los Angeles, California 90024. Projective homomorphisms of division rings.

Let \( K, E \) be division rings, and \( F \) a subfield of the center of \( E \). A map \( x \rightarrow x^\sigma \) of \( K \) into \( E \) is a projective homomorphism if \( (x + y)^\sigma = ax^\sigma + by^\sigma \) and \( (xy)^\sigma = cx^\sigma y^\sigma \), where \( a, b, c \) are in \( F^* \) and may depend on \( x, y \). It is shown that if \( [K^\sigma : F] \geq 3 \), then there is a homomorphism \( x \rightarrow x^\alpha \) of \( K \) into \( E \) and a map \( x \rightarrow x^\beta \) of \( K \) into \( F^* \) such that \( x^\sigma = x^\alpha x^\beta \). This theorem has an application to difference sets. (Received October 3, 1963.)
For an infinite sequence \( \{x_k\} \) of real numbers \( \epsilon[0,1] \) denote by \( s_n = \langle (1/n) \sum_{k=1}^{n} e(mx_k) \rangle^2 \) where \( m \neq 0 \) is an integer and \( e(x) = e^{2\pi i x} \). Call a sequence strongly equidistributed if for every integer \( m \neq 0 \) there exist the two conditions hold: \( \lim s_n = 0 \) and \( \sum |(s_n - s_{n+1})|s_n| < \infty \). \( \{x_k\} \) is strongly equidistributed if for every integer \( m \neq 0 \) there exists a divergent series \( \sum c_n(s_n \geq 0) \) which fulfills the conditions: \( \sum c_n s_n < \infty \) and \( |s_n - s_{n+1}| \leq c_n \). Let \( \{f_k(x)\} \) be a sequence of real functions bounded and integrable on \([a,b]\) and denote by \( s_n = s_n(x) = \langle (1/n) \sum e(mx_k(x)) \rangle^2 \) and \( I(n) = \int_a^b s_n dx \). If there exists a divergent series \( \sum c_n (c_n \geq 0) \) for every integer \( m \neq 0 \) such that \( |s_n - s_{n+1}| \leq c_n \) almost everywhere and \( \sum c_n I(n) < \infty \) then \( \{f_k(x)\} \) is strongly equidistributed mod 1 for almost all \( x \in [a,b] \). This is a generalization of a theorem due to Davenport-Erdős-LeVeque. These results remain valid if our considerations are based on an arbitrary compact topological space with countable basis. As an application some theorems are proved concerning equidistribution on compact groups. (Received October 7, 1963.)

The November Meeting in Madison, Wisconsin
November 30, 1963

Let \( \Gamma \) be an \( H \)-group and consider automorphic forms on \( \Gamma \) of dimension \( -r-2 \) (\( r \) a positive even integer), with multiplier system identically one. Only forms regular in the upper half-plane and at the parabolic cusps are considered. If \( G(\tau) \) is such a form and \( H(\tau) \) is an \( (r + 1) \)st order primitive of \( G(\tau) \), then \( H(\tau) \) has the functional equation \( H(\tau) = (c\tau + d)^r H(V\tau) = h(\tau) \) for each \( V = (**|cd) \in \Gamma. \) Here the \( h(\tau) \) are polynomials of degree \( \leq r \), the period polynomials of \( H(\tau) \). The following is proved: Theorem. \( G(\tau) \) has an \( (r + 1) \)st order primitive all of whose period polynomials have real coefficients if and only if \( G(\tau) \) is a linear combination of Eisenstein series with real coefficients. The crux of the proof is the observation that if \( G(\tau) \) is a cusp form and the \( h(\tau) \) all have real coefficients, then \( G(\tau) \equiv 0 \). The theorem generalizes a result of H. Cohn (Trans. Amer. Math. Soc. 82 (1956), 117-127) in which \( \Gamma \) is a subgroup of finite index in the modular group and \( -r - 2 = -4 \). (Received October 2, 1963.)
The Annual Meeting in Miami and Coral Gables, Florida
January 23-26, 1964

608-1. G. W. SACKS, White Hall, Cornell University, Ithaca, New York. A maximal set which
is not complete.

Stanley Tennenbaum (Abstract 587-128, These Notices 8 (1961), 608-609) announced that every
maximal set has degree 0'. This is far from true. **Theorem.** A set is recursive if and only if it is
recursive in every maximal set. In addition, the set of degrees of maximal sets is infinite. **Corollary.**
There exists a nonrecursive, recursively enumerable set which is not contained in any complete set.
(Received April 26, 1963.)

608-2. J. S. MAC NERNEY, University of North Carolina, Chapel Hill, North Carolina. A non-
linear integral operation.

Most of the results of Integral equations and semigroups [Illinois J. Math. 7 (1963), 148-173]
are extended as follows. **Let** S be a set, with linear ( ≤ ) ordering O; \{G, +, \| \} be a complete
normed Abelian group, with zero element 0; H be the class of all functions from G to G to which
\{0,0\} belongs, with identity function 1; OA be the class of all functions V from S X S to H such that
(1) if \{x,y,z\} is an O-subdivision of \{x,z\} then V(x,y)P + V(y,z)P = V(x,z)P for each P in G, and
(2) there is a member α of OA⁺ [ibid., p. 150] such that if \{x,y\} is in S x S then ∥V(x,y)P - V(x,y)Q∥
≤ α(x,y)∥P - Q∥ for each \{P,Q\} in G X G; and OM be the class of all functions W from S x S to
H such that (1) if \{x,y,z\} is an O-subdivision of \{x,z\} then W(x,y)W(y,z)P = W(x,z)P for each P in G, and
(2) there is a member μ of OM⁺ [ibid., p. 150] such that if \{x,y\} is in S x S then ∥W(x,y) - 1\]P
- [W(x,y) - 1]Q∥ ≤ μ(x,y) - 1]∥P - Q∥ for each \{P,Q\} in G X G. **Theorem.** These are equivalent:
(1) V is in OA and W is defined by W(x,z)P = x²[1 + V]P; (2) W is in OM and V is defined by V(x,z)P
= x²[W - 1]P; (3) V is in OA and W is such a function from S X S that if \{c,P\} is in S x G then
W( ,c)P is a function u from S to G, of bounded variation on each O-interval of S, such that
u(x) = P + (R) \sum xCV·u for each x in S. (Received June 24, 1963.)

608-3. N. S. MENDELSOHN and A. L. DULMAGE, University of Manitoba, Winnipeg, Manitoba,
Canada. The exponent set of primitive matrices.

For an n x n primitive matrix A, the exponent γ(A) of A is defined to be the least integer with
the property that A^t is positive for t ≥ γ(A). Let S be the set of all exponents of n x n primitive
matrices. The set S contains gaps. In particular, if n is odd there is no matrix A such that
n² - 3n + 4 < γ(A) < (n - 1)² or n² - 4n + 6 < γ(A) < n² - 3n + 2. If n is even there is no matrix A
for which n² - 4n + 6 < γ(A) < (n - 1)². The methods which lead to this result can be applied to the
following problem in number theory. Let a₁, a₂,..., aₙ be a set of relatively prime positive integers
and let P(a₁,a₂,...,aₙ) be the largest integer, not expressible in the form x₁a₁ + x₂a₂ + ... + xₙaₙ,
where xᵢ are non-negative integers. In many cases the exact value for P can be found, and in other
cases good bounds on the magnitude of P can be obtained. (Received July 11, 1963.)

A. L. S. Corner has proven that there exist indecomposable homogeneous torsion free abelian groups of every cardinality less than that of the first inaccessible cardinal. Call a torsion free abelian group $G$ purely indecomposable if every pure subgroup of $G$ is indecomposable, and $p$-purely indecomposable ($p$ a prime) if every $p$-pure subgroup of $G$ is indecomposable. **Theorem:** If $G$ is a homogeneous purely indecomposable group, then $|G|$ does not exceed the cardinality of the continuum.

**Theorem:** A $p$-reduced torsion free abelian group $G$ is $p$-purely indecomposable if and only if $G$ is isomorphic to a $p$-pure subgroup of the $p$-adic integers. (Received July 31, 1963.)

608-6. MAURICE HOROWITZ, The Magnavox Company, Bueter Road, Fort Wayne, Indiana.

**Mean random path across a square.**

Let a point $P$ be picked at random on the perimeter of an area. From $P$ let a direction for a straight path across the area be picked at random, and let the termination point of the path be $Q$. An investigation has been made of the average value of the length $PQ$ called the mean random path and denoted by $L_{\text{rms}}$, as well as the rms value of the length $PQ$, denoted by $L_{\text{rms}}$. For example, results for the circle of unit radius are easily obtained, with $L_{\text{rms}} = 1.273 = 4/\pi$, and $L_{\text{rms}} = 1.414$; moreover, it can be readily shown that for any simply connected shape of area $A$, $L_{\text{rms}} = \sqrt{2A/\pi}$. Thus, for a square of side unity, $L_{\text{rms}} = \sqrt{2/\pi} = 0.798$. However, the value of $L_{\text{rms}}$ for a square is somewhat more tedious to find, but a closed integral expression has been obtained. The integral was evaluated using an IBM 1620 digital computer, and it was found $L_{\text{rms}} = 0.71$. This result was confirmed by a Monte Carlo computation of $L_{\text{rms}}$ using 4000 samples and a random number generator of A. C. Reynolds, Jr., referred to in his patent no. 3089125. The figure for $L_{\text{rms}}$ differs slightly from the mean free path ordinarily used in room acoustics, which is $\pi/4 = 0.785$ for a square. The difference arises from a difference in the assumptions used about the random model. (Received August 5, 1963.)

608-7. WITHDRAWN.


Given a set $X$ in an $n$-dimensional real linear space $E$, say that $p \in \text{int}_d X$ if and only if there exists a $d$-dimensional flat $F$ through $p$ such that $p$ is relatively interior to $F \cap X$. In The generation of convex hulls, Math. Ann. 149 (1963), Bonnice-Klee prove that if $p \in \text{int}_d X$, then $p \in \text{int}_d \text{con} U$ for some subset $U$ of $X$ of cardinality at most $\max (n + 1, 2d)$. A well-known theorem of E. Steinitz (see Bonnice-Klee) asserts that if $p \in \text{int}_n \text{con} X$, then $p \in \text{int}_n \text{con} U$ for some subset $U$ of $X$ and either $\card U \leq 2n - 1$, or else $\card U = 2n$ and $U$ consists of $2n$ points collinear in pairs with $p$, and $X$ is contained in the union of the $n$ lines thus formed. Bonnice-Klee ask whether a similar characterization of $U$ exists when $d < n$ and $\card U = \max (n + 1, 2d)$ in the above theorem. Conditions are given which assure the existence of a subset $U$ of $X$ of cardinality at most $2d$ such that $p \in \text{int}_d \text{con} U$.
Under these conditions, either \( \text{card } U \leq 2d - 1 \) or else \( U \) may be characterized as in Steinitz's theorem, thus solving Bonnice-Klee's problem. (Received August 21, 1963.)

608-9. WITHDRAWN.

608-10. A. H. ZEMANIAN, State University of New York at Stony Brook, Long Island Center, Stony Brook, New York. Necessary and sufficient conditions for a matrix distribution to have a positive-real Laplace transform.

Let \( w(t) \) be an \( n \times n \) matrix distribution and let \( W(s) \) be its Laplace transform. \( W(s) \) is said to be positive-real if for \( \text{Re } s > 0 \) it is analytic, it is real for real \( s \), and \( W(s) + W^*(s) \) is the matrix of a nonnegative definite hermitian form. (The asterisk denotes the complex-conjugate transpose.)

**Theorem:** The necessary and sufficient conditions for \( W(s) \) to be positive-real are the following:

(i) \( w(t) = A \delta^{(1)}(t) + w_0(t) \) where \( A \) is a real symmetric nonnegative-definite constant matrix, \( \delta^{(1)}(t) \) is the first derivative of the Dirac delta function, and \( w_0(t) \) is a real matrix distribution of zero order whose support is contained in \( 0 \leq t < \infty \). (ii) For every \( n \times 1 \) constant vector \( y \), \( y^* [w(t) + w^T(-t)] y \) is a nonnegative-definite distribution, where \( w^T \) is the transpose of \( w \). (Received August 23, 1963.)

608-11. J. D. HALPERN, California Institute of Technology, Pasadena, California, and AZRIEL LÉVY, Hebrew University, Jerusalem, Israel. The ordering theorem does not imply the axiom of choice. Preliminary report.

Using the methods of Paul Cohen's independence proofs in set theory (article to appear in the Proceedings of the National Academy of Sciences) and additional methods of Feferman one can prove that if the Zermelo-Fraenkel set theory is consistent then it stays consistent when the following two axioms (a) and (b) are added to it. (a) The set of the real numbers cannot be well-ordered, (b) every set is equinumerous with a subset of the power set of some ordinal (see Kinna and Wagner, Fund. Math. 42 (1955), 75-82). (b) implies that every set can be ordered. (Received August 29, 1963.)

608-12. R. S. VARGA, Case Institute of Technology, University Circle, Cleveland, Ohio 44106. Minimal Gerschgorin sets.

Let \( A = (a_{i,j}) \) be a fixed \( n \times n \) complex matrix, and let \( \Omega \) be the set of \( n \times n \) matrices \( B = (b_{i,j}) \) with \( b_{i,i} = a_{i,i}, |b_{i,j}| = |a_{i,j}| \) for \( i \neq j \). For any \( n \)-vector \( \mathbf{x} > 0 \), it is known that the union \( G(\mathbf{x}) \) of the Gerschgorin disks \( G_i(\mathbf{x}) = \{z | |z - a_{i,i}| \leq (\sum_{j \neq i} |a_{i,j}| |x_j|)/x_i \equiv \wedge_i(x)\} \) contains all the eigenvalues of any \( B \in \Omega \). Let \( G(\Omega) = \Diamond_{x>0} G(\mathbf{x}) \) be called the minimal Gerschgorin set. It is shown that each boundary point of \( G(\Omega) \) is an eigenvalue of some \( B \in \Omega \), so that \( G(\Omega) \) is also optimal. This answers affirmatively a conjecture by A. S. Householder. If \( A \) is irreducible, each boundary point \( \sigma \) of \( G(\Omega) \) can be characterized as the intersection of \( n \) Gerschgorin circles: \( |\sigma - a_{i,i}| = \wedge_i(x) \) for \( 1 \leq i \leq n \) for some \( x > 0 \). These results make use of theorems by Olga Taussky [Amer. Math. Monthly 56 (1949), 672-676] and Ky Fan [Duke Math. J. 25 (1948), 441-445]. Results for disconnected minimal Gerschgorin sets are also obtained. (Received August 30, 1963.)
Consider the elliptic partial differential equation \(- (p u_x)_x - (q u_y)_y = f\) in a convex domain with polygonal boundary, where \(p\) and \(q\) are positive \(C^1\) functions and \(u\) vanishes on the boundary. After approximating the derivatives by finite differences, the Peaceman-Rachford method (J. Soc. Indust. and Appl. Math. 3 (1955), 28-41) is applied, generating from a guess vector \(x_0\) a sequence of vector iterates \(x_n\) converging to the finite difference solution \(x\). Let \(x_0\) come from evaluating on the mesh a \(C^2\) function which vanishes on the boundary. Let \(\|\varepsilon_n\|\) denote a discrete \(L^2\) norm of \(x_n - x\).

In the following theorems, \(N\) and \(K\) are independent of the mesh increments.

**Theorem 1.** If \(p = q := 1\), given \(\varepsilon > 0\) there exists an \(N > 0\) such that \(n > N\) implies \(\|\varepsilon_n\| < \varepsilon\). Enclose the convex domain in a square with side length \(w\) and let \(\sigma > 0\) be a lower bound of \(p\) and \(q\).

**Theorem 2.** If the Peaceman-Rachford parameter \(r < \sigma/w^2\), there exists a \(K > 0\) such that \(\|\varepsilon_n\| < K/n\) for all \(n\). Let \(\delta_n\) be the Cesaro sum of \(\varepsilon_m\). **Theorem 3.** There exists a \(K > 0\) such that \(\|\delta_n\| < K/n\) for all \(n\). (Received September 3, 1963.)

In any finite ring \(R\) of \(q\) elements, let \(\{H_i; i = 1, 2, \ldots, s\}\) be a collection of \(s\) subsets of \(R\), each \(H_i\) containing \(h_i\) members. Let \(D(H_i)\) denote the set of all differences \(d' - d''\) with \(d', d''\) from \(H_i\), including \(d' = d''\). Further, suppose for an integer \(r < s\) that \(h_1 h_2 \ldots h_s > q^r\). Then the box principle implies the theorem. The system of \(r\) equations in \(s\) unknowns \(\sum_{j=1}^{s} a_{ji} x_i = 0 \quad (j = 1, 2, \ldots, r)\), where \(a_{ji} \in R\) or is an integer, has a nontrivial solution such that \(x_i \in D(H_i)\) \((i = 1, 2, \ldots, s)\). This generalizes a result of A. Brauer and T. L. Reynolds (Canad. J. Math. 3 (1951), 367-374) for \(R = \mathbb{Z}_q\), the integers, mod \(q\). One of several applications is the determination of bounds for primitive roots and nth power nonresidues in an arbitrary Galois field. (Received September 3, 1963.)

**Assumptions:** A continuous time branching process is considered whose size (i.e., number of particles) at time \(t\) is denoted by \(X_N(t)\). \(P[X_N(0) = N] = 1\). All particles "act independently" of each other. The conditional probability that a particle born at time \(t' \leq t\) and alive at time \(t\) (dies and) "splits" into \(k\) new particles during \([t, t + h]\) is \(\lambda_k \phi_N(t) h + o(h)\) and does not split (and does not die) during \([t, t + h]\) with probability \(1 - \lambda_k \phi_N(t) + o(h)\), where \(k = 0, 1, 2, \ldots, \lambda_k \geq 0\), all \(o(h)'s\) are uniform in \(t\) and do not depend on \(t'\), \(0 < \lambda = \sum \lambda_k < \infty\), \(\sum k^2 \lambda_k < \infty\), and \(\phi_N(t) > 0\) depends only on \(N\) and \(t\) and is continuous for all \(t\). Each \(\lambda_k\) and \(\lambda\) are functions of \(N\) only, such that if \(N \rightarrow \infty\), then \(N \lambda_k \rightarrow r_k \geq 0\), \(N \rightarrow \nu > 0\) (where \(\nu = \sum \nu_k < \infty\)), and \(\phi_N(t) \rightarrow \phi(t) > 0\) uniformly over every bounded interval.

**Results:** The exact distribution is obtained of the number of particles that split during \([0, t]\). The limiting distribution of the stochastic process \(X_N(\cdot)\) as \(N \rightarrow \infty\) is shown to be that of a process with independent increments whose characteristic function at \(t\) is \(\exp \Phi(t) \sum e^{iu(k-1)} 1) r_k\), where \(\Phi(t) = \int_0^t \Phi(w)dw\). (Received September 6, 1963.)
Let $\theta$ be a finitely generated free semi-group (with identity) and let $X$ be a subset of $\theta$. $X$ is said to be bounded if there exists a finite set of words $w_1, \ldots, w_n$ in $\theta$ such that for every word $w$ in $X$ there exist nonnegative integers $i_1, \ldots, i_n$ such that $w = w_1^{i_1} \cdots w_n^{i_n}$. The purpose of this paper is to consider the bounded ALGOL-like languages, study their structure, and show that certain questions about them are recursively solvable. Among the results established are the following: (1) Bounded languages are decomposed into "simpler" ones. (2) Bounded languages are characterized as the smallest family of languages closed with respect to three "elementary" operations. (3) A generalized sequential machine transforms a bounded language to a bounded language. (4) A decision procedure is given for determining of an arbitrary ALGOL-like language whether it is bounded. (5) It is solvable to determine of arbitrary ALGOL-like languages $L_1$ and $L_2$, one of them bounded, whether $L_1 \subseteq L_2$ and whether $L_2 \subseteq L_1$. (The same problems with the boundedness condition removed are recursively unsolvable.) (Received September 11, 1963.)

Let $Z_n(z)$ be any solution of Bessel's equation for integral order $n$, and $m$ any non-negative integer. The theorem is proved: any function of the form $e^{\pm iz^m}Z_n(z)$ can be written as the total differential of a finite collection of cylindrical functions, exponentials, and powers of $z$. Some explicit representations for various $m$ and $n$ will be exhibited. (Received September 18, 1963.)

WITHDRAWN.

The author demonstrates the application of the integral operator concept to certain types of
ordinary differential equations. The concept is widely and successfully applied in the theory of partial differential equations. It is characterized by two features: the recursion formulas for the variable coefficient functions appearing in the infinite series expressing the nth coefficient function in terms of (n - 1)th coefficient, and the integration operation. A few particular examples show the procedure whose convergence is assured. (Received September 23, 1963.)


If G is an additively written abelian group, \( G^1 = \bigcap_{n=1}^{\infty} nG \) is set. If H is a subgroup of G which is maximal with respect to \( H \cap G^1 = 0 \), then H is said to be a high subgroup of G. All high subgroups of the group G are pure in G (see Irwin and Walker, Pacific J. Math., 11 (1960), 1363-1374). A group G is said to be a \( \Sigma \)-group if all high subgroups of G are direct sums of cyclic groups. Lemma. If \( D \) is minimal divisible containing \( G^1 \), then G is a subdirect sum of \( G/G^1 \) and \( D \). Theorem 1. A group H with \( H^1 = 0 \) is a direct summand of every group in which it is a high subgroup if and only if the torsion subgroup of H is isomorphic to the torsion subgroup of an algebraically compact group. Theorem 2. If some high subgroup of G splits, then \( G/G^1 \) splits, Theorem 3. If G is a \( \Sigma \)-group, then every high subgroup of G is an endomorphic image of G. Theorem 4. If G is a \( \Sigma \)-group with \( |G^1| \leq K_0 \), then \( G/G^1 \) is a direct sum of cyclic groups, Theorem 5. If G is a torsion \( \Sigma \)-group with \( |G^1| \leq K_0 \), then every subgroup of G is a \( \Sigma \)-group. The condition \( |G^1| \leq K_0 \) in Theorems 4 and 5 cannot be removed. (Received September 23, 1963.)


A Baer semigroup is a multiplicative semigroup with \( 0 \) having the property that the left (right) annihilator of each element is a principal left (right) ideal generated by an idempotent, A quasi-orthomodular lattice is a lattice \( L \) with \( 0 \) and \( 1 \) such that each \( e \in L \) admits (not necessarily unique) complements \( f \) and \( g \) such that \( (e,f) \) and \( (g,e) \) form modular pairs, while \( (f,e) \) and \( (e,g) \) form dual modular pairs. Conditions are given which guarantee that the set of right annihilators of elements of a Baer semigroup forms a quasi-orthomodular lattice, and it is shown that every quasi-orthomodular lattice arises in this manner. An example is given of a Baer semigroup whose lattice of right annihilators is not complemented. (Received September 25, 1963.)


In this paper the Maximum Principle of Pontryagin et al., is established with the limiting cone \( K_{t_1} \) replaced by a convex cone constructed at the endpoint \( x(t_1) \) of the trajectory \( x(t) \). Goddard Problem: given the equations of motion of a rocket in vertical flight, drag, \( D(h,v) \), initial mass at rest, find the piecewise continuous thrust program so that a summit altitude is attained with least fuel expenditure. Applying the Maximum Principle of Pontryagin et al., to the Goddard Problem establishes that the solution consists of coasting, intermediate, and full thrust subarcs. Lemma 1. Along an
optimal trajectory the adjoint variable \( \psi(t) > 0 \) in \( 0 \leq t < t_1 \). \textbf{Lemma 2.} For a class \( D_1 \) defined by \( D_{vv} + kD_v \geq 0 \), \( k \) = positive constant, an intermediate arc cannot be interrupted by a coasting arc except in the final stage. This second lemma extends one of the several results of Garfinkel given in BRL Report 1194 (1963). Garfinkel, admitting infinite thrust into his analysis establishes that the solution does not contain any coasting arcs except in the final stage provided \( D_v, D_{vv} \) are non-negative. From Lemmas 1, 2 follows the \textbf{Theorem.} 3 nonempty class \( D'_1 \subseteq D_1 \) for which the solution to the Goddard Problem with bounded thrust contains at most three subarcs. (Received September 12, 1963.)


The Martin boundary theory for Markov chains, developed by Doob and others [J. Math. Mech. 8, (May 1959), pp. 433-458], is applied to a specific random walk which is intimately associated with the \((C,r)\) summability of Fourier series. In particular it is shown that the Martin exit boundary \( R' \) is \( C(0;1) \), the circumference of the unit circle, and that the minimal positive regular functions are the \((C,r)\) kernels \( \{K_n^r(x - t) : t \in C(0;1)\} \). The "Martin representation" is equivalent to the following classical result: A trigonometric series is a Fourier-Stieltjes series if and only if its \( n \)-th \((C,r)\) partial sums are non-negative for all \( n \). Using methods of Doob, Brelot and Nairn, \( R' \) is shown to be resolutive, the fine topology at \( R' \) is constructed and the corresponding "fine limit theorems" are proved. The following theorem links the classical limit theorems with the fine topology. \textbf{Theorem.} If a trigonometric series is \((C^*,r)\) summable to the value \( a \), then \( a \) is a fine cluster value. For the notion of \((C^*,r)\) summability see [Marcinkiewicz-Zygmund, Trans. Amer. Math. Soc. 50, 407-453]. (Received September 30, 1963.)


Friedberg [J. Symbolic Logic 23, 309-316] showed that the class of all recursively enumerable \((r.e.)\) sets is \( r.e. \) without repetition. Pour-El and Putnam have shown that this does not hold for every \( r.e. \) class. The authors obtain a structural criterion for an \( r.e. \) class \( F \) to be \( r.e. \) without repetition. This result is achieved by placing certain structural requirements on the class \( H \) of finite subsets of members of \( F \)—requirements which are embodied in a "height function" \( h \) defined on \( H \) with values in the natural numbers. \textbf{Definition.} \( h \) is a \textit{height function} for \( F \) if \( h \) is defined on \( H \) and satisfies the following conditions. (1) \textbf{Monotonicity:} If \( A \subseteq B, A \subseteq H, B \subseteq H \) then \( hA \leq hB \). (2) \textbf{Ascending Chain Condition:} Given any ascending sequence of finite subsets of a fixed member of \( F, A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots \) the associated sequence of heights \( h(A_0) \leq h(A_1) \leq \ldots \leq h(A_n) \leq \ldots \) eventually becomes constant. (3) Given any \( A \subseteq H \), there is a \( B \subseteq H \) such that \( h(A) \neq h(B) \).

\textbf{Theorem 1.} Let \( F \) be an \( r.e. \) class. If \( F \) has a partial recursive height function then \( F \) is \( r.e. \) without repetition. The proof is by the "priority" method. (Received October 2, 1963.)
The author presents a closed form solution to the following integral equation: 

\[ F_{n,m}(k, \lambda, b) = \int_0^\infty x^m \exp(-bx^2) I_{m-1}(0x) I_n(\lambda x^2) \, dx \]

where \( n \) and \( m \) are integers and \( k, \lambda, \) and \( b \) are positive constants. This equation may be considered to be a second generalization to Weber's second exponential integral which can only be placed in closed form for special cases. The integral may be related to the following trigonometric integral. 

\[ A_{n,m}(B, D) = \frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \left\{ \frac{1}{(1 - B \cos \theta)} \right\}^m \exp[D/(1 - B \cos \theta)] \, d\theta \]

where \( B \) and \( D \) are positive constants. In essence \( A_{n,m}(B, D) \) may be considered to be the \( n \)th Fourier coefficient in the Fourier series expansion of the function enclosed by the braces. It is shown that the solution to this equation involves a finite sum of imaginary Bessel functions where the coefficient of each term in the sum is represented by a particular member of the associated Legendre functions. Discussed are physical situations where this integral is encountered.

(Received October 2, 1963.)

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On the faithfulness of homology functors.

A (free abelian) chain complex is determined to within equivalence by its homology group. What homology information determines a chain map, a chain endomorphism, a pair of chain complexes, etc.? Call a functor \( \mathcal{F} \) faithful if every object and every map of \( \mathcal{F} \) is (essentially) an image under \( \mathcal{F} \), and if a map \( f \) is 0 when \( Tf = 0 \); call it semi-faithful if the last condition is replaced by: \( f \) is an equivalence when \( Tf \) is. Then an object of \( \mathcal{X} \) is determined to within equivalence by its image under \( \mathcal{F} \) if \( \mathcal{F} \) is surjective, it is semi-faithful if those endomorphisms \( f \) of any object of \( \mathcal{X} \) with \( Tf = 0 \) lie in the Jacobson radical of the endomorphism ring of this object. The homology functor on the category \( \mathcal{C} \) of chain complexes and homotopy classes of chain maps is semi-faithful: indeed its kernel is nilpotent of class 2. In any such case, a semi-faithful functor on the derived category is constructed: a chain map \( f \) is determined by \( f_* \) and an element of \( \text{Ext}(\ker f, \text{coker } f) \). A detailed investigation of mapping-cylinder functors, introducing the idea of complex category, deduces from this result the semi-faithfulness of the functor: pair of chain complexes \( \rightarrow \) exact homology sequence. The "invariants" of a chain endomorphism can be found similarly and are different from those of \( f \) considered as a chain map. Various other functors have been and are being investigated in this way.

(Received October 4, 1963.)

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Contractive projections on an \( L_1 \)-space

Let \((X, \mathcal{A}, \mu)\) be a probability space. Let \( E(\cdot / \mathcal{J}) \) denote the conditional expectation operator for the \( \sigma \)-subalgebra \( \mathcal{J} \) of \( \mathcal{A} \). A function \( k \in L^1(X, \mathcal{A}, \mu) \) is said to be a weight function for \( \mathcal{J} \) if \( E(k, \mathcal{J}) \) \( = X_T \) (characteristic function) for some \( T \in \mathcal{J} \). The weighted conditional expectation operator is \( E(\cdot / \mathcal{J}, k) = k \cdot E(\cdot / \mathcal{J}) \). For each subset \( S \subset \mathcal{A} \), let \( U_S \) denote the operator \( U_S f = (2X_S - 1)f \). Let \( R[P] \) denote the range of \( P \). A projection \( P \) on \( L_1 \) is said to satisfy (*) if \( P \{ f : \| f \| \cdot R[P] = 0 \} = 0 \).

**Theorem 1.** An operator \( P \) on \( L_1 \) is a contractive (i.e., \( \| P \| \leq 1 \)) projection satisfying (*) iff \( P = \)}
\( \text{Corollary. An operator } P \text{ on } L_1 \text{ is a conditional expectation iff (1) } P^2 = P, \text{ (2) } \|P\| \leq 1, \text{ and (3) } P1 = 1. \text{ Theorem 2. For a closed subspace } \mathcal{M} \text{ of } L_1 \text{ the following are equivalent: (1) } \mathcal{M} \text{ is the range of a contractive projection, (2) } U_S \mathcal{M} \text{ is a vector lattice for some subset } S \subset \mathcal{J}, \text{ and (3) } \mathcal{M} \text{ is isometrically isomorphic to some } L_1 \text{ space. (Received October 7, 1963.)} \)


It has been conjectured by P. Ungar that if two perpendicular chords intersect in a point inside the domain bounded by a plane convex curve \( \theta \), and if they divide \( \theta \) into four arcs of equal length, then the sum of the lengths of the two chords is at least equal one half the perimeter length of \( \theta \) and equality holds only if \( \theta \) is a rectangle. Here we consider the more general problem in which the chords need not be perpendicular. It is shown that the ratio of the sum of the lengths of such chords over the length of \( \theta \) is at least \((\sqrt{5} - 2)^{1/2}\), and that equality holds only if \( \theta \) is a certain isosceles triangle. It is also shown that if the above ratio has a local extremum, for a given convex curve which is not a triangle, the extremum is \( \geq 1/2 \). (Received October 7, 1963.)

608-29. P. T. CHURCH, Syracuse University, Syracuse 10, New York. Factorization of differentiable maps with small branch set dimension.

Let \( M^n \) and \( N^n \) be \( C^m \) n-manifolds, and let \( f: M^n \to N^n \) be \( C^m \). The branch set of \( f \), denoted by \( B_f \), is the set of points in \( M^n \) at which \( f \) fails to be a local homeomorphism. The map \( f \) is called proper if, for each compact set \( X \subset N^n \), \( f^{-1}(X) \) is also compact (e.g., if \( M^n \) is compact, \( f \) is proper); \( f \) is monotone if, for each point \( y \in N^n \), \( f^{-1}(y) \) is connected and compact. Theorem. If \( f \) is \( C^m(m \geq n) \) and proper, and \( \dim(B_f) \leq n - 3 \), then there exists a factorization \( f = hg \) such that (1) \( g: M^n \to K^n \) is a \( C^m \) monotone map onto the \( C^m \) n-manifold \( K^n \), and \( h: K^n \to N^n \) is a k-to-1 \( C^m \) diffeomorphism. Moreover, (3) the factorization is unique in the sense that, if \( f = \tilde{h}\tilde{g} \) is another factorization with intermediate manifold \( L^n \), then there is a \( C^m \) diffeomorphism \( \alpha \) of \( K^n \) onto \( L^n \) such that diagrams commute, i.e., \( \tilde{g} = \alpha g \) and \( h = \alpha h \). Corollary. If \( f: S^n \to S^n \) is \( C^n \) and degree \( f \neq \pm 1 \), then \( \dim(B_f) \geq n - 2 \). (Received October 9, 1963.)


The limiting properties of the solutions of \( \rho w'' = z^2 a(z, \rho)w' + b(z, \rho)w \) as \( \rho \to 0 \), where \( z \) and \( \rho \) are complex variables, \( a \) and \( b \) are holomorphic at \((0,0)\) and \( a(0,0)b(0,0) \neq 0 \), are investigated. A solution \( w(z, \rho) \) of period 12\( \pi \) in \( \arg \rho \) is constructed, images of which: \( w(z, \rho e^{2\pi ik}) \), \( k = 0, 1, 5 \), chiefly characterize the asymptotic behavior of solutions. The methods used hinge upon the treatment of certain geometrical phenomena associated with the algebroid function of two variables defined by \( \lambda^2 = z^2 a(z,0)\lambda + \rho b(0,0) \). The problem of determining classical asymptotic formulas.
which become meaningless at $z = 0$) is solved for solutions satisfying initial conditions at $z = 0$.
(Received October 8, 1963.)


Let $f(x) \sim \sum_{n=0}^{\infty} a_n L_n(x)$ where $L_n(x)$ is the Laguerre polynomial defined by $(n!)e^{-x}L_n(x) = D^n(e^{-x}x^n)$. Then $a_n = \int_0^{\infty} e^{-tf(t)}L_n(t)dt$. The convergence of this series has been carefully studied (G. Szegő, Orthogonal polynomials, Amer. Math. Soc., Providence, R. I., (1959), 244). The restriction on $f(x)$ at $x = \infty$ for convergence is such that $f(x) = e^{ax}$ for $1/2 < a < 1$ will not satisfy the conditions although the coefficients $\{a_n\}$ are well defined. Try to find a summability method which will permit the summability of the Laguerre series for at least this class of functions. Define $\sum_{n=0}^{\infty} b_n = s, (B,1)$ if $\int_0^{\infty} e^{-t(\sum_{n=0}^{\infty} (n!)^{-1}b_n^n)}dt = s$. This is a Borel method. **Theorem.** Let $\exp[-t + a \sqrt{t}] \cdot f(t) \in L(0,\infty)$ for every $a > 0$ and let $f$ be of B.V. in some neighborhood of $x > 0$. Then the Laguerre series of $f$ is summable $(B,1)$ to $(1/2)[f(x +) + f(x -)]$. The point $x = 0$ as in the case of convergence seems to be exceptional. This result can be applied to the problem of the inversion of the Laplace transform. (Received October 9, 1963.)

608-32. Y. M. CHEN, Purdue University, Lafayette, Indiana. Some special results on the asymptotic behavior of the solution of the initial-boundary value problem of the two-dimensional wave equation for a large time.

Although many results on the decay and the asymptotic behavior of the solution of the exterior initial-boundary-value problem for the three-dimensional wave equation are known (Wilcox, Morawetz, Lax and Phillips, Buchal, and Zachmanoglou), very little has been done on the two-dimensional case. The following theorem gives some special results for the latter case. This theorem can be proved by first taking the Laplace transform with respect to time, then constructing explicitly the solution of the transformed system and finally examining the singularities of the constructed solution. **Theorem.** Let $R$ be the region exterior to the circle $x^2 + y^2 = a^2$ and $B$ be its boundary. Let $U(x,y,t)$ be the solution of the wave equation $\nabla^2 U(x,y,t) - \frac{\partial^2 U(x,y,t)}{\partial t^2} = 0$ in $R$ for $t \geq 0$, such that $U(x,y,0) = \partial U(x,y,0)/\partial t = 0$, and for $(x,y) \in B$, either (a) $U(x,y,t) = -H(t - x - a)$ or (b) $\partial U(x,y,t)/\partial n \partial t = 0$ at $(x,y) \in \partial B$. Then, in case (a) $U(x,y,t) \rightarrow 1 + O(\log t)^{-1}$ as $t \rightarrow \infty$, and in case (b) $U(x,y,t) \rightarrow O(t)^{-1}$ as $t \rightarrow \infty$. (Received October 10, 1963.)

608-33. R. T. SANDBERG, University of Arizona, Tucson, Arizona. On the compatibility of the uniform integral. II.

In a previous abstract (On the compatibility of the uniform integral, these Notices 10 (1963), 265) it was announced that the uniform and Lebesgue integrals are compatible on the class of functions which are equal almost everywhere to Riemann integrable functions. The general question of compatibility, however, was left open. For bounded functions this question has now been settled. Using a privately communicated result of J. C. van der Corput, it can now be shown that if $f$ is bounded
and both uniformly and Lebesgue integrable on the interval \([a,b]\), then the two integrals are equal.

From van der Corput's result, it also follows that if \(K\) is a measurable set, \(c > 0\), and \(\mu(K; c)\) the number of multiples of \(c\) in \(K\), then \(\lim_{c \to 0} \int_0^1 c \mu(K; c) dt = m(K)\) where \(m(K)\) is the Lebesgue measure of \(K\), and the integral is taken in the Lebesgue sense. (Received October 10, 1963.)

608-34. GUSTAVE SOLOMON and J. J. STIFFLER, Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, California. New bound on \((n,k)\) binary group codes.

Let \(A\) be an \((n,k)\) binary group code, i.e., a \(k\)-dimensional subspace of \(V_n(F)\), the vector space of dimension \(n\) over the field \(F\) of two elements. The Hamming metric on \(V_n(F)\), \(d(a,b)\), is the number of ones in the vector \((a + b)\). Let \(d_0 = \min\{d(a,b) : a \neq b \in A\}. \textbf{Theorem 1.}\) The minimum value for \(n\) for which it is possible to obtain an \((n,k)\) group code with minimum distance \(d_0\) is greater than or equal to \(d_0 + d_1 + \ldots + d_{k-1}\) where \(d_i = d_{i-1}/2\) if \(d_{i-1}\) is even and \(d_i = (d_{i-1} + 1)/2\) if \(d_{i-1}\) is odd.

\(\textbf{Corollary 1.}\) The smallest value of \(n\) for which an \((n,k)\) code with distance \(d = 2k-1 - \sum_i (2^{i-1} - 1)\) can exist is \(n \geq (2^{k-1} - 1) - \sum_i (2^{i-1} - 1)\). \textbf{Theorem 2.}\) If \(\sum_i d_i \leq k\), the bound obtained in Theorem 1 can be achieved. (Received October 11, 1963.)

608-35. JOSEPH NEGGER, Florida State University, Tallahassee, Florida. Derivations on \(\mathbb{p}\)-adic fields.

Let \(K\) be a \(\mathbb{p}\)-adic field with ring of integers \(R\), valuation \(V\) and residue field \(R/(p) = k\). Let \(\textbf{K} = [x\,\|\,k]\), \([\textbf{K} : k] = e\), with ring of integers \(\textbf{R}\), valuation \(V\) and residue field \(R/(\mathbb{p}) = k\). Let \(G_m = \{T : T : R \to R (a\mapsto a)\} , G_m = \{T : T : R \to R (a\mapsto (a)_{m+1})\} . \textbf{Theorem 1.}\) Let \(H : R \to R/(p)\). Then \(\Delta(a) = n\) if and only if for some \(i\) \(\textbf{Theorem 2.}\) The following statements are equivalent: (i) \(\textbf{K} \setminus \textbf{K} \) \(> 0\); (ii) every derivation \(D : R \to R\) can be extended to \(D^* : R^* \to R^*\); (iii) every derivation \(D^* : R^* \to R^*\) is a derivation, \(D^*(\pi) \in \pi\); (iv) every derivation \(D : R \to k\) can be lifted to a derivation \(D^* : R^* \to R^*\); (v) for some \(n \geq 2e/2p - 3\), \(\textit{G}_n/\textit{G}_{n+1}\) is isomorphic to the additive group of derivations on \(k\); (vi) for all \(n \geq 2e/2p - 3\), \(\textit{G}_n/\textit{G}_{n+1}\) is isomorphic to the additive group of derivations on \(k\); (vii) if \(n \geq 2e/2p - 3\) and \(T \in \textit{G}_n\), then \(T = T_{n+1}D_n\), where \(T_{n+1} \in \textit{G}_{n+1}\) and \(D_n = I + \sum_{i=1}^{\infty} \pi_i^n D(i)\), where \(D : R^* \to R^*\) is a derivation, \(D(\pi) \in \pi\). (Received October 14, 1963.)

608-36. R, F. JOLLY, University of California, Riverside, California. On families of functions characterized by kernels.

Suppose that \(F\) is a family of functions defined on \(I\) where \(I\) is one of the number intervals \([0,1], [0,\infty)\) or \((-\infty,\infty)\). The author discusses conditions under which \(F\) will be determined by a kernel \(K(x,t)\) in the sense that \(f\) belongs to \(F\) if, and only if, there exists a nondecreasing function \(\phi\) such that \(f(x) = \int_0^1 K(x,t) d\phi(t)\). The kernel is not necessarily continuous or symmetric. Various forms of the integral are considered and for some families a product or exponential integral is necessary.
Modifications of the restrictions on $\phi$ are also studied such as requiring $\phi(0) = \phi(0 +)$ or allowing $\phi$ to be of bounded variation. Specific kernels are given for families of solutions (with additional restrictions) of certain functional inequalities. In particular, the techniques of this paper are applied in an investigation of the subadditive functions on $I$. (Received October 14, 1963.)


Since $C_{x,n} = \sum_{i=0}^{n} T_{i}^{n}(b) x^{i}$ and $x_{n}^{A}(b) = \sum_{i=0}^{n} R_{i}^{n}(b) C_{x,i}$, $n! T_{i}^{n}(b)$ is a generalized Stirling number of the first kind and $(i!)^{-1} R_{i}^{n}(b)$ is a generalized Stirling number of the second kind [by virtue of the definition in Abstract 589-6, Notices Amer. Math. Soc. 9 (1962), 105]. It is also noted that $f(x) = \sum_{i=0}^{n} D_{i}(f)(x - x)_{i}$ implies that $i! D_{i}$ is the "bi-derivative" of $f$ at $x$. Thus $i! T_{i}^{n}(b)$ is the "bi-derivative" of $C_{x,n}$ at zero. (Received October 16, 1963.)


A technique for the calculation of characteristic values of arbitrary matrices based on the progressive reduction of the superdiagonal norm of the matrix by means of unitary transformations was described and illustrated in previous papers; see Quart. Appl. Math. 14 (1956), 267-275, and ibid. 17 (1959), 237-244. It is now shown that the application of this technique to certain matrices of the circulant type reduces such matrices to diagonal form by means of only a finite number of unitary transformations. (Received October 17, 1963.)

608-39. R. D. SINKHORN, University of Houston, Houston 4, Texas. Factor sets for doubly stochastic operators on a Hilbert space.

Let $H$ be a complex Hilbert space and let $u \in H$, $\|u\| = 1$. Define $S_{u} = \{T \in [H] | Tu = T*u = u\}$. The members of $S_{u}$ are said to be doubly stochastic relative to $u$. Let $\phi = \{T \in [C \Theta H, H] | T(1 \Theta 0) = u, T*u = 1 \Theta 0\}$ where $C$ is the complex plane. Using the notation $X^{A} = \{T | T \in X\}$ for any set of mappings $X$, there exists the Theorem, $S_{u} = \phi^{A}, \phi^{A} = 1 \Theta [H]$ where $I$ is the identity function in $[C]$. If $\Sigma \subseteq [C \Theta H, H]$ is such that $S_{u} = \Sigma^{A}, \Sigma^{A} = 1 \Theta [H]$, then there exists a rotation $e^{i\alpha}$ such that $\Sigma \subseteq e^{i\alpha}$. The inclusion may be proper. (Received October 18, 1963.)


The equations governing the quasi-one-dimensional flow of an ideal, inviscid, perfectly conducting compressible fluid, subjected to an oblique magnetic field, are written in characteristic form.
If the analysis is limited to small disturbances to an otherwise uniform, isentropic flow in a uniform duct, two integrals are obtained immediately from the characteristic system. Then, linearizing the remaining equations in the neighborhood of a (known) isentropic, uniform flow in a uniform duct, a system of linear equations, for which general solutions are given, is obtained. The solutions for the corresponding problems in conventional gas dynamics and for the case of a transverse magnetic field (Quasi-one-dimensional magnetohydrodynamic flow with small heat addition, Z. Angew. Math. Phys. 14 (1963), 294) are contained as special cases. (Received October 18, 1963.)

608-41. M. W. LODATO, The Mitre Corporation, Bedford, Massachusetts. On topologically induced generalized proximity relations, II.

Smirnov [Mat. Sb. N. S. 31 (73), (1952), 543-574] proved that a set $X$ with a binary relation, $\delta$, on its power set is a proximity space iff there exists a compact $T_2$ space $Y$ in which $X$ can be topologically imbedded so that $1) A \delta B$ in $X$ iff $A \cap B \neq \emptyset$ in $Y$. Earlier [Proc. Amer. Math. Soc. to appear] the author displayed a set of axioms for a binary relation $\delta$ such that the system $(X, \delta)$ satisfies the axioms iff there is a $T_1$ space $Y$ such that $X$ is regularly dense in $Y$ and in which $X$ can be imbedded so that $1)$ holds. The clusters of Leader [Fund. Math. 47 (1959)] were used here. The present paper generalizes this notion and thus relaxes the condition that $X$ be regularly dense in $Y$. (Received October 18, 1963.)


Throughout $E$ and $F$ denote locally convex topological linear spaces and operator denotes continuous linear operator. In locally convex spaces there are three distinct notions of a strictly singular operator. An operator $T: E \to F$ is said to be strictly singular if for each infinite dimensional subspace $M$ of $E$, $T|M$ is not an isomorphism. An operator $T: E \to F$ is said to be $b$-strictly singular if it is bounded and strictly singular. An operator $T: E \to F$ is said to be $B$-strictly singular if for each infinite dimensional subspace $M$ of $E$, $T|M$ does not have a bounded inverse. It is shown that both $b$-strictly singular operators and $B$-strictly singular operators form a subspace. The question as to whether or not strictly singular operators form a subspace remains open. Under certain restrictive conditions on the spaces it is shown that if $T'$ is strictly singular (respectively $b$-strictly singular or $B$-strictly singular), then so is $T$. This is not true for general spaces even if they are Banach spaces. There are several theorems relating to strictly singular operators and their conjugates. (Received October 21, 1963.)

608-43. E. C. POSNER and H. C. RUMSEY, JR., Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, California. An uncertainty function arising in search theory.

Consider a probability density function $P = \{P(i)\}$ on a set of $n$ points $i$, $1 \leq i \leq n$. Suppose that for each point $i$, there is associated a Gaussian process with independent increments of known
variance $\sigma^2$. Furthermore, it is known that exactly one of these $n$ processes has known mean $\mu > 0$, and the others have mean 0. The probability that point $i$ has the positive mean is given by $P(i)$. The most likely $i$ is looked at by integrating its process, changing the density $P$ by Bayes' rule. When a different point becomes more likely, it is looked at instead. Given $\epsilon > 0$, let $T(P; 1 - \epsilon)$ denote the expected time until the largest probability first reaches $1 - \epsilon$, starting from density $P$ at time 0. It is then shown that as $\epsilon \to 0$, $T(P; 1 - \epsilon) = (2\sigma^2/\mu^2)(\log(1/\epsilon) + F(P) + O(\epsilon \log \epsilon))$. Here $F(P)$ depends only on the density $P$ and not on $\mu$ or $\sigma$. This function $F(P)$ is the "uncertainty function" of the title. Furthermore, $F(P)$ is found in closed form. The maximum of $F$ is achieved only for the flat distribution $B:F(B) = n - 2) + \log (n - 1)$. (Received October 21, 1963.)

608-44. R. J. DISTLER, Department of Electrical Engineering, University of Kentucky, Lexington, Kentucky. The domain of univalence of certain classes of meromorphic functions.

Let $K$ be a closed set of points in the complex plane and let $\mathcal{F}(K)$ be the family of functions of the form $f(z) = \sum_{k=1}^{N} A_k(z - a_k)^{-1}$, $A_k > 0$, $k = 1,2,\ldots,N$ where all the $a_k$ lie in $K$. The director set of $K$ is the collection $D$ of all points $P$ such that $K$ subtends an angle of $\pi/2$ at $P$. Whenever $D$ is a simple curve it will divide the plane into two domains, one of which will contain $K$, and the other, which we will be denoted by $U(K)$, will consist of all points at which $K$ subtends an angle less than $\pi/2$. The main result is: $U(K)$ is the domain of univalence of the class of functions $\mathcal{F}(K)$. Stated in another form this is: Let $C(t)$ be any curve that lies in $K$ for $t \in [-1,1]$, and let $\alpha(t)$, $t \in [-1,1]$ be an nondecreasing function with at least one point where it is increasing. Then $U(K)$ is the domain of univalence of the class of all functions of the form $f(z) = \int_{-1}^{1} [z - C(t)]^{-1} \alpha(t)$. (Received October 21, 1963.)


Let $X_n$ be a discrete parameter Markov process on a measurable space $(X, \Sigma)$ with stationary transition probabilities $P^n(x, E)$. $\Sigma$ is assumed separable, Harris (Third Berkeley Symposium, 1956) proved that if there exists a $\sigma$-finite measure $m$ on $\Sigma$ such that $m(E) > 0$ implies $P(X_n \subseteq E$ for infinitely many $n|X_0 = x) = 1$ for all $x \in X$, then there exists a $\sigma$-finite stationary measure $Q$ for the process: $\int P(t, E)Q(dt) = Q(E)$ for all $E \in \Sigma$. Call the process $m$-singular if, for each $x \in X$ except for an $m$-null set, there exists a set $L_x$, $m(L_x) = 0$ and $P^n(x, L_x) = 1$ for all $n$. In the contrary case call the process $m$-nonsingular. The author proves: If the process is $m$-nonsingular and $m(E) > 0$ implies $P(X_n \subseteq E$ for infinitely many $n|X_0 = x) = 1$ for almost all $x(m)$, then there exists a $\sigma$-finite stationary measure for the process. The method of proof relies to a large extent on the methods of Harris. (Received October 22, 1963.)


Let $A = \langle A; F \rangle$ a finite algebra; i.e., $A$ is a finite set and $F$ is a finite collection of finitary operations on $A$. Let $SP(A)$ denote the class of subdirect powers of $A$. In many known cases
SP(Δ) ∈ EC, i.e., it can be characterized by a single first order sentence. Let X = {x_1,...,x_n}; an equation over Δ is f(y_1,...,y_m) = y, where y_1,...,y_m ∈ A ∪ X. Let S_1(X_1) and S_2(X_2) be two systems of equations over Δ. Let S_1(X_1) ⊨ S_2(X_2) mean that any equation in S_2(X_2) can be proved from the equations in S_1(X_1) and the valid relations in Δ. Theorem. Suppose Δ is a finite algebra and Δ satisfies condition (U_N): if S_1(X_1) has no solutions in Δ then there exist an S_2(X_2) such that S_1(X_1) ⊨ S_2(X_2) and |X_2| ≤ N. Then SP(Δ) ∈ EC. Problems: Find necessary and sufficient condition for SP(Δ) ∈ EC, SP(Δ) ∈ ECΔ. (Received October 21, 1963.)

608-47. P. R. YOUNG, Reed College, Portland 2, Oregon. A recursively enumerable splinter which is not a cylinder.

Myhill has asked (Recursive digraphs, splinters and cylinders, Math. Ann. 138 (1959), 211-218) whether every infinite r.e. splinter is a cylinder. Def. A set, A, is a weak cylinder if there is a total recursive function f such that x ∈ A implies f(x) ∈ A - {x} and x ∈ A implies f(x) ∈ A - {x}. A priority argument is used to construct a nonunit set S, which is a splinter but which is not a weak cylinder. Since every cylinder is a weak cylinder, and since every recursive set which does not have exactly one member is a weak cylinder, there are infinite splinters which are not cylinders. (Received October 23, 1963.)

608-48. LEOPOLD FLATTO, Yeshiva University, St. Nicholas Avenue and 183rd Street, New York, New York. Partial differential equations and difference equations.

Let f(x) be a continuous real-valued function defined in an n-dimensional region and let let be a generalized solution of the system P_1(∂/∂x)f = 0 (1 ≤ i ≤ m) where P_1(x) is a homogeneous polynomial. It is shown that all such solutions satisfy an equation (1) ∑ N_{i=1}^N μ_i f(x + ty_i) = 0, x ∈ R, 0 < t < ε_x (the μ_i's are real with ∑ N_{i=1}^N μ_i = 0) if and only if the ideal P = (P_1,...,P_m) contains a polynomial which splits into real homogeneous linear factors. In the case of one equation P_1(∂/∂x)f = 0 there is obtained a geometric criterion on μ guaranteeing that all solutions satisfy (1). Also obtained is a geometric criterion on μ guaranteeing that (1) be equivalent to P(∂/∂x)f = 0. (Received October 24, 1963.)


In [Math. Z. 29 (1929), 549-640] Pólya remarked that no proof had been given of the following: Let f(z) = ∑ N_{j=1}^N P_j(z)exp(α_jz) where the P_j are polynomials; let L be the perimeter of the convex hull of the set {α_j}; and let n(r) be the number of zeros of f of modulus at most r. Then n(r) = Lr/(2π) + O(1). This theorem is proved using the methods of C. E. Wilder [Trans. Amer. Math. Soc. 18 (1917), 415-442], Schwengeler [Dissertation, Zurich, 1925], and Langer [Bull. Amer. Math. Soc. 37 (1931), 213-239]. The result is a special case of a similar theorem concerning the zeros of f when the P_j are simply analytic for large |z| with P_j(z)/A_j z^m_j → 1 as z → ∞ for some A_j and integers m_j. (Received October 24, 1963.)
608-50. WITHDRAWN.


The abelian groups $G$ and $H$ are quasi-isomorphic if there are subgroups $G' \subseteq G$, $H' \subseteq H$ such that $G' \cong H'$, $nG \subseteq G$, $mH \subseteq H'$ for positive integers $m$ and $n$. For any ordinal $\alpha$, $f_G(\alpha)$ is the $\alpha$th Ulm invariant of the abelian $p$-group $G$. Theorem 1. Let $G$ and $H$ be countable $p$-groups. Then $G$ and $H$ are quasi-isomorphic if and only if (i) there exists a non-negative integer $k$ such that $\sum_{j=n+k}^{n+r} f_G(j) \leq \sum_{j=n}^{n+r} f_H(j)$, $\sum_{j=n+k}^{n+r} f_H(j) \leq \sum_{j=n}^{n+r} f_G(j)$ for all non-negative integers $n$ and $r$; and (ii) $f_G(\alpha) = f_H(\alpha)$ for all $\alpha \geq \omega$. Theorem 2. Let $G$ and $H$ be $p$-groups which are direct sums of cyclic groups. Then $G$ and $H$ are quasi-isomorphic if and only if $G$ and $H$ satisfy condition (i) of Theorem 1. These results are extendable to torsion groups in an obvious way. (Received October 25, 1963.)


Let $L$ be a solvable Lie algebra over a field of characteristic zero. Let $L^n = [L^{n-1}, L]$, and let $L^\infty = \cap_{n=1}^{\infty} L^n$. It has been shown by Borel and Mostow that any two Cartan subalgebras of $L$ are conjugate by an automorphism of $L$ given by $\exp(\text{Ad} x)$, $x$ in $L^\infty$. Let $G$ be a semisimple group of automorphisms of $L$. Then Mostow has shown that $G$ will leave invariant a Cartan subalgebra of $L$. Here it is shown: Any two $G$-invariant Cartan subalgebras of $L$ are conjugate by an automorphism of $L$ given by $\exp(\text{Ad} x)$, where $x$ is a fixed point of $G$ in $L^\infty$. (Received October 25, 1963.)


Let $Y$ be a complex Banach space and $I$ be either the additive group $\mathbb{R}$ of reals or the additive semigroup $\mathbb{R}^+$ of non-negative reals. Let $A$ be a linear continuous operator from $Y$ into itself. Let $f(x,y)$ from $I \times Y$ into $Y$ satisfy Lipschitz's condition with respect to $y$ with a constant $L$. In the case when $I = \mathbb{R}^+$, let the function $f(x,y_0)$ be asymptotically almost periodic for any fixed $y_0$ in $Y$ and the spectrum of the operator $A$ lie in the left half-plane of the complex plane. In the case when $I = \mathbb{R}$, let the function $f(x,y_0)$ be almost periodic for any fixed $y_0$ in $Y$ and the spectrum of the operator $A$ not intersect the imaginary axes of the complex plane. There exists a constant $C$ depending only on $A$ such that if $L < C$ then the differential equation $y' + Ay = f(x,y)$ has an asymptotically almost periodic or almost periodic solution respectively. The proof makes use of contraction mapping theorem and
some topological construction which establishes an isomorphism and an isometry of the space of almost periodic functions with values in \( Y \) and a certain space of continuous functions on a compact semigroup or group respectively. (Received October 23, 1963.)


Let \( G \) be a finite Abelian group of order \( v \). Let \( A \) and \( B \) be sets of elements from \( G \). Denote the cardinal of a set, \( S \), by \( |S| \), and the set theoretic complement by \( S^c \). A is a component of \( C \) if there is a set \( B \) such that \( A + B = \{ a + b | a \in A, b \in B \} = C \). Theorem 1. Let \( C \subseteq G \). Let \( \tilde{C} = \{ \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_r \} \). Then \( A \) is a component of \( C \) if and only if for every \( k \in D \) it is true that \( A \cup D \). Corollary: \( \{0, a\} \) is a component of \( C \) if and only if \( 2a = \tilde{c}_i - \tilde{c}_j \) implies \( a = \tilde{c}_j - \tilde{c}_m \) for some \( m \). Corollary: Let \( \tilde{C} \) be a difference set with parameters \( v = |G| \), \( r = |\tilde{C}| \), and \( \lambda \). Let \( r < v \). If \( 2a = 0 \) for \( a \neq 0 \), then \( \{0, a\} \) is not a component of \( C \). If \( (\lambda, v) = 1 \), then there does not exist an \( a \in G \), \( a \neq 0 \), such that \( \{0, a\} \) is a component of \( C \). If \( (\lambda, v) = d > 2 \), if \( H \) is a cyclic subgroup of \( G \) such that \( |H| = d \), if there are exactly \( \lambda/d \) cosets of \( H \) contained in \( \tilde{C} \), and if a \( E H \) such that a is of order \( d \), then \( \{0, a\} \) is a component of \( C \). (Received October 28, 1963.)

608-55. I. D. BERG, Yale University, Box 2155 Yale Station, New Haven, Connecticut. Perturbations of linear operators and summability. Preliminary report.

This paper deals primarily with questions concerning the effects of perturbations of conservative matrices on the summability domains of such matrices. Let \( \Gamma \) denote the Banach algebra of conservative matrices and \( \Delta \) the subalgebra of triangular conservative matrices. Consider \( A \in \Gamma \) as an operator from \( c \) into \( c \). \( A \in \Gamma \) is said to have property B if there exists a bounded divergent sequence \( x \) such that \( Ax \subseteq c \). Theorem 1. \( A \) has property B if and only if there exists a sequence \( A_n \rightarrow A \) such that for each \( n \), \( A_n \in \Gamma \) and \( \bigcup_{j=n}^{\infty} \text{kernel} (A_j) \) contains an infinite set of linearly independent elements of \( c \). Theorem 2. Let \( A \) be a Nörlund matrix corresponding to a quadratic polynomial with at least one zero outside the unit circle. Let \( B \in \Delta \) be invertible, then there is an open set \( N \subset \Delta \) such that \( AB \in \Delta \) and also for each \( D \in N \) there is an \( x \in \ell_1 \) such that \( D^*x = 0 \). Finally, consideration of operators with closed range makes it possible to present open sets in \( \Gamma \) consisting of matrices with property B. (Received October 28, 1963.)

608-56. D. R. LICK, New Mexico State University, University Park, New Mexico. Sets of convergence of Dirichlet series.

Theorem 1. If \( b \) is any real number, and \( M \) is a denumerable set on the line \( L: \sigma = b \), where \( s = \sigma + it \), then there exists an ordinary Dirichlet series \( \sum a_n n^{-s} \) which converges on \( L - M \) and diverges on \( M \). The proof is obtained by using circuits of the type \( C_q(i\tau) = t_q^{-1} \exp(-in_q \tau/2q) \cdot \sum_{j=0}^{t_q} \exp(-ij\tau/2q) - \sum_{j=1}^{t_q} \exp(-ij\tau/2q) \), where \( t_q \) and \( n_q \) are properly chosen positive integers, to construct an ordinary Dirichlet series with the desired properties. It is well known that the sets of convergence of ordinary Dirichlet series are of type \( F_{\sigma \delta} \). If \( M \) in Theorem 1 is taken to be dense on \( L \), then \( M \) is not of type \( G_\delta \) and \( L - M \) is not of type \( F_\sigma \). Hence, not every set of convergence of an ordinary Dirichlet series is of type \( F_\sigma \). (Received October 28, 1963.)
A generalization of Axt's primitive recursive hierarchy. Preliminary report

In Abstract 597-182 Axt constructs a primitive recursive hierarchy, an \( \omega \)-string of classes \( K_n \). In the generalization to be considered two results are preserved (I) \( K_n \subseteq K_{n+1} \), properly and (II) the universal function for class \( n \) is an element of class \( 2n+k \), for some fixed \( k \); (These results also hold for a relativization of Axt's or the generalized hierarchy.) If \( \prec \) is any well-ordering we define classes \( K_n^\prec \) as follows: \( K_n^\prec \) is the smallest class of functions containing the successor, constant, and identity functions which is closed under substitution and the operation of primitive recursion or \( \prec \)-ordinal recursion from functions in class \( K_n^\prec \). This immediately gives a hierarchy of the \( k \)-fold recursive functions (Péter, *Rekursive Funktionen*, Budapest, 1951). Now for any general recursive function \( f(x) \) there exists a well-ordering \( \prec \) over which \( f \) is ordinal recursive, where \( \prec \) has the properties: \( | \prec | = \omega \); \( \prec \) is primitive recursive; the predecessor function for \( \prec \) is primitive recursive; primitive recursion is reducible to \( \prec \)-ordinal recursion and; every \( \prec \)-ordinal recursive function is primitive recursive in \( \lambda x\mu zT_1(\epsilon, x, z) \), where \( \epsilon \) is any fixed Gödel number of \( f \). Thus classes \( K_n^\prec \) may be used to subdivide quite general classes of general recursive functions. (Received October 28, 1963.)

A translation on a normed algebra \( A \) is any linear isometric onto map with \( T(xy) = (Tx)y \); \( G_A \) is the group of translations on \( A \). A QCG algebra is any Banach algebra \( A = \phi(L^1(H)) \subseteq M(H)/N \), with \( H \) a compact group, \( N \) a \( \text{wk}^* \)-closed ideal in \( M(H) = C(H)^* \), and \( \phi: M(H) \to M(H)/N \) the canonical homomorphism. Compact group algebras are QCG algebras; conversely, most properties of the translations on a compact group algebra are equally true for all QCG algebras. QCG algebras can be characterized as abstract Banach algebras, reducing the characterization problem for compact group algebras to one of identifying QCG algebras which are actually group algebras. Theorem. A Banach algebra \( A \) is a QCG algebra if and only if \( A \) is a semisimple dual algebra with central approximate identity of norm \( 1 \) and: (A) if \( \|a\| \leq 1 \), then \( L_a(L_\varepsilon x \to ax) \) is a strong operator limit of convex sums in \( \text{co}(G_A) \). Axioms other than (A) can also be used. Let \( R_A \) be the finite dimensional representations of \( G_A \) equivalent to direct sums of the representations gotten by letting \( G_A \) act on finite dimensional left ideals of \( A \). Theorem. A QCG algebra \( A \) is a group algebra if and only if it has a nonzero multiplicative functional and \( R_A \) contains all representations \( V_3 \otimes (V_2)^* \otimes V_3 \) with \( V_1 \ldots V_3 \) in \( R_A \). (Received October 25, 1963.)

A limitation of uniform cluster operators.

Let \( H^\infty \) denote the Banach algebra of bounded analytic functions in the open unit disc \( D \). Let \( \Gamma \) be the unit circle. For each \( P \subseteq \Gamma \), \( f \in H^\infty \), let \( R(f, P) \) be the set of values achieved by \( f \) in every neighborhood of \( P \) and let \( C(f, P) \) be the (full) cluster set of \( f \) at \( P \). Let \( K \) be a uniform cluster operator on \( H^\infty \times \Gamma \), i.e., for each \( P \subseteq \Gamma \) there is a compact subset \( K_P \) of the fiber of the maximal ideal space of \( H^\infty \) lying above \( P \) such that \( K(f, P) = \hat{f}[K_P] \) for every \( f \in H^\infty \) (\( \hat{f} \) is the Gelfand representation of \( f \)). Theorem. If \( K(f, P) \) does not contain the boundary of \( C(f, P) \) for every \( f \) and \( P \), then there are
Q ∈ Π and g ∈ H∞ such that g[D] = D and C(g,Q) = D ∪ Π while Π ⊂ C(g,Q) - K(g,Q) - R(g,Q). Example. There is g ∈ H∞ with C(g,1) = D ∪ Π while Π - R(f,g) is of zero capacity. Therefore, if one writes R(ℓ,P) = C(ℓ,P) - K(ℓ,P) - M(ℓ,P) ∪ N(ℓ,P) with M(ℓ,P) = C(ℓ,P) - (R(ℓ,P) ∪ K(ℓ,P)) and N(ℓ,P) = R(ℓ,P) ∩ K(ℓ,P), it is possible for either M(ℓ,P) or N(ℓ,P) to be large (in the sense of linear measure). Thus nonuniform cluster operators are required for any reduction in the sizes of M(ℓ,P) and N(ℓ,P).

Example. There is g ∈ H∞ with C(g,1) = D ∪ Π while Π - R(f,g) is of zero capacity. Therefore, if one writes R(ℓ,P) = C(ℓ,P) - K(ℓ,P) - M(ℓ,P) ∪ N(ℓ,P) with M(ℓ,P) = C(ℓ,P) - (R(ℓ,P) ∪ K(ℓ,P)) and N(ℓ,P) = R(ℓ,P) ∩ K(ℓ,P), it is possible for either M(ℓ,P) or N(ℓ,P) to be large (in the sense of linear measure). Thus nonuniform cluster operators are required for any reduction in the sizes of M(ℓ,P) and N(ℓ,P).

(Received October 29, 1963.)

608-60. V. LAKSHMIKANTHAM, University of Alberta, Calgary, Alberta, Canada. **Parabolic differential equations and Lyapunov like functions.**

One of the most important techniques in the theory of nonlinear differential equations is the direct method of Lyapunov and its extensions. It depends basically on the fact that a function satisfying the inequality m(t) ≤ w(t,m(t)), m(t₀) ≤ t₀, is majorized by the maximal solution of the equation r' = w(t,r), r(t₀) = t₀. Using this approach, some problems of parabolic differential equations are studied. For example, the results include bounds on the solutions, uniqueness, stability and boundedness of solutions. It is also indicated that using Lyapunov like vector functions is useful in some cases. Examples are given to illustrate some of the results. (Received October 29, 1963.)

608-61. D. R. HENNEY, University of Maryland, College Park, Maryland. **A structure theorem of set-valued additive functions.**

Let Y denote a Banach space and let C(Y) be the space of all nonempty, bounded, closed and convex sets of the space Y. The space C(Y) forms a semi-linear space under the operation of algebraic addition of sets and algebraic multiplication of a set by a scalar. If V is a neighborhood of zero in a locally convex topology of the space Y then the family of sets N = {B: A ⊂ B + V and B ⊂ A + V} constitutes a base of neighborhoods for the set A in C(Y). This topology is said to be the weak (or strong) topology of the space C(Y) if it is generated by the weak (or strong) topology of the space Y.

**Theorem.** If A(t) is an additive function defined on the set of positive numbers S with values in the space C(Y) then the following conditions are equivalent: (a) A(t) is bounded in an open interval (c,d); (b) A(t) is continuous at a point t ∈ S in the weak topology on C(Y); (c) A(t) is continuous at a point t ∈ S in the strong topology on C(Y); (d) A(t) is continuous at every point t ∈ S for the strong topology on C(Y); (e) A(t) is of the form A(t) = tA(1), for all t ∈ S. To prove this theorem one uses an analogous theorem for additive real-valued functions and the theorem that every closed, convex set in a Banach space is the intersection of all closed half-spaces containing it. (Received October 29, 1963.)

608-62. MIKLÓS CSÖRGÖ, Princeton University, Princeton, New Jersey. **Some Smirnov type limit theorems of probability theory.**

Let x₁, x₂, ..., xₘ be independent observations on the random variable x₁ having a continuous distribution function F(x). Let x₂, x₃, ..., xₘ be independent observations on the random variable x₂ having a continuous distribution function H(x). Let Fₙ(x) and Hₘ(x) be the empirical distribution functions of the above two random samples respectively. Assuming that F(x) ≡ H(x), the limit dis-
distribution of the supremum of \( \{ F_n(x) - H_m(x) \} / F(x) \) is derived under specified conditions. Let \( F_{n+m}(x) \) be the empirical distribution function of the sample gained by pooling the two samples of size \( n \) and \( m \). Assuming again that \( F(x) = H(x) \), the limit distribution of the supremum of \( \{ F_n(x) - H_m(x) \} / F_{n+m}(x) \) is derived under specified conditions. (Received October 29, 1963.)

608-63. G. G. LORENTZ, Syracuse University, Syracuse, New York. **Inequalities and the saturation classes of Bernstein polynomials.**

If \( B_n(f, x) \) is the Bernstein polynomial of \( f \) and if \( |B_n(f, x) - f(x)| \leq (2n)^{-1}Mx(1 - x) \), \( n = 1, 2, ... \) holds for some subinterval \( a < x < b \) of \([0, 1]\), then on this subinterval, \( f' \) exists and belongs to the class \( \text{Lip}_1 \). This allows the determination of the saturation class for the Bernstein polynomials on the interval \([0, 1]\); the critical measure of approximation, \((2n)^{-1}Mx(1 - x)\) depends on \( x \) and \( n \), and the saturation class consists of all functions \( f \) with \( f' \in \text{Lip}_1 \) on the interval \([0, 1]\). The proof depends on inequalities, in terms of \( L \), for the derivatives of the polynomials of the form \( \sum_{k=0}^{n} a_k C_{n,k} x^k(1 - x)^{n-k} \), \( |a_k| \leq L \). (Received October 30, 1963.)

608-64. D. M. TOPPING, University of Chicago, Chicago, Illinois 60637. **States and ideals in Jordan operator algebras.**

Let \( A \) be a uniformly closed Jordan algebra of self-adjoint operators containing the identity. A linear subspace \( I \) of \( A \) is a **quadratic ideal** if \( a \in I \) and \( b \in A \) imply \( aba \in I \). **Theorem 1.** The intersection of all maximal quadratic ideals is zero. The **kernel** of a state \( \omega \) is \( K = \{ a \in A : \omega(a^2) = 0 \} \); it is the largest quadratic ideal in the null space of \( \omega \). Call \( \omega \) **maximal** if its kernel is. **Theorem 2.** Each closed quadratic ideal is the intersection of the kernels of all pure states annihilating it. There are enough maximal states (necessarily pure) to distinguish positive operators from zero. Call \( \omega \) **central** if \( \omega(ab^2a) = \omega(ba^2b) \), and a **character** if \( \omega(a^2) = (\omega(a))^2 \), all \( a, b \in A \). **Theorem 3.** The characters are exactly the central pure states. Now assume \( A \) is weakly closed and let \( L \) be its projection lattice. **Theorem 4.** There is a one-to-one correspondence between the norm-closed quadratic ideals of \( A \) and the norm-closed lattice ideals of \( L \). **Theorem 5.** A state is completely additive if and only if it is strongly continuous on the unit ball (this requires \( |\omega(ab + ba)|^2 \leq 4\omega(ba^2b) \)). Using an idea of S. Sakai leads to **Theorem 6 (R - N).** Let \( \Phi \leq \Psi \) be completely additive states. Then \( \phi(a) = \psi(ah + ha)/2 \) for some \( h \in A \), \( 0 \leq h \leq 1 \), all \( a \in A \). (Received October 30, 1963.)

608-65. L. L. LININGER, State University of Iowa, Iowa City, Iowa 52240. **Some results on crumpled cubes.**

A compact continuum \( C \) is a crumpled cube if and only if \( C \) is homeomorphic to a 2-sphere and its bounded complementary domain in \( \mathbb{R}^3 \). **Theorem 1.** If \( C \) is a crumpled cube and \( B \) is the unit ball in \( \mathbb{R}^3 \) then there exists an upper semi-continuous decomposition \( G \) of \( B \) with the following properties: (1) the intersection of each nondegenerate element of \( G \) with the boundary of \( B \) exists and is exactly one point, (2) each nondegenerate element of \( G \) is pointlike, and (3) \( B/G \) is homeomorphic to \( C \). **Theorem 2.** Suppose \( S \) is the 2-sphere constructed by Bing in Duke Math. J. 28 (1961), 1-15, and \( C \) is the one point compactification of the closure of the bounded complementary domain.
of $S$. Then the union of two copies of $C$ sewn together by the identity on $S$ is topologically $S^3$.

(Received October 31, 1963.)

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608-66. F. J. SANSONE, Case Institute of Technology, Cleveland 6, Ohio. On a class of sums of regressive isols.

Let $T$ and $U$ be infinite regressive isols. For an a recursive function of $n$, define

$$
\sum_U(T - a_n) = \text{Req} \bigcup_{n=0}^{\infty} \bigcup_{n'=0}^{\infty} (u_n - a_n),
$$

where $a_n = (t_n, t_{a_n+1}, t_{a_n+2}, \ldots)$ and $t_n$, $u_n$ are any regressive functions ranging over sets in $T$ and $U$ respectively. **Theorem 1.** Let $T, U \subseteq \Lambda_R - \epsilon$. If $a_n$ is a recursive function of $n$ such that $\langle \forall n \rangle [a_n \leq n + 1]$, then $U \leq * T \Rightarrow \sum_U(T - a_n) = T \cdot U - \sum U a_n$

[see Dekker, Recursive function theory, Amer. Math. Soc., 1962, 77-96].

**Theorem 2.** Let $T, U \subseteq \Lambda_R - \epsilon$. If $a_n$ is an increasing, recursive function such that $\langle \forall n \rangle [a_n \leq n + 1]$, then $T \leq * U \Rightarrow \sum U(T - a_n) = \sum V(T - a_n)$.

**Theorem 3.** Let $T, U \subseteq \Lambda_R - \epsilon$. For an a strictly increasing, recursive function, $\sum U(T - a_n) = \sum \min(\phi a_n(T), U)(T - a_n) = T \cdot \min(\phi a_n(T), U)$.

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608-67. SEYMOUR SHERMAN, Wayne State University, Detroit, Michigan 48202. Fluctuation and periodicity.

Let $X_1, X_2, \ldots$ be independent random variables with a common discrete distribution function. Let $\pi_n$, the "number of periods", be the largest integer such that $(X_1, X_2, \ldots, X_n)$ is just $(X_1, X_2, \ldots, X_n; \pi_n)$ repeated an integer number of times. Consider the least concave majorant of the broken line graph with vertices $(0, S_0), \ldots, (n, S_n)$. Let $J_n$ be an appropriate "number of sides" of the least concave majorant. **Theorem.** If $|t| < 1$, then \[ \sum_{n=0}^{\infty} E(\exp(i \lambda J_n))t^n = \exp \sum_{k=1}^{\infty} k^{-1} E(\exp(i \lambda R_k)). \]

For other "number of sides", $H_n$ and $K_n$, which E. S. Andersen considered to be measures of fluctuation, [On the distribution of the random variable $H_n$, Aarhus Tech. Note No. 1, 1959] and whose definitions depend on whether any $(j, S_j)$ is permitted to be an inner point of a side) analogous formulas are obtained. The proof is via the noncommutative Witt identity [Schützenberger, M. P., and Sherman, S., On a formal product over the conjugate classes of a free group, to appear in J. Math. Anal. Appl.].

The analogy to Spitzer's formula is not accidental since that formula may be derived in the same way.

(Received October 31, 1963.)

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Only real matrices are considered. Without loss of generality only matrices $A$ with positive determinants, and with elements $a_{ij}$, $i, j = 1, 2, \ldots, n$, such that $|a_{ij}| \leq 1$ are considered. For every matrix $A$, there exists a matrix $B$, with the same limitations and with $a_{in} = 1, i = 1, 2, \ldots, n$ and with $\det B \geq \det A$. The $n$ rows of $B$, excluding the last column, give the coordinates in $n - 1$ space of the
vertices of a simplex. Det $B$ represents $(n - 1)!$ times the content of this simplex. The sphere in $n - 1$ space circumscribing the hyper cube with vertices having coordinates $(\pm 1, \pm 1, \pm 1, \ldots, \pm 1)$ has radius $\sqrt{(n - 1)}$. The simplex with maximum content inscribed in this sphere is a regular simplex with content $\frac{n^{n/2}}{(n - 1)!}$. Thus if the simplex represented by matrix $B$ is such that $\text{Det } B$ has the value given by Hadamard's limit, then the simplex associated with it is regular, and therefore its $n$ vertices are vertices of the hyper cube previously mentioned. For this to be true, it is proved that $n$ must be a power of 2. J. H. E. Cohn in his paper On the value of determinants (Proc. Amer. Math. Soc. 14 (1963), 581-586) gives a matrix having a determinant of this value. An example of such a simplex is a tetrahedron having its 4 vertices of a cube, with each pair diagonally opposite each other in a face of the cube. (Received October 31, 1963.)


Let $\langle A, \leq, A, V, '0, 1 \rangle$ be a Boolean Algebra, and let $\langle D, \leq, A, \ast, 1 \rangle$ be an implicative semi-lattice. (H. B. Curry, Foundations of math. logic, New York, 1963). Let $f: A \times D \rightarrow D$. For $a \in A$, $d \in D$, let $f_a: D \rightarrow D$ be defined by $f_a (d) = f(a, d)$. It is said that $f$ is "admissible" iff for every $a \in A$, $f_a$ is an endomorphism on $D$, and (i) $f(a \land b, d) = f(a, f(b, d))$, (ii) if $a \leq b$, then $f(a, d) \geq f(b, d)$, (iii) $f(0, d) = 1$, $f(1, d) = d$, for $a, b \in A$, $d \in D$. Let $\theta$ be the equivalence relation on $A \times D$ defined by $\langle a, d \rangle \theta \langle b, e \rangle$ iff $a = b$, and $f(a, d) = f(a, e)$, $a, b \in A$, $d, e \in D$. Let $[a, d]$ denote the equivalence class of $\langle a, d \rangle$, and for $[a, d]$ and $[b, e]$ elements of $(A \times D)/\theta$, let $[a, d] \land [b, e] = [a \land b], [d \land e], [a, d] \ast [b, e] = [a \lor b], f(a, d \ast e)$. Then, with these operations, $(A \times D)/\theta$ is a bounded implicative semi-lattice, the closed algebra of which is isomorphic to $A$, and the dense filter of which is isomorphic to $D$. Also, for any $a \in A$, and $d \in D$, $f(a, d)$ corresponds to the $\ast$ composition of the image of $a$ in $(A \times D)/\theta$ with that of $d$. Furthermore, if $L$ is a bounded implicative semi-lattice with dense filter $D$, and if $f: L^* \times D \rightarrow D$ is defined by $f(a, d) = a \ast d$, then $f$ is admissible, and $L$ is isomorphic to $(L^* \times D)/\theta$. (Received October 31, 1963.)

608-70, R. D, BROWN, University of Kansas, Lawrence, Kansas. Reflection laws of fourth order elliptic differential equations.

Let $L(u) = 0$ be a fourth order linear elliptic partial differential equation in two real variables with constant coefficients. Assume that the principal part of the equation factors into four distinct (complex) linear factors and that a solution $u(x, y)$ is defined and analytic in a convex domain $D$ adjacent to an open segment $\sigma$ of the x-axis on the side $y < 0$. Assume also that $u$ and its derivatives up to and including those of fourth order are defined and continuous in $D \cup \sigma$ and that on $\sigma$, $u(x, 0), u_y(x, 0)$ are equal to the two analytic functions $f(x), g(x)$ respectively. If $f(x + iz), g(x + iz)$ can be extended as analytic functions of the complex variable $x + iz$ into all of an explicitly defined simply connected domain $\Gamma$, then $u(x, y)$ can be analytically extended as a solution of $L(u) = 0$ into all of $D \cup \sigma \cup \hat{\sigma}$, where $\hat{\sigma}$ is an explicitly defined convex domain adjacent to $\sigma$ on the side $y > 0$. Both $\Gamma$ and $\hat{\sigma}$ are independent of the particular solution $u$ and the given functions $f$ and $g$. (Received October 31, 1963.)

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608-71. WITHDRAWN.


Here, all subspaces of topological spaces are Q-spaces. Let \( S(X) \) be the semiring of all real-valued continuous functions \( f \) with domain \( D(f) \subseteq X \) where \( f + g = \{ (x, f(x) + g(x)) : x \in D(f) \cap D(g) \} \), \( fg = \{ (x, f(x)g(x)) : x \in D(f) \cap D(g) \} \). Let \( I \) be the identity of \( S(X) \), \( I = \{ (x, 1) : x \in D(I) \} \), and \( 0_H = \{ (x, 0) : x \in H \} \). Let \( \sigma \) be a homomorphism from \( S(X) \) into \( S(Y) \) and for \( f \in S(X) \), let \( H_f = \{ y \in D(\sigma(f)) : \sigma(I)(y) \neq 0 \} \). **Theorem.** If \( H_1 = \emptyset \), there is a mapping \( \alpha \) from the power set of \( X \) into the power set of \( Y \) such that for any \( f \in S(X) \), (i) \( H_f \subseteq H_1 \) (ii) \( \sigma(f)(y) = 0 \) for \( y \in D(\sigma(f)) \) \( H_f \) and \( \sigma(f)(y) = f \circ h(y) \) for \( y \in H_f \). **Definition.** A homomorphism \( \phi \) from a semiring \( A \) into a semiring \( B \) is a regular homomorphism if there exists no homomorphism \( \phi^* \) distinct from \( \phi \) such that \( \phi(a) + \phi(a) = \phi(a) + \phi^*(a) \) for all \( a \in A \). **Theorem.** A mapping \( \phi \) from \( S(X) \) into \( S(Y) \) is a regular isomorphism such that \( \phi(1) = 1 \) iff there is a continuous function \( h \) mapping \( Y \) onto \( X \) such that \( \phi(f) = \{ (y, f \circ h(y)) : y \in h^{-1}(D(f)) \} \). **Theorem.** \( \phi \) is an isomorphism onto \( S(Y) \) iff there is a homomorphism \( h \) from \( Y \) onto \( X \) such that \( \phi(f) = \{ (y, f \circ h(y)) : y \in h^{-1}(D(f)) \} \). (Received November 12, 1963.)


A set, \( H_k(G) \), of endomorphisms of a nonabelian group \( G \) which has an abelian member \( G_k \) in its lower central series was introduced by J. C. Howarth in (Proc. Glasgow Math. Assoc. (IV) IV (1960), 204-207) where he shows it to be a multiplicative semigroup of endomorphisms and, in case \( G \) is a finite p-group, he shows that it is a p-subgroup of the automorphism group of \( G \). Here, it is shown that: if \( G \) is residually nilpotent, then \( H_k(G) \) is included in the monomorphisms of \( G \); if \( G \) is periodic and locally nilpotent, then \( H_k(G) \) is included in the automorphism group of \( G \); if \( G \) is of finite exponent and locally nilpotent, then \( H_k(G) \) is a periodic subgroup of the automorphism group of \( G \); if \( G \) is a Černikov p-group, then \( H_k(G) \) is included in the automorphism group of \( G \) but, as is shown by an example, it may fail to be a subgroup. Two descriptions of \( H_k(G) \) are given. The first relates \( H_k(G) \) to the integral group ring of \( G_{k-1} \) and the second relates it to the crossed homomorphisms from \( G \) into \( G_k \). (Received November 1, 1963.)

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608-74. ALBERT MARDEN, Institute of Technology, University of Minnesota, Minneapolis 14, Minnesota. Weakly reproducing differentials on open Riemann surfaces.

If $\Omega$ is a bordered Riemann surface and $c$ is an arbitrary cycle (compact cycle or relative cycle) on $\Omega$, the weakly reproducing differential $\delta(c)$ for $c$ is uniquely determined in the class $R_0^*\alpha(\Omega)$ by the conditions $\langle \omega, \delta(c) \rangle^\ast = \int_\Omega \omega$ and $\int_\partial \delta(c) = c \times d$ (intersection number) for all $\omega$ in $R_0^\ast$ and all compact cycles $d$ in $\Omega$. Suppose $W$ is an open surface and $\gamma$ is an arbitrary cycle on $W$. For an exhaustion $\{ \Omega_n \}$ of $W$, let $\delta_n(\gamma \cap \Omega_n)$ denote the weakly reproducing differential for $\gamma \cap \Omega_n$ in $\Omega_n$. It follows that $\lim \delta_n(\gamma \cap \Omega_n) = \delta(\gamma)$ exists in $R_0^\ast$ if $\| \delta(c) \|$ is uniformly bounded. Otherwise set $\| \delta(\gamma) \| = \infty$.

Theorem 1. The extremal length of the class of cycles weakly homologous to $\gamma$ is $\| \delta(\gamma) \|^2$. Theorem 2. There exists a harmonic differential in $R_0$ with periods $a_i, b_i$ over a canonical homology basis $\{ A_i, B_j \}$ if and only if $\lim_{n \to \infty} \| \delta_n(\Sigma a_i B_i - b_i A_j) \|$ exists in $R_0^\ast$. Thus these theorems provide a connection between the theory of differentials and the geometry of $W$. (Received November 1, 1963.)


Ackermann (Math, Ann, 99, 118-133) considers a sequence of primitive recursive functions $f_n(x, y)$. Modifying his definition trivially, let $f_0(x, y) = x + 1$, $f_1(x, y) = x + y$, $f_2(x, y) = xy$. For $n \geq 2$, $f_{n+1}(x, y + 1) = f_n(x, f_{n+1}(x, y))$. Define $F_n$ to be the smallest class of functions including successor, zero, the identity functions and $f_n(x, y)$ which is closed under substitutions and limited recursion (primitive recursion in which the function introduced must be bounded by a function in the class). Grzegorczyk (Rozprawy Mat, 1953) defines classes $\mathcal{E}^n$ and asks "Can the same theorems be proved for the classes $F^n$ as for the classes $\mathcal{E}^n$?" where $\mathcal{E}^n$ is a trivial variant of $F_n$.

The answer is affirmative. The classes $F_n$, $F^n$, and $\mathcal{E}^n$ are equal for every $n$, and further if $f(x)$ is in $F_n$, then there is a $k$ such that $f(x) < f_{n+1}(x, k)$ for all $x \leq 2$ (cf. Grzegorczyk's Theorem 4.8). Further, using methods of Robinson (Bull. Amer. Math. Soc, 53, 925-942), the class of one argument functions in $\mathcal{E}^n$, for $n \geq 2$, is shown to be the smallest class containing successor, $[x^{1/2}], x - [x^{1/2}]^2$ and $f_n(x, x)$ which is closed under addition, multiplication, substitution and limited pure recursion (functions of one variable). (Received November 1, 1963.)

608-76. DUANE SATHER, Institute of Technology, University of Minnesota, Minneapolis 14, Minnesota. A maximum property of Cauchy's problem in $n$-dimensional space-time.

Let $Mu = Lu + Lu + cu$ where $Lu = \partial^2 u / \partial t^2 - Du$, $Du = g^{1/2} \sum_{i,j=1}^n \partial(g^{1/2} g^{ij} \partial_i u / \partial x^j) / \partial x^i$, $g = [\det(g^{ij})]^{-1}$ and $Lu = b \partial u / \partial t + \sum_{i=1}^n a^i \partial u / \partial x^i$. The matrix $g^{ij}$ is uniformly positive definite. The functions $g^{ij}, a^i, b$ and $c$ are independent of $t$. Let $K(Q; \delta)$ denote the Riemannian curvature with respect to the two-section $\delta$ of the metric defined by the inverse of $g^{ij}$. Suppose $K(Q; \delta) \leq k^2$ for every two-section $\delta$ and for every point $Q$. Let $n$ be even and $N = (n - 2)/2 \geq 0$. Theorem. Suppose that $\partial^k(Mu) / \partial t^k \leq 0$ and $\partial^k u(0, x) / \partial t^k = 0 (x \sim (x^1, x^2, \ldots, x^n))$ for $k = 0, 1, \ldots, N$ and $\partial^{N+1} u(0, x) / \partial t^{N+1} \leq 0$. Then there is a $T_0$ depending only on the operator $M$ such that $0 < T_0 \leq \epsilon - k^2$ and $u \leq 0$ when $0 \leq u \leq T_0$. Corollary. Let $B \geq 0$ be a constant such that $\partial^k(Mu - Mu_0) / \partial t^k \leq B t^{N-k}/(N-k)!$ for $k = 0, 1, \ldots, N$ where $u_0 = \sum_{k=0}^{N+1} k^k \partial t^k (t^k / k!)$. Then for $T_0$ as above $u \leq u_0 + B t^{N+2}/(N+2)!$ when
0 \leq t \leq T_0. By descent these results are valid for any \( n \geq 2 \) if only \( N \geq (n - 2)/2 \). (Received October 28, 1963.)

608-77. WITHDRAWN.

608-78. PAWEL SZEPTYCKI, University of Kansas, Lawrence, Kansas. On restrictions of functions of the spaces \( P^{a,p} \) and \( B^{a,p} \).

\( P^{a,p}(\mathbb{R}^n) \) with \( a > 0 \) and \( 1 \leq p \leq \infty \) denotes the space of all Bessel potentials of functions in \( L^p(\mathbb{R}^n) \), i.e., functions of the form \( u(x) = \int G_a(x-y)f(y)dy = G_a f(x) \), \( f \in L^p(\mathbb{R}^n) \), defined wherever the integral exists and is finite. The exceptional class \( \mathcal{A}_{a,p} \) for \( P^{a,p} \) consists of all sets \( A \) such that \( A \subset \{ x : G_a f(x) = + \infty \} \) for some \( f \in L^p(\mathbb{R}^n) \). The norm \( \| \cdot \|_{a,p} \) in \( P^{a,p} \) is defined by

\[ \| f \|_{a,p} = \| f \|_{L^p} + \int \| G_a^a u \|_{L^p} |t|^{-n-\alpha} dt, \quad 0 < a < 1, \quad 1 \leq p < \infty. \]

\( \mathcal{A}_{a,p} \) denotes the exceptional class for \( B^{a,p} \). (For details cf. Aronszajn, Mulla, Szeptycki, On potentials connected with \( L^p \) spaces, to appear in Ann. L'Inst. Fourier.)

Let \( n = n' + n'' \), \( n', n'' \) positive integers, and denote

\[ x' = (x_1, \ldots, x_{n'}), \quad x'' = (x_{n'+1}, \ldots, x_{n''}), \quad x = (x', x'') \]

Then for \( a, p \in (0, \infty) \), \( a < p < \infty \).

Theorem. Let \( A \in \mathcal{A}_{a,p} \) then \( A \subset \mathcal{A}_{a',p} \) for \( 0 \leq a' < a \) if also \( A \subset \mathcal{A}_{a,p} \) for \( 0 < a < \infty \) and \( A \subset \mathcal{A}_{a,p} \) for \( 0 \leq a' < a \). If \( u \in B^{a,p}(\mathbb{R}^n) \) then \( u \subset B^{a',p}(\mathbb{R}^n) \) for \( 0 \leq a' \leq a \). (Received November 4, 1963.)

608-79. J. R. TELLER, Tulane University, New Orleans, Louisiana 70118. On the extensions of lattice-ordered groups.

Let \( A, \Delta, G \) be partially ordered abelian groups. If \( G \) is a p.o., extension of \( A \) by \( \Delta \) then for \( a \in A + \Delta \), let \( Q_a = \{ a \in A | r(a) + a \geq 0 \} \) where \( r(a) \) is the representative of the coset determined by \( a \). Then for \( a, \beta \in A + \Delta \), \( Q_a \cap Q_\beta = A + \beta \) and \( Q_a + Q_\beta + f(a-\beta, \beta) \leq Q_{a+\beta} \). If \( A \) and \( \Delta \) are lattice-ordered groups (\( \ell \)-groups) then \( G \) is an \( \ell \)-extension iff (1) for \( a, \beta \in A + \Delta \), \( Q_a \cap [Q_\beta + b + f(a-\beta, \beta)] \) has a smallest element for all \( b \in A \) and (2) for \( a, \beta \in A + \Delta \), \( Q_a + Q_\beta + f(a-\beta, \beta) \leq Q_{a+\beta} \). If \( G \) is an \( \ell \)-extension of \( A \) by \( \Delta \) where \( A \) is a lexicographic extension of an \( \ell \)-ideal \( B \) then, (i) \( G \) contains an \( \ell \)-ideal \( H \) \( \cong A + \Delta \), (ii) \( \Delta/J \) is an \( \ell \)-group, (iii) \( G \) is an \( \ell \)-extension of \( H \) by \( \Delta/J \). If \( A \) is an \( \ell \)-group with a finite basis then every \( \ell \)-extension of \( A \) is determined by a meet preserving homomorphism of \( \Delta^+ \) to the semi-group of cardinal summands of \( \Delta \) such that \( f(a, \beta) \in H_{a+\beta} \). (Received November 4, 1963.)


Let \( X \) denote a function on the interval \([a, b]\) of reals into an interval \( I \) in \( k \)-dimensional space \( \mathbb{R}^k \), which is nondecreasing and continuous from the right (each component). Let \( G \) be a real-valued function of bounded variation on \([a, b]_k \), continuous from the left, with \( G(a) = 0 \). A function from \( \mathbb{R}^k \).
into the reals will be said to have nondecreasing increments if \( a \in \mathbb{R}^k, b \in \mathbb{R}^k, h \in \mathbb{R}^k, h \geq 0, a \leq b \) imply \( f(a + h) - f(a) \leq f(b + h) - f(b) \). **Theorem.** 

Necessary and sufficient conditions that 
\[
\int_{[a,\beta]} f(x) dG(x) \geq f[\int_{[a,\beta]} x dG(x)] \text{ for every continuous function } f \text{ on } I \text{ with nondecreasing increments are: (1) } G(\beta) = 1; \text{ (2) } \int_{[a,t]} G dX \geq 0 \text{ for every left interval } [a,t] \subset [a,\beta], \text{ and } 
\]
\[
\int_{[t,\beta]} [1 - G] dX \geq 0 \text{ for every right interval } [t,\beta] \subset [a,\beta]. \text{ This theorem extends, under less restrictive conditions, one and two dimensional theorems of Ciesielski (Annales Polonici Mathematici 4 (1957-1958), 269-274), and extends also one dimensional theorems of Brunk (Proc. Amer. Math. Soc. 7 (1956), 817-824) and of Olkin (Proc. Nat. Acad. Sci. 45 (1959), 230-231). (Received November 4, 1963.)}

608-81. T. S. CHIHARA, Seattle University, Seattle 22, Washington. **On recursively defined orthogonal polynomials.**

Continuing an earlier study [Trans. Amer. Math. Soc. 104 (1962), 1-16], the chain sequences of Wall are used to study the orthogonal polynomials defined by the classical recursion
\[
(1) \ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_nP_{n-1}(x), \ P_0(x) = 0, \ c_n \text{ real, } \lambda_{n+1} > 0 \ (n = 1,2,\ldots). 
\]
Primary attention here is directed towards the case where the support of the distribution, \( d\psi(x) \), with respect to which the \( P_n(x) \) are orthogonal is a denumerable set, bounded below and having \( \infty \) as its only accumulation point. Assuming that the associated Hamburger moment problem (HMP) is determined, it is shown, for example, that the support of \( d\psi(x) \) will have the above property if
\[
\lim_{n \to \infty} (c_n c_{n+1} - 4\lambda_{n+1})/(c_n + c_{n+1}) = \infty. \] 

The question of the determinacy of the HMP as decidable on the basis of (1) is also briefly considered. For example, it is shown that the associated HMP will be determined if \( c_n > 0, \ \{\lambda_{n+1}/(c_n c_{n+1})\} \) is a chain sequence and \( \sum n^2/c_n \) diverges. (Received November 4, 1963.)

608-82. D. T. HAIMO, 7201 Cornell Avenue, St. Louis 30, Missouri. **Inversion for Hankel convolutions.**

Let \( \gamma \) be a fixed positive number and let \( \mu(x) = x^{2\gamma+1/2} / \Gamma(\gamma+3/2) \). Define \( \mathcal{F} = 2^{\gamma - 1/2} / \Gamma((\gamma + 1)/2) \) \( J_{(\gamma - 1)/2}(x) \) is the ordinary Bessel function of order \( \gamma - 1/2 \). Let \( G(x,y) = \int_0^\infty \mathcal{F}(\xi) \gamma(\xi)(\xi^2/a_k) d\mu(\xi), 0 < a_1 \leq a_2 \leq \ldots, \) and \( \sum_{k=1}^\infty a_k = \infty. \) Denote by \( \Delta_x \) the differential operator \( \Delta_x h(x) = h^{(n)}(x) + [2\gamma/x] h'(x). \) **Convergence theorem.** Let \( \phi \) be a function integrable in any finite interval, and let \( \int_0^\infty G(x_0,y) \phi(y) d\mu(y) = \lim_{x \to \infty} \int_0^T G(x_0,y) \phi(y) d\mu(y), x_0 > 0, \) converges conditionally. Then \( \int_0^\infty G(x,y) \phi(y) d\mu(y) \) converges conditionally for all \( x > 0 \) and uniformly for \( x \) in any finite interval. **Inversion theorem.** Let \( \phi \) be a function integrable in every finite interval and let \( f(x) = \int_0^\infty G(x,t) \phi(t) d\mu(t), 0 < x < \infty, \) converges conditionally. Then \( \lim_{N \to \infty} \int_{x=0}^N (1 - \Delta_x/a_k^2) f(x) = \phi(x), \) if \( \lim_{h \to 0} 1/h \int_x^{x+h} \phi(t) - \phi(x) d\mu(t) = 0, \) a condition which holds almost everywhere. The behavior of the kernels \( G(x,y) \) plays a central role in the development of the inversion theory. (Received November 4, 1963.)
Mixed problems for the wave equation in a time dependent domain. Preliminary report.

Consider a semi-infinite string, the position of one of its ends being a prescribed function \( \beta(t) \) of time. Problem. To find a solution of the nonhomogeneous wave equation in the region \( x > \beta(t) \) that satisfies zero initial data and vanishes for \( x = \beta(t) \). (A similar situation is investigated by N. L. Balazs, J. Math. Analysis and Applications 3 (1961), 472-484.) One generalization of this problem is to seek a solution of the nonhomogeneous wave equation in the region exterior to a smooth convex body undergoing a prescribed rigid motion \( \hat{\beta}(t) \), satisfying zero initial data, and vanishing on the boundary. Results of Lions (Equations différentielles opérationnelles et problèmes aux limites, Springer, Berlin, 1961) are used to prove the following theorems. If the data are sufficiently regular and bounded uniformly away from the wave velocity, then there exists a unique classical solution to the above problem. If the data are \( C^0 \) functions, then so is the solution. \( L^2 \) bounds on the solution are also obtained. (Received November 4, 1963.)

Tame 2-complexes in high-dimensional manifolds.

The following results hold for imbeddings in a combinatorial manifold \( M^n \) of dimension \( n \geq 6 \). In Theorems 1 and 2, \( K \) is a compact 2-complex topologically imbedded in \( M^n \). Theorem 1. If \( K \) is locally tame except possibly at countably many points, then \( K \) is tame in \( M^n \). Theorem 2. \( K \) is tame in \( M^n \) if and only if every simplex of \( K \) is tame. Corollary 1. If the closed 2-manifold \( P \) in \( M^n \) is locally flat except possibly at countably many points, then \( P \) is tame. Corollary 2. If the 2-sphere \( S \) is the union of two tame disks intersecting in their common boundary, then \( S \) is tame in \( M^n \). Using the Homma-Gluck-Greathouse results, Theorems 1 and 2 reduce to the following two lemmas:

Lemma 1. If the disk \( D \) in \( M^n \) is locally polyhedral except at a single interior point, then \( D \) is flat.

Lemma 2. If the disk \( D \) in \( M^n \) is locally polyhedral modulo its boundary \( J \), with \( J \) being a simple closed polygon, then \( D \) is flat. (Received November 4, 1963.)

The genus of the n-cube.

By the \( n \)-cube, \( Q_n \), the graph of \( 2^n \) points is meant, each a binary \( n \)-place number, in which two points are adjacent whenever they differ in exactly one place. As usual, the genus \( \gamma(G) \) of a graph \( G \) is the minimum genus of any orientable surface in which \( G \) is embeddable. Theorem. \( \gamma(Q_n) = (n - 4)2^{n-3} + 1 \). That this number is a lower bound for the genus of \( Q_n \) follows easily from Euler's polyhedron formula. The proof that this is also an upper bound is based on a construction which shows that \( \gamma(Q_n) \leq 2\gamma(Q_{n-1}) + 2^{n-3} - 1 \). (Received November 4, 1963.)

On the thickness of the complete graph.

The thickness \( t(G) \) of a graph \( G \) is the minimum number of planar subgraphs whose union is \( G \). The complete graph \( K_p \) with \( p \) points has all \( p(p - 1)/2 \) lines. It is well known that \( t(K_p) = 2 \) when
It has recently been verified (Battle, Harary and Kodama, Bull. Amer. Math. Soc. 69 (1962), 569-571) that $t(K_9) = 3$. It is straightforward to verify that $t(K_p) = \frac{(p + 7)}{6}$ also for $p = 1, 2, 5 \mod 6$. The proof follows from a combination of Euler's classical polyhedron formula and an appropriate construction. Euler's formula implies that $t(K_n) \geq \frac{(p + 7)}{6}$ for every $n$. A construction is devised which shows that $t(K_{6n+2}) \leq n + 1$. A confrontation of these two inequalities establishes the theorem.

For $p = 3$ or $4 \mod 6$ the problem remains unsolved. (Received November 4, 1963.)

608-87. R. B. KELLOGG, Bettis Atomic Power Laboratory, Westinghouse Electric Corporation, P.O. Box 1468, Pittsburgh 30, Pennsylvania. **Alternating direction methods for operator equations.**

On a Hilbert space $H$ consider closed operators $A_j$ with dense domains $D_j$ which satisfy

$$\text{Re}(A_j u, u) \geq 0, \quad u \in D_j, \quad \text{Re}(A_j^* u, u) \geq 0, \quad u \in D_j^* = \text{domain of } A_j^*, \quad j = 1, 2.$$ If $f \in H$, $u^0 \in D_2$, $\rho > 0$, the alternating direction iterative scheme $u^{n+1/2} = (\rho I + A_1) u^n + f$, $u^{n+1} = (\rho I + A_2)^{-1} [u^{n+1/2} + f]$, $n = 0, 1, \ldots$, has meaning and $u^n \subset D_2$, $u^{n+1/2} \subset D_1$. Let $u \in D_1 \cap D_2$, and let $(A_1 + A_2) u = f$. **Theorem 1.** Suppose $v \in D_1^* \cap D_2^*$, $(A_1^* + A_2^*) v = 0$ implies $v = 0$. Then the Cesaro sum of each of the sequences \{${u^n}$\}, \{${u^{n+1/2}}$\} converges to $u$. **Theorem 2.** Suppose $A_j$ is Hermitian and has spectrum contained in $[u, \infty)$, $u > 0$, $j = 1, 2$. If $0 < \rho < u$, each of the sequences \{${u^n}$\}, \{${u^{n+1/2}}$\} converges to $u$. These theorems apply to the equation $- (pux, x) - (quy, y) = f$ to give "continuous analogues" of some results of W. H. Guilinger, Jr., and to the one group neutron transport equation in two independent variables to give a new iterative method for solving this equation. Further details are in the report WAPD-TM-391, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania. (Received November 4, 1963.)

608-88. ARNE PERSSON, University of Kansas, Lawrence, Kansas. **Summation methods on locally compact spaces.**

A unified summability theory is developed by considering integral transformations $(Mf)(t) = \int f(x) d\mu(t, x)$ mapping locally bounded Baire functions $f$ on a locally compact space $X$ onto functions $Mf$ on another locally compact space $T$. $\int f(x) d\mu(t, x)$ is defined as $\lim \int_{K_a} f(x) d\mu(t, x)$, where $(K_a)_{a \in A}$ is a given fundamental system of neighborhoods of $\infty$. In particular a wide class of such transformations, called uniform, is considered. It is shown that the field of a uniform method is a Fréchet space and many known theorems for summability of sequences are generalized and sharpened.

For instance, a regular uniform method is perfect for each $f$ in the field for which $\sup_{a \in A} \sup_{t \in T} \left| \int_{K_a} f(x) d\mu(t, x) \right| < \infty$, in particular for each bounded function. The classical bounded consistency theorem is also shown to hold true. The Abel-Laplace transformation \( \int_0^\infty e^{-x/t} f(x) dx \) and the Riesz mean \( (a/t)^{T} \int_0^T (t - x)^{a-1} f(x) dx \) of order $a > 0$ are examples on uniform methods. They are both seen to be perfect. (Received November 4, 1963.)
For every locally compact group $G$, define an algebra $\mathcal{A}(G)$, called the big group algebra. (Due to space limitations, the definition cannot be given here.) $\mathcal{A}(G)$ is an abstract $C^*$-algebra in which analogues of the operator topologies (weak, $\sigma$-weak, strong and $\sigma$-strong) are defined. The traditional group algebras, $L^1(G)$, $C^*(G)$ and $M(G)$ are canonically embedded, one-to-one and norm decreasing, in $\mathcal{A}(G)$. These group algebras are weakly dense in $\mathcal{A}(G)$. The group $G$ appears canonically and isomorphically as a group of unitary elements in $\mathcal{A}(G)$, the topology of $G$ being given by the weak topology of $\mathcal{A}(G)$. The group $G$ generates $\mathcal{A}(G)$ in the sense that linear combinations of group elements are weakly dense in $\mathcal{A}(G)$. $\mathcal{A}(G)$ has an isomorphic (and $\sigma$-weak homeomorphic) representation as a von Neumann algebra. As a Banach space, $\mathcal{A}(G)$ is isomorphic to the second conjugate space of $C^*(G)$.

The separable strongly continuous unitary representation theory of $G$ is completely equivalent to the separable normal $*$-representation theory of $\mathcal{A}(G)$, in the sense that every such unitary representation of $G$ has a unique extension to a normal $*$-representation of $\mathcal{A}(G)$. This correspondence preserves all the usual representation theoretic properties. (Received November 4, 1963.)

Let $A_L$ be the automorphism group of a finite dimensional semi-simple Lie algebra $L_\mathbb{C}$ over the complex field $\mathbb{C}$. Let $A$ be the automorphism group of the corresponding classical Lie algebra $L$ over an arbitrary field $K$ of characteristic $\not\equiv 2,3$. $A_L$ has a decomposition into connected (algebraic) components: $A_1 \cup \ldots \cup A_r = \sigma_1 A_1 \cup \sigma_2 A_1 \cup \ldots \cup \sigma_r A_1$, where $A_1$ is the component of 1 and the $\sigma_i$ are graph automorphisms ($\sigma_1 = 1$). $A$ has a corresponding decomposition $\sigma_1 G \cup \sigma_2 G \cup \ldots \cup \sigma_r G$, where $G$ is the Chevalley group of $L$ (Steinberg, Pacific J. Math. 11 (1961), 1119-1129). Define $n(A_i)$ to be the minimal multiplicity of 1 as characteristic root and $m(A_i)$ the minimal dimension of fixed point spaces for elements of $A_i$. Similarly define $n(\sigma_i G)$, $m(\sigma_i G)$. Then $n(A_i) = m(A_i)$, and these numbers can be determined (Jacobson, Pacific J. Math, 12 (1962), 303-315). Theorem, $n(\sigma_i G) \geq m(\sigma_i G) \geq n(A_i)$, $1 \leq i \leq r$. If $K$ is infinite, $m(\sigma_i G) = m(A_i)$, and under a mild additional restriction $n(\sigma_i G) = n(A_i)$. (Received November 4, 1963.)

Consider the system of nonlinear difference equations $(\ast)y(x + 1) = f(x, y(x))$, where $x$ is a complex variable, $y$ is an $n$-dimensional vector, and $f$ is an $n$-dimensional vector with components holomorphic in the region $|x| \geq R$, $\|y\| \leq \delta_0$. If $f(\infty, 0) = 0$ and one is not an eigenvector of the matrix $f_y(\infty, 0)$, there exist a unique formal solution of $(\ast)$, $y = \sum_{m=1}^{\infty} x^{-m} p_m$. Laplace transform techniques are used to prove the existence of analytic solutions with the formal solution as asymptotic representation in sectors of the $x$-plane. These sectors, in general, form a covering of a neighborhood of infinity. Further, these solutions have a convergent inverse factorial series representation. (Received November 4, 1963.)
608-92. T. L. HAYDEN, University of Kentucky, Lexington, Kentucky and E. P. MERKES, University of Cincinnati, Cincinnati, Ohio 45221. Chain sequences and univalence.

Let $A = \{A_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that $A_{mp} = 0$ for $p > 0$ whenever $A_m = 0$. For a fixed positive integer $a$, let $\pi(A; a)$ denote the class of formal power series $f(z)$ for which $zf'(z)/f(z) - 1 - a_1 z^a/1 - a_2 z^{a^2}/1 - \cdots$, where $|a_n| \leq A_n$, $n = 1, 2, \ldots$. The radius of univalence and the radius of starlikeness of the class $\pi(A; a)$ are equal to the sup $r$ for which $\{A_r^m\}$ is a chain sequence. In particular, it is shown that the radius of starlikeness $\rho_v$ of $F_v(z) = z^{1-v} J_v(z)$, where $J_v(z)$ is a Bessel function of order $v$, $x = \Re v > 1$, is not less than the smallest positive zero of $F_v^m(z)$ and asymptotic equality as $v \to \infty$ is obtained. (Received November 4, 1963.)

608-93. AMNON JAKIMOVSKI, Tel-Aviv University, Tel-Aviv, Israel and M.S. RAMANUJAN, University of Michigan, Ann Arbor, Michigan. Theorem on uniform approximation and its applications.

The following generalized version of a theorem of Bohman [Ark. Mat. 2 (1952), 43-56] is proved. For $-\infty < a < b < \infty$ and for $0 \leq n \leq n(m)$, let $a_{mn}$ be defined with $c \leq a_m < a_m + 1 < \ldots a_m, n(m) \leq d$, where $[a, b] \subset [c, d]$. Let the functions $c_{mn}(x)$ be defined for $0 \leq n \leq n(m)$ and be finite for $x \in [a, b]$. Suppose that for some $n_0$, $c_{mn}(x) \geq 0$ for $n \leq n_0 \leq n(m)$ and $x \in [a, b]$. Assume also that the following conditions hold, uniformly in $x$, for $a \leq x \leq b$: (i) $\lim_{m \to \infty} c_{mn}(x) = 0$, $n = 0, 1, \ldots, n_0 - 1$; (ii) $\lim_{m \to \infty} \sum_{n=0}^{n(m)} c_{mn}(x) = 1$; (iii) $\lim_{m \to \infty} \sum_{n=0}^{n(m)} c_{mn}(x) x^{r-1} = 0$, $r = 1, 2$. Then for $0 \leq x \leq b$, the function $f(x)$, which is bounded in $[c, d]$ and is continuous in $[a, b]$, is uniformly approximated by the sequence $C_f(m) = \sum_{n=0}^{n(m)} c_{mn}(x) f(a_m)$. This theorem yields the following theorem of the Bernstein type. Let $a$ be real and let $\beta = \inf_{m,n} \{(1-a)/[(m+n)+a]\}$, $(n \leq m)$. Let $B_m(a,f,x) = \sum_{n=0}^{m} C_{m+n-a} ((1-x)^{m+n-a} f([n-a]/(m+n)+a))$. Then if $f(x)$ is continuous in $0 < a \leq x \leq b \leq 1$ and is bounded in $[\beta, 1]$, then $B_m(a,f,x)$ converges uniformly to $f(x)$ for $x \in [a, b]$. This last result is applied to solve moment problems of the type $\mu_n = \int_0^1 x^{n+a} f(x)dx$, $x \in BV[0, 1]$ or $\mu_n = \int_0^1 x^{n+a} f(x)dx$ where $f$ belongs to a suitable function space. (Received November 4, 1963.)


This paper is a follow-up of a preliminary report and presents three new integral relations which were obtained by using the notion of simultaneous separability in solving the Schrödinger equation, $V^2 u + \phi u = 0$ [Abstract 589-33, Amer. Math. Soc. Notices, vol. 9, no. 2]. The three new integrals relate functions of the hypergeometric class. The integrands are composed of products of two or three Whittaker functions. (Received November 4, 1963.)

608-95. J. W. MOELLER, Case Institute of Technology, Cleveland 6, Ohio. A norm bound for the inverse shift operator.

Let $H$ denote the Hilbert space of all analytic functions $h(z) = \sum_{n=0}^{\infty} h_n z^n$ which satisfy the property that $\sum_{n=0}^{\infty} |h_n|^2 < \infty$. Let $S: H \to H$ be the one-sided shift operator defined by $(Sh)(z) = z^{-1}(h(z) - h(0))$. A closed subspace $L \subseteq H$ is said to be left translation invariant if $S(L) \subseteq L$. According to a well-known theorem of Beurling, the orthogonal complement of $L$ has the form $G(z)H$, 83
where $G(z)$ is a bounded analytic function defined on the interior of the unit disc which is uniquely determined up to multiplication by constants of modulus one. Call $G(z)$ the characteristic function for the space $L$. **Theorem.** Let $G(z)$ be the characteristic function for the left translation invariant space $L$, and let $T$ denote the restriction of $S$ to $L$. If $G(0) \neq 0$, then $\|T^{-1}\|^2 \leq 1 + |G(0)|^2$. This inequality has applications to prediction theory. (Received November 4, 1963.)

608-96. F. C. HUCKEMANN, University of Tennessee, Knoxville, Tennessee. **On extremal decompositions of a quadrilateral.**

**Notation.** $Q$ is the rectangle $0 < R_z < M$, $0 < I_z < 1$, where $M > 0$. $\Gamma$ is the boundary, $\overline{Q}$ is the closure of $Q$. $a_1$ and $a_2$ are the subsets of $\Gamma$ on $R_z = 0$ and $R_z = M$ respectively. $M$, incidentally, is the modulus of $Q$ with respect to the sides $a_1$, $a_2$. $\beta'$ and $\beta''$ are the subsets of $\Gamma - a_1 - a_2$ on $I_z = 0$ and $I_z = 1$ respectively. $z^*$ is a fixed point of $\beta''$. A decomposing set $C$ is a closed connected subset of $Q$ with (i) $C \cap (a_1 + a_2) = \emptyset$, (ii) $z^*_C \subseteq C$, (iii) $C \cap \beta' \neq \emptyset$. For $i = 1, 2$: $Q_i$ is the component of $Q - CQ$ which has $a_i$ as a boundary segment, $C_i$ is the set of points common to $C$ and the boundary of $Q_i$, $M_i$ is the modulus of $Q_i$ with respect to the sides $a_i$, $C_i$. **Problem 1.** Given $\delta \in (0,1)$, select $C$ such that $M_1/M_2 = \delta/(1 - \delta)$ and that $M_1$ as well as $M_2$ are maximal. **Problem 2.** Given $\eta \in (0,1)$, select $C$ such that $\eta M_1 + (1 - \eta)M_2$ is maximal. It is shown that both problems have a unique solution which is explicitly given. (Received November 4, 1963.)

608-97. D. R. TRAYLOR, University of Houston, Houston, Texas. **Some cardinality conditions for metrization of locally compact, normal Moore spaces.**

For each cardinal number $\kappa$, denote by $\Sigma(\kappa)$ the collection to which the normal locally compact Moore space belongs if and only if each discrete collection $G$ of mutually exclusive closed compact point sets such that $\mathcal{G} < \kappa$ is screenable in $S$. A space is pointwise $\kappa$-paracompact if for each open covering $H$ of $S$ there is a refinement $V$ of $H$ such that if $x$ is a point of $S$ and $U$ is the collection consisting of those domains of $V$ which contain $x$, then $\mathcal{U} < \kappa$. **Theorem.** If $S$ is in $\Sigma(\kappa)$ then $S$ is metrizable if $S$ satisfies one of the conditions: (1) if $H$ is a collection of mutually exclusive domains in $S$ then $\mathcal{H} < \kappa$, and the generalized continuum hypothesis is true, (2) if $H$ is a collection of mutually exclusive domains in $S$ such that $\mathcal{H} \geq \kappa$, then $B = (\mathcal{H}^* - H^*)$ is such that $\mathcal{B} < \kappa$, (3) if $R$ is a region and $H$ is a collection of domains, each of which intersects $R$, then there is a subcollection $V$ of $H$ such that $\mathcal{V} < \kappa$ and $V^*$ contains $H^*$, (4) there is a cardinal number $\alpha < \kappa$ such that $S$ is pointwise $\alpha$-paracompact. (Received November 5, 1963.)

608-98. SEYMOUR LIPSCHUTZ, Polytechnic Institute of Brooklyn, 333 Jay Street, Brooklyn 1, New York. **The root problem in one-eighth groups.**

Greendlinger solved the conjugacy problem for the set $\mathcal{G}$ of one-eighth groups (Comm. Pure Appl. Math. 13 (1960), 641–677). Let $W$ be an arbitrary element in a group $G$ in $\mathcal{G}$. It is shown that there can be at most a finite number of elements $V_1, \ldots, V_m$ such that $W = V_i^{n_i}$. Furthermore, an algorithm is given which finds such elements. (Received November 5, 1963.)
608-99. W. KUYK, Ottawa University, Ottawa, Ontario, Canada. An algebraic application of the wreath product.

Let A and B be finite groups, A satisfying the condition: there exists an isomorphic representation of A as a transitive permutation group A_\times on a set X of indeterminates together with a field k such that the subfield k_\text{A} of k(x) of all invariants under A_\times is purely transcendental with respect to k.

The following result is obtained: Theorem. Let L/K and M/L be finite galoisian field extensions with Galois groups B and A respectively. Let G denote the Galois group of the field M/K, and let S(B,A) denote the set of all groups G that may occur if extensions M/K of this type are considered. Then for any G \in S(B,A) there exists an injection g into the wreath product A \circ B of the groups A and B. A \circ B is an element of S(B,A).

(Received November 6, 1963.)

608-100. JOHANN MARTINEK and H. P. THIELMAN, United Electrodynamics, Inc., 300 North Washington Street, Alexandria, Virginia. Circle theorem for the biharmonic equation. (Interior problem.)

Theorem. Let w_0(r,\theta) be defined in the plane \mathbb{D}, except possibly in some subset D (not including the origin) inside the circle r = a, and satisfy the biharmonic equation \Delta^2 w_0(r,\theta) = 0 in \mathbb{D} - D.

Let w(r,\theta) have the same singularities as w_0(r,\theta) and satisfy the biharmonic equation and the boundary conditions (\Delta w)_r + ((1 - \sigma)/a^2) [w_{\theta\theta r} - (1/a)w_{\theta r}] = 0, \sigma w + (1 - \sigma)w_{rr} = 0 on r = a, where \sigma is a constant. Then w(r,\theta) can be constructed uniquely from w_0(r,\theta). (An explicit formula for w(r,\theta) is given.)

(Received November 6, 1963.)

608-101. MISHAEL ZEDEK, University of Maryland, College Park, Maryland. Continuity and location of zeros of linear combinations of polynomials.

The following theorem about the continuous dependence of the zeros of a polynomial on its coefficients is proved: Theorem 1. Given a polynomial P_n(z) = \sum_{k=0}^{n} a_k z^k, an integer m \geq n and a number \epsilon > 0, there exists a number \delta > 0 such that whenever the m + 1 complex numbers b_k, 0 \leq k \leq m, satisfy the inequalities |b_k - a_k| < \delta for 0 \leq k \leq n and, if m > n, |b_k| < \delta for n + 1 \leq k \leq m, then the zeros \beta_k, 1 \leq k \leq m of the polynomial q_m(z) = \sum_{k=0}^{m} b_k z^k can be labeled in such a way so as to satisfy with respect to the zeros \alpha_k, 1 \leq k \leq n, of P_n(z) the inequalities |

\beta_k - \alpha_k| < \epsilon for 1 \leq k \leq n and, if m > n, |\beta_k| > 1/\epsilon for n + 1 \leq k \leq m. This theorem is used to prove Theorem 2.

Let f_m(z) = z^m + ... + a_0 and g_m(z) = z^n + ... + b_0 be two polynomials whose zeros lie, respectively, in the disks D(c_1,R_1) and D(c_2,R_2) (center c_1, radius R_1) and suppose m > n \geq 1. For a fixed \lambda let F(z,\lambda) = f_m(z) + \lambda g_n(z). I. The m zeros of F(z,\lambda) lie in D(c_1,R_1 + \rho_1), where \rho_1 is the unique positive root of the equation x^m - |\lambda|(x + |c_2 - c_1| + R_1 + R_2)^n = 0. II. Setting L = m^m n^{-n} (m - n)^{-n - m} \cdot (|c_2 - c_1| + R_1 + R_2)^m, if |\lambda| \geq L at least n zeros of F(z,\lambda) lie in the disk D(c_2,R_2 + \rho_2), where \rho_2, \rho_3 (\rho_2 \leq \rho_3) are the two positive roots of the equation |\lambda|x^n - (x + |c_2 - c_1| + R_1 + R_2)^n = 0. This result is extended to linear combinations of a finite number of polynomials. (Received November 6, 1963.)
608-102. J. D. BUCKHOLTZ, University of North Carolina, Chapel Hill, North Carolina.

Power series whose sections have zeros of large modulus.

Given a power series $\sum_{n=0}^{\infty} a_n z^n$ for which $a_0 \neq 0$, for each positive integer $n$ let $r_n$ denote the least modulus of a zero of $\sum_{n=0}^{\infty} a_n z^n$. It follows easily from Hurwitz's theorem that $r_n \to \infty$ if and only if $\sum_{n=0}^{\infty} a_n z^n$ is an entire function which has no zero. Various properties of the sequence $\{r_n\}$ are studied under the assumption that $\sum_{n=0}^{\infty} a_n z^n = \exp\{g(z)\}$, where $g(z)$ is an entire function. It is shown that the order of $g(z)$ is equal to the limit superior of $(\log \log n)/\log r_n$; for $g(z)$ of positive finite order, a similar result is obtained for the type of $g(z)$. An asymptotic formula for $\log r_n$ (in terms of the maximum modulus of $g(z)$) is obtained under the hypothesis that $g(z)$ is of finite order; somewhat stronger conditions on $g(z)$ yield an asymptotic formula for $r_n$. The following theorem gives a relation between $\{r_n\}$ and the coefficient sequence $\{a_n\}$. Theorem. If $\sum_{n=0}^{\infty} a_n z^n$ is of infinite order and has no zero, then $\lim \sup r_n |a_n|^{1/n} = 1$. (Received November 6, 1963.)

608-103. E. O. THORPE, New Mexico State University, University Park, New Mexico and W. E. WALDEN, Los Alamos Scientific Laboratory, Los Alamos, New Mexico. A winning bet in Nevada baccarat.

A winning strategy is developed for the nine to one side bet on a Banker natural nine. Let $n$ be the number of cards that remain for play. Let $t$ be the number of nines that remain for play. Let $p(n,t)$ be the probability of a natural nine when $n$ and $t$ are given. $p(n,t)$ is greater than 0.1 frequently enough to make the counting of $n$ and $t$ the basis for a winning strategy. The Kelly Gambling System is used to determine betting amounts. The strategy was tested both on a computer and in actual play in the casinos. Similar strategies are developed for the side bets on Banker natural eight, Player natural nine, and Player natural eight. The relationships between the four side bets are analyzed. Capital doubles every 25 playing hours under the conditions observed in practice. The long-standing question of whether Baccarat, as it is generally played throughout the world, admits favorable strategies based on card-counting techniques, is answered. It is shown that the main (roughly 1:1) bets on Banker and Player occasionally favor the player. There are theoretical favorable strategies but none exists which is currently practical. (Received November 6, 1963.)


In (Monatshefte für Math. u Physik, pp. 1-19), Arens showed that the second conjugate space of a Banach algebra can be endowed with a multiplication so that it is an algebraic (as well as topological) extension of the original Banach algebra. The result is extended in the following way. Let $E$ be a locally multiplicatively-convex topological algebra (abbreviated "lmc algebra"). Let $E^{**}$ be the second conjugate space of $E$, with the topology of uniform convergence on bounded sets of $E^*$. Theorem 1. If $E$ is an lmc algebra, then one can endow $E^{**}$ with an Arens multiplication which makes $E^{**}$ an algebraic extension of $E$ and which makes $E^{**}$ itself into an associative algebra.

Theorem 2. $E^{**}$ is an lmc algebra if and only if $E$ is an lmc algebra under the topology of uniform convergence on bounded sets of $E^*$. Various other results pertaining to $E^{**}$ are also obtained. (Received November 6, 1963.)
K. W. WESTON, University of Notre Dame, Notre Dame, Indiana. 

ZA groups satisfying an Engel condition.

Denote the upper central series of a group $G$ by $1 = Z_0 < Z_2 < \ldots < Z_n < Z_{n+1} = G$. If $(Z_{n+q}, G) \leq Z_a$ for $a + q \leq n$ then $G$ is said to be a ZA(q) group. It is easily shown that ZA(1) contains all ZA groups whose class is not a limit ordinal and that ZA(1) > ZA(2) > ZA(3). There are metabelian ZA(3) groups of exponent 4 which satisfy the third Engel condition but are not nilpotent.

However every ZA(3) group which satisfies the Engel condition of class $m$ is periodic modulo $Z_k$ for $k = \prod_{i=0}^{m-2} (c_{m-1, m-2})^{i}$, and the periods divide powers of $k$. Therefore if a periodic ZA(3) group satisfies the Engel condition of class $m$ and if the prime divisors of the periods are larger than those of $m$, $G$ must be nilpotent. The proof of the above theorem associates to every ZA(3) group $G$ an associative ring $\Gamma$. If $G$ satisfies the Engel condition of class $m$ then $\Gamma$ is all of index $m$. It is shown that $kr^{(m-1)} = 0$ and the theorem can be shown to follow from this. (Received November 6, 1963.)

Aaron Siegel, State University of New York at Buffalo, Buffalo 14, New York.

Summability C on the $(k - 1)$-dimensional hypersphere.

Necessary and sufficient conditions for the Cesaro summability of series of surface spherical harmonics on the unit $(k - 1)$-dimensional hypersphere $\Omega$ in Euclidean $k$-space $k \geq 3$ are obtained. The results are extensions of those given by the author for 3-space (Summability C of series of surface spherical harmonics, Trans. Amer. Math. Soc., 104 (1962), 284-307). A definition of the $r$th generalized Laplacian $(r > 0)$ at an arbitrary point $P$ on $\Omega$ of a function $f$, denoted $\Delta_r f(P)$, is presented and the major result is stated as follows: Theorem. Let $\sum_{n=0}^{\infty} Y_{n}(Q)$ be a series of surface spherical harmonics with $Y_{n}(Q) = O(n^c)$ uniformly on $\Omega$ for some $c$. A necessary and sufficient condition that $\sum_{n=0}^{\infty} Y_{n}(Q)$ be summable to $s$ at an arbitrary point $P$ on $\Omega$ is that there exist a non-negative integer $r$, $r > (c + 1)/2$, such that $\Delta_r Y_{r}(P)$ exists and equals $s$ where $F_{r}(Q) = \sum_{n=1}^{\infty} Y_{n}(Q)[\sum_{n=1}^{\infty} Y_{n}(Q)]^{T}$ with $\lambda = (k - 2)/2$. (Received November 6, 1963.)

H. L. Johnson, Purdue University, Lafayette, Indiana 47907. On the existence in the large of a solution to a nonlinear wave equation with mixed boundary conditions.

The dynamics of a tethered balloon cable engenders the BVP. I. (1) $u_{yy} - vu_{xx} + 2u_{x}v_{x} = 0$, $v_{xx} - v(u_{x})^{2} + (u_{y})^{2} = 0$, $v_{x}(0,y) = \sin(u(0,y))$, $v_{x}(0,y) = \cos(u(0,y))$, (2) $u(1,y) = \theta_{0}(y)$, $v(1,y) = \theta_{1}(y)$; where $u, v, (u_{x}, u_{y})$ denote nondimensional, arclength, time, angle of inclination, and tension respectively. A solution to these equations is sought in the form $I_{II}$, $u_{y} = w^{1}(u,v)$, $u_{x} = w^{2}(u,v)$, $v_{y} = w^{3}(u,v)$, $v_{x} = w^{4}(u,v)$. In $(u,v)$ space the equations are reduced to an equivalent integral form and classical analysis is used to prove Theorem 1. If (i) $\theta^{T}(y)$ is sufficiently large, (ii) $\theta_{0}^{T}$ and $\theta_{1}^{T}$ are "sufficiently well behaved," then a classical solution of I of the form II exists in the large. The phrase "sufficiently well behaved" is made definite. One consequence of the bounds that are obtained in the proof is the fact that a perfectly flexible cable in which the tension is sufficiently large at a point behaves "asymptotically like a rigid body." (Received November 6, 1963.)

A detailed qualitative study of the system (1) \( \frac{dx}{dt} = x(x + y + 1) \), \( \frac{dy}{dt} = y(ax + by + c) \), which is encountered in astrophysics, chemical kinetics and nonlinear oscillations, has been given by N. N. Serebrilakova (PMM, vol. 27, no. 1, 1963), and N. N. Bautin has proved (PMM, vol. 18, no. 1, 1954) the absence of limit-cycles for the system (1). Transforming (1) into polar form and applying the Bendixon criterium one can prove the non-existence of limit-cycles for the system (1). It can also be shown that if the parameters \( a \) and \( b \neq 1 \), the equator is an integral curve of (1). (Received November 6, 1963.)


The following theorem is due to Koch (Pacific J. Math. vol. 9, 723-728). Let \( X \) be a compact Hausdorff space endowed with a continuous partial order such that \( \{ y | y \preceq x \} \) is connected for each \( x \in X \) and such that there is a unique minimal element \( 0 \in X \). Then each point \( x \) of \( X \) lies in a connected chain with 0 and \( x \) as endpoints. In particular, if \( X \) is metrizable then \( X \) is arcwise connected. Koch's proof, while ingenious, is quite long and complicated. A short proof is given whose essence is the following. There exists a continuous partial order \( \Gamma \) on the space which is a subset of the given partial order and which is minimal with respect to satisfying the hypotheses of the theorem. If the theorem fails for the given partial order, it must fail with respect to \( \Gamma \) and one may extract a smaller partial order \( \Delta \subset \Gamma \) which contradicts the minimality of \( \Gamma \). An example is given, showing that Koch's theorem fails if the partial order is assumed only to be semi-continuous. Some related unsolved problems are proposed. (Received November 6, 1963.)

608-110. ALFRED GRAY, University of California, Los Angeles, California and S. M. SHAH, University of Kansas, Lawrence, Kansas. Holomorphic functions with gap power series.

Let \( f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^{n_k}}{\beta(1) \ldots \beta(n_k)} \), \( |z| < R \lesssim \infty \), where \( \{n_k\} \) is a strictly increasing sequence of positive integers such that \( (n_k+1 - n_k)/(n_k - n_{k-1}) \to a \text{ as } k \to \infty \), and \( 0 < \beta(1), \ldots, \beta(n) \). Define \( \mu(z) = z^{n_k}/\beta(1) \ldots \beta(n_k) \) for \( \rho(n_k) < |z| < \rho(n_{k+1}) \). It is proved that if \( z = \rho(n_{m-1}) \exp(\text{Re } W/(n_{m+1} - n_m)) \) where \( 0 \leq \text{Re } W \leq \omega(1/2 < \omega < 1, \omega \text{ fixed}) \), then (i) \( \lim_{m \to \infty} f(z)/\mu(z) = \sum_{j=0}^{\infty} \exp[-(\phi/j)(j + 1 - 2W)] \), when \( a = 1 \); and (ii) \( \lim_{m \to \infty} f(z)/\mu(z) = \sum_{j=0}^{\infty} \exp[-(\phi/(a - 1)j) \cdots (a+1 - j) \cdots (a - 1)] \) when \( a > 1 \). With the help of these results, some inequalities for \( \lim \sup_{r \to \infty} [\inf] \mu(r)/M(r) \) and \( \lim \sup_{r \to \infty} [\inf] \mu(z)(\rho(z))/\mu(\rho(z)) \) are obtained. (Received November 6, 1963.)

608-111. ALFRED GRAY, University of California, Los Angeles, California and S. M. SHAH, University of Kansas, Lawrence, Kansas. On entire functions and a conjecture of Erdos.

Let \( \mu(r) \) denote the maximum term of an entire function \( f \), and \( M(r) \) the maximum modulus of \( f \). The following two theorems are proved. (1) Given numbers \( \rho, \lambda \) and \( U \) satisfying \( 0 \leq \lambda \leq \rho \leq \infty \), and

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$0 \leq U \leq 1$, there exists an entire function $f$ of order $\rho$, and lower order $\lambda$ such that
\[ \limsup_{r \to \infty} \frac{\mu(r)}{M(r)} = U. \]
Also the function $f$ can be so chosen that each term of $f(z)$ becomes a maximum term on some interval, (2). Let $\{n_k\}$ be a given sequence of strictly increasing positive integers. Given $\rho$ satisfying $0 \leq \rho \leq \infty$, there exists an entire function of given order $\rho$ of the form $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ for which each term is a maximum term on some interval and
\[ \liminf_{r \to \infty} \frac{\mu(r)}{M(r)} = 0. \]
(Received November 6, 1963.)

608-112. WITHDRAWN.


An intrinsic function $f$ on a linear algebra $\mathbb{B}$, over the field $\mathbb{K}$, is a function from $\mathbb{B}$ to $\mathbb{B}$ such that $f(\Omega a) = \Omega f(a)$ for any $\Omega$ in the group of all automorphisms and antiautomorphisms of $\mathbb{B}$ which leave $\mathbb{K}$ invariant. For $Q$, the algebra of real quaternions, Rinehart has shown that if $f(z) = u(x,y) + iv(x,y)$ is such that $f(\overline{z}) = \overline{f(z)}$ then $f(\xi) = f(x_0 + p \xi) = u(x_0,p) + v(x_0,p)i \mu^2 = -1$ is an intrinsic function on $Q$ and all intrinsic functions on $Q$ are of this form. $f(z)$ is called the stem function of $f(\xi)$ and $f(\xi)$ is an analytic intrinsic function if its stem function is analytic. For its content, if not for its properties, this class has more appeal than the class of regular functions on $Q$ studied by R. Fueter and his students. In this paper results analogous to Cauchy's integral theorem and the Cauchy integral formulas of classical complex analysis are obtained for analytic intrinsic functions on $Q$. It has been necessary to impose certain conditions of symmetry on the surfaces integrated over, but this symmetry is of the same type as is possessed by the domains of the functions.
(Received November 7, 1963.)
Middle nucleus = center in a simple Jordan ring.

Let R be any Jordan ring, M its middle nucleus, C its center. In a paper to appear, entitled The collineation groups of division ring planes, I: Jordan division algebras, R. H. Oehmke and Reuben Sandler have shown that if R is also simple, finite-dimensional and of characteristic \( \neq 2,3 \) then \( M = C \). Their proof depends on the classification of simple Jordan algebras. A. A. Albert (On the nucleus of a simple Jordan algebra, Proc. Nat. Acad. Sci. 50 (1963), 446-447) has given a second proof of this result, valid also for characteristic 3, using results on trace functions. These results have been generalized so that they are valid when R is simple, characteristic \( \neq 2 \). Since little is known about simple Jordan rings, a direct proof is necessary. Starting with Albert's result that the additive subgroup generated by elements of the form \((m,x,y)\), where \( m \) is a fixed element of \( M \), and \( x, y \) are variable elements of \( R \), forms an ideal \( I \) of \( R \), one assumes \( I = R \). By direct computation one verifies that \( (m, (m, x, y), (m, w, z)) = 0 \), thus proving \( (m, R, R) = 0 \), a contradiction. Hence \( I = 0 \), and \( M = C \). This result is of interest in studying infinite-dimensional Jordan division-ring planes. (Received November 7, 1963.)

**608-115. R. B. KELMAN, University of Maryland, College Park, Maryland. Steady-state diffusion thru an impervious finite cylinder into a porous semi-infinite cylinder: an exact solution. Preliminary report.**

A continuous solution \( u(r, x) \) is sought for the equation \( u_{xx} + u_{rr} = - \frac{u_r}{r} \) in the region \( \{ 0 < r < 1; - \lambda < x < \infty \} \) satisfying the boundary conditions: \( \int_0^1 r u(r, - \lambda^+) - G(r)^2 \, dr = 0 \) (G given and \( r^{1/2} G \in L^2(0,1) \)); \( \frac{\partial u}{\partial r} = 0 \) on \( \{ r = 0; x > - \lambda \} \) and \( \{ r = 1; - \lambda < x < 0 \} \); \( \frac{\partial u}{\partial r} = h u \) on \( \{ r = 1; x > 0 \} \) where \( h \) is a positive constant. Seek square summable vectors \( \{ j_n \} \) and \( \{ p_n \} \) such that \( u(r, 0^-) = j_0^{1/2} + \sum_{1}^{\infty} j_n^{1/2} J_0(a_n r)/J_0(a_n) \) and \( u(r, 0^+) = \sum_{0}^{\infty} p_n^{1/2} b_n J_0(b_n r) \). Formal solutions are obtained for the two cylindrical regions. Setting \( u(r, 0^-) = u(r, 0^+) \) and \( u_x(r, 0^-) = u_x(r, 0^+) \) and employing the orthogonality properties of \( J_n \) one obtains a system of equations (in \( L^2 \)): \( j = Lk, (I + D)k = g \) where \( g' \in L^2 \) is known, \( L \) is a known infinite diagonal matrix and \( d_{kn} = 4h^2 \left[ \tanh a_k \tanh a_n \right] \left[ a_k a_n \right]^{1/2} \). Regularity of \( u \) is established using Weyl's lemma. The system can be put in an absolutely continuous form and numerical results obtained by reduction. Besides its obvious technical appications this problem occurs in neural physiology. (Received November 7, 1963.)


Sequences \( f \) of real numbers are identified with their unique continuous extensions over the Stone-Cech compactification \( \beta N \) of the (discrete) space \( N \) of positive integers. A regular \( b \)-multiplicative summability method \( \Phi \) is a multiplicative linear functional on a uniformly closed subspace \( \mathcal{A}(\phi) \) of the space \( B \) of bounded sequences that sends convergent sequences to their limits. If \( A \) is a positive regular matrix, \( f \in B \), and \( \sum_j a_j |f(j) - a| \rightarrow 0 \), let \( \psi_A(f) = a \). Theorem. Every \( \psi_A \)
equivalent for bounded sequences to a regular matrix summability method and every regular $b$-multiplicative matrix method is a $\psi_A$. There is a closed subset $F(\psi_A)$ of $\beta \mathbb{N} \sim \mathbb{N}$ such that $\{ f \in A \mid f \text{ is constant on } K \}$ iff $f$ is constant on $K$ (Math. Z. 71 (1959), 427-435). \textbf{Theorem.} If $f \in B$ vanishes on $F(\psi_A)$, then $f$ vanishes on a neighborhood in $\beta \mathbb{N} \sim \mathbb{N}$ of it. \textbf{Corollary.} $\psi_A(f) = a$ iff there is a $Z \subset \mathbb{N}$ such that $f$ converges to $a$ on $Z$ and $\lim_{j \to \infty} \sum |a_j; j \in Z| = 1$. \textbf{Theorem.} The interior in $\beta \mathbb{N} \sim \mathbb{N}$ of $F(\psi_A)$ is an $F_{\sigma}$ (possibly empty); $F(\psi_A)$ intersects a pairwise disjoint open and closed sets of $\beta \mathbb{N} \sim \mathbb{N}$; $F(\psi_A)$ has no isolated points. Every zero-set that is an $F(\psi_A)$ is open. There are two sets $F(\psi_A)$ with empty interior that are not carried to each other by any homeomorphism of $\beta \mathbb{N} \sim \mathbb{N}$ onto itself. (Received November 7, 1963.)

608-117. A. A. GOLDSTEIN, University of Texas, Austin, Texas and T. I. SEIDMAN, Boeing Scientific Laboratory, P.O. Box 3981, Seattle, Washington. Fuel-optimal controls.

The control equations for a rocket moving in a plane, assuming thrust is proportional to rate of fuel consumption and a priori bounded in magnitude but arbitrary in direction, take the form

$$
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= [F(x) + au] / m, \\
\dot{m} &= -|u|^2, \\
F &= \text{the external (e.g., gravitational) force.}
\end{align*}
$$

It is shown that if there exists a control function $u(t)$ with $\|u(t)\| \leq k$ carrying the rocket from given initial $(x, y, m)$ to given terminal $(x, y)$ in time $T$, then there exists one which minimizes the total fuel consumption, $c[u] = \alpha \int |u(t)| dt$, and certain further characterization of this fuel-optimal control can be given. (Received November 7, 1963.)


Given a simply connected set $\Omega$ in the plane, let $K[\Omega]$ denote the set of all entire functions of exponential type whose Borel transforms are regular on $\Omega$, the complement of $\Omega$. Let $\Omega$ be a domain on which a given function $W$ is regular, and let $\{ L_n \}$ be defined on $K[\Omega]$ by $L_n(f) = (2\pi)^{-1} \int_{\Gamma} [W(t) t^n F(t)] dt$ where $F$ is the Borel transform of $f$ and $\Gamma$ is a simple contour in $\Omega$ enclosing all singularities of $F$. It is proved that $K[\Omega]$ is a uniqueness class for $\{ L_n \}$ (i.e., $f \in K[\Omega]$ and $L_n(f) = 0$, $n = 0, 1, \ldots$, implies $f = 0$) if and only if $W$ is univalent on $\Omega$. This is a stronger form of a theorem proved by Buck [Trans. Amer. Math. Soc. 64 (1948), 283-298], which had the additional restrictions that $\Omega$ contain the origin, $W(0) = 0$, and $W'(0) = 0$. The theorem of Buck was an improvement of a result of Gelfond [Mat. Sb. (N.S.) 46 (1938), 115-147]. The proof is simple and elementary. It depends on showing that $L_n(f) = 0$, $n = 0, 1, \ldots$, implies $F$ is regular on $\Omega$ and since it is also regular outside $\Omega$ and at $\infty$ and $F(\infty) = 0$, $F = 0$, so that $f = 0$. (Received November 8, 1963.)

608-119. B. E. GOODWIN, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland. On the realization of the eigenvalues of integral equations whose kernels are entire or meromorphic in the eigenvalue parameter.

If the $\lambda_h (h = 1, 2, \ldots)$ are the eigenvalues of the integral equation $u(x) - \lambda \int_0^1 K(x, y; \lambda) u(y) dy = f(x)$, where $\Re(x, y; \lambda) = \sum_{h=1}^{N} \left( f_h(\lambda)/g_h(\lambda) \right) X_h(x) Y_h(y) + \sum_{h=0}^{\infty} (T(x, y)) \, \lambda_h^m$, and the $\left( \lambda_h(\lambda)/g_h(\lambda) \right)$ are meromorphic functions, the $X_h(x) Y_h(y)$ are in $L_2$, and the $(T(x, y))_h$ are in $L_2$; a procedure is given for finding the $\sum_{h=0}^{\infty} \lambda_h^m (m = 1, 2, \ldots)$. In particular for $\Re(x, y; \lambda) = f(\lambda)/g(\lambda) X(x) Y(y) + T(x, y)$, the eigenvalue equation
\[ g(\lambda)D_T(\lambda) = M(\lambda)D_T(\lambda) \int_0^1 X(x)Y(x)dx + \lambda^2 f(\lambda) \int_0^1 D_T(x,z;\lambda) X(z)Y(x)dx dz = 0 \]

is obtained, where \( D_T(x,y;\lambda) / \lambda(\lambda) \) is the Fredholm resolvent for \( T(x,y) \). If \( \lambda_h^* \) are the roots of \( g(\lambda) \) and \( \lambda_h^{**} \) are the roots of \( f(\lambda) \), then \[ \sum_{h=1}^m \lambda_h^{-1} = \int_0^1 T(x,x)dx + \int_0^1 X(x)Y(x)dx, \]

with rather more complicated formulae for the \[ \sum_{h=1}^m (m = 2,3,\ldots) \] which depend on the \( \lambda_h^{**} \) as well. The procedure generalizes to kernels having the form of \( K(x,y;\lambda) \). Applications of the technique to the symmetrizations of non-symmetric kernels and to the transverse vibrations of a rotating beam carrying a tip mass will be found in the paper described here. (Received November 8, 1963.)

608-120. MARTIN TIERNEY, P. E. WALTMAN and G. M. WING, Sandia Laboratory, Albuquerque, New Mexico. On some problems in the optimal design of shields and reflectors in particle physics.

Using the rod model and the invariant imbedding formulation of the transport problem, several problems in the optimal design of particle shields and reflectors are considered. The dynamic programming technique is utilized to construct algorithms for the computation of solutions. A typical question is the following: Given \( n \) materials in slabs of thickness \( \Delta \), construct a reflector of thickness \( L \) in such a way as to obtain a reflection function \( R(L) \geq R^*, R^* \) prescribed, and so that the total weight of the reflector is a minimum. (Received November 8, 1963.)

608-121. J. B. DIAZ and F. T. METCALF, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland. Stronger forms of inequalities of Kantorovich and Strang for operators in a Hilbert space.

A "Blitzbeweis" is given of the following inequality (*):
\[ (Cx,x) + m \text{M}(C^{-1}x,x) \leq (M + m)(x,x), \]

where \( C \) is a self-adjoint operator on a Hilbert space \( H \) to itself, subject only to the inequalities \( 0 < M \leq C \leq ME \), and \( x \) is any element of \( H \). Since \( 0 \leq \| (Cx,x) \|^2 - \| [m \text{M}(C^{-1}x,x)] \|^2 \), inequality (*) is actually sharper than an inequality of Kantorovich (see, e.g., Greub and Rheinboldt, Proc. Amer. Soc. 10 (1959), 407-415). The inequality (*) is used to prove the following inequality (**):
\[ \| (Tx,y) \| + m \text{M}[(x,T^{-1}y)] \leq (M + m) \| (x,y) \|^{1/2}, \]

where \( T \) is an arbitrary operator on \( H \) to itself, subject only to the inequalities \( \| T \| \leq M \) and \( \| T^{-1} \| \leq 1/m \), and \( x \) and \( y \) are elements of \( H \). Since \( 0 \leq \| (Tx,y) \| - (m \text{M})^{1/2} \| (x,T^{-1}y) \|^{1/2} \|^2 \), inequality (**) is actually stronger than an inequality of Strang (see Proc. Amer. Math. Soc. 11 (1960, 468). (Received November 8, 1963.)

608-122. D. H. TUCKER, University of Utah, Salt Lake City, Utah 84112. A representation theorem for a continuous linear transformation on a space of continuous functions.

Suppose each of \( X \) and \( Y \) is a linear normed space, \( C \) is the space of continuous functions \( f \) from \([0,1]\) into \( X \) with \( \| f \| = \max_{x \in [0,1]} \| f(x) \|_X \), and \( T \) is a continuous linear transformation from \( C \) into \( Y \). Denote by \( B = B[X,Y] \) the space of all continuous linear transformations from \( X \) into \( Y \) and denote by \( B^+ \) the space of equivalence classes of weakly Cauchy sequences in \( B \) endowed with the norm \( \| [p_n] \| = \sup_{n \to \infty} \| p_n \| \leq 1 \). The elements of \( B^+ \) are also elements of \( B[X,Y^+] \).

Theorem: There exists a number \( W > 0 \) such that if \( 0 = t_0 < t_1 \ldots < t_n = 1 \) and \( x_0,\ldots,x_{n-1} \) are points in \( X \), then \[ \sum_{i=0}^{n-1} \| p_i \| - \| p_{i+1} \| \| x_i \|_Y \| \leq W \max \| x_i \|_X. \]

The minimum such \( W \) is denoted \( W^1_0 \). Fur-
thermore, $W^1_0K = \|T\|$ and for each $f \in C$, $T(f) = \int_0^1 dK(t) \cdot f(t)$ where the convergence of the integral is in the norm of $Y^*$. The question of the uniqueness of $K$ is also treated. (Received November 8, 1963.)

608-123. L. C. KURTZ and D. H. TUCKER, University of Utah, Salt Lake City, Utah 84112. Vector-valued summability methods on a linear normed space.

Suppose $X$ and $Y$ are linear normed spaces. A summability method $M$ from $X$ to $Y$ is a matrix $(f_{mn})$ of bounded linear transformations from $X$ to $Y$. $M$ is convergence preserving if it maps convergent sequences in $X$ into weakly convergent sequences in $Y$, $M$ is regular with respect to a linear transformation $L$ if a sequence convergent to $x$ in $X$ is mapped into a sequence weakly convergent to $L(x)$. Theorems of the Toeplitz type are established. It is shown that if $X$ and $Y$ are complete, then the Hausdorff moment problem is equivalent to the representation theorem of the previous abstract. (Received November 8, 1963.)


Let $T$ be a closed linear transformation whose domain and range lie in some complex Banach space $X$. Let $\mathcal{M}(T)$ be the class of all complex-valued functions $f$ such that (1) the domain of $f$ is some open set containing $\sigma_e(T)$, the extended spectrum of $T$; (2) $f$ is meromorphic on its domain; (3) if $a$ is a pole of $f$, then $a$ is not an eigenvalue of $T$. An operational calculus is obtained for $\mathcal{M}(T)$ analogous to the operational calculus of Dunford and Taylor for analytic functions. If $f$ is in $\mathcal{M}(T)$, then the operational calculus for $\mathcal{M}(T)$ yields a closed (not necessarily bounded) linear transformation $f(T)$ and a generalization of the fine structure spectral mapping theorem in Hille and Phillips is obtained. See Hille and Phillips, Functional analysis and semi-groups, p. 204, Amer. Math. Soc. Colloq. Publ. Vol 31 Amer. Math. Soc., Providence, R.I., Rev. ed. 1957. (Received November 8, 1963.)


Let $K$ be a subcomplex of a piecewise-linear $n$-manifold $M$ and let $N$ be a regular neighborhood of $K$ in $M$. (1) If $K$ is cellular in $M$, $\pi_1(\partial N) = 1$, and, consequently, $\partial N$ is a homotopy $(n-1)$-sphere, (2) $K$ is cellular in $M$, $n \geq 6$, if and only if $N$ is an $n$-cell. Let $Cell(n)$ be the conjecture: If $K$ is cellular in $M$, then $N$ is an $n$-cell. Let $PC(n)$ be the Poincaré conjecture: If $S$ is a homotopy $n$-sphere, then $S$ is a topological $n$-sphere. There is the following relationship: $PC(n-1)$ implies $Cell(n)$, $PC(n) + Cell(n)$ implies $PC(n-1)$, $n \geq 5$. (Received November 8, 1963.)

608-126. C. E. BURGESS, University of Utah, Salt Lake City, Utah. Pairs of 3-cells with intersecting boundaries in $E^3$.

Let $S$ and $S'$ be 2-spheres in $E^3$ such that (1) $S \subset S' + \text{Int } S'$, (2) $S' + \text{Int } S'$ is a 3-cell, and (3) $S$ is locally tame from $\text{Int } S$ at each point of $S \cdot S'$. Then $S + \text{Int } S$ is a 3-cell. This can be generalized where $S$ and $S'$ are connected 2-manifolds that are tame from one side in a 3-manifold. (Received November 8, 1963.)

This answers a question raised by S. Smale, and also presented by J. Milnor in the List of Problems of the Seattle Topology Conference (summer, 1963). The problem is to determine the maximal number of linearly independent commuting vector fields on the three-dimensional sphere $S^3$. The answer is one. The proof uses: the action $\phi$ of $\mathbb{R}^2$ on $S^3$ induced by the alleged fields, a theorem of Haefliger forbidding that all $\phi$-orbits be simply-connected, a Poincaré-Bendixson type argument to obtain a torus from a cylindrical $\phi$-orbit, and the observation that, on a solid torus in $\mathbb{R}^3$, one cannot have a field of 2-frames which reduces to the standard one on the boundary surface. (Received November 8, 1963.)

608-128. C. F. KENT, Case Institute of Technology, University Circle, Cleveland 6, Ohio. Reduction of ordinal recursion.

Nested ordinal recursion is defined in W. Taits paper (Math. Ann. 143 (1961)). Tait shows that nested ordinal recursion may be replaced by unnested ordinal recursion at the expense of raising the order type of the well-ordering over which the recursion takes place. The main result of the present paper is a procedure for lowering the order type again (to order type $\omega$ if desired) while preserving the unnested recursion, and making the resulting order primitive recursive in the original. Thus, any function definable by nested ordinal recursion can also be defined by unnested ordinal recursion over a well-ordering primitive recursive in the original and of order type $\omega$. The result is closely related to one of Myhill (J. Symb. Logic 18 (1953)), Liu (Proc. Amer. Math. Soc 11 (1960)) and Routledge (Proc. Cambridge Philos. Soc. 49 (1953)), but the procedure possesses independent interest. (Received November 8, 1963.)


The class of compact spaces having no infinite projective quotient space (subspace) is closed under formation of direct products (countable direct products). The class of proximity spaces having no infinite projective quotient space, resp. having a unique compatible uniformity, is closed under formation of direct products. (Received November 8, 1963.)


In problems concerning the guidance of space vehicles, one frequently encounters a type of differential equations boundary-value problem. If a particular numerical solution to the problem is known, it is shown that a general solution can be represented by a convergent Taylor’s series in several variables in a neighborhood of the known solution. A system of first-order differential equations $\dot{y}_i = f_i(t, y_1, \ldots, y_m, y_{m+1}, \ldots, y_{m+n})$, $i = 1, \ldots, m+n$, is considered along with end conditions $F_j(t, y_1, \ldots, y_m, y_{m+1}, \ldots, y_{m+n}) = 0$, $j = 1, \ldots, n+1$. Initial values $y_i(t_0) = a_i$ ($i = 1, \ldots, m$) are given and initial values $y_{i+m}(t_0) = b_i$ ($i = 1, \ldots, n$) are to be determined so that there exists a solution to the system satisfying the conditions $F_j = 0$ for some $t_1 > t_0$. The initial values $b_i$ and terminal time $t_1$
can be considered as functions of the parameters \(a_1, \ldots, a_m\); say \(b_i = \beta_i(a_1, \ldots, a_m)\) and \(t_1 = \tau(a_1, \ldots, a_m)\).

With certain continuity and analyticity conditions on the functions \(f_i\) and \(F_j\) and by the use of imbedding theorems, implicit function theorems and a method for obtaining the required partial derivatives, the functions \(\beta_i\) and \(\tau\) can be expanded in a Taylor's series about the parameters \(a_1^*, \ldots, a_m^*\) corresponding to the known solution. (This research has been sponsored by the Aero-Astrodynamics Lab, Marshall Space Flight Center, NASA.) (Received November 8, 1963.)

608-131. R. C. O'NEIL, Rice University, Houston 1, Texas. Interpolation of operators for Lip spaces.

Theorem 1. If \(T\) is a linear operator which maps \(\text{Lip } a_0\) boundedly into \(\text{Lip } \beta_0\) and also maps \(\text{Lip } a_1\) boundedly into \(\text{Lip } \beta_1\) then \(T\) maps \(\text{Lip } a\) boundedly into \(\text{Lip } \beta\) where \(a = a_t = a_0(1 - t) + a_1t\) and \(\beta = \beta_t = \beta_0(1 - t) + \beta_1t\) for \(0 \leq t \leq 1\). Theorem 2. If \(T\) is a linear operator and maps \(\text{Lip } a_0\) and \(\text{Lip } a_1\) boundedly into \(\text{Lip } a_\alpha\) and \(\text{Lip } a_\beta\) where \(1/p = (1 - t)/p_0 + t/p_1\) and \(a = a_0(1 - t) + a_1t\) for \(0 \leq t \leq 1\). The proofs use only elementary real variable theory. Using the same techniques a number of other results on the interpolation of linear operators may be established. (Received November 8, 1963.)


Let \(-\infty < a < b < \infty\) and let \(\Phi\) denote the set of all functions, continuous and strictly monotone in \([a,b]\). Let \(\psi\) and \(x\) be elements of \(\Phi\). (*): \(M_{\psi} \leq M_x\) if for every finite sequence \(x_1, x_2, \ldots, x_n\) of points of \([a,b]\), and every \(q_1, q_2, \ldots, q_n\) (positive \(\sum_{\nu=1}^n q_\nu = 1\)), \(\psi^{-1} \left( \sum_{\nu=1}^n q_\nu \psi(x_\nu) \right) \leq x^{-1} \left( \sum_{\nu=1}^n q_\nu x(x_\nu) \right)\).

Let \(\psi\) and \(x\) be differentiable in \((a,b)\) and let \(\psi^\prime > 0\) there. Then a necessary and sufficient condition for (*): to hold is that \(x^\prime/\psi^\prime\) be nondecreasing in \((a,b)\) if \(\psi\) and \(x\) are monotone in the same sense, and that \(x^\prime/\psi^\prime\) be nonincreasing there if \(\psi\) and \(x\) are monotone in opposite senses. From this one gets a simple representation of \(x\) in terms of \(\psi\) which holds if (*) holds. Furthermore, given a real \(\psi\), continuous in \([a,b]\), differentiable in \((a,b)\) and satisfying \(\psi^\prime \neq 0\) there, one can easily construct functions \(x\) belonging to \(\Phi\) for which (*) holds. One of the other questions studied is the special type of inequalities \(M_{\psi^\prime} \leq M_\psi\). (Received November 8, 1963.)

608-133. J. H. BRAMBLE and B. E. HUBBARD, University of Maryland, College Park, Maryland, A finite difference analog of the Neumann problem for Poisson's equation.

A finite difference problem on a square mesh, size \(h\), is posed which results from discretizing the Laplace operator in a bounded two-dimensional region, \(R\), and the normal derivative on the boundary, \(C\). The local error is \(O(h^2)\) at "regular" interior points and on \(C\) and is \(O(h)\) at interior points near \(C\). It is shown that a maximum principle can be applied to study the order of convergence (which in this case is \(O(h^2|\ln h|)\)). (Received November 8, 1963.)
A base of open sets at the point x in a space X is contractable if each element U is contractable and $U - x$ is 0-connected and 1-connected. X is a strong homology n-manifold, or n-shm, if, for $n \geq 3$, it is a singular homology n-manifold with a contractable basis at each point, and, for $n < 3$, it is an n-manifold. Theorem 1. An n-shm over Z (the integers) is a homotopy n-manifold. Theorem 2. A homotopy n-manifold with a contractable basis at each point is an n-shm over Z. Theorem 3. If X is an n-shm and Y is an m-shm, then $X \times Y$ is an $(n + m)$-shm. Corollary 1. The product of polyhedral homotopy manifolds is a homotopy manifold. (Received October 30, 1963.)

A one-dimensional Schrodinger equation $\psi''(x) + (E_n - V(x)) \psi(x) = 0$ with two simple turning points $h_n$, $k_n$ may be estimated by Bohr's quantization rule:

$$\int_{h_n}^{k_n} \sqrt{E_n - V(x)} \, dx \approx \frac{n(1/2)^{1/2}}{(n + 1/2)^{1/2}}$$

for large n. A modification of this rule gives the $E_n$ exactly in those problems for which complete solutions of the differential equation are known. (Received November 12, 1963.)

Let G be an abelian lattice-group. Definition: A net $(x_i)_{i \in I}$ $a$-converges to x, if x is the only element of G satisfying the property: $x = \bigvee_{i \in I} (x_i \wedge x) = \bigwedge_{i \in I} (x_i \vee x)$ for every $i \in I$. In archimedean 1-groups this is equivalent to: for every $a, b \in G$, $(a \wedge x) \rightarrow (a \vee x) \wedge b$. However, in an arbitrary abelian 1-group it is weaker and, in many respects, more natural than most other purely lattice-theoretic convergences so far investigated. It is shown that it satisfies the essential properties of a decent convergence concept and that the operations $+, \cdot, V, \Lambda$ in G are continuous relative to a-convergence. Other properties are also investigated. (Received November 12, 1963.)

An o-topological lattice which cannot be regularly extended to a $\sigma$-complete o-topological one.

As is known, a lattice L is said to be o-topological if $x_i \rightarrow (L)_x$ implies $(*) x_i \rightarrow y (\rightarrow (L)_{x \wedge y}$ and dually. We shall call L sequentially o-topological if this property is satisfied for sequences $(x_n)$. A lattice is constructed which is o-topological, meet-complete, conditionally join-complete but cannot be extended to a $\sigma$-complete and sequentially o-topological lattice with preservation of existing countable joins and meets. (Received November 12, 1963.)
Fermat's last theorem is true for any exponent up to 25,000.

A computational method was developed by the Lehmers and Vandiver (Proc. Nat. Acad. Sci. 40 (1954), 25-33) to prove the impossibility of $x^n + y^n = z^n$ if $n$ is any multiple of a given odd prime $\ell$. Their method has been programmed for the IBM 7090 and this has been a standby problem at the UCLA Computing Facility. The results are complete for all $\ell < 25000$, and the work is continuing on a low priority basis. The largest degree of irregularity so far found is four. Thus Fermat's last theorem is true for any exponent which has an odd prime factor below 25000 or which is divisible by 4.

(Received November 12, 1963.)

Spectral sequences and Frobenius groups.

$G$ denotes a Frobenius group of type $(h,m)$ (W. Feit, On the structure of Frobenius groups, Canad. J. Math. 9 (1957), 587-596). $N$ denotes the set of elements of $G$ whose orders divide $m$, and $H$ stands for a subgroup of $G$ of order $h$. We put $G = G(0)$ ($H = H(0)$) and denote the commutator subgroup of $G(i)$ by $G(i+1)$ (of $H(i)$ by $H(i+1)$) for $i \geq 0$; $e$ stands for the smallest integer such that $H(e) = H(e+1)$.

If $A$ is a $G$-module, the relative cohomology groups $H^r(G:H, A)$ for $r \geq 0$ are related to the classical cohomology groups $H^r(G,A)$ by a spectral sequence (E. Snapper, Cohomology of permutation representations, J. Math. Mech., 13 (1964)). In the present paper we investigate the "Frobenius triple" $(N,G,H)$ by investigating this spectral sequence. We show that the lift-restriction sequence

$$0 \rightarrow H^r(G;H,A) \rightarrow H^r(G,A) \rightarrow H^r(H,A) \rightarrow 0$$

is exact for $r \geq 1$, and that $H^2(G;H, Z) = 0$. It follows that:

1. If $H$ is solvable, $G(i)/G(i+1) \cong H(i)/H(i+1)$ for $0 \leq i \leq e - 1$; furthermore, $G(e) = N$ which shows that $N$ is a group. (2) If $H$ is not solvable, $G(i)/G(i+1) \cong H(i)/H(i+1)$ for all $i \geq 0$; furthermore, $N \subseteq G(e)$ and $(N,G(e), H(e))$ is again a Frobenius triple. (Received November 12, 1963.)

On the numerical solution of the Dirichlet problem for $u + ku = F$.

The Dirichlet problem for the equation $\Delta u + ku = F$ is considered in an $N$-dimensional bounded connected region $R$. The symbol $\Delta$ denotes the Laplace operator, $k$ is a sufficiently smooth, bounded function $|k| \leq \overline{k}$ and $F$ is given. Assuming existence and uniqueness and sufficient smoothness in this problem, it is shown that the solutions of the standard finite difference approximations to this problem involving the 5-point approximation of $\Delta$ at interior points, tend to the exact solution uniformly, just as in the case $k = 0$. In general the matrix of the linear system does not have "diagonal dominance", and its inverse is not non-negative so that these problems are not of the so called "monotone type". (Received November 12, 1963.)

A torsion theory for Abelian categories.

$C$ denotes a complete Abelian category whose subobject lattices are sets. A class $A$ of objects is called a left kernel if there exists a class $B$ of objects such that $A$ and $B$ are complete
with respect to the relation $\text{Hom}(A, B) = 0$. In this case $A$ is said to be a left kernel of $B$. Right kernels are similarly defined. The class $A$ is a left kernel iff $A$ is closed under (i) images, (ii) infinite sums, and (ii) extensions. The dual result holds. It follows that any class of objects is contained in a unique smallest left (right) kernel. The left kernels of the category $C$ form a complete lattice whose dual is the lattice of right kernels. If $\text{Hom}(A, B) = 0$, then $A$ and $B$ are complete with respect to this property iff for each $M$ of $C$ there is an exact sequence $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$ with $A \subseteq C, B \subseteq C$. Let $J$ be the smallest left kernel containing the class of simple objects of $C$. Then $J$ forms a natural class of torsion objects which is closed under taking subobjects. For modules over a commutative Noetherian ring $J$ has a primary decomposition theorem, and for modules over Dedekind domains $J$ coincides with the usual torsion modules. For arbitrary Abelian categories $J$ does not have a primary decomposition. (Received November 12, 1963.)

608-142. NACHMAN ARONSZAJN, University of Kansas, Lawrence, Kansas. The Banach algebra of multipliers on a functional Banach space.

Let $F$ be a functional Banach space on a basic set $\mathcal{E}$ rel. an exceptional class $\mathfrak{A}$. The class $\mathfrak{A}_1$ of unessential sets if formed by sets $A$ such that each function $f \in F$ vanishes on $A$ exc. $\mathfrak{A}$. A function $\phi(x)$ is a multiplier on $F$, $\phi \in \mathfrak{M}_F$, if it is defined on $\mathcal{E}$ exc. $\mathfrak{A}$ and if $f \in F$ implies $\phi f \in F$.

The following statement is always true: (I) For $\phi \in \mathfrak{M}_F$ the linear operator $f \rightarrow \phi f$ is bounded on $F$; its bound $|\phi|$ satisfies ess. sup$_x |\phi(x)| \leq |\phi|$. Corollary. $\mathfrak{M}_F$ is a commutative Banach algebra without radical; it is a functional space rel. $\mathfrak{A}_1$. Consider now $F = P^0(D) \equiv$ the class of Bessel potentials of order $\alpha$ in a domain $D \subseteq \mathbb{R}^n$. For $\phi \in \mathfrak{M}_F$ the set $V_\phi$ of essential values of $\phi$ is formed by all complex $\iota$'s such that for every $\epsilon > 0$, the set where $|\phi(x) - \iota| < \epsilon$ does not belong to $\mathfrak{A}_1$. (II) For $F = P^0(D)$ and $\phi \in \mathfrak{M}_F$, [the spectrum of $\phi$ in $\mathfrak{M}_F$] = $V_\phi$; for $\iota \notin V_\phi$, $|(|\iota - \phi|^{-1}) \leq C[\text{dist.} (\iota, V_\phi)]^{-\alpha}$ with $C$ independent of $\iota$. (For a non-integer we need here a slight regularity hypothesis on $\partial D$.) The set of maximal ideals for $\mathfrak{M}_F$, $F = P^0(D)$, can be completely described. (Received November 12, 1963.)


Consider the two combinations of Bessel functions $f_\nu(kz) = J_\nu(z)Y_\nu(kz) - Y_\nu(z)J_\nu(kz)$ and $g_\nu(kz) = J'_\nu(z)Y'_\nu(kz) - Y'_\nu(z)J'_\nu(kz)$. Gray and Mathews (Bessel functions, MacMillan, London, 1922, 82) established that for real $\nu$ and positive $k$ the $z$-zeros of $f_\nu(kz)$ are all real and simple. In this paper it is shown that these same results hold for $g_\nu(kz)$ also. Furthermore, by contour integrations it is proved that for large integer $s$ there are precisely $2s$ and $2s + 2$ such zeros of $f_\nu$ and $g_\nu$ respectively within the circle $|z| = (s + 1/2)\pi/k - 1]$. One may then conclude that the asymptotic expansions of McMahon (Ann. of Math., 9 (1894), 23-30) for the larger $z$-zeros actually give the $s$th and $(s + 1)$st zeros of $f_\nu$ and $g_\nu$ respectively. Regarding the $r$-zeros, for given real $z$ and $k > 0$ it is established by classical techniques that both $f_\nu$ and $g_\nu$ have a countable number of zeros of which only at most a finite number are real, the remainder being purely imaginary. All of these zeros are shown to be simple except for that occurring at the origin ($r = 0$) which can appear only as a zero of the second order. Asymptotic expressions are derived for the larger of the imaginary $r$-zeros. (Received November 12, 1963.)
The problem of solving Maxwell equations in a domain bounded by two infinitely long concentric circular cylinders of perfect conductivity with a finite number of slots of finite width and infinite length in the outer cylinder, for a finite number of given axial line sources, is reduced to the Dirichlet and Neumann problems for the two dimensional wave equation. The solutions of these problems are represented by Fourier-Bessel series with unknown coefficients, and dual series equations for these coefficients are derived. By means of approximation of the Bessel and Hankel functions, dominant parts of the dual series equations are picked out, and the dual series equations are converted into singular integral equations with Cauchy type kernel, the dominant terms of the latter coming from that of the former. The integral equations are then solved by methods described by Muskhellishvili. (Received November 12, 1963.)

Orbits about an oblate planet, II.

The purpose of this paper is to establish the existence and stability properties of the periodic orbits which occur at critical inclination. In addition, the behavior of nearby orbits is investigated by means of the second order method of averaging. (Received November 12, 1963.)

Approximation of maps of inverse limit spaces by induced maps.

Let \((X, f)\) and \((Y, g)\) be inverse limit systems of (finite) polyhedra. Assume that (1) each bonding map \( f^\alpha: X^\alpha \rightarrow X^\alpha \) is onto; (2) \((Y, g)\) is a solenoidal sequence — This means that the index set for \((Y, g)\) is the positive integers and that each bonding map \( g^n: Y_n \rightarrow Y_1 \) is a regular covering map. Let \(M(X^\infty, Y^\infty)\) be the space of all maps from the limit space \(X^\infty\) into the limit space \(Y^\infty\), with the topology of uniform convergence \((Y^\infty\) is compact, metrizable). Call a member \( \Phi \) of \(M(X^\infty, Y^\infty)\) an induced map if \( \Phi \) is induced from a system map \( \Phi: (X, f) \rightarrow (Y, g) \). (See Eilenberg-Steenrod, Foundations of algebraic topology, p. 213.) Theorem. Under the above assumptions on \((X, f)\) and \((Y, g)\), the induced maps are dense in \(M(X^\infty, Y^\infty)\). (Received November 12, 1963.)

On representing groups by permutations.

Let \( \{ \pi_\alpha \} \) be the set of all representations of a group \( G \) such that \((G)\pi_\alpha\) is a permutation group. Denote by \( \Phi_1 \) the subset of \( \{ \pi_\alpha \} \) such that \( \pi \in \Phi_1 \) if and only if \((G)\pi\) is transitive. An equivalence relation, called weak equivalence, is now introduced on \( \{ \pi_\alpha \} \) in a natural way. Say that \( \pi_1 \) is \( w \)-equivalent to \( \pi_2 \), \( \pi_1 \approx \pi_2 \), if and only if \((G)\pi_1\) and \((G)\pi_2\) are permutation isomorphic. It can be shown that "equivalence" (Burnside) implies "weak equivalence" but not conversely. Necessary and sufficient conditions for the weak equivalence of two members of \( \Phi_1 \) are established on the subgroups of \( G \) inducing them. (Received November 12, 1963.)
Kuratowski has shown that one can construct at most fourteen sets from a given set using the closure and complement operators. Similar results can be obtained by considering other pairs of operators. The operators considered here are the following: the closure, the interior, the isolated, the scattered, and the border. (Received November 12, 1963.)

A commutative ring $R$ of characteristic $p > 0$ is perfect if $x \mapsto x^p$ is an automorphism. A scheme $X$ over the field $\mathbb{Z}/p$ is perfect if the stalks of the structure sheaf of $X$ are perfect local rings. The functor $T \mapsto \text{Mor}(T, X)$ on the category of perfect schemes $T$ is shown to be representable by a scheme $X^p \cdot 00$; this is the perfect closure of $X$. (Received November 12, 1963.)

Let $\{x(t), x(0) = 0, 0 \leq t \leq 1\}$ be the continuous sample functions of a Brownian motion process. A stochastic process $\{y(t), y(0) = 0, 0 \leq t \leq 1\}$ is defined by means of the continuous extension in the uniform topology of $dy(t) = f(y(t))dt + \sigma(t, y(t))dx(t)$. (Received November 12, 1963.)

One of the problems arising in pattern recognition is the following: Given two subsets $X$ and $Y$ of the Euclidean $n$ space $\mathbb{R}^n$ and a positive integer $k$, to find, if possible, $k$ vectors $w_1, \ldots, w_k$ each in $\mathbb{R}^n$ and $2k + 1$ real numbers $a_1, a_k, a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_k}$ such that $w_i z + a_{\mu_i} \neq 0$, $i = 1, 2, \ldots, k$, $\forall z \in X \cup Y$, and such that $\sum_{i=1}^{k} \mu_i \text{sgn}(w_i z + a_{\mu_i}) > 0$, $\forall z \in X \cup Y$. In the case where $X$ and $Y$ are finite sets, it is shown that the problem is solvable for some $k$, if and only if $X$ and $Y$ are disjoint. The proof is constructive. Although the construction need not yield a minimal $k$, it generates a solution with $k$ no larger than $2 \min \{ |X|, |Y| \}$. The algorithm simultaneously yields a solution to the so-called "majority", "parallel plane", and "parity" separation problems. (Received November 12, 1963.)

Let $E$ and $F$ be locally convex Fréchet-spaces and let $A$ be a continuous linear mapping from $E$ into $F$. Define on $E'$ the topology $T_c(E)$ of uniform convergence on the compact subsets of $E$, then: $A$ is open if and only if $A'$ is continuous and open for the topologies $T_c(F)$ and $T_c(E)$ on $F'$ resp. $E'$.

A consequence is the following open mapping theorem: Let $E$ and $F$ be Fréchet-Montel-spaces, $E'$ and $F'$ their strong duals. A continuous linear mapping $A$ from $F'$ into $E'$ is open iff $A(F')$ is sequentially closed in $E'$. (Received November 12, 1963.)
In a ternary ring with operation $T(a,b,c)$ and distinguished elements $0, 1$ a double loop is defined by the two binary operations $a + b = T(a,1,b)$ and $ab = T(a,b,0)$. It is shown that any infinite double loop $D$, satisfying the necessary condition "$x + a = xb$ uniquely solvable for $a \neq 0$, $b \neq 0, 1$" can be canonically embedded in a ternary ring. The proof uses transfinite induction to successively embed $D$ in a $3,4,\ldots, m,\ldots$ -loop, where $m$ ranges over the ordinals less than the initial number corresponding to the cardinal of the set $D$. The same method serves to show that any infinite additive or multiplicative loop can be canonically embedded in a double loop of the kind mentioned above, and hence in a ternary ring. As a third application of the method it is shown that any infinite additive loop $L$, satisfying simple sufficient conditions which cover the case where $L$ is a group, can be canonically embedded in a linear ternary ring (a ternary ring with $T(a,b,c) = ab + c$). (Received November 12, 1963.)

Such a function is constructed as the boundary values of a function $f(z)$, analytic for $|z| < 1$ and continuous for $|z| \leq 1$. It is well known that if $f(z) = \sum a_n z^n$ for $|z| < 1$, then $\sum a_n e^{in\theta}$ is the Fourier series of $f(e^{i\theta})$, which is continuous on $[-\pi, \pi]$. Thus it suffices to find such an $f(z)$ with $\sum a_n e^{in\theta}$ divergent everywhere. Theorem 1. There exists a function $f(z)$, analytic for $|z| < 1$, continuous for $|z| \leq 1$, whose Taylor series diverges everywhere on $|z| = 1$. Let $P_n(z)$ be the $n$th Fejer polynomial. Polynomials $f_n(z)$ are constructed in terms of the $P_k(z)$ so that (1) $f_n(z)$ converges uniformly on $|z| \leq 1$, and (2) if $\sum a_n z^n$ is the power series obtained from $\sum f_n(z)$ by removing parantheses, it converges absolutely for $|z| < 1$ and $\sum a_n e^{in\theta}$ diverges everywhere. The main technique is to construct a series $\sum c_n e^{ir_n}$, $r_n$ rational, with essentially the above properties, and then find a power series "close" to this series. (Received November 12, 1963.)

Theorem: Let $T$ be a bounded linear map from $X$ to $BC(S)$, where $S$ is an arbitrary nonvoid topological space and $BC(S)$ is the Banach space of all bounded continuous functions on $S$ with the sup norm. Then there is a map $p: S \rightarrow X'$ which is continuous with the weak* topology on $X'$ and so that $p(S)$ is bounded and $Tx(s) = p(s)(x)$, $x \in X$, $s \in S$; $\|T\| = \sup \{\|p(s)\|; s \in S\}$. Conversely, if $p: S \rightarrow X'$ is continuous with the weak* topology on $X'$ and if $p(S)$ is bounded, then the $T$ above is continuous. $T$ is weakly compact iff in addition $p(S)$ is conditionally weakly compact in $X'$. $T$ is compact iff in addition $p(S)$ is a conditionally norm compact subset of $X'$. The theorem generalizes. A similar theorem [Trans. AMS 1955, pp. 55-56], although stated for this situation, is valid only when $S$ is compact. (Consider $X = BC(S) = m$.) The proof is brief. The only significant information about $BC(S)$ which is used is that implicit in the theorem "If $K$ is a weakly compact subset of a $B$-space, then $\overline{\text{coK}}$ is weakly compact". Numerous detailed and satisfactory particular representation
theorem follow. Much of Tables VIA., B., C. in Dunford-Schwartz follows, including many spaces with no theorems listed. (Received November 12, 1963.)

608-156. P. D. HILL, Emory University, Atlanta 22, Georgia, and C. K. MEGIBBEN, Texas Technological College, Lubbock, Texas. Minimal pure subgroups in primary Abelian groups.

Let G denote a p-primary abelian group. A subsocle of G is a subgroup S of G such that pS = 0. Topological references are to the p-adic topology. Theorem 1. Let S be a dense subsocle of G, $S = G[p]$. If H is maximal in G with respect to $H[p] = S$, then H is pure and dense in G.

Theorem 2. Let L be a subgroup of G. If H is a minimal pure subgroup of G containing L, then $H = A + K$ where A is bounded and $K[p] = L[p]$. The latter theorem enables us to characterize the groups G in which each subgroup is contained in a minimal pure subgroup. Also, a new generalization of Fuchs' Problem 4 is posed and solved. (Received November 12, 1963.)

608-157. SHLOMO HALFIN, University of California, Berkeley 4, California. First integrals of prime differential ideals.

Let R be a differential field with an algebraically closed field of constants and let I be a prime ideal in $R[y_1, \ldots, y_n]$. Let A and B be polynomials such that $c_1A + c_2B \subseteq I$ implies $c_1 = c_2 = 0$ for any constants $c_1, c_2$ in R, and $A'B - AB' \subseteq I$. Then A, B are called components of a first integral of I. It is shown that there exists a unique prime ideal $I^*$ in $R_u[y_1, \ldots, y_n]$, where $R_u$ is the field generated by the adjunction of a transcendental constant u to R, such that $I^*$ includes I and the polynomial $A - uB$, and contracts to I. $I^*$ possesses a characteristic set which is lower than the corresponding set for I. Zeros of $I^*$ can be specialized to zeros of I by replacing u by constants from R. Existence of such specializations is proved. (Received November 12, 1963.)


Denote by $A^k$ the Kronecker product of A with itself k times, and by $A \times B$ the Kronecker product of A with B. Kronecker polynomials $P[A]$ and $P[A,B]$ are defined. Using Williamson's theorem concerning the characteristic roots of matrices with simultaneously triangularizable blocks, results are obtained concerning the characteristic roots of $P[A]$ and $P[A,B]$. A block Kronecker product $A \Theta B$ is defined and the characteristic roots of $A \Theta B$ are found in case A and B have simultaneously triangularizable blocks. Polynomials are defined using block Kronecker products, and results are obtained concerning the characteristic roots of these polynomials. Using Kronecker powers as described in Bellman's Introduction to matrix analysis, a polynomial is defined. Characteristic roots of $A * B = (a_{ij} b_{ij})$ are found for some special matrices. An investigation is made of solutions to matrix equations where the products involved are Kronecker products. (Received November 12, 1963.)
E. G. P. GERLACH, University of Kansas, Lawrence, Kansas.

608-159. Eigenfunction expansions in proper functional Hilbert spaces.

For any separable Hilbert space $H$ and selfadjoint operator $A$ with resolution of identity $\{E_\lambda\}$ and spectral measure $\mu$ one can find $\mu$-null sets $\Lambda_v$, for every $v \in H$, so that $\text{d}E_\lambda \mu$ exists for every $\lambda \notin \Lambda_v \cup \Lambda_w$. Let $H = F$ be a proper functional Hilbert space of functions $f$ on a basic set which can be identified with a total subset $E \subset E$ so that $f(x) = (f, x)$ for $x \in E$; its reproducing kernel is $K(x,y) = (y,x)$. $F$ is called $A$-expansible if $\mu(\bigcup_{x \in E} \Lambda_x) = 0$, and totally expansible if it is expansible for every $A$. Theorem. If $F$ is $A$-expansible then: 1. For every $\lambda \notin \bigcup_{x \in E} \Lambda_x$ the derivatives $(\text{d}E_\lambda^y, \text{d}E_\lambda^x)/\text{d}\mu(\lambda)$ exists for $x \in A \Lambda_x$. Theorem. If $F$ is $A$-expansible then: 2. The direct integral $\int F(\lambda) \text{d}\mu(\lambda)$ is canonically determined, and canonically isomorphic to $F$. 3. To $f \in F$ corresponds $\mu(\lambda) = 0$, and for $\lambda \in \Lambda^f, f(x;\lambda) \in F(\lambda)$ so that $f(x;\lambda) = (\text{d}E_\lambda f, \text{d}E_\lambda x)/\text{d}\mu(\lambda).$ The $f(x;\lambda)$ are generalized eigenfunctions of $A$ corresponding to the eigenvalue $\lambda$. Results of G. I. Kac lead to: $F$ is totally expansible if $E$ is contained in the range of a Hilbert-Schmidt operator in $F$. Examples are given. The considerations may be extended to measurable Hilbert spaces with pseudo-reproducing kernels. (Received November 12, 1963.)

608-160. HARUO MURAKAMI, University of Kansas, Lawrence, Kansas. Regularity of domains for semi-linear partial differential equations of parabolic type.

Let $D$ be a region in $(n + 1)$-dimensional $(t,x)$-space which has an upper base $S$ in the hyperplane $t = b$. We denote $\mathcal{S} = D \cup \mathcal{S}$, and $\partial_p \mathcal{S} = \partial \mathcal{S} - \mathcal{S}$, where $\mathcal{S}$ is the $n$-dimensional interior of $S$. Consider $\mathcal{R}_u \equiv \sum_{i,j=1}^n a_{ij}(t,x) \partial^2 u/\partial x_i \partial x_j - \partial u/\partial t = f(t,x,u,Vu)$ in $D$. We assume that $a_{ij}(t,x)$ are bounded and continuous on $\mathcal{S}$, that the matrix $a_{ij}(t,x)$ is uniformly positive definite on $D$, and that for any $M > 0$ there exist $B$ and $\Gamma$ such that $|f(t,x,u,p)| \leq B |p|^2 + \Gamma$ for $(t,x) \in D$. $u \leq M$ and $|p| < \infty$. Suppose that for $(t^0,x^0) \in \partial_p \mathcal{S}$ there exists a function $\xi(t,x) \in C^2$ in $\mathcal{S}$ such that $\mathcal{R}_\xi \leq -1$ in $\mathcal{S}$, $\lim_{(t,x) \to (t^0,x^0)} \xi(t,x) = 0$ and $\lim_{(t,x) \to (t^0,x^0)} (\frac{\partial^2 \xi}{\partial x_i \partial x_j}(t,x)) \equiv 0$ for any $(t^1,x^1) \in \partial_p \mathcal{S}$. Then, for any bounded function $\beta(t,x)$ given on $\partial_p \mathcal{S}$, we can construct a suitably defined barrier for our equation at the point $(t^0,x^0)$, and this barrier is used for proving the existence of a solution of our equation with boundary condition $u = \beta$ on $\partial_p \mathcal{S}$. (Received November 12, 1963.)


Let $V$ be the complex elliptic curve $yz^2 = x(x - z)(x - k^2 z)$ with $0 < k < 1$. It is well known that $V$ can be parametrized by theta functions in such a way that the torus they define is mapped in a one-to-one manner onto the curve, so that $V$ acquires the structure of abelian variety. This procedure is carried out here explicitly; also for the curve $y^2 z = (x^2 - z^2)(x^2 - k^2 z^2)$. The novel feature consists in the adjustment of the parameters in the theta functions to match the constant $k$, in a way avoiding the use of modular functions. (Received November 12, 1963.)

The complex zeros $v_n(z)$, $n = 1, 2, 3, \ldots$ of $J_\nu(z)$, $dJ_\nu(z)/dz$ and $dJ_\nu(z)/dz + ZJ_\nu(z)$ are investigated. Here $J_\nu(z)$ is the Bessel function of order $\nu$ and argument $z$, the functions are to be considered as functions of $\nu$ with $z$ fixed, and $Z$ is given constant. These zeros are the eigenvalues of certain non-self-adjoint boundary value problems which arise in finding alternative representations for the solution of the reduced wave equation in cylindrical and spherical coordinates. Formulas for $v_n(z)$ are obtained for both large and small values of $|z|$ and for large values of $n$. Analogous problems for Hankel functions have been treated recently by Keller, Rubinow, and Goldstein (J. Mathematical Physics 4 (1963), 829-832), who stress the importance of the zeros as poles of scattering amplitudes. (Received November 12, 1963.)

608-163. B. A. JOHNS, JR., University of Kansas, Lawrence, Kansas. On the absolute Norlund summability of a Fourier series.

In this note the following theorem is proved: Suppose that $\{p_n\}_{n=0}^\infty$ is a sequence of bounded variation, with $p_0 > 0$ and $p_n \geq 0$ (n = 1, 2, ...). Let $P_n = \sum_{k=0}^{n} p_k$ (n = 0, 1, ...). Suppose that $f_j$ (j = 1, 2) is a function of class $L(0,2\pi)$, periodic with period $2\pi$; and let its Fourier series be $\sum_{n=0}^\infty A_n^j(t)$. Suppose that $f_1 = f_2$ on some neighborhood of a point $x$ and suppose that at least one of (a) - (d) below is true:

(a) $\sum_{n=0}^\infty |A_n^1(x)|/P_n < \infty$ (j = 1, 2); $\sum_{n=0}^\infty |A_n^2(x)|/P_n < \infty$ (j = 1, 2);
(b) $\sum_{n=0}^\infty |A_n^1(x)|/P_n^2 < \infty$ (j = 1, 2);
(c) $\sum_{n=0}^\infty |A_n^1(x)|/P_n < \infty$ (j = 1, 2); $\sum_{n=0}^\infty P_n/n^2 < \infty$;
(d) $\sum_{n=0}^\infty |A_n^1(x)|/P_n < \infty$ (j = 1, 2); $\sum_{n=0}^\infty P_n/n^2 < \infty$; $\limsup_{n \to \infty} P_n/n < \infty$.

Then both $\sum_{n=0}^\infty A_n^1(x)$ and $\sum_{n=0}^\infty A_n^2(x)$ are summable $[0,p_n]$, or neither is summable $[0,p_n]$. (Confer S. N. Bhatt, Proc. Nat. Inst. of Sci, India 28 (1962), 787-794.) (Received November 12, 1963.)

608-164. R. C. BUCK, Van Vleck Hall, University of Wisconsin, Madison, Wisconsin. The solution of systems of functional equations.

Using the characterization theorem for extreme points [Bull. Amer. Math. Soc. 65, 130-133], the extreme measures which have norm 1, are supported on a compact set $K$, and annihilate the space of functions of the form $F(x,y) = A(x) + B(y)$, can be determined. (See Abstract 571-138, Notices Amer. Math. Soc.) In particular, this yields conditions on $K$ under which every continuous function $f$ can be uniformly approximated on $K$ by functions $F$. Unexpectedly, these results now yield necessary and sufficient conditions for the existence of (approximate) continuous solutions $\phi$ of a general system of simultaneous functionals of the form $\phi(a_j(x)) - \phi(b_j(x)) = u_j(x)$, $j = 1, 2, \ldots, n$, where the functions $a_j$, $b_j$, and $u_j$ are unrestricted continuous functions on $[0, 1]$. Corresponding results are obtained for functions of several variables. (Received November 12, 1963.)

A 3-tuple of mappings \((f,g,h)\) of \(H\) into \(G\) is called a homotopism of the groupoid \(H\) into the groupoid \(G\) if for all \(x,y \in H\), \(f(x)g(y) = h(x \cdot y)\) and then \(G\) can be called homotopic to \(H\). **Theorem 1.** Iff \((f,g,h)\) is a homotopism of an abelian semigroup \(S\) into a group \(G\), then there exists a homomorphism \(k\) of \(S\) into \(G\) so that \(f(x) = ak(x)\), \(g(y) = k(y)b\), \(h(x \cdot y) = ak(x \cdot y)b\), \((0)\) holds. **Theorem 2.** Iff \((f,g,h)\) is a homotopism of a transitive \((G \cdot S = S)\) abelian semigroup \(S\) into \(G\) (a group enlarged by a zero-element), then either there exists a homomorphism \(k\) of \(S\) into \(G\) so that \((0)\) hold, or the homotopism is trivial \((g = h = 0, f\) arbitrary or \(f = h = 0, g\) arbitrary). Counter-examples show that Theorem 2 is not true anymore for nontransitive semigroups. The problem of determining all homotopisms in this latter case is still open. (Received November 12, 1963.)

W. A. HARRIS, JR. and YASUTAKA SIBUYA, University of Minnesota, Institute of Technology, Minneapolis 14, Minnesota. General solution of nonlinear difference equations.

A method for obtaining the general solution of a system of nonlinear difference equations of the form \(y(x + 1) = f(x,y(x))\) is presented, where \(x\) is a complex variable, \(y\) is an \(n\)-dimensional vector, and \(f(x,y)\) is an \(n\)-dimensional vector with components holomorphic in the region \(\text{Im} x \geq R, \|\| \leq \delta\). It is assumed that \(f\) admits a uniformly asymptotic expansion \(f(x,y) \approx \sum_{k=0}^{\infty} x^{-k} k(y)\) for \(\|\| \leq \delta\) as \(x \to -\infty\) in \(\text{Im} x \geq R\); \(f_k(y)\) are holomorphic for \(\|\| \leq \delta\); \(f_0(0) = 0\); \(1 < |\lambda_1| < |\lambda_2| < \ldots < |\lambda_n|, \prod_{i=1}^{n} |\lambda_i|^{p_i} \neq |\lambda_j|, j = 1, \ldots, n\), where \(\lambda_j\) are the eigenvalues of \(f_0y(0)\) and \(p_i\) are nonnegative integers. (Received November 12, 1963.)

A. E. DANENSE, State University of New York, Buffalo, New York. Some properties of Jacobi polynomials.

Monotonicity properties of the Turán function: \(\left[\frac{P_n(a,\beta)(x)}{P_{n+1}(a,\beta)(x)}\right]^2 \cdot \frac{P_{n+1}(a,\beta)(x)}{P_n(a,\beta)(x)}\), where \(P_n(a,\beta)(x)\) is the Jacobi polynomial of degree \(n\), are studied and these are used to establish a number of convexity theorems for the zeros of the Jacobi polynomials. (Received November 12, 1963.)

J. T. DARWIN, JR., Auburn University, Auburn, Alabama. A representation theorem for linear operators.

Suppose \(E\) is a subset of the complex numbers, and \(S\) is a complete inner product space, with inner product \(<\cdot, \cdot>\) and norm \(\|\cdot\|\), such that each point of \(S\) is a complex valued function from \(E\). Further, suppose that there exists a real valued function, \(R\), from \(E\) such that \(|f(z)| \leq \|f\| \cdot R(z)\) for each number \(z\) in \(E\) and each function \(f\) in \(S\). Then, if \(T\) is a bounded linear operator on \(S\), there exists a function, \(\Phi\), from \(E \times E\) such that \((1)\) for each number, \(z\), in \(E\), \(\Phi(z, \cdot)\) is in \(S\) and \((2)\) \((T\Phi)(z) = <f, \Phi(z, \cdot)>\) for each \(f\) in \(S\) and \(z\) in \(E\). (Received November 12, 1963.)
A note on counting isotropy subgroups.

An elementary p-group of rank k is a group isomorphic to the direct product of k copies of \( \mathbb{Z}_p \), the additive group of integers modulo a prime p. The purpose of this note is to show that if such a group \( G \) of homeomorphisms acts effectively on the n-sphere \( S^n \), then the number of distinct isotropy subgroups cannot exceed \( 2^{\lfloor (n+1)/2 \rfloor} \), the same bound which exists for orthogonal actions. The proof proceeds by first observing that every maximal isotropy subgroup (contained properly in no isotropy subgroup with the possible exception of \( G \)) must be of rank \( k - 1 \). A formula of Borel's (Seminar on transformation groups, Annals of Mathematics Studies, p. 175) is then utilized to show that the number of maximal isotropy subgroups cannot exceed \( \lfloor (n + 1)/2 \rfloor \) and that, moreover, each isotropy subgroup of rank \( k - i \) \((2 \leq i \leq k - 1)\) is the intersection of i maximal isotropy subgroups. Noting that \( k \leq \lfloor (n + 1)/2 \rfloor \), a theorem of Smith's, and allowing for the isotropy subgroups \( G \) and \( \{e\} \), the result follows. (Received November 12, 1963.)

This paper presents a unified theory of completions which is applicable to uniform spaces, topological groups and linear topological spaces. The development is based on the concept of convergence on filters and the related topologies (see J. W. Brace, Abstract 587-32, these Notices, No. 7 (1961)). The desired completion of an object \( X \) is obtained by densely embedding \( X \) in a space of continuous functions where the domain of the functions is a space \( F(X) \) of functions defined on \( X \), \( F(X) \) being topologized by convergence on a specified class of filters in \( X \). For example, if \( X \) is a locally convex space and \( X' \) has the topology \( t \) of convergence on the Cauchy filters of \( X \), then the topology \( t \) is the coarsest topology such that \((X', t')\) is the completion of \( X \), \((X', t')\) having the topology of uniform convergence on equicontinuous subsets of \( X' \). In a similar way the second adjoint \( X'' \) is obtained as the weak completion of the bounded subsets of \( X \). For a uniform space \( X \) the parallel situation is the dense embedding of \( X \) in a function space which is the "smallest" space for which a specified collection of subsets of \( X \) has the property that the closure of each set is complete. (Received November 12, 1963.)

Let \( A \) be an integral domain and \( B \) a ring between \( A \) and its quotient field. \( B \) is said to be a generalized quotient ring (GQR) of \( A \) if \( B \) is a flat \( A \)-module. **Theorem 1.** \( B \) is a GQR of \( A \) iff for every prime ideal \( P \) of \( A \) either \( PB = B \) or \( B \subseteq A_P \). **Theorem 2.** \( B \) is a GQR of \( A \) iff \( B_M = A_M \cap A \) for every maximal ideal \( M \) of \( B \). **Corollary.** If \( B \) is a GQR of \( A \) then \( B = \bigcap A_P \), where \( P \) ranges over some set of prime ideals of \( A \). GQR's enjoy the standard properties of quotient rings, e.g. with respect to integral closure and extensions of ideals. For unique factorization domains the two concepts coincide. Nontrivial examples are available for \( A \), a Dedekind domain whose class group is not torsion. (Received November 12, 1963.)
A topology $T$ on a set $X$ is said to be minimal Hausdorff iff there exists no Hausdorff topology on $X$ strictly weaker than $T$. **Theorem 1.** The product of minimal Hausdorff spaces is minimal Hausdorff. **Corollary 1.** The product of compact Hausdorff spaces is compact. **Corollary 2.** There exist minimal noncompact Hausdorff spaces of arbitrary infinite cardinality. **Theorem 2.** Let $(X, T)$ be a topological space. If there exists a collection $\{T_a\}$ of minimal Hausdorff topologies on $X$ such that $T$ is the smallest topology containing all the $T_a$, then $(X, T)$ can be embedded in a minimal Hausdorff space. (Received November 12, 1963.)

**Almost-Gaussian domains.**

Let $\mathcal{O}, m)$ be a normal local domain with quotient field $K$, let $K^0 = K \setminus \{0\}$ and let $I$ be the collection of all essential valuations of $\mathcal{O}$. If $f$ is a group homomorphism from $K^0$ into the additive group of real numbers that is non-negative on $K^0 \cap \mathcal{O}$ and zero on $\mathcal{O} \setminus m$, then it is shown here that $\mathcal{F}$ is a non-negative, real-valued function $F$ defined on $I$ such that $f(x) = \sum_{v \in I} F(v)x(x)$, for all $x \in K^0$, thus extending a result of Samuel (Multiplicités de certaines composantes singulières, Ill. J. Math, 3 (1959), 319-327). Any such function $F$ is called a representation function for $f$. It is further shown that if $f$ and $g$ are homomorphisms of the type described above such that $f$ has a unique representation function, then also $g$ has a unique representation function provided the linking number $\ell(f, g) = \inf \{|f(x)/g(x); x \in K^0 \cap \mathcal{O}, g(x) \neq 0\}$ is nonzero. In Samuel's terminology, this means that if $w$ is a rank 1 discrete valuation which dominates $\mathcal{O}$ such that $\mathcal{O}$ is almost-Gaussian (presque-factoriel) relative to $w$, then also $\mathcal{O}$ is almost-Gaussian relative to each rank 1 discrete valuation $w'$ such that $\ell(w, w') \neq 0$. (Received November 12, 1963.)

**A new class of self-adjoint differential operators of the pure wave type.**

Consider the linear partial differential operator $\mathcal{L}_m u = u_{tt} - \sum_{i=1}^{m-1} u_{x_ix_i} + c(t)u$ with $c(t)$ analytic on some open, connected interval of the real $t$-axis. Assume that $\mathcal{L}_m$ is of the pure wave type, that is, $\mathcal{L}_m$ satisfies Huygens' principle in the sense of "Hadamard's minor premise."

**Theorem 1.** In the special case $m = 8$, the complete set of functions $c(t)$ for which $\mathcal{L}_m$ is of pure waves is essentially $\{0, -2/\tau^2, -6/\tau^2, -6\tau(2 + \tau^3)/(1 - \tau^3)^2\}$. **Theorem 2.** Let $\mathcal{L}_m$ be a fixed pure wave operator. There exists a family $F$ of linear first order differential operators, depending on $\mathcal{L}_m$ but independent of $m$, which has the following properties: (i) For each element $A \in F$ there is a differential operator $\mathcal{L}_m^A u = u_{tt} - \sum_{i=1}^{m-1} u_{x_ix_i} + c^*(t)u$, depending on $A$, for which $\mathcal{L}_m^A = \mathcal{L}_m^A$; (ii) The operator $\mathcal{L}_{m+2}$ is again of pure waves. The class of pure wave operators obtainable in this manner contains as special cases all previously known pure wave operators of the form under consideration. In addition, a slight modification of the argument permits the construction of a class of operators of the form $L_m u = u_{tt} - \sum_{i=1}^{m-1} u_{x_ix_i} + c(x)u (x = (t, x_1, ..., x_{m-1}))$ which again includes all previously known pure wave operators of this type. (Received November 12, 1963.)
A generalization of the EHP-sequence.

Let \( F \to E \to B \) be a fibration and let \( r: E \cup CF \to B \) extend the projection by mapping \( CF \) to the base-point. Then, the fibre of \( r \) has the homotopy type of the join \( F \star A \). This yields a new proof of a theorem of Serre concerning the homomorphism \( H_q(E,F) \to H_q(B) \). It is also possible to generalize the classical results on the homology suspension. Dually, let \( A \to X \to B \) be a cofibration, let \( F \) be the fibre of \( X \to B \), let \( e: A \to F \) be the canonical embedding, and let \( \phi \) be the fibre of \( A \to X \). Then there is an \( (m + n - 1 + \text{Min}(m,n)) \)-connected map \( A \to F/A \) if \( A \) is \( (n - 1) \)-connected and \((X,A)\) is \( m \)-connected. This yields a generalization of G. W. Whitehead's EHP sequence. Using this result it is possible to generalize a theorem by Sugawara obtaining a result relating the connectivity of \( X \), \( \text{nil} X \) and \( \text{cocl} X \). (Received November 12, 1963.)

Bauer fields of values of a matrix.

F. L. Bauer (Num. Math. 4 (1962), 103-111) has defined the field of values of a complex matrix \( A \) subordinate to a strictly homogeneous vector norm \( \| \cdot \| \) as follows: \( F[A,\| \cdot \|] = \{ (y, Ax) : (y,x) \} = \{ y \| \cdot \| : \| x \| = 1 \} \), where \( y \) is a vector of the (conjugate) dual space and \( \| \cdot \|' \) is the norm dual to \( \| \cdot \| \). Let \( F^0[A,\| \cdot \|] \) be the convex closure of \( F[A,\| \cdot \|] \). If \( P[A] \) is the convex hull of the eigenvalues of \( A \), it is shown that \( P[A] \) is the intersection of \( F^0[A,\| \cdot \|_G] \) over all nonsingular diagonal \( G \), where \( \| x \|_G = \| Gx \| \). For a wide class of norms (which includes in particular all Hölder norms) it is proved that there exists a \( G \) such that \( P[A] = F^0[A,\| \cdot \|_G] \) if and only if all eigenvalues on the boundary of \( P[A] \) have simple elementary divisors. For the case of the Euclidean norm \( \| \cdot \| \) these results are known and due to Givens (Proc. Amer. Math. Soc. 3 (1952), 206-209). (Received November 13, 1963.)

Invariant subspaces of continuous functions.

Let \( X \) be the unit circle \( \{ z : |z| = 1 \} \) in the complex plane. A space \( E \) of complex-valued functions on \( X \) is called simply invariant if \( zE \subseteq E \) but \( zE \neq E \). It is shown that any uniformly closed simply invariant subspace \( E \) of \( C(X) \) is of the form \( qL^\infty(X) \cap Z(K) \), where \( q \subseteq L^\infty(X) \) and \( |q| = 1 \) a.e. on \( X \), and \( Z(K) \) is the space of functions in \( C(X) \) which vanish on a closed set \( K \) in \( X \) of Lebesgue measure zero. Conversely, any nontrivial \( qL^\infty(X) \cap Z(K) \) is a uniformly closed simply invariant subspace of \( C(X) \). The function \( q \) is determined by \( E \) uniquely up to a constant multiple of modulus one. For any closed \( K \) of Lebesgue measure zero in \( X \), \( qL^\infty(X) \cap Z(K) \) is nontrivial if and only if \( qL^\infty(X) \cap C(X) \) is nontrivial. A necessary and sufficient condition on \( q \) for nontriviality of \( qL^\infty(X) \cap C(X) \) is given. These results may be regarded as a generalization of a theorem of Beurling-Rudin on the structure of closed ideals of the disc algebra. A characterization of the weak-star closed simply invariant subspaces of the dual \( M(X) \) of \( C(X) \) is also obtained. (Received November 13, 1963.)
Let \( H \) be the set of \( \infty \)-smooth homeomorphisms of \( S^2 \) with the \( C^r \)-topology, \( 1 \leq r < \infty \). A point \( x \) of \( S^2 \) is called a rank 1 critical point of \( f (f \in H) \) iff the differential of \( f \) has rank 1 at \( x \). Theorem, if \( C \) is a closed set of rank 1 critical points of \( f \), and if \( U \) is a neighborhood of \( C \), then there is a path \( t \to f_t \) in \( H \) such that \( f_0 = f \), \( f_t(x) = f(x) \) for \( x \in (S^2 - U) \), and \( f_1 \) has no rank 1 critical points in a neighborhood of \( C \). The proof uses a theorem of Palais (Natural operations on differential forms, Trans. Amer. Math. Soc. 92 (1959), 125-141) to reduce the problem to the following: Suppose that \( f \) is an \( \infty \)-smooth homeomorphism of the unit square \( I^2 \) which leaves a neighborhood of the boundary pointwise fixed. If, for \( x \) in a neighborhood of \( x_0 \in I^2 \), the symmetric factor of the differential \( f'(x) \) has distinct eigenvalues, and the orthogonal factor of \( f'(x) \) is constant, then there is a neighborhood \( U \) of \( x_0 \) and a family of weights \( t \to W_t(y,x) \) such that \( f_t(x) = f(x + y)W_t(y,x)dy \) satisfies the conclusions of the statement of the theorem with \( S^2 \) replaced by \( I^2 \). (Received November 13, 1963.)

Let \( \mathcal{M} \) be a Riemannian manifold of class \( C^{(k,1)} \) and such that the metric tensor is of class \( C^{(k-1,1)} \). For \( a \leq k + 1 \) we define \( P^a_{10c} (\mathcal{M}) \) as the class of functions on \( \mathcal{M} \) which are in \( P^a_{10c} (H(U)) \) where \( H(U) \subset R^n \) is the \( C^{(k-1,1)} \) homeomorphic image of an arbitrary coordinate patch \( U \). For \( m \) an integer, \( 0 \leq m \leq k + 1 \), we define the class \( \tilde{P}^m (\mathcal{M}) \) as the subspace of \( P^m_{10c} (\mathcal{M}) \) on which the norm

\[ |u|_{m,\mathcal{M}}^2 = \sum_{k=0}^{m} C_{m,k} \int |u_k(x)|^2 dx \]

where \( u_k(x) \) is the norm of the tensor of \( k \)th order covariant derivatives of \( u \) at \( x \in \mathcal{M} \). If \( 0 \leq m < a < m + 1 \leq k + 1 \), we define \( \tilde{P}^a (\mathcal{M}) \) by quadratic interpolation between \( \tilde{P}^m (\mathcal{M}) \) and \( \tilde{P}^{m+1} (\mathcal{M}) \) with parameter \( t = a - m \). If \( \mathcal{M} \) is a compact manifold and \( \{U_k\}_{k=1}^{m} \) is a covering of \( \mathcal{M} \) by coordinate patches then \( |u|_{a,\mathcal{M}}^2 \) is equivalent to \( \sum_{k=1}^{m} |u|_{a,H_k(U_k)}^2 \) where \( |u|_{a,H_k(U_k)} \) is the standard \( a \)-norm of Bessel potentials. A similar equivalent norm can be constructed for a class of open manifolds which admit infinite coverings by coordinate patches satisfying suitable uniformity conditions. In particular, this class of manifolds contains the boundaries of unbounded domains in \( R^n \) which are sufficiently regular. (Received November 13, 1963.)

Denote by \( \mu(r) \), \( 0 < r < 1 \), the conformal modulus of the doubly connected domain determined by the interior of the unit disc minus the slit \( 0 \leq Re z \leq r \), \( Im z = 0 \). Let \( 1 \leq K < \infty \) and let \( w(z) \) be a sense-preserving homeomorphism of a plane domain \( D \). It is shown that \( w(z) \) is \( K \)-quasiconformal if and only if for each Jordan domain \( J \subset D \), each boundary arc \( a \) of \( J \), and each point \( z_0 \) in \( J \),

\[ \mu(\sin \omega/2) \leq K \mu(\sin \omega/2) \]

where \( \omega \) is the harmonic measure of \( a \) at \( z_0 \) with respect to \( J \), and \( \omega ' \) is the corresponding harmonic measure in the image plane. It is also shown that \( w(z) \) is \( K \)-quasiconformal if and only if for each simply connected subdomain \( G \) of hyperbolic type, and each pair of points \( z_0 , z_1 \in G \), \( \mu(e^{-2h}) \geq K \mu(e^{-2h}) \), where \( h \) is the hyperbolic distance between \( z_0 \) and \( z_1 \) with respect to \( G \), and \( h ' \) is the hyperbolic distance in the image plane. The necessity for both re-
sults has been shown by J. Hersch (Comment, Math. Helv. 30 (1956), 1-19). The second result leads directly to certain distortion theorems in the theory of quasiconformal mappings. (Received November 13, 1963.)


A surface of type \((p, l', r)\) will be a nonsingular, compact, orientable surface tamely embedded in the three sphere \(S^3\) with \(r\) components, \(l'\) 1-spheres as boundary, and \(p\) is the sum of the genera of the components of \(S\). Finally, each component of \(S\) is required to have nonempty boundary.

**Theorem 1.** If \(L\) is a tame link in \(S^3\) with \(l'\) components and of genus \(p\), then there is a surface of type \((p, l', r)\) with boundary \(L\) such that the inclusion \(i: S^3 - S \subset S^3 - L\) induces a monomorphism \(\pi_1(S^3 - S) \to \pi_1(S^3 - L)\). If \(S'\) is any surface of type \((p, l, r)\) with boundary \(L\), since there is a surface \(S\) of type \((p, l', 1)\) with boundary \(L\), one may speak of the genus of a tame link \(L\) in \(S^3\) with \(l\) components to be the smallest number \(p\) such that \(L\) can be spanned by a surface \(S\) of type \((p, l, r)\) where the inclusion \(i: S^3 - S \subset S^3 - L\) induces a monomorphism \(i_*: \pi_1(S^3 - S) \to \pi_1(S^3 - L)\). Let the image of \(i_*\) in \(G = \pi_1(S^3 - L)\) be \(H\). **Theorem 2.** If \(H \cap [G, G]\) is trivial and \(L\) is not splittable, then \(L\) has genus zero. (Received November 13, 1963.)

608-182. R. E. PEINADO, State University of Iowa, Iowa City, Iowa 52240. Ancestral rings and left-idealizers.

A ring \(R\) is called an Ancestral ring, written \(A\)-ring, if all nonzero subrings of \(R\) have a specified property. Let \(K\) be a subring of \(R\). The left-idealizer, \(I_1(K)\), is the largest subring of \(R\) in which \(K\) is a left-ideal. Right idealizers, \(I_r(K)\), and idealizers, \(I(K)\), are analogously defined. For \(K\) and \(S\) arbitrary subrings of \(R\): (i) \(I_1(K) \cap I_1(S) \subseteq I_1(K \cap S)\), (ii) \(I_1(K \cup S) \subseteq I_1(K) \cup I_1(S)\), (iii) \(I_1(K) \subseteq I_1(K^R)\). Examples are given to show (i) and (ii) are the best possible results. Every subring and every homomorphic image of an \(A\)-division ring is an \(A\)-division ring. A similar result is not true, in general, for an arbitrary \(A\)-ring. Let \(R\) be a ring such that \(I_1(K) = K\) for every nonzero subring in \(R\). Then the same result holds in every homomorphic image of \(R\) and in every subring of \(R\) considered as a ring. A result similar to the above is obtained if \(I_1(K) \neq K\) for every nonzero subring \(K\) of \(R\). **Theorem 1.** A ring \(R\) is an \(A\)-division ring if and only if \(R\) is a field of prime characteristic, that is an algebraic extension of its prime subfield. **Theorem 2.** \(I_1(K) = K\) for every nonzero subring of \(R\) if and only if \(R\) is an \(A\)-division ring. Other results include: if for some integer \(n\), \(I_1(K^n) = R\), then \(I_1(K) \neq K\). If \(R\) is a finite ring and \(I_1(K) \neq K\), then there exists an \(n \in \mathbb{Z}\) such that \(I_1(K^n) = R\). (Received November 13, 1963.)


Let \(K\) be an equational category (= equationally defined class of abstract algebras, cf. Slominski, Rozprawy Mat. 18, Warsaw, 1959). **Notation:** \(F_y, |A|, \text{Hom}_K(A, B), S, \text{Struc}(K, K)\) denote the free \(K\)-algebra on one generator, the underlying set of the \(K\)-algebra \(A\), the set of \(K\)-algebra homomor-
The following three conditions on $K$ are equivalent: (1) each operation in each $K$-algebra is a $K$-algebra homomorphism; (2) there is a functor $h_K: K^* \times K^* \to K$ satisfying both (2a) $|h_K| = \text{Hom}_K(\bullet, \bullet)$ and (2b) $h_K(F, \cdot) \cong \text{id}_K$; (3) $\text{Hom}_K(A, B)$ is a subalgebra of $B^A$, for all $A$ and $B$. Remark: The functor $h_K$ of (2) is uniquely determined by (2a) and (2b); that (2b) is essential is shown by any category where the canonical projection $\pi: \text{Struct}(K, K)$ has at least two sections. Theorem 2: If $\pi$ has exactly one section (in particular, if $\pi$ is an equivalence), then the conditions of Theorem 1 are valid and $h_K$ is determined uniquely by (2a) alone. Open questions: Is $\pi$ an equivalence if it has exactly one section? Has $\pi$ always at most one section? (Received November 13, 1963.)

Ascoli's theorem deals with continuous functions and states that the space of bounded, equi-continuous functions is compact. The present paper extends it to the measurable functions. The space of bounded "equi-measurable functions" is compact, and it contains the bounded equi-continuous functions as a subset. The above theorem is applied to two problems in the theory of optimal control: (1) to give an existence proof of optimal control among allowed control functions which are measurable and enter the system equations in a nonlinear manner, (2) to derive a necessary condition for optimal control in bounded phase space. The method used in deriving the latter is as follows: In place of the rigid bound, a cost function with a multiplier $K$ is introduced for regions beyond the boundaries in phase space. It is shown that in the limit of the multiplier $K$ approaching infinity, both the added cost and the maximum excursion of the optimal path beyond the boundaries approach zero, and the condition for optimal control is thus derived. The generalized Ascoli's theorem is useful in both the existence proof of optimal control, and in proving the existence of a limit as $K$ approaches infinity. (Received November 13, 1963.)

R. P. Gilbert, Institute for Fluid Dynamics, University of Maryland, College Park, Maryland. Concerning the location of singularities of solutions to certain classes of elliptic partial differential equations in four variables.

The classical results of Hadamard and Mandelbrojt concerning the Taylor coefficients of an analytic function of one complex variable and the number and location of its singularities has just been generalized by S. Bergman (The coefficient problem in the theory of a system of linear partial differential equations, to appear) to the case of functions of two complex variables. In this paper we develop analogous results for harmonic functions of four variables, and certain other elliptic partial differential equations by using the method of integral operators, and generalizations of certain three-variable operators introduced earlier by S. Bergman (Integral operators in the theory of linear partial differential equations, Ergeb, Math, 23, Springer, 1960). Other results concerning the location of singularities are obtained using a classical result of Weierstrass combined with an earlier result of the author (On harmonic functions of four variables with rational associates, Pacific J. Math. 13, No. 1 (1963)). (Received November 14, 1963.)

N. R. Zitron, Purdue University, Lafayette, Indiana. Multiple scattering of elastic waves by cylindrical cavities.

A perturbation method employed previously by Zitron and Karp [J. Math. Phys. 2, 394 (1961)] for multiple scattering of scalar with harmonic time dependence $e^{-i\omega t}$ applicable to the acoustic and electromagnetic cases has now been extended to the case of elastic waves. Under the appropriate conditions of differentiability and linearity, the difficulties arising in the elastic case from the coupling of the vector and scalar wave functions at boundaries between different media may be overcome. The perturbation method employed previously can then be used to obtain an asymptotic solution for two widely spaced cylinders of arbitrary shape. (Received November 14, 1963.)
For each integer \( n > 0 \) let the non-negative integer \( p_n \) be defined by \( i_\ast (\pi_{4n-1}(SO(2n + 1))) = 2^{p_n}(\pi_{4n-1}(SO)) \), where \( i : SO(2n + 1) \rightarrow SO \) is the inclusion. Let \( a_n = 2 \) if \( n \) is odd, and \( a_n = 1 \) if \( n \) is even. **Theorem.** If \( n > 1 \), and if \( \Delta \) is the transgression in the bundle \( (SO(2n + 1), S_{2n}, SO(2n)) \), the Samelson product \( \langle \Delta_{2n}, \Delta_{2n} \rangle \) is either of order \( 2^{p_n}a_n(2n - 1)!/8 \) or of order \( 2 \cdot 2^{p_n}a_n(2n - 1)!/8 \). This refines a result of I. M. James, *Products on spheres*, Mathematika 6 (1959), 1-13. (Received November 14, 1963.)

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**608-190. WITHDRAWN.**

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**608-191. A. W. ADLER, Purdue University, Lafayette, Indiana. Universal Laplacians.**

Let \( M \) be a Riemannian manifold with Riemannian metric \( g \). An isometric imbedding of \( M \) in the unit sphere of a high-dimensional Euclidean space, together with the spherical image mapping of Gauss, define a nonsingular mapping \( \phi \) of \( M \) into a suitable Grassmann manifold. There exists on this Grassmann manifold a natural operator \( \Delta \) of degree 0 (independent of \( M \) and of \( \phi \)) with the following property: If \( M \) is a complex-analytic manifold and if \( g \) is the real part of a Kahler-metric on \( M \), then \( \phi \ast \Delta \ast \phi^{-1} \ast \) is the Laplacian of \( M \). (Received November 14, 1963.)

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**608-192. N. J. DIVINSKY, University of British Columbia, Vancouver, Canada and A. SULINSKI, University of Warsaw, Warsaw, Poland. Kurosh radicals of rings with operators.**

Jacobson proved that for any ring \( A \) which is an algebra over a commutative ring with unity, the Jacobson radical of \( A \), thought of as an algebra, is precisely the same as the Jacobson radical of \( A \), thought of merely as a ring. This result is extended to: **Theorem.** If \( A \) is any ring with operator domain \( T \), and \( P \) is any radical property in the sense of Kurosh, then the \( P \)-radical of \( A \) thought of as a ring with operators \( T \), exists and is precisely the same as the \( P \)-radical of \( A \), thought of merely as a ring. (Received November 14, 1963.)

The minimal prime ideals of the ring \( C(X, \mathbb{Z}) \) of continuous integer-valued functions is homeomorphic with \( 2^X \). (See R. S. Pierce's paper in Trans. Amer. Math. Soc. 100 (1961), 371-395 for definitions.) The associated residue class domains \( \mathcal{B} \), when not \( \mathbb{Z} \), are non-Archimedean, totally ordered, nonstandard models of \( \mathbb{Z} \) which are near \( \eta_1 \)-sets. (See Alling, Trans. Amer. Math. Soc. (to appear) for definitions.) The local ring \( \mathcal{B}_N \) of \( \mathcal{B} \) by a maximal ideal \( N \) of \( \mathcal{B} \) is a valuation ring in the quotient field \( \mathbb{Q} \). \( \mathbb{Q} \) is a nonstandard model of the rationals which is a near \( \eta_1 \)-set. If \( N \) is not finite then the value group \( G \) of \( \mathcal{B}_N \) is a nonstandard model of the group \( \mathbb{Z} \) that is a near \( \eta_1 \)-set. (Compare with the meromorphic function case in Alling, Acta. Math. (to appear).) Thus the prime ideals in \( C(X, \mathbb{Z}) \) below a maximal ideal form a chain, anti-isomorphic to the lower sets of the value set of \( G \). Further, maximal ideals in \( C(X, \mathbb{Z}) \) contain a prime-valued function iff the corresponding ideal in \( \mathcal{B} \) is principal. Finally, maximal ideals in \( \mathcal{B} \) are either principal or uncountably generated. (Received November 14, 1963.)

608-194. J. R. RETHERFORD, Florida State University, Tallahassee, Florida. \( w^* \)-bases of subspaces.

A study of \( w^* \)-Schauder bases of subspaces (\( w^* \)-Sbos) in the first and second conjugate spaces of a Banach space \( X \) is made (for definitions see Notices 10 (1963) 644, where a basis of subspaces is called a decomposition). A norm-Sbos \( \{M_i, E_i\} \) for \( X \) is shrinking if \( |\text{R}(E_i^*)| \), where \( \text{R}(E_i^*) \) denotes the range of the adjoint of the projection \( E_i \), is a norm-Sbos for \( X^* \). Among the theorems proved are the following: (1) A \( w^* \)-Sbos \( \{N_i, P_i\} \) is a norm-Sbos for the closed linear span of \( \bigcup_{i=1}^{\infty} N_i \), (2) If \( \{N_i, P_i\} \) is both a norm- and \( w^* \)-Sbos for \( X^* \), then \( \{\text{R}(J^{-1} P_i J)\} \), where \( J \) is the canonical embedding map of \( X \) into \( X^{**} \), is a shrinking norm-Sbos for \( X \). (3) While the bounded \( w^* \)-topology for \( X^* \) is different from the \( w^* \)-topology in infinite dimensional spaces, the notions of \( w^* \)-Sbos and bounded \( w^* \)-Sbos are equivalent. (Received November 14, 1963.)

608-195. J. V. RYFF, 2 Divinity Avenue, Cambridge 38, Massachusetts. Orbits of L-functions under doubly stochastic transformations.

Denote by \( \mathcal{B} \) the class (semigroup) of doubly stochastic operators acting on \( L^1 = L^1(0,1) \). To each \( f \in L^1 \) there corresponds a weakly compact, convex set, \( \Omega(f) = \{Tf : T \in \mathcal{B}\} \) called the orbit of \( f \). Theorem 1. \( \Omega(f) = \Omega(g) \) if and only if \( f \) and \( g \) are equimeasurable (equally distributed). If \( f \in L^1 \), let \( f^* \) be the decreasing rearrangement of \( f \). Then define \( g \preceq f \) whenever \( \int_0^s f^* \leq \int_0^s g^* \), with equality holding when \( s = 1 \). Theorem 2. \( g \preceq f \) if and only if \( g \in \Omega(f) \). Furthermore, it is shown that if \( f \in L^1 \) then a measure preserving transformation \( \sigma \) exists such that \( f = f^* \circ \sigma \). The transpose of the operator induced by \( \sigma \) carries \( f \mapsto f^* \). Theorem 3. If \( g \) is equimeasurable with \( f \), then \( g \) is an extreme point of \( \Omega(f) \). (Received November 14, 1963.)
A theorem is given to the effect that, under certain hypotheses on the coefficients of the nonhyperbolic equation (1) \[ \sum_{i,j=1}^{n} a_{ij}(x_1 \frac{\partial u}{\partial x_j}) \frac{\partial u}{\partial x_i} + a(x_1 \frac{\partial u}{\partial x_0}) = 0 \] and on the smoothness of the boundary of the region under consideration, the normal derivative \( \frac{\partial u}{\partial n} \) is negative at a point at which the solution of equation (1) attains a maximum value. From this theorem, a strong maximum principle for a quasilinear parabolic equation is deduced. (Received November 14, 1963.)

The result can be reformulated more economically. Let \( X \) be a Hausdorff space, \( P \) a closed relation on \( X \), \( S \) a collection of compact subsets of \( X \) and let \( f \) be a function on \( S \) to \( X \) satisfying:

(i) \( f(A) \subseteq A \)
(ii) \( f(A) \notin \emptyset \) and \( f(A) \notin P \)
(iii) \( A \notin f(A) \cup f(A) \) does not intersect \( P \).

One concludes that the function \( f \) is continuous in the Frink topology. As in the Capel-Strother paper, applications are made to fixed-point theorems, but there are also applications to relation-theory and to topological semigroups. It should be noted that the proposition has been used by L. W. Anderson and that his methods can be extended somewhat to partially ordered spaces. S. P. Franklin has shown (oral communication) that, with additional hypotheses, the Capel-Strother theorem has a useful converse. It is important to notice that in the present formulation one need not assume that \( X \) is compact or that \( P \) is a partial order. (Received November 14, 1963.)

Let \( E \) be a nonvoid set and \( r : 2^E \rightarrow 2^E \) a closed relation on \( E \), \( S \) a collection of compact subsets of \( E \) and let \( f \) be a function on \( S \) to \( E \) satisfying:

(i) \( f(A) \subseteq A \)
(ii) \( f(A) \notin \emptyset \) and \( f(A) \notin S \)
(iii) \( A \notin f(A) \cup f(A) \) does not intersect \( S \).

The space \( (S, \mathcal{B}) \) is finer than the nbd space \( (E, \mathcal{B}) \) iff \( a \in \mathcal{B} \), \( a \in S \). Every topological space is a nbd space but the converse is not true. The nbd space \( (E, \mathcal{B}) \) is a nbd product space. (Received November 14, 1963.)
On extreme Banach limits. Preliminary report.

The shift $T_n = n + 1, n \in \mathbb{N} = \{1, 2, \ldots\}$ extends to an into homeomorphism $\hat{T}: \beta\mathbb{N} \to \beta\mathbb{N}$ and the Banach limits (of bounded sequences) become Baire probability measures on $\beta\mathbb{N}$ invariant under $\hat{T}$. The extreme Banach limits are ergodic, and it is desired to decompose $\beta\mathbb{N} - \mathbb{N}$ into disjoint supporting sets, one for each extreme Banach limit. The following preliminary result has been obtained. The second conjugate $C^{**}(\beta\mathbb{N})$ is isometrically algebraically isomorphic to $C(X)$, with $\beta\mathbb{N}$ a continuous image of compact Hausdorff $X$. Shift $T$ extends to an into homeomorphism $\hat{T}: X \to X$, and the extreme invariant Baire probability measures on $X$ live on disjoint closed subsets of $X$. (Received November 14, 1963.)

Improvement of the error term in an asymptotic formula.

Let $\pi(r; s, t)$ denote the number of monic irreducibles in $GF[x, q]$ of degree $r$ having given first $s$ and last $t$ coefficients. According to a previous announcement [Abstract 603-59, Notices Amer. Math. Soc., 10 (1963), 448], $\pi(r; s, t) = q^r/(r^s t^{r-t} (q-1)) + O(q^{r/2}/r)$ for some $\theta$ with $1/2 \leq \theta < 1$. Using the Riemann hypothesis for function fields proved by A. Weil [Proceedings of the National Academy Sciences, 27 (1941), 345-347], it is shown that the above asymptotic formula is valid with $\theta = 1/2$. This result is a special case ($M = x^4$) of the following more general Theorem. Let $s$ field elements be given. Let $M$ be a polynomial in $GF[q, x]$ not divisible by $x^q - x$, and let $A$ be prime to $M$. If $\pi(r)$ denotes the number of monic irreducibles in $GF[q, x]$ of degree $r$ which (1) have the given field elements as first $s$ coefficients and (2) are congruent to $A$ modulo $M$, then $\pi(r) = q^r/(r^s \phi(M)) + O(q^{r/2}/r)$, where $\phi(M)$ is the number of polynomials in a reduced residue system modulo $M$. The proof makes use of the Riemann hypothesis for function fields. (Received November 14, 1963.)

Normed modules and almost periodicity.

A normed module is a pair $(A, M)$ such that $A$ is a commutative Banach algebra, $M$ is a Banach space and also a module over $A$, and the norms on $M$ and $A$ are related by the inequality: 
\[ \|ax\| \leq \|a\| \|x\| \quad (a \in A, x \in M). \] An element $x \in M$ is said to be almost periodic if the linear map $a \mapsto ax$ from $A$ to $M$ is compact; $AP$ will denote the set of almost periodic elements in $M$. If every closed $A$-submodule of $AP$ is a topological direct sum of one-dimensional submodules which it contains, the approximation theorem is said to hold for $(A, M)$. A sufficient condition for the approximation theorem to hold can be given in terms of "approximate identities" and "multipliers." If $G$ is a locally compact abelian group, then $L^p(G)$ is a normed module over $L^1(G)$, multiplication being convolution, and the above mentioned condition holds. The fundamental theorem for almost periodic functions follows from this as a special case ($p = \infty$). The above mentioned condition also holds for any unitary normed module over $C(S)$, where $S$ is compact Hausdorff. Applications of the latter result are considered. (Received November 14, 1963.)

A Hausdorff ring topology on the ring \( \mathbb{Z} \) of integers is called an ideal topology if there exists a basis, for the neighborhood system of zero, consisting of ideals. If no such basis exists the topology is called a nonideal topology. It is shown that uncountably many nonideal topologies exist on \( \mathbb{Z} \). This result is used to show that there exists a Hausdorff ring topology on \( \mathbb{Z} \) which is not first countable. (Received November 14, 1963.)

608-203. LEONARD CARLITZ, Duke University, Durham, North Carolina. Rings of arithmetic functions.

Consider the set \( S \) of arithmetic functions defined over the positive integers with values in a fixed field \( F \). It is familiar that with respect to the Dirichlet product \(*\) the system \((S, +, *)\) is a domain of integrity while with respect to the ordinary product \(\cdot\) defined by \((f \cdot g)(n) = f(n)g(n)\), the system \((S, +, \cdot)\) is a commutative ring with divisors of zero. The purpose of the present paper is to give an abstract characterization of such systems. The characterization makes use of minimal functions. A function \( f \in S \) is minimal provided there exists an integer \( k \) (depending on \( f \)) such that \( f(n) = 0, n \neq k; f(k) \neq 0 \). (Received November 15, 1963.)

608-204. ISIDORE EISENBERGER, 4800 Oak Grove Drive, Pasadena, California. Tests of hypotheses using quantiles.

Techniques for testing hypotheses concerning distribution parameters are well known when all the observations are available. However, it is highly desirable in the case of a space probe to limit the amount of data transmitted back to Earth and, at the same time, extract the maximum amount of information possible from such data that is received. Accordingly, in this paper tests of hypotheses are considered using a small number of sample quantiles when the sample size is large. The population is assumed to be normally distributed and the simple hypothesis that the mean \( \mu = \mu_1 \) and variance \( \sigma^2 = \sigma_1^2 \) is tested against the composite alternative hypothesis that \( \mu = \mu_2 \neq \mu_1 \) and \( \sigma^2 = \sigma_2^2 \neq \sigma_1^2 \). Best critical regions for given significance levels are determined, as well as the orders of one, two and four quantiles which maximize the power of the test. The efficiency of the tests is determined by comparing the power with that of the optimal tests using the entire sample. (Received November 15, 1963.)


Let \( p_j \) be a homogeneous polynomial of degree \( m_j \). Let \( \Gamma \) be the half space \((x, \theta) \geq 0\). Theorem. If the \( p_j \) have no common zero \( \xi = \xi + i\theta \neq 0 \) with \( \xi \) and \( \tau \) real then \( u(x) = \sum_j \int_\Gamma u(x + y)K_j(y)dy \) for every \( u \in C_0^\infty(\mathbb{R}^n) \). The kernel \( K_j \) is homogeneous of degree \( m_j - n \), is very regular except at 0, and has support in \( \Gamma \). This result follows an earlier one (Bull. Amer. Math. Soc., 67 (1961)) in which it is assumed that the \( p_j \) have no common complex zero
Let $\mathcal{A}$ be a topological algebra of functions which contains the polynomials. An $\mathcal{A}$-operational calculus $(\mathcal{A} \circ \mathcal{O})$ is a continuous representation $T(\cdot)$ of $\mathcal{A}$ on a Banach space $X$, such that $T(1) = I$ and $T(\cdot)$ has compact support. A bounded operator $T$ is of class $\mathcal{A}$ ($T \in (\mathcal{A})$) if there exists an $\mathcal{A}$-o.c. $T(\cdot)$ such that $T(\cdot) = T$. Suppose $T$ has real spectrum. If $\mathcal{A}$ is a Banach algebra which satisfies certain conditions, then $T \in (\mathcal{A})$ implies $T \in (\mathcal{C}^n)$ for some finite $n$. For $X$ reflexive and $n$ finite, a certain subclass of $(\mathcal{C}^n)$ coincides with a certain class of spectral operators of type $n$. Various characterizations of $(\mathcal{C}^n)$ are obtained. For example, $T \in (\mathcal{C}^n)$ iff $v_n(T; J) < \infty$ for some compact interval $J$ ($v_n(T; J)$ is the sup of $\| \int f(t)e^{itT} dt \|$ over a certain class of functions $f$ which depends on $n$ and $J$).
ABSTRACTS FOR SPECIAL SESSIONS

608-211. D. A. BUCHSBAUM and DOCK SANG RIM, Brandeis University, Waltham, Massachusetts 02154. Cramer's rule, complexes and multiplicity.

A study of the Koszul complex, multiplicity, (and Cramer's rule) leads to generalizations of these notions and a study of their interrelationships. (Received November 29, 1963.)

608-212. MAURICE AUSLANDER, Brandeis University, Waltham, Massachusetts 02154 and ARMUND BRUMER, Boston University, Boston, Massachusetts. Brauer groups of fields with discrete rank one valuations.

A field $K$ is said to be shallow if there exists an integer $n > 0$ such that $nB(K) = 0$ where $B(K)$ is the Brauer group of $K$. If $K$ is a function field in one variable with constant field $k$, then we show that $K$ is shallow if and only if $k$ is a real closed or algebraically closed field. Further, it is shown that $B(K) = 0$ if and only if $k$ is algebraically closed. These results are based on the following facts about a field $K$ with discrete rank 1 valuation ring $R$ and residue class field $F$. (a) If $K$ is shallow then $F$ is perfect; (b) If $F$ is perfect then we have an exact sequence $0 \rightarrow B(R) \rightarrow B(K) \rightarrow \chi(F) \rightarrow 0$ where $B(R)$ is the Brauer group of $R$ and $\chi(F)$ is the character group of the galois group of the algebraic closure of $F$. Finally an example is given of a field $K$ with trivial Brauer group but which has finite extensions with nontrivial Brauer group. (Received November 29, 1963.)

608-213. RICHARD SWAN, University of Chicago, Chicago 37, Illinois. Minimal resolutions for finite groups.

The deficiency $d(G)$ of a finitely generated group $G$ is defined to be the maximum of $d - r$ over all presentations of $G$, where $d$ is the number of generators and $r$ the number of relations in the presentation. Theorem 1. If $[G:H] < \infty$, then $d(G) - 1 \leq n^{-1}(d(G) - 1)$. Using this result it is easy to construct finite groups of order $3 \cdot 7^k$ with trivial Schur multiplier and arbitrarily large $-d(G)$. This answers negatively a question of B. H. Neumann, On some finite groups with trivial multiplicator, Publ. Math. Debrecen 4 (1955), 190-194. Let $\ldots \rightarrow F \rightarrow F \rightarrow F \rightarrow Z \rightarrow 0$ be a free resolution of $Z$ over $ZG$, $F_i$ being free on $f_i$ generators, and let $d'(G)$ be the maximum of $1 - f_0 + f_1 - f_2$ over all such free resolutions. Then $d(G) \leq d'(G)$. This can be used to give further upper bounds for $d(G)$.

It is possible to calculate $d'(G)$ explicitly for any finite group $G$. Theorem 2. If $G$ is a finite $p$-group, there is a free resolution of $Z$ over $ZG$ with $f_i = \dim H^i(G,Z_p)$ for all $i$. This is clearly the best possible result. Corollary. If $G$ is a finite $p$-group then $d'(G) = \dim H^1(G,Z_p) - \dim H^2(G,Z_p)$. I do not know whether $d(G) = d'(G)$. There are similar but more complicated results for arbitrary finite groups. (Received November 29, 1963.)
Abstract theorems and concrete problems.

A review of several representation theorems that require the abstract context of abelian categories, and some computational interpretations in a number of specific cases. (Received November 29, 1963.)

Grothendieck determinants.

A "determinant" is a map from automorphisms of finitely generated projective $A$-modules to an abelian group, which is multiplicative (for composition) and additive (for exact sequences). Grothendieck tells us how to manufacture the most general one, and a group, here called $K^1(A)$, receiving it. Of this functor a celebrated theorem of Bott leads one to predict that $\ker(K^1(A[z,z^{-1}]) \to K^1(A))$, $z$ an indeterminate, is isomorphic to the Grothendieck group, $K^0(A)$, of projective $A$-modules. This is true, but for a discrepancy which vanishes in good cases. This result, proved jointly with A. Sheller and R. Swan, yields, (i) a (weak) generalization, to several variables, of the division algorithm for polynomials in one variable, and (ii) the fact that simple homotopy type $\cong$ homotopy type for spaces with free abelian $\pi_1$. (Received November 29, 1963.)

On the nonexistence of certain generalized polygons.

Generalized polygons have been defined by J. Tits (Publ. Math. I.H.E.S. Paris 2 (1959), 58-59).

Theorem 1. Let $P_n$ be a finite nondegenerate $n$-gon. Suppose that every line of $P_n$ contains exactly $s + 1$ points and every point of $P_n$ lies on exactly $t + 1$ lines. Let $k$ be the square free part of $st$. Then either $P_n$ is an ordinary polygon or $n = 2, 3, 4, 6, 8$ or 12. Furthermore if $s > 1$ and $t > 1$ then $n \neq 12$, $k = 1$ if $n = 3$ or 6 and $k = 2$ if $n = 8$. J. Tits has also defined the concept of a $(B,N)$ pair (C. R. Acad. Sci. Paris 254 (1962), 2910-2912). Theorem 2. Let $(G; B,N)$ be a finite $(B,N)$ pair whose Weyl group $W$ is a group generated by reflections. Then $W = W_1 \times W_2$, where $W_1$ is a direct product of dihedral groups of order 16 and $W_2 = 1$ or $W_2$ is the Weyl group of a semi-simple Lie group. After some elementary properties of $(B,N)$ pairs have been developed, Theorem 2 is a direct consequence of Theorem 1. Let $A$ be the incidence matrix of $P_n$ and let $M = A'A$. The proof of Theorem 1 is based on an analysis of the properties of $M$. If $n$ is odd, then it is easily seen that $s = t$. This and a knowledge of the characteristic roots of $M$ leads to a proof in this case. If $n$ is even, then it can be shown that certain polynomials in the $n$th roots of unity and $\sqrt{n}$ must be rational. At this point the proof of Theorem 1 follows from elementary properties of roots of unity. (Received November 14, 1963.)

Groups of exponent 4.

Groups of exponent 4 were shown to be locally finite by Sanov (Leningrad State Univ. Ann. 10 (1940), 166-170) and a $k$ generator group was shown to have class at most $3k - 1$ by C. R. B. Wright
The Burnside groups $B(4,k)$ have not been completely determined except for $k = 2$. This paper introduces methods applicable to any nilpotent Burnside group, and specific application is made to the groups of exponent 4. Among the results found so far are an identity on six arguments of weight six modulo $G_7$ which is not a consequence of any identity on five arguments and the Engel-type condition $(x,y,z,w,w,w) = 1 \pmod{G_7}$. A computer program has been written which has general application to relations in groups, and it is expected that this will determine at least $B(4,3)$ exactly. (Received November 16, 1963.)

708-218. WITHDRAWN.


Scattering theory.

The scattering problem for the wave equation in the exterior of a reflecting obstacle can be given an abstract formulation which reveals an intimate connection with prediction theory. The solution can be represented by a group of unitary operators $U(t)$ acting on a Hilbert space $H$ which corresponds to a function space with an energy norm. Two orthogonal subspaces $D_+$ and $D_-$ are singled out to characterize the outgoing and incoming spectral representations of $U(t)$; in the application $D_+$ (respectively $D_-$) consists of all initial data whose solutions vanish for $|x| < t$ ($|x| < -t$). In the outgoing (incoming) representation $D_+(D_-)$ maps onto $H_2(N)$ (respectively $H_2(N')$), the Hardy class of $N$-valued functions analytic in the upper (lower) half plane; here $N$ is an auxiliary Hilbert space which in the application turns out to be the sphere integrable functions on the unit sphere. To each $t \in H$ there are associated two functions $k_-$ and $k_+$, the respective incoming and outgoing spectral representors of $f$. The mapping $k_-$ into $k_+$ defines the scattering operator $S$. It can be represented as a left multiplier operator, $\mathcal{A}(z)$, unitary-valued (on $N$) having an analytic extension in the lower half-plane; the latter being a consequence of the fact that $SD_-$ is orthogonal to $D_+$. The operator $\mathcal{A}(z)$ is closely connected with the semigroup $Z(t) = P_+ U(t) P_-$ acting on $K = H \Theta D_+ \Theta D_-$; here $P_+$ and $P_-$ denote the orthogonal projections onto the orthogonal complements of $D_+$ and $D_-$ respectively. In fact $\mathcal{A}(z)$ is the inner factor associated with the translation invariant subspace $K$ and the spectral properties of $Z(t)$ and its infinitesimal generator $B$ are determined by the invertibility of $\mathcal{A}(z)$ in the lower half-plane. If for some $T$, $|Z(T)| = a < 1$, then $\mathcal{A}(z)$ can be continued analytically into the strip $\text{Im } z < -\log a/T$. For the wave equation in the exterior of a star-shaped obstacle this has been demonstrated by C. Morawetz and the authors, proving at the same time the exponential decay of energy for such configurations. In the case of the wave equation for the general exterior problem it can be shown that $Z(T)(B - \lambda)^{-1}$ is compact. From this it follows that $\mathcal{A}(z)$ can be continued into the entire upper-half plane as a meromorphic function. Finally in the case of the exterior problem for the wave equation the eigenvalues of $B$ can be characterized as follows. A solution $u$ of the reduced wave equation $\Delta u - \mu^2 u = 0$ in the exterior domain is called outgoing if the free space solution of the wave equation with initial data $u = u, u_t = -\mu u$ vanishes for $|x| < t - \rho$ for $t > \rho$, where $\rho$ is the diameter of the obstacle. This definition is equivalent to the Sommerfeld definition of outgoing when $\mu$ is imaginary. Now $\mu$ is an eigenvalue of the generator $B$ of $Z(t)$ if and only if there exists an outgoing solution of the reduced wave equation satisfying the boundary conditions. The above ideas can
also be used to easily obtain an explicit description of the incoming and outgoing spectral representations of $U(t)$ from which an explicit representation of the scattering operator $\mathcal{E}(z)$ can be found.  

(This one abstract covers two of the papers being presented in the Special Session, one by each of the authors.) (Received November 26, 1963.)


Can every (non-Desarguesian) projective plane be imbedded (in some natural, geometric fashion) in a (Desarguesian) projective space? The question is new but important, for, if the answer is yes, two entirely separate fields of research can be united. This paper provides a conceptually simple geometric construction which yields an affirmative answer for a broad class of planes. A plane $\pi$ is given by the construction precisely when $\pi$ is a translation plane with a coordinatizing right Veblen-Wedderburn system which is finite-dimensional over its left-operator skew-field. The condition is satisfied by all known translation planes, including all finite translation planes. (Received November 14, 1963.)

608-222. D. G. ARONSON, Institute of Technology, University of Minnesota, Minneapolis 14, Minnesota. Removable singularities for the equation of heat conduction.

Let $u = u(x,t)$ be a solution of (*) $u_t - \Delta u = 0$ in $D - K$, where $D$ is an open set in $(n + 1)$-dimensional $(x,t)$-space and $K$ is a compact set. Serrin has recently obtained a general removable singularity theorem for Laplace's equation (to appear, Bull. Amer. Math. Soc.); here we extend Serrin's result to (*). Let $U(K)$ be the class of smooth functions $\psi = \psi(x,t)$ such that $\psi \equiv 1$ on $K$, $\psi$ has compact support in $\mathbb{E}^{n+1}$, and $0 \leq \psi \leq 1$. $K$ is said to be a $(p,q)$-null set if

$$\inf_{U(K)} \int_{-\infty}^{+\infty} \int_{\mathbb{E}^n} \left( |\nabla \psi|^2 + |\nabla \psi|^q \right) |\Delta \psi|^p |\partial \psi|_t = 0.$$  

The $(p,q)$-null sets are an extension to (*) of the notions of zero capacity for Laplace's equation. Let $S_\eta = S_{\eta(t = \eta)}$ for arbitrary sets $S$, and let $I$ be a $t$-interval containing $D$. \textbf{Theorem.} If $K$ is a $(p,q)$-null set for $2 \leq p \leq n$ and $q \geq 2$ and if $u(x,t) \in L^q \left( I: L^q \{D - K\} \right)$, where $n/2q + 1/\beta \leq n(p - 2)/2p + (q - 2)/q$ ($=$ only if $p = q = 2$), then $u$ can be defined on $K$ so that the resulting function satisfies (*) throughout $D$. In particular, if $u \in L^\infty(D - K)$ and $K$ is a $(2,2)$-null set, then $K$ is removable. A single point is removable if $u \in L^q \left( I: L^q(D_t) \right)$ for $n/2q + 1/\beta < n/2$; here the constant $n/2$ is best possible, as shown by the example of the Green's function. An analogous theorem holds for uniformly parabolic equations $u_t - \sum_{ij} a_{ij}(x,t)u_{x_i}u_{x_j} = 0$. (Received November 7, 1963.)


Let $G$, $H$ be finite groups and $R$ a ring of algebraic integers. If $x \in G$, let $\{x\}$ denote the conjugate class of $x$. \textbf{Theorem:} If $RG \cong RH$, then $G$ and $H$ have the same character table. Furthermore there exists a bijection $\phi$ from the classes of $G$ to those of $H$ such that if $\phi\{x\} = \{u\}$, then $|\{x\}| = |\{u\}|$, the order of $x$ = the order of $u$, and more generally $\phi\{x^a\} = \{u^a\}$. Finally if $X$ is an
absolutely irreducible character of G, it is also one of H and if \( \phi(x) = \{u\} \), then \( X(x) = X(u) \).

(Received August 22, 1963.)

608-224. DANIEL GORENSTEIN, Clark University, Worcester 10, Massachusetts and J. H. WALTER, University of Illinois, Urbana, Illinois. The classification of finite groups with dihedral Sylow 2-subgroups.

The following result is proved: Theorem 1. If G is a finite group with dihedral Sylow 2-subgroups and if \( O(G) \) denotes the maximal normal subgroup of G of odd order, then either (i) \( G/O(G) \) is isomorphic to a subgroup of \( \text{P} \text{S}L(2,q) \) containing \( \text{PSL}(2,q) \); q odd; (ii) \( G/O(G) \) is isomorphic to the alternating group \( A_7 \); or (iii) \( G/O(G) \) is isomorphic to a Sylow 2-subgroup of G. As an immediate corollary, we have Theorem 2. If G is a simple group with dihedral Sylow 2-subgroups, then either G is isomorphic to \( \text{PSL}(2,q) \), q odd, \( q \equiv 1 \pmod{8} \), or to \( A_7 \). In particular, if G is a simple group of order \( 4g' \), \( g' \) odd, then G is isomorphic to \( \text{PSL}(2,q) \), \( q \equiv 3, 5 \pmod{8} \). The proof relies heavily upon the techniques of Chapter IV of the paper Solvability of groups of odd order by W. Feit and J. G. Thompson (Pacific J. Math. vol. 13, no. 3), which can be adapted with suitable modification; and also upon arithmetic methods based upon various formulas for the order of G which are obtained by means of the theory of modular characters. (Received December 12, 1963.)

608-225. L. AUSLANDER, University of California, Berkeley 4, California. Algebraic groups attached to complete compact locally affine spaces.

Let M be a compact complete locally affine manifold of dimension n. Then if \( A^n \) denotes the n-dimensional affine plane and \( A(n) \) the group of affinities of \( A^n \), we can imbed the fundamental group \( \Gamma \) of M as a subgroup of \( A(n) \) in such a way that the orbit space \( A^n/\Gamma \) is homeomorphic to M. Now if \( S \) is the algebraic hull of \( \Gamma \) with identity component \( S_0 \), \( S_0 \) is solvable and hence \( S_0 = U \cdot T \), where \( T \) is an abelian group of semisimple elements and \( U \) is the maximal unipotent subgroup of \( S_0 \). In addition we can show that \( S = U \cdot K \) and \( M = K[S]/\Gamma \), where \( T \) is the algebraic group attached to \( M \). This definition is justified by the following result. Let \( M_1 \) and \( M_2 \) be compact complete locally affine manifolds with fundamental groups \( \Gamma_1 \) and \( \Gamma_2 \). Let \( S_1 = U_1 \cdot K_1 \) be the attached algebraic group to \( M_1 \), \( i = 1,2 \), and let \( \psi \) be an isomorphism of \( \Gamma_1 \) onto \( \Gamma_2 \). Then \( \psi \) is uniquely extendable to an isomorphism \( \psi^* \) of \( S_1 \) onto \( S_2 \) such that \( \psi^*(U_1) = U_2 \). Hence \( \psi^* \) can be used to induce a homeomorphism of \( M_1 \) onto \( M_2 \). This announcement amplifies results already announced in the Bull. Amer. Math. Soc. 69 (1963), 242-243. (Received December 12, 1963.)

608-226. L. W. GREEN, Institute of Technology, University of Minnesota, Minneapolis 14, Minnesota. Geodesic inequalities.

"Geodesic inequalities" is a loose term referring to various relations between geometrically defined quantities of the following kind: Let \( p \in M \), a compact \( C^0 \) Riemannian manifold. Set \( C(p) = \text{inf} C(p) \), \( J = \text{inf} J(p) \), and set H equal to half the length of the smallest nontrivial closed geodesic on M. Klingenberg has proved that \( C = \min(J,H) \) and, for M simply-connected, even dimensional, and of positive curvature,
For dimension two, equality in this last relation implies constancy of curvature. Further inequalities involving \( J \) and the volume of \( M \) are discussed, with particular emphasis on their similarity to results of \( P_1 \). (Received December 12, 1963.)


Let \( R \) be real or complex euclidean space of dimension \( n \) with coordinates \( x = (x_1, \ldots, x_n) \). Let \( D_i \) (\( i = 1, 2, \ldots, r \)) be linear partial differential operators with constant coefficients on \( R \). A Cauchy Problem for the system \( D_i f = 0, i = 1, 2, \ldots, r \) is a method of parametrizing a space \( W \) of solutions in terms of their restrictions and the restrictions of certain derivatives to certain subvarieties \( T^j \) of \( R \).

In case \( r = 1 \), only one \( T^1 \) is necessary, but for \( r > 1 \) several may be needed. An analog of the Cauchy-Kowelski theorem is proven in case \( R \) is complex and \( W \) is the space of entire solutions on \( R \).

This leads to the concept of noncharacteristic families \( \{ T^j \} \) in terms of which an analog of Holmgren's uniqueness theorem can be formulated and proven. The method applies even to certain characteristic families and leads to an extension of Taeklind's uniqueness theorem. (Received November 21, 1963.)

608-228. HERBERT BUSEMANN, University of Southern California, University Park, Los Angeles 7, California. Length preserving maps.

A map of a metric space into itself preserves length or is equilong if each curve goes into a curve of the same length. A region of injectivity is a maximal open connected set on which the map is injective. The map is locally finite if any bounded set intersects only a finite number of regions of injectivity. For many important spaces the regions of injectivity of equilong maps are convex, so that the map, if locally finite, yields a division of the space into convex polyhedral regions, from which the map is easily reconstructed. For the \( n \)-dimensional (\( n \geq 2 \)) elementary, i.e. the euclidean, hyperbolic, and spherical spaces the divisions into convex polyhedral regions belonging to equilong maps are characterized by the following property: The number of \( (n - 1) \)-faces (of the regions) with a common \( (n - 2) \)-face is even and if \( a_1, \ldots, a_{2k} \) are the angles formed by these \( (n - 1) \)-faces in cyclic order then \( \sum_{i=1}^{k} a_{2i-1} = \sum_{i=1}^{k} a_{2i} \). General Minkowski spaces and the nonelementary hermitian and quaternion elliptic or hyperbolic spaces possess no other equilong maps than motions. (Received December 12, 1963.)

608-229. MICHIO SUZUKI, 369 Altgeld Hall, University of Illinois, Urbana, Illinois. A characterization of 3-dimensional unitary groups.

The three dimensional projective unitary group \( G = U_3(q) \) over a finite field of \( q \) elements with odd \( q \) is characterized by the following properties: (1) \( G \) is doubly transitive on \( 1 + q^3 \) points, (2) the stabilizer \( H \) of one point in \( G \) contains a normal subgroup \( Q \) of order \( q^3 \) which is regular on the rest of points, and (3) \( H/Q \) is cyclic. A similar characterization in case \( q \) is even has been given elsewhere. Using this characterization, one can also characterize \( U_3(q) \) in terms of the properties of the centralizer of an involution in \( U_3(q) \). (Received December 16, 1963.)
A two-dimensional manifold is said to be minimally embedded in $\mathbb{R}^n$ if no hyperplane which meets it locally in a single point meets it outside a neighborhood of the point. (Convex bodies in $\mathbb{R}^n$ have this property.) A manifold is substantially embedded in $\mathbb{R}^n$ if it does not lie in any hyperplane. Kuiper has shown that if $M$ is a differentiable 2-manifold minimally and substantially embedded in $\mathbb{R}^n$ then $n \leq 5$. This result is not true for polyhedra! In this paper, the following results are proved:

1. It is possible to construct a sequence $M(n)$ of polyhedral 2-manifolds such that $M(n)$ is minimally and substantially embedded in $\mathbb{R}^n$.
2. There is an upper bound on the dimension of the receiving space in terms of the Euler-Poincaré characteristic of the manifold. Specifically, if the 2-manifold $M$ is embedded minimally and substantially in $\mathbb{R}^n$, $n \geq 6$ then $n < (1/2)(7 + \sqrt{49 - 24\chi(M)})$.
3. This is the best possible bound for the torus and the projective plane. (Received November 18, 1963.)
ABSTRACTS PRESENTED BY TITLE


Let \( \mu \) be a probability measure on the group \( G \) of integers, whose support generates the whole group. Its \( n \)-fold convolution is \( \mu^n = \mu \ast \ldots \ast \mu \) and its Fourier transform is \( \hat{\mu}(\lambda) = \sum_{x \in G} \mu(x) \exp(ix\lambda), \) \( \lambda \in \hat{G} \) is the circle group. **Theorem.** \( \sum_n \mu^n \) is finite on \( G \) if and only if the real part of \( (1 - \hat{\mu})^{-1} \) is integrable on \( \hat{G} \). In other words, the random walk on the integers with displacement distribution \( \mu \) is transient if and only if \( \text{Re}[(1 - \hat{\mu})^{-1}] \in L^1(\pi,\mu) \). The proof employs parts of the potential theory of recurrent random walk. It generalizes immediately to discrete Abelian groups \( G \), but the general case, when \( \hat{G} \) is not compact, seems difficult. (Received September 25, 1963.)

64T-2. WILLIAM REDDY, Syracuse University, Syracuse 10, New York. Sufficient conditions that a group not act expansively.

Let \( X \) be a compact metric (d) space such that for each \( \epsilon > 0 \) there exist points \( x \) and \( y \) of \( X \) with \( 0 < d(x,y) < \epsilon \), and let \( G \) be a generative subgroup of the total homeomorphism group, \( H \), provided with the compact-open topology. By exploiting several formulations of equicontinuity and the trivial remark that if \( G \) is equicontinuous, it does not act expansively, one obtains the following sufficient conditions that \( G \) not act expansively: (1) \( \hat{G} \) is the union of compact subgroups. (2) \( H \) possesses a compact normal subgroup \( C \) such that \( H/C \) is finite. (3) \( G \) is almost periodic. Another such condition is that \( G \) not be discrete, or, if \( d \) is the metric of the previous abstract, that for each \( \epsilon > 0 \), there exist elements \( g, h \) of \( G \) such that \( 0 < d[g,h] < \epsilon \). It follows that if \( G \) is pseudo-compact it does not act expansively. Examples show that none of these conditions is necessary. (Received September 25, 1963.)

64T-3. WILLIAM REDDY, Syracuse University, Syracuse 10, New York. Upper equicontinuity in measure.

Let \( G \) be a set of homeomorphisms of a compact metric (d) measure (m) space \( X \) such that (1) \( m(X) = 1 \), (2) the open sets of \( X \) are measurable, and (3) for each \( \epsilon > 0 \) there exists \( R_\epsilon > 0 \) such that all open spheres of radius \( \epsilon \) have measure greater than \( R_\epsilon \). Then \( G \) is a metric space with metric \( d(g,h) = \max d(xg, xh) \), and \( d \) yields the compact-open topology on \( G \). The function \( d'(g,h) = \int_X d(xg, xh) \) is also a metric for \( G \). **Definition.** \( G \) is upper equicontinuous in measure if there exists \( \epsilon_1 > 0 \) such that for \( \epsilon_1 > \epsilon > \delta > 0 \) there exists \( k(\epsilon, \delta) > 0 \) such that \( d[g, id] > \epsilon \) implies \( m\{y: |yg, y] > \delta\} > k \), for each element \( g \) of \( G \). **Theorem.** If \( G \) is equicontinuous, then it is upper equicontinuous in measure. **Theorem.** Let \( G \) be a group. \( G \) is upper equicontinuous in measure iff \( d \) and \( d' \) are equivalent. It is shown by examples that this weakened form of equicontinuity is neither a necessary nor a sufficient condition that \( G \) not act expansively. (Received September 25, 1963.)
64T-4. WILLIAM REDDY, Syracuse University, Syracuse 10, New York. Remark on expansive autohomeomorphisms.

Let $f$ be autohomeomorphism of a compact metric space $X$. It is said that the positive semi-orbit of $x$ diverges or converges as the sequence $f^n(x)$, $n = 1, 2, 3, \ldots$, does, and similar terminology is used for negative semi-orbits. **Theorem.** If $f$ is expansive, the set of points for which both semi-orbits converge is a countable set. **Corollary.** The closed unit interval admits no expansive autohomeomorphism, since all semi-orbits converge. The corollary is due to B. F. Bryant. (Received September 25, 1963.)

64T-5. L. L. LININGER, State University of Iowa, Iowa City, Iowa 52240. The union of a crumpled cube and a 3-cell is topologically $S^3$.

**Theorem.** If $C$ is homeomorphic to a 2-sphere and its bounded complementary domain in $E^3$, then there exists a homeomorphism $h$ from $C$ into $S^3$ with the property that the closure of the complement of $h[C]$ is a 3-cell. (Received September 26, 1963.)


**Existence Theorem.** Let $T$ be a t-norm in the sense of Menger-Sklar-Schweizer. Then there exists a function $f: [0,1] \to [0, +\infty]$ with a left inverse $f^*: [0, +\infty] \to [0,1]$ such that $T(x,y) = f^*(f(x) + f(y))$, for all $x, y \in [0,1]$. Such a function $f$ is called an additive generator for the t-norm $T$.

**Proof.** Let $H$ be a Hamel basis for the vector space of the real numbers over the field of the rational numbers. All the elements of $H$ are assumed to be positive. Let $g: [0,1] \to H \cup \{0, +\infty\}$ be a 1-1 correspondence such that $g(0) = +\infty$ and $g(1) = 0$. Define $f: [0,1] \to [0, +\infty]$ to be $j \circ g$, where $j: H \cup \{0, +\infty\} \to [0, +\infty]$ is the injection map. Let $K = \{f(x) + f(y), \text{ for all } 0 \leq x, y < 1\}$, then $H \cap K = \emptyset$. A left inverse $f^*: [0, +\infty] \to [0,1]$ is defined as follows: $f^*(z) = \{f^{-1}(z), \text{ if } z \in H; T(x,y), \text{ if } z = f(x) + f(y) \in K; \text{ any element of } [0,1], \text{ if } z \notin H \cup K\}$. Due to the linear independence of Hamel basis, the definition of $f^*$ is unambiguous. The verification that $T(x,y) = f^*(f(x) + f(y))$ for all $x, y \in [0,1]$ is immediate. q.e.d. (Received September 26, 1963.)


An M-computer is defined to be an effective list of all partial recursive functions $\phi_0, \phi_1, \phi_2, \ldots$, and a corresponding list of partial recursive functions $\Phi_0, \Phi_1, \Phi_2, \ldots$, called measure functions with the properties: (1) For all $i$ and $x$, $\phi_i(x)$ converges iff $\Phi_i(x)$ converges. (2) There exists a total recursive function $a$ such that $a(i,x,y) = 1$ if $\Phi_i(x) = y$ and $a(i,x,y) = 0$ otherwise. For instance, $\Phi_i(x)$ may measure the time required to compute $\phi_i(x)$ or the amount of tape used in the computation. **Theorem.** To each total recursive function $g$ there corresponds a total recursive function $h$ with $h(x) > g(x)$ for all $x$, and a 0-1 valued total recursive function $f$ such that: (1) If $i$ is any index for $f$ ($\phi_i = f$), then $\Phi_i(x) > h(x)$ for almost all $x$. (2) There exists an index $k$ for $f$ such that $h(x + 1) > \Phi_x(x) > h(x)$ for almost all $x$. **Theorem.** Let $g$ be a total recursive function of two variables. Then there exists a
0.1 valued total recursive function \(f\) such that to every index \(i\) for \(f\) there corresponds an index \(j\) for \(f\) such that \(\Phi_j(x) > g(x, \Phi_j(x))\) for almost all \(x\). (Received September 26, 1963.)


Let \((S)\) denote the class of functions \(f(z) = z + \sum_{k=2}^{\infty} a_k z^k\) regular and schlicht for \(|z| < 1\), with image domain \(D(f)\). Let \(\gamma \notin D(f)\). Define \((1/f(z)) = (1/z) + b_0 + b_1 z + \cdots\); \(F(z) = (1/f(z)/\gamma - f(z)) = z + d_2 z^2 + \cdots\); \(\psi(n) = (n + 1) + \sum_{k=1}^{n-1} (-2 - n + k + 2^{n-k+1})|b_k|\) for \(n = 2, 3, \ldots\) and \(\psi(1) = 2\).

Let \(|d_k| \leq k\) for \(k = 2, 3, \ldots, (n + 1)\). Let \(\phi(n) = \psi^{2/n}(n) - |a_2|^2\). The following theorem is proven:

For \(n = 1, 2, 3, \ldots\) (a) \(|a_2 + (1/\gamma)|^n \leq \psi(n)\), (b) \(|a_2|^{2n} \leq \psi(n)\), (c) \(D(f)\) contains the circles with centers at \(C_n = \frac{a_2}{2}\psi(n)\) and radii \(R_n = \psi^{1/n}(n)/\psi(n)\), (d) \(D(f)\) contains the circles \(|w| < 1/(\psi^{1/n}(n) + |a_2|^2)|\); (e) for \(n, n = 1, 2, 3, \ldots\) there exist the coefficient inequalities \((1/\psi(n))|1 - (|a_2|/\psi^{1/n}(n))^{2N}| \leq 1 + \sum_{k=2}^{N} |a_k|^2\) and \((1/\psi(n)) \leq 1 + \sum_{k=2}^{N} |a_k|^2\). All results are sharp. Properties of the function \(\psi(n)\) are obtained, and it is conjectured that \(\psi(n) \leq 2^n\) for starlike functions. (Received September 27, 1963.)


Let \(N(T)\) be the normal system of Post which corresponds to the semi-Thue system, \(T\), as in Davis, Computability and unsolvability, pp. 98-100, Theorem 1. For any recursively enumerable degree of unsolvability, \(D\), there exists a normal system, \(N(T_D)\), such that the decision problem for \(N(T_D)\) is of degree \(D\). Proof. Combine Davis' Lemma 3, p. 99, and Boone's theorem, Bull. Amer. Math. Soc. 68, 616-623. Define a generalized normal system as a normal system without initial assertion. For such a GN the decision problem is to determine for any enunciations \(A\) and \(B\) whether or not \(A \downarrow \text{GN}(T) B\). Thus the generalized system corresponds more naturally to algebraic problems.

Theorem 1. For any recursively enumerable degree of unsolvability, \(D\), there exists a generalized normal system, \(\text{GN}(T_D)\), such that the decision problem for \(\text{GN}(T_D)\) is of degree \(D\). Lemma 1. If \(A\) and \(A^*\) are associates (in Davis' sense), then \(A \downarrow \text{GN}(T_D) B\) iff \(A^* \downarrow \text{GN}(T_D) B\). Consider enunciations of \(\text{GN}(T)\) as factored into words on primed and unprimed letters. Lemma 2. Every \(A\) in \(\text{GN}(T_D)\) has an associate of the form \(A_1 A_2^* \ldots A_{n-1}^* A_n\) (\(n\) odd). Further \(A \downarrow \text{GN}(T_D) B\) iff \(A\) and \(B\) have associates \(A_1 A_2^* \ldots A_{n-1}^* A_n\) and \(B_1 B_2^* \ldots B_{n-1}^* B_n\) such that \(A_1 \downarrow T_D B_1, A_2 \downarrow T_D B_2, \ldots, A_n \downarrow T_D B_n\). (Received September 30, 1963.)

64T-10. George Glueberman, University of Wisconsin, Madison, Wisconsin. On relatively prime automorphism groups.

Let \(G\) be a finite group and \(A\) a subgroup of the automorphism group of \(G\) such that \(|A|, |G| = 1\). Let \(T = C_G(A)\) be the fixed point subgroup of \(G\). Let \((G,A)\) be the group generated by \(g^{-1} g^a, g \in G, a \in A\). Theorem 1. Let \(U\) be a normal subgroup of \(T\) such that \(|T:U|, |G:T| = 1\). Then \(G\) possesses a normal subgroup \(N\) such that \(NT = G\) and \(N \cap U = T\). Corollary. If \(T\) is a Hall subgroup of \(G\), then \((G,A)\) is a normal complement for \(T\). Theorem 2. Suppose \(A\) is the direct product of two groups of (possibly equal) prime orders. Then \(A\) acts faithfully on some nilpotent subgroup of \(G\).

Theorem 3. Suppose \(G\) is solvable; \(A = B \times C\) for \(B\), \(C\) of distinct prime orders \(r,s\); and
Then either $(G,B)$ is nilpotent; or $r = 2, 2s - 1$ is a power of some prime $q$, and $(G,B)$ modulo its Fitting subgroup is a non-Abelian special $q$-group of exponent $q$. **Theorem 4.** Let $\pi$ contain every prime divisor of $[G:T]$. Let $m$ and $n$ be the number of $\pi$-elements in $G$ and $T$ respectively. Then $m = [G:T]n$. (Received October 1, 1963.)

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64T-11. GEORGE GLAUBERMAN, University of Wisconsin, Madison, Wisconsin, Correspondence of characters in relatively prime automorphism groups.

Assume the notation of the previous abstract and let $p$ be a prime. Suppose that $A$ is solvable, $A$ fixes a (irreducible complex) character $\lambda$ of $G$ if $\lambda(g^a) = \lambda(g)$ for all $g \in G, a \in A$. Then there exists a one-to-one correspondence $\lambda \leftrightarrow \lambda'$ of the characters of $G$ fixed by $A$ with the characters of $T$ so that there results: **Theorem 1.** $\lambda(1)$ divides $[G:T]\lambda'(1)$. If $G$ is solvable, $\lambda'(1)$ divides $\lambda(1)$. **Theorem 2.** If $\lambda'$ and $\lambda''$ lie in the same $p$-block of $T$ then $\lambda$ and $\lambda''$ lie in the same $p$-block of $G$. **Theorem 3.** Every $p$-defect group $D'$ of $\lambda'$ is contained in some $p$-defect group $D$ of $\lambda$. Moreover, if $D \subseteq T$, then $D = D'$; $p$ does not divide $[G:T] \lambda'(1)/\lambda(1)$; and $A$ fixes every character in the $p$-block of $G$ containing $\lambda$.

**Theorem 4.** Suppose $A$ is a $q$-group for some prime $q$. For some generalized integral character $\lambda = \lambda(T)$ of $T$ and some $\epsilon = \pm 1$, $\lambda|_T = \epsilon \lambda' + q\Delta$. Moreover, the number of Brauer modular irreducible (respectively indecomposable) characters of $G$ fixed by $A$ equals the number of $p$-regular conjugate classes of $T$, and these characters are linearly independent, modulo $q$, on these classes. (Received October 1, 1963.)

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64T-12. WITHDRAWN


The following are corollaries of Theorem 1 stated in Structural criterion for recursive enumeration without repetition, I. 1. Friedberg's theorem - the class of all r.e. sets is r.e. without repetition. 2. Let $F$ be a r.e. class of finite sets. Suppose that for every $R \subseteq F$ there is an $S \subseteq F$ such that $R \subsetneq S$. Then $F$ is r.e. without repetition. 3. Any r.e. class of finite sets which is linearly ordered under set inclusion is r.e. without repetition. 4. Let $F$ be a r.e. class. Suppose that for every $x$ there is an $R \subseteq F$ such that $\not\exists y < x \subseteq R$. Suppose further that $N \notin F$. Then $F$ is r.e. without repetition. 5. Let $F$ be a r.e. class. Suppose that if $A \subseteq H, B \subseteq H$ then $A \cup B \subseteq H$. Suppose further
∪ F ⊈ F. Then F is r.e., without repetition. In contrast there is Theorem 2. There exists a r.e., class linearly ordered under set inclusion which is not r.e. without repetition. (Received October 2, 1963.)

64T-14. G. J. RIEGER, Bernt-Notke-Weg 26, 8000 Munich 61, West Germany. On the number of those integers below a positive bound and in a prime residue class which can be represented as the sum of two squares.

Denote by \( B(x; k, l) \) the number of all integers \( n \leq x \) of the form \( n = u^2 + v^2 \) (\( u, v \) integers) with \( n \equiv l \mod k \). Using Brun's method, the following result is proved: Theorem. For \( k \leq x \) and \( (k, l) = 1 \) there exists an absolute positive constant \( C \) such that \( B(x; k, l) < C(x/k)(\log(2x/k))^{1/2}(1 + p^{-1}) \) \( + x^{1/2} \) where the product is extended over all primes \( p \) with \( p \mid k \) and \( p \equiv 3 \mod 4 \). If \( 0 < \sigma < 1/2 \), \( k \leq x^\sigma \), and \( (k, l) = 1 \) this implies that there exists a positive constant \( C(\sigma) \), depending on \( \sigma \) only, such that \( B(x; k, l) < C(\sigma)(x/\phi(k))(\log 2x)^{-1/2} \) where \( \phi \) denotes Euler's function. (Received October 3, 1963.)

64T-15. WITHDRAWN


It is proved that every product of absolutely closed spaces (that is, "H-closed" in the terminology of Alexandroff-Hopf) is absolutely closed, and deduced that every product of minimal Hausdorff spaces is minimal Hausdorff. This answers a question of M. P. Berri (Trans. Amer. Math. Soc. 108 (1963), 105). It is shown that under certain conditions a product of feebly compact spaces is feebly compact, and that every product of \( \aleph_1 \) sequentially compact spaces is countably compact.
(Received October 9, 1963.)
64T-17. G. L. LEVIN, University of Chicago, Chicago 37, Illinois. A condition for regularity in local rings.

Let $R$ be a local ring with maximal ideal $M$. Then for any ideal $J$, the homological dimension of $R/MJ$ is finite if and only if $R$ is regular or $MJ = 0$. In particular, this result is true when $J$ is a power of $M$. (Received October 10, 1963.)


Let $M$ be a compact, connected 2-manifold, orientable or not, with or without boundary. Let $I$ be the unit interval, and suppose $M \times I$ and $M$ to be triangulated. Let $f$ be a piecewise linear homeomorphism of $M$ into $M \times I$ such that $f(M)$ separates $M \times 0$ from $M \times 1$, and such that $f(Bdy(M)) = Bdy(f(M)) = f(M) \cap (Bdy(M) \times I)$ whenever $Bdy(M)$ is nonempty. Under these hypotheses there exists a piecewise linear homeomorphism $g$ of $M \times I$ onto $M \times I$ such that $gf(M) = M \times 1/2$ and such that $g$ is the identity homeomorphism on $M \times 0 \cup (M \times 1)$. If $C$ is a component of $Bdy(M)$ and if $f(M) \cap (C \times I) = C \times 1/2$, then $g$ may be chosen to be the identity homeomorphism on $C \times I$ as well. (Received October 11, 1963.)


All integrals discussed are Hellinger type limits of the appropriate sums. Theorem: if each of $r, s, g$ is a real-valued nondecreasing function on the number interval $[a,b]$ such that for $a \leq x \leq b$, $\int_{[a,x]} \min\{|dr,Kds| \rightarrow g(x)$ as $K \rightarrow \infty$, then $\int_{[a,b]} \int (dr)^p(ds)^{1-p} \rightarrow 0$ as $p \rightarrow 1$ for $0 < p < 1$. (Received October 11, 1963.)

64T-20. C. C. ELGOT, IBM Research Center, P.O. Box 218, Yorktown Heights, New York and M. O. RABIN, Hebrew University, Jerusalem, Israel. Decision problems of extensions of second order theory of successor.

The (monadic) second (resp. weak second order theory (language) of successor (SS) (resp. WSS) is the interpreted calculus with equality having individual variables ranging over elements of the set $N$ of nonnegative integers, second order variables ranging over subsets (resp. finite subsets) of $N$, and nonlogical constants to denote zero and the successor function. WSS was suggested by Tarski and discussed by R. M. Robinson (Proc. Amer. Math. Soc. 9 (1958), 238-242), J. R. Büchi (Zeit. für Math. Logik und Grund. der Math. 6 (1960), 66-92), and C. C. Elgot (Trans. Amer. Math. Soc. 98 (1961), 21-51). Consider extensions of these calculi by addition of a fixed function $f: N \rightarrow N$ or set $P \subseteq N$. Theorems 1-2. The theory resulting from WSS (or SS) by adding a function $f: N \rightarrow N$ satisfying (1) $f^{-1}(n) \text{is infinite for every } n \in N$, or (2) if $n < m$, then $1 < f(m) - f(n)$, is undecidable. Theorem 3. The theory resulting from SS (or WSS) by adding the set $P = \{n! | n \in N\}$ is decidable. Similar results hold for each of the sets $P_n = \{m^n | m \in N\}$ and $Q_m = \{m^n | n \in N\}$. (Received October 15, 1963.)
64T-21. C. C. ELGOT, IBM Research Center, P.O. Box 218, Yorktown Heights, New York and M. O. RABIN, Hebrew University, Jerusalem, Israel. On the first order theory of generalized successor.

The theory of generalized successor (GS) is the first order interpreted calculus with equality having individual variables ranging over elements of the set $B$ of all finite words on the alphabet $\{0,1\}$, and having nonlogical constants to denote the empty word, the functions $f: B \rightarrow B$ and $g: B \rightarrow B$ such that $f(x) = x0$ and $g(x) = xl$, the relation $\text{length}(x) = \text{length}(y)$, and the relation $x$ is a prefix (initial segment) of $y$. This theory was first considered by J. C. Shepherdson. The correspondence which maps $n \in \mathbb{N}$ into $\omega^* n \in B$ and each finite $\alpha \in \mathbb{N}$ into the shortest $u \in B$ such that $u$ has a 1 in the $k$th place iff $k \in \alpha$ is used to construct natural interpretations of WSS in GS and vice versa. Using these interpretations, the theorems of the previous abstract immediately yield undecidability and decidability results about GS, of which the following are typical. Theorem 1. Adding to GS the predicate (set) $D = \{0^n \mid n \in \mathbb{N}\}$ makes it undecidable. Corollaries 1-3. Adding to GS each of the predicates $y = x$, $x = x$, $y = xx$ ($x$ denotes the word $x$ read backwards) makes it undecidable.

Theorem 2. The theory resulting from GS by adding the predicate $H = \{0^n \mid n \in \mathbb{N}\}$ is decidable. Thus, GS is not a "maximally decidable" interpreted calculus. (Received October 15, 1963.)


Let the function $f(z) = a_0 + a_1 z + \ldots$ be analytic at $z = 0$, and for fixed $n$ and $\nu$ let $R_{n\nu}(\epsilon,z)$ denote the function $(s_0 + s_1 z + \ldots + s_n \epsilon^n)/(t_0 + t_1 z + \ldots + t_n \epsilon^n)$, such that $f(z)$ is of best Chebycheff approximation to $f(z)$ on $|z| \leq \epsilon (\epsilon > 0)$. If $\Delta_n, \nu \neq 0$ (notation of Padé), then $R_{n\nu}(\epsilon,z)$ approaches $f(z)$ uniformly on any closed bounded set containing no pole of $P_{n\nu}(z)$. If $f(z)$ is also meromorphic with precisely $\nu$ poles in $D_0: |z| < \rho(\infty)$, then for fixed $\epsilon$ the geometric degree of convergence $(n \rightarrow \infty)$ of the Padé functions $P_{n\nu}(z)$ to $f(z)$ on closed bounded subsets of $D_0 - \delta$ is the same as that of the functions $R_{n\nu}(\epsilon,z)$ to $f(z)$. (Received October 17, 1963.)

64T-23. GUIDO STAMPACCHIA, University of Pisa, via Lavagna n. 30, Pisa, Italy. An interpolation theorem.

Let $T$ be a linear operation which maps functions $f$ defined in a cube $Q_0$ of $E^n$. Assume that (i) $\|Tf\|_{L^1} \leq K_1 \|f\|_{L^1}$ (for $C_\lambda$ being the Hölder space with exponent $\lambda$) and that (ii) $\|Tf\|_{L^2} \leq K_2 \|f\|_{L^2}$ where $a_1 > a_2 \geq 1$, $\beta_2 > a_2$. Then setting: $1/a = (1 - t)/a_1 + t/a_2$ and $\tau = \lambda_1/(n/\beta_2 + \lambda_1)$ there is;

(1) $\|Tf\|_{C_\lambda} \leq K \|f\|_{L^1}$, $\lambda = -tn/\beta_2 + (1 - t)\lambda_1$ for $0 < t < \tau$;
(2) $\|Tf\|_{L^1} \leq K \|f\|_{L^1}$ for $t = \tau$;
(3) $\|Tf\|_{L^2} \leq K \|f\|_{L^2}$, $\beta_2 = (1/\beta_2)(t - \tau)/(1 - \tau)$ for $\tau < t < 1$, where $K$ and $\xi$ independent of $f$, depend on $a_1 > a_2$, $\beta_2$, $K_1, K_2, t$ and $Q_0$. This is a special case of an interpolation theorem between some spaces which contain either Hölder spaces or $L^p$-spaces. (Received October 17, 1963.)
Consider \( y'(x) = f(x, y(x), y(n(x))) \), \( y(a) = y_0 \) with \( a \leq x \leq b \). For \( h > 0 \), \( x_n = a + nh \), let

\[
[a(x_n) - a]/h = q_n + r_n
\]

with integer \( q_n \) and \( 0 \leq r_n < 1 \). Compute \( y_{m+1} \) from the one-step method:

\[
y_{m+1} = y_m + h\Phi_2 y_m (h) + \Phi_2^2 y_m (h)
\]

where \( \Phi_1 = \Phi_1(h) = \Phi_1^2 y_m (h) \) for the true solution \( y(x) \).

Let \( A_1 \) mean that \( f, a, \) and \( \Phi \) are sufficiently differentiable in \( [a,b] \). Let \( A_2 \) mean that in \( [a,b] \) \( a \) is absolutely continuous and \( a' \) has finitely many zeros of maximum local order \( \sigma - 1 \). Let \( g(x) \) be \( \Phi_2 \) partial w.r.t. the \( k \)th variable evaluated for \( y(x) \).

**Theorem 1 (a priori error bound):** If \( A_1 \) holds and if \( p = p_2 \leq p_1 + 1 \), then \( y_n = y(x_n) + O(h^p) \) for \( x_n \in [a,b] \). **Theorem 2.** (asymptotic error formula): If \( A_1 \) holds and if \( p = p_2 = p_1 + 1 \), then \( y_n = y(x_n) + e(x_n)h^p + O(h^{p+1}) \) for \( x_n \in [a,b] \) where \( e'(x) = g_2'(x) + g_3'(x)/p + 2 \) with \( e(a) = 0 \). **Theorem 3.** (simplified one-step method): If \( A_1 \) and \( A_2 \) hold and \( p = p_2 = p_1 + 1 \), then \( y_n = y(x_n) + e(x_n)h^p + O(h^{p+1/\sigma}) \) for \( x_n \in [a,b] \) where \( e'(x) = g_2(x) + e(a(x))g_3(x)/p + 2 \) with \( e(a) = 0 \). The author is developing a similar theory for multistep methods. (Received October 21, 1963.)

**64T-25.** D. J. RODABAUGH, Box 1631, Vanderbilt University, Nashville, Tennessee. **Algebras with all commutators in the center.** Preliminary report.

Algebras are studied where all commutators are in the center. If simple, they are either commutative or have the idempotent decomposition of associative algebras. The following theorem is proved: Let \( M \) be a subalgebra such that \( (M,A) = 0 = (A,M,A) \) for \( A \) a simple algebra. If \( (M,A) \) or \( (A,A,M) \) is a subset of \( M \) then either \( M = 0 \) or there is a unit in \( A \). Consequently, if all commutators are in the center of a simple algebra \( A \), then \( A \) has a unit. A simple not commutative, not associative example exists. (Received October 21, 1963.)

**64T-26.** G. L. KRABBE, Purdue University, Lafayette, Indiana. **All symmetric operators have a weakly-continuous functional calculus on \( (BV) \).**

Let the space \( P \) (of all complex polynomials) be endowed with the topology \( G \) induced by the space \( (BV) \) (see Abstract 606-28, these Notices 10 (1963), 659). If \( A \) belongs to the algebra \( \mathcal{A} \), it has the usual functional calculus \( \{ p \mapsto p(A) \} \) defined on \( P \). If the mapping \( \{ p \mapsto p(A) \} \) is continuous, then it can be extended by continuity to a unique weakly-continuous representation \( \{ g \mapsto g(A) \} \) of \( (BV) \) into \( \mathcal{A} \), and there exists a function \( f \in F_0 \) such that \( g(A) = \int g \cdot df \); moreover, the spectrum \( \sigma(g(A)) \) is the closure of \( g(\sigma(A)) \). In particular: if the mapping \( \{ p \mapsto p(A) \} \) is continuous, there is a uniquely determined \( f \in F_0 \) such that \( A = \int \lambda \cdot df(\lambda) \); this extends a result obtained by D. R. Smart [J. Austral. Math. Soc. 1 (1960), 319-333] in the case where \( \mathcal{A} = \mathcal{L}(X,X) \) and \( X \) is a reflexive Banach space (the locally convex algebra \( \mathcal{L}(X,X) \) is endowed with the strong operator-topology). The mapping \( \{ p \mapsto p(A) \} \) is continuous for several nonpathological choices of \( A \) and \( \mathcal{A} = \mathcal{L}(X,X) \). The assertion in the title of this abstract is a very special consequence of the following theorem: the mapping \( \{ p \mapsto p(A) \} \) is continuous when \( A \) is a real spectral element of the locally convex algebra \( \mathcal{A} \). (Received September 20, 1963.)

By \([\mathcal{M}, \mathcal{D}, \phi]\) is meant a set \(\mathcal{M}\), a mapping \(\mathcal{D}\) of \(\mathcal{M}\) into the integers \(\geq 0\), and a superassociative operation \(\phi\) such that (1) if \(\mathcal{D}F = n\), then \(\phi[F, G_1, ..., G_n]\) is an element in \(\mathcal{M}\) for which 
\[
\mathcal{D}[\phi[F, G_1, ..., G_n]] = \mathcal{D}G_1 + ... + \mathcal{D}G_n;
\]
(2) there is \(I \subset \mathcal{M}\) for which \(\phi[I, F] = F\) and \(\phi[F, I, ..., I] = F\) for every \(F \in \mathcal{M}\). If \(F, G_1, ..., G_n\) are functions over a set \(S\) of \(n, m_1, ..., m_n\) places, by \(\phi[F, G_1, ..., G_n]\) is meant the function \(F(G_1, ..., G_n)\) whose value for \(x_1, ..., x_m\) is \(F(G_1(x_1), ..., G_n(x_{m_1}, ..., x_{m_n}))\) cf. Menger, The axiomatic method, ed. Henkins et al., 1959, 461). If \(\mathcal{M}\) is a set of functions over \(S\) containing the identity function, if \(\mathcal{D}F = \) place number of \(F\), and if \(\mathcal{M}\) is closed under \(\phi\), then \([\mathcal{M}, \mathcal{D}, \phi]\) satisfies the assumptions (1) and (2). Theorem 1. \([\mathcal{M}, \mathcal{D}, \phi]\) is isomorphic to a set of functions \(\mathcal{M}'\) over \(\mathcal{M}\). The mapping is given by \(F' = [G_1, ..., G_n] = \phi[F, G_1, ..., G_n]\). Call \(F^{-1}\) the inverse of \(F\) if \(\phi[F, F^{-1}j] = \phi[F, F, F^{-1}] = I\). The mapping \(G \leftrightarrow \phi[F, \phi[G, F^{-1}, ..., F^{-1}]]\) of \(\mathcal{M}\) into \(\mathcal{M}\) is an automorphism, which shall be called an inner automorphism. Theorem 2. Abstract 607-11, is also true under the present assumptions. Every automorphism of the set of all multplace functions over \(S\) is an inner automorphism. (Received September 27, 1963.)


Let \(m_1, ..., m_n\) be a (possibly finite) sequence of distinct integers; \(\mathcal{M}_1, ..., \mathcal{M}_n\), a collection of disjoint sets; and \(\phi_1, ..., \phi_n\), a collection of mappings such that (1) \(\phi_1\) maps \(\mathcal{M}_1 \times (\mathcal{M}_j^{m_j})\) into \(\mathcal{M}_2 \times (\mathcal{M}_j^{m_j})\); (2) \(\phi_1(F, \phi(G_1, ..., G_{m_1}, ..., G_j, ..., G_{m_j})) = \phi(F, G_1, ..., G_{m_1}, ..., G_j, ..., G_{m_j})\); and (3) each \(\mathcal{M}_1\) includes \(I_1, ..., I_{m_1}\) such that \(\phi(F_{k_1}, ..., F_{k_{m_1}}) = F_k\) if \(k = 1, ..., n_\) and \(k = 1, ..., m_1\) and \(\phi(F, F_1, ..., F_{m_1}) = F\). Theorem 3. If \(\mathcal{M}_1\) can be imbedded in a system of functions mapping \((\mathcal{M}_j^{m_j})\) into \(\mathcal{M}_j(1, ..., n, ...)\), Theorem 2. If for fixed \(j\) and \(i\) and any distinct \(F', F'' \subset \mathcal{M}_j\) there exist \(G_1, ..., G_{m_j} \subset \mathcal{M}_j\) such that \(\phi(F', G_1, ..., G_{m_j}) \neq \phi(F'', G_1, ..., G_{m_j})\), then \((\mathcal{M}_j, \phi)\) is isomorphic to a subset of the \(m_j\)-place functions over \(\mathcal{M}_j\). Theorem 4. Every automorphism of the entire set of multplace \((n\)-place\) functions over \(S\) is an inner automorphism. (Received September 27, 1963.)

64T-29. C. J. EVERETT, 1334 43 Street, Los Alamos, New Mexico. An inequality on doubly stochastic matrices.

If \(n \geq 2\), \(0 \leq x_j \leq 1\) for \(j = 1, ..., n\), and \(\sum x_j = x \leq 3/2\), then \(\sum x_j \{(1 - x_1) \ldots (1 - x_n)\} \geq x(1 - (x/n)^{n-1})\), where the brace omits the factor \((1 - x_j)\). In case \(x = 1\), this implies \(\text{Per}(X) \geq n!/n^n\) for the permanent of the special doubly stochastic matrix with one arbitrary row \(x_1, ..., x_n\), and all remaining rows \((1 - x_1)/(n - 1), ..., (1 - x_n)/(n - 1)\). For terminology cf. H. J. Ryser, Combinatorial mathematics, Carus Math. Monograph No. 14, Wiley, New York. (Received October 18, 1963.)
A mathematical system W which is a Riemannian geometry and is described in the nomenclature of physics has as elemental quantities a scalar ψ, a vector φ₁₄, and a tensor gₐβ. These are complex functions of xa (α = 1,...,4) and are related by the postulated equations: φ₁₄ + φ₁₄ = 0, Rₐβ - Rgₐβ/2 = Tₐβ, and gₐβφₐβ = ± 4π²m²c²ψ/h². A complex mass is defined by m = n + ie/√7 where m is mass and e is charge. The quantities ψ, φ₁₄, and gₐβ are each combined gravitational (real) and electromagnetic (imaginary) quantities, where ψ is a probability density function, and gₐβ is a metric. A construction for the general system W is given. A steady-state solution W₁ is obtained assuming Tₐβ = 0 and spherical symmetry. Calculations lead directly to the well-known physical equation: ψ = 2π²me²z²/n²h³, where n is an integer, for the frequencies of a hydrogen-like ion. The motion of a complex mass in W₁ correctly mirrors physical laws for combined mass and charge in a combined gravitational and electromagnetic field gₐβ. A generalization of the Dirac matrices is suggested. Further refinements are given in the postulates for a mathematical world W which lead to quantum-electrodynamics. (Received October 18, 1963.)

Bessaga and Pelczynski (Proc. Symp. on Topology, Prague (1961), 87-90) ask whether two homeomorphic Banach spaces are also uniformly homeomorphic. In the present paper it is shown that the answer to this problem is negative. An example of two homeomorphic separable Banach spaces X and Y is given such that for every map F from X onto Y there exist sequences {uₙ}ₙ=₁ and {vₙ}ₙ=₁ in X such that either ||uₙ - vₙ|| → 0 while ||Fuₙ - Fvₙ|| → ∞ or ||uₙ - vₙ|| → ∞ while ||Fuₙ - Fvₙ|| → 0. The proof of this result is based on the following lemma. For every p with 1 < p < ∞ there exists a separable metric space Xₚ containing the Banach space lₚ such that there is no projection P from Xₚ onto lₚ which satisfies a Lipschitz condition of order > 1/p. (Received October 21, 1963.)

Let B(X,Y) be the Banach space of all bounded linear operators from the Banach space X into the Banach space Y. Let P(X,Y) be the subset of B(X,Y) consisting of all operators T for which there is an x ∈ X satisfying ||x|| = 1, ||Tx|| = ||T||. The following problem, which was raised by Bishop and Phelps (Bull. Amer. Math. Soc. 67 (1961), 97-98), is treated here: For which X and Y is P(X,Y) dense in B(X,Y) (in the norm topology for operators)? Among the results obtained are: (1) If X is reflexive then P(X,Y) is dense in B(X,Y) for every Y. (2) If X is separable and if P(X,Y) is dense in B(X,Y) for every Y then the unit cell of X is the closed convex hull of its strongly exposed points. As a corollary the following refinement of the Krein Milman theorem is obtained:
Every weakly compact convex set in a separable Banach space is the closed convex hull of its strongly exposed points. (Received October 21, 1963.)


Analytic polygenic functions are complex-valued functions of a single complex variable \( z \) which are representable as power series in \( z \) and \( \overline{z} \). Elementary properties of such functions are considered. Cauchy's formula for the radius of convergence of a power series in \( z \) and \( \overline{z} \) is deduced. An analogue of Cauchy's inequalities is established, containing as a special case Cauchy's inequalities for direct holomorphic, reverse holomorphic, and harmonic functions. The partial derivatives of \( f \) with respect to \( z \) and \( \overline{z} \) are introduced, in terms of which the coefficients of the power series for \( f \) are expressed. Necessary and sufficient conditions that a polygenic function \( f \) be an analytic polygenic function are given in terms of inequalities on the partial derivatives of \( f \) with respect to \( z \) and \( \overline{z} \). Various analogues of the principle of permanance for analytic polygenic functions are obtained. For example, if \( f \) is an analytic polygenic function defined on some open connected set \( D \) and if \( f \) vanishes for an infinite number of points on some real analytic curve \( C \) in \( D \) with a finite point of accumulation on \( C \), then \( f \) vanishes identically on \( C \). (Received October 24, 1963.)

64T-34. T. G. McLAUGHLIN, University of Illinois, Urbana, Illinois. A theorem on indecomposable sets.

Theorem. Let \( \beta \) be a nonrecursive r.e. set and \( \{ \omega_{\phi(j)} \} \) an (effective) \( K_0 \)-component Friedberg-type decomposition of \( \beta \). Then there exist a cohesive set \( a \subseteq \beta \), and a sequence \( \{ K_j \} \) of cohesive sets, such that \( (1) K_j \subseteq \omega_{\phi(j)} \) for every \( j \); \( (2) a \cup K_j \) is indecomposable for every \( j \); and \( (3) a \cup ( \bigcup K_j ) \) is indecomposable and is not split by any r.e. superset of \( a \). Corollary 1. The conjunction of all of the propositions announced in 63T-304 (these Notices, August, 1963) and in 63T-387 (these Notices, October, 1963). Corollary 2. There exist indecomposable number sets with infinite retraceable subsets (none of which can be retraced by a total recursive function). (Received October 24, 1963.)


Physical systems \( S_k, k \neq -1 \), of \( \omega^3 \) trajectories of a bicentral positional field of force in the Euclidean plane are studied. Elliptic and bipolar coordinates are introduced. Various formulas concerning positional fields of force and physical systems \( S_k, k \neq -1 \), in the Euclidean plane are expressed in elliptic and bipolar coordinates. Conditions for conservative, solenoidal, and Laplacian fields of force are deduced in terms of elliptic and bipolar coordinates. A certain type of bicentral positional field of force, which is an analogue of the Newtonian bicentral positional field of force, is studied. For this type of field of force, the \( \omega^3 \) trajectories \( C \) of a physical system \( S_k, k \neq -1 \), are obtained by means of quadratures. For these physical systems, an analogue of the description of the areal velocities under a Newtonian bicentral field of force is obtained. Also an extension of Liouville's potential function to bicentral fields of force is given for physical systems \( S_k, k \neq -1 \), in the Euclidean plane. (Received October 24, 1963.)
A theorem on definitions of the Zermelo-Neumann ordinals.

Let $S$ be the system obtained from the predicate calculus with equality by adding extensionality, Aussonderung, and self-adjunction, i.e., $(x)(Ey) y = x'$ or $(Ey)(y = x \cup \{x\})$. Let $STI$ for $On$ be:

$$Fx \text{ if } On(x), \quad (v)(Fv :: \neg Fv'), \quad \text{and} \quad (w)(w \in F :: \neg F(w)).$$

Theorem. If $Q$ is a subsystem of $ZF$ containing $S$ and $On(a)$ is any formula of $ZF$, and $STI$ for $On$ can be proved in $Q$, then $\{Q On(x) \supset Od(x), \quad Od \text{ being} \quad \text{the current definition of Zermelo-Neumann ordinals; hence, if every ordinal does have the property} \quad \text{On, On gives in} \quad Q \text{ an adequate definition of ordinals. This is applied to give genetic definitions of ordinals and, in particular, finite ordinals. The theorem for DI in a previous abstract (A genetic definition of the class $On$ of ordinals) follows easily; a similar theorem is also proved for another definition DI*, viz., }$ On(x) \text{ if, for every } u, \quad (Ew)(:::w \in u A w \in xA(x\wedge w = 0) \text{ whenever } x \in u \quad \text{and} \quad (v)(v' \in u \supset v \in u).$ (Received October 24, 1963.)

An algorithm for the solution of a word problem.

Let $G$ be a group and let $H$ be a subgroup of finite index generated by a finite set of words in $G$. The following problem is solved. Find a set of representatives for the cosets of $H$ and a new set of generators for $H$. Also given any word in $G$, express this word as a product of the new generators times a coset representative. The solution is practical for a digital computer. It combines the Coxeter-Todd coset enumeration procedure with a process due to Schreier. Included is a formal proof that the Coxeter-Todd process must close after finitely many steps. (Received October 25, 1963.)

Dual integral equations, Bessel kernels.


$$\int_0^{\infty} y^{N} f(y) J_\mu(xy) dy = g(x) \quad (0 < x < 1); \quad \int_0^{\infty} y^{N} f(y) J_\nu(xy) dy = h(x) (1 < x).$$

Here the Bessel functions in the kernel are of different order. The procedure is analogous and the result is a similar, although somewhat more complicated, expression. (Received October 28, 1963.)

Identity and uniqueness theorems for automorphic functions.

Let $C$ and $D$ denote the unit circle and the unit disk, respectively, and let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between the points $z_1$ and $z_2$ in $D$. Theorem. Let $\mathcal{G}$ be a Fuchsian group of the first kind and let $f(z)$ be a meromorphic function automorphic with respect to $\mathcal{G}$. Let $\{z_n\}$ be a sequence of points in $D$ with at least two limit points on $C$, such that $|z_n| \rightarrow 1$ and $\rho(z_n, z_{n+1}) \rightarrow 0$. If $f(z_n) \rightarrow c$, then $f(z) \equiv c$. A corresponding uniqueness theorem is also given. However, if $\mathcal{G}$ is a Fuchsian group of the second kind, there exists a nonconstant function automorphic with respect to $\mathcal{G}$, together with a set of points $\{z_n\}$ in $D$ with at least two limit points on $C$ such
that \(|x_n| \rightarrow 1\), \(\rho(x_n, x_{n+1}) \rightarrow 0\), and \(f(x_n) \rightarrow 0\). The construction of this function involves a modification of a Poincaré theta series. (Received October 28, 1963.)


In a complex inner product space \(V\), a real symmetric inner product \(\langle \cdot, \cdot \rangle\) is the real part of the complex inner product \(\langle \cdot, \cdot \rangle\). The angle \(\theta\) between two vectors \(A\) and \(JL\), with \(\|A\| > 0\), \(\|JL\| > 0\), is defined by the relation \(\langle A, JL \rangle = \|A\| \|JL\| \cos \theta\), where \(0 \leq \theta \leq \pi\). In the associated complex affine space \(\Sigma\) of \(V\), an isocline \(\pi\) is a real two-dimensional vector space obtained from a complex one-dimensional vector space. If \(\pi\) and \(\sigma\) are two distinct isoclines in \(\Sigma\), then the angle \(\theta\) between any vector \(A\) with \(\|A\| > 0\), on \(\pi\), and its real orthogonal projection \(JL\) with \(\|JL\| > 0\), on \(\sigma\), is independent of the direction of \(A\) in \(\pi\), and depends only on the real two-dimensional directions of \(\pi\) and \(\sigma\). It is given by \(\langle (\lambda, \mu) \rangle = \|\lambda\| \|\mu\| \cos \theta\), with \(0 \leq \theta \leq \pi/2\).

In an affine space \(\Sigma_{2n}\) there are \(n\) isoclinal coordinate planes. The direction cosines \(\cos a_i\) of an isocline \(\pi\) relative to these obey the identity \(\sum_{i=1}^{n} \cos^2 a_i = 1\). If \(\theta\) is the angle between two isoclines \(\pi\) and \(\tau\) with direction angles \(a_i\) and \(\beta_i\), the inequality \(\sum_{i=1}^{n} \cos a_i \cos \beta_i \leq \cos \theta\) is established. The area \(K\) in \(\pi\) is equal to the sum of the projected areas on the coordinate isoclinal planes. Finally, these concepts are extended to the classical types of complex Hilbert space.

(Received October 28, 1963.)


Let \(X = \{x_1, x_2, \ldots\}\) denote a countably infinite completely regular space, let \(C^*(X)\) denote the set of all bounded continuous real-valued functions on \(X\), and let \(A\) be a regular (summability) matrix. Theorem. If \(A\) sums every element of \(C^*(X)\), then there is an absolutely convergent series \((a_n)\) with sum 1 such that \(A\) sums \(f\) to \(\sum a_n f(x_n)\) for every \(f \in C^*(X)\). This is applied to prove that no regular matrix can sum all of \(C^*(X)\) if \(X\) is extremally disconnected. This latter result was obtained by W. Rudin (Duke Math. J. 25 (1958), 197-204) in case \(A\) is the matrix of \(c_1\)-summability. (Received October 28, 1963.)

64T-42. A. L. WHITEMAN, University of Southern California, Los Angeles 7, California, Theorems on Brewer and Jacobsthal sums.

Let \(p\) denote an odd prime and \(\chi(s)\) the Legendre symbol \((s/p)\). The sum \(B = \sum_{s=0}^{p-1} \chi(s(s^4 - 5s^2 + 5))\) has been studied by Brewer (Trans. Amer. Math. Soc. 99 (1961), 241-245). Brewer obtained the following results: If \(p \equiv 3 \pmod{4}\) or if \(p \equiv \pm 2 \pmod{5}\), then \(B = 0\). If \(p = 20f + 1 = u^2 + 5v^2 = x^2 + 4y^2\) with \(x \equiv 1 \pmod{4}\) (mod 4), then \(B = 0\) if \(5|x\) and \(B = -4u\) if \(5|x\) and \(u \equiv x \pmod{5}\) (mod 5). If \(p = 20f + 9 = u^2 + 5v^2 = x^2 + 4y^2\) with \(x \equiv 1 \pmod{4}\) (mod 4), then \(B = 0\) if \(5|x\) and \(B = 4u\) if \(5|x\) and \(u \equiv x \pmod{5}\) (mod 5). In this paper \(B\) is evaluated by a new method. The following theorems are also established, if \(p = 20f + 1\), then \(C_{10f,f} C_{10f,3f} \equiv 4u^2 \pmod{p}\) (Cauchy). Furthermore \(C_{10f,f} \equiv C_{10f,3f}\) or \(-C_{10f,3f}\) (mod p) according as \(5|x\) or \(5|x\). If \(p = 20f + 9\), then \(C_{10f+4,f} \equiv C_{10f+4,3f+1} \equiv 4u^2 \pmod{p}\).
Furthermore \( C_{10f+4,3} = C_{10f+4,3}^+ \) or \( C_{10f+4,3}^+ = C_{10f+4,3} \) according as \( 5 \not| \ x \) or \( 5|\ x \). (Received October 30, 1963.)

64T-43. WITHDRAWN.

64T-44. K. J. DAVIS, University of Tennessee, Knoxville 16, Tennessee. A generalization of the Dirichlet product of arithmetical functions.

For each integer \( k \geq 1 \), let \( L_k \) represent the set of positive integers \( n \) whose prime divisors have multiplicity \( \geq k \), and let \( Q_k \) be the set of all \( n \) whose prime divisors have multiplicity \( < k \). Every \( n \) can be written uniquely in the form \( n = n_1 n_2 \) where \( n_1 \in L_k \), \( n_2 \in Q_k \) and \( (n_1, n_2) = 1 \).

If \( m \) and \( n \) are positive integers with decompositions, \( m = m_1 m_2 \), \( n = n_1 n_2 \) of this form, then \( m \) and \( n \) are said to be relatively \( k \)-prime (notation: \( (m,n)_k = 1 \)) provided \( (m_2, n_2) = 1 \). The generalized convolution \( h \) of arithmetical functions \( f, g \) is defined by \( h(n) = \sum \{f(d)g(\delta) \mid do = n, (d, \delta)_k = 1\} \), the summation being over \( d, \delta \) such that \( d \delta = n \). In case \( k = 1 \), this definition reduces to the ordinary Dirichlet product.

Analogues of certain well-known arithmetical functions are introduced with the Dirichlet product replaced by the generalized convolution. Using elementary methods, asymptotic formulas involving these functions are proved. In particular, results are obtained for generalized divisor and totient functions and for a generalization of the square-free integers. (Received September 19, 1963.)


Theorem. Suppose \( \alpha \) is a number set such that each of \( \alpha \), \( \overline{\alpha} \) is regressive. Then either \( \alpha \) or \( \overline{\alpha} \) is recursively enumerable. Corollary (Mansfield). Complementary retraceable sets are recursive.

The proof of the theorem proceeds by hindsight, in that one constructs enumerating functions based on the conditions necessary for various reductio ad absurdum hypotheses to hold. The paper also includes (a) a generalization to regressive sets (and to discrete arrays with rows of bounded cardinality) of a certain retraceability lemma of Mansfield, and (b) the result that if \( \alpha \) is hypersimple, \( \overline{\alpha} \) is regressive, and \( p \) (specially) regresses \( \alpha \), then domain (\( p \)) must be a finite extension of \( \alpha \).

(Received October 31, 1963.)

64T-46. E. W. CHENEY, University of California, Los Angeles 24, California, and A. A. GOLDSTEIN, University of Texas, Austin, Texas. Existence theorems for Tchebycheff approximation, I.

The theorem below answers the following question. For what sets in a Banach space is it true that every bounded functional on that set possesses a best Tchebycheff approximation by a bounded linear functional? Theorem. If \( X \) is a subset of a real Banach space such that \( 0 \) lies in the interior of the convex closure of \( Xu - X \) relative to the linear hull of \( X \), then to each bounded functional \( \phi \) on \( X \) there corresponds a bounded linear functional \( f \) minimizing \( \sup \{||f(x) - \phi(x)|| : x \in X\} \). A similar result for operators is as follows. Theorem. Let \( E_1 \) and \( E_2 \) be Banach spaces. If \( X \) is a subset of \( E_1 \) such that \( 0 \) is interior to the convex closure of \( Xu - X \), then to each bounded map \( \phi \) of \( X \) into \( E_2 \) there corresponds a bounded linear map \( F \) of \( E_1 \) into \( E_2^* \) which minimizes \( \sup ||Fx - \phi x|| : x \in X \).

(Received November 4, 1963.)
64T-47. B. R. KRIPKE and R. T. ROCKAFELLAR, University of Texas, Austin, Texas. Existence theorems for Tchebycheff approximation, II.

Theorem. If $X$ is a bounded subset of a real Banach space, $B$, such that $0$ does not lie in the interior of the convex closure of $X_0 - X$ relative to the linear hull of $X$, then there exists a bounded functional $\phi$ on $X$ such that the minimum of $\sup \{ |f(x) - \phi(x)| : x \in X \}$ is not attained as $f$ ranges over the set of bounded linear functionals on $B$. In fact, $\phi$ may be taken to be a linear functional on the linear hull of $X$ which is continuous. See the preceding abstract for a converse to this theorem. (Received November 4, 1963.)

64T-48. E. W. CHENEY, University of California, Los Angeles 24, California, and A. A. GOLDSTEIN, University of Texas, Austin, Texas. Characterization theorems for Tchebycheff approximation.

The theorems below answer the following question. What characterizes those continuous linear functionals which are best Tchebycheff approximations to a nonlinear functional? Theorem 1. Let $X$ be a bounded subset of a locally convex space $E$ (real or complex), and let $\phi$ be a bounded functional on $X$. In order that an element $f \in E^*$ minimize the expression $N(f) = \sup \{ |f(x) - \phi(x)| : x \in X \}$ it is necessary and sufficient that for each $\epsilon > 0$ the origin lie in the convex closure of $X_\epsilon = \{ f(x) - \phi(x)x : x \in X \text{ and } |f(x) - \phi(x)| \leq N(f) - \epsilon \}$. Theorem 2. If $X$ is compact and $\theta$ is continuous then the condition is that $0$ lie in the convex closure of $X_0$. (Received November 4, 1963.)

64T-49. D. T. HAIMO, 7201 Cornell Avenue, St. Louis 30, Missouri. Representation for Hankel convolutions.

Let $\gamma$ be a fixed positive number and let $\mu(x) = x^{2\gamma+1} / \Gamma(\gamma + 1/2) \Gamma(\gamma + 3/2)$. Define $f_\gamma(x) = x^{1/2-\gamma} J_{1/2-\gamma}(x)$ and $G_\gamma(x) = x^{1/2+\gamma} I_{1/2+\gamma}(x)$, where $J_{1/2-\gamma}(x)$ is the ordinary Bessel function of order $\gamma - 1/2$ and $I_{1/2+\gamma}(x)$ is the Bessel function of imaginary argument.

Let $G(x,y) = \int_0^\infty G_\gamma(xt) G_\gamma(yt) dt \Phi_\gamma(t) + \sum_{k=1}^\infty (1 + t^2/a_k^2) \Phi_k(t)$, $0 < a_1 \leq a_2 \leq \ldots$, and $\sum_{k=1}^\infty 1/a_k^2 < \infty$. Denote by $\Delta_x$ the differential operator $\Delta_x h(x) = h''(x) + x^{2\gamma} h'(x)$. Representation theorem. Necessary and sufficient conditions that a function $f$ be given by $f(x) = \int_0^\infty G_\gamma(x,y) \Phi(y) dy$ with $\Phi(y) \uparrow$ are that (i) $f(x) \in C^\infty$, $0 \leq x < \infty$, (ii) $f^{(2k+1)}(0) = 0$, $k = 0, 1, \ldots$, (iii) $f(x) = \Phi(a_1 x)$, $x \rightarrow \infty$, and (iv) $\prod_{n=1}^\infty (1 - \Delta_x/a_k^2) f(x) \geq 0$, $0 < x < \infty$, $1 = N_0 < N_1 < \ldots$. The theorem depends on the fact that a function $f$ satisfying assumptions (i) - (iv) has the fundamental representation $f(x) = \int_0^\infty G_\gamma(x,y) \Phi(y) dy \Phi_k(t) + \sum_{k=1}^\infty (1 + t^2/a_k^2) \Phi_k(t)$, $0 < x, y < \infty$, $N = 1, 2, \ldots$. This basic result in conjunction with an application of Helly's theorem and an appeal to Tauberian theorems serves to establish the principal representation theorem. (Received November 4, 1963.)


Let $G$ be a compact group and $L(G)$ the group algebra of $G$ with convolution multiplication. The terminology is that used in the book Normed rings by M. Naimark. Theorem. A function $\phi$ on $G$
is an elementary continuous positive definite function if and only if \( \phi \) is an irreducible hermitian idempotent of \( L_2(G) \) for some positive \( \lambda \). This result is well known for \( G \) abelian, because such a function \( \phi \) is necessarily some positive multiple of a character on \( G \). (Received November 4, 1963.)

64T-51. JOHN DeCICCO and S. T. RAMCHANDRAN, Illinois Institute of Technology, 3301 South Dearborn, Chicago 16, Illinois. \textbf{The initial rates of departure in a directional field of force.}

If the angular rate \( \lambda \neq 1 \), at an admissible lineal element \( E \) of the Euclidean plane, then in a neighbourhood of \( E \), the given directional field of force \( F \) possesses \( \omega^\rho \) Faraday lines of force \( C_1 \). If a particle is constrained to move along a trajectory \( C \) of a physical system \( S_k^m \) and if it starts from rest at the initial point \( P \) of the lineal element \( E \), then both the trajectory \( C \) and the Faraday line of force \( C_1 \) are initially tangent in the direction of the lineal element \( E \). If \( m \) and \( n \) are the orders of contact of \( C_1 \) and \( C \) with the line of the lineal element \( E \) at the initial point \( P \), then \( m \geq n - 1 \). This is a new phenomenon, since \( m = n \) for the positional case. When \( m = n \geq 1 \), the ratio \( \rho \) of the initial rates of departure of the rest trajectory \( C_1 \) and the Faraday line of force \( C_1 \) is \( \rho = (1 + k)(1 - \lambda)/[2n + (1 + k)(1 - \lambda)] \). If \( \lambda = 0 \), the direction of the force vector depends only on the position of the point \( P \). Here the formula for \( \rho \) is \( \rho = (1 + k)/(2n + 1 + k) \). This is an extension of the theorem of Kasner in the positional case. (Received November 3, 1963.)

64T-52. ECKFORD COHEN, The University of Tennessee, Knoxville 16, Tennessee. \textbf{Remark on a set of integers.}

Let \( k \) be an integer \( > 1 \) and let \( S_k(x) \) denote the number of positive integers \( n \leq x \) such that the maximal exponent to which each prime factor of \( n \) divides \( n \) is not a multiple of \( k \). It was proved previously (Acta Sci. Math. (Szeged) 22 (1961), 223-233) that \( S_k(x) = a_k x + O(x^{1/k}) \), where \( a_k \) is a positive constant depending on \( k \). The proof was elementary but made use of generating functions. In this note the above formula is proved without the use of generating functions. (Received November 4, 1963.)

64T-53. Y. F. WONG, Box 216, Atlanta University, Atlanta 14, Georgia. \textbf{A theorem on manifolds of the same homotopy type. Preliminary report.}

All manifolds considered here are to be compact, oriented and differentiable. \textbf{Theorem.}

Let \( M_1^{2k-1} \) and \( M_2^{2k-1} \) be two simply connected manifolds, with \( k \) greater than 3, satisfying the following: (1) There is a simply connected manifold \( N^{2k} \) with boundary \( \partial N = M_1 \cup - M_2 \), such that \( \pi_q(N,M_2) = 0 \) for \( q = 0,1,2,\ldots,k - 1 \). (2) There is a continuous map \( g: N \rightarrow M_2 \) such that \( g|\partial N = \text{identity} \) and \( g|M_1 \) is a homotopy equivalence. Then there is a homotopy sphere \( S^{2k-1} \) bounding a \((k-2)\)-connected handle body, such that \( M_1 \) is diffeomorphic to the connected sum of \( M_2 \) and \( S \).

\textbf{Proof.} One obtains from (2) the splitting homotopy and homology exact sequences of \( (N,M_2) \). A \((k-2)\)-connected handle body which represents \( \pi_k(N,M_2) \) can be realized in \( N \). The boundary of the handle body is a homotopy sphere \( S^{2k-1} \). One removes the handle body and a smooth curve connecting the handle body and \( M_2 \). The resultant manifold is denoted by \( N' \). Using the Mayer-Vietoris sequence, one concludes that \( \pi_i(N,M_2) \) vanishes for all \( i \). One of Smale's h-cobordism
theorems implies that the two boundary components of N' are diffeomorphic. The boundary components are correspondingly M_1 and M_2 connected with S. (Received November 5, 1963.)


Assume k is the Fourier transform of a real-valued integrable function \( \hat{k} \) and denote by \( \lambda_1 \geq \lambda_2 \geq \ldots \) the positive eigenvalues of the kernel \( k(x - y) \) on \((-1, 1)\). (I) If \( \log \hat{k}(\xi) \sim -a|\xi| \) as \( \xi \to \pm \infty \) \((0 < a < \infty)\) then \( \log \lambda_n \sim -\pi(K'/K)(\tanh \pi a) \) as \( n \to \infty \), where \( K \) is the complete elliptic integral of the first kind. (II) If \( k \) is bounded, vanishes outside \((-\beta, \beta)\), but has positive g.l.b. on each closed subinterval, then \( \log \lambda_n = -2n \log n + 2n(1 + \log \beta/4) + o(n) \). Both theorems are proved from special cases \((k(x) = \text{sech}^2(2\alpha x) - 1)\) and \((k(x) = \sin(\beta x)/\beta x, \text{respectively})\) where the eigenfunctions are the eigenfunctions of certain differential equations. [See J. A. Morrison, On the commutation of finite integral operators, with difference kernels, and linear self-adjoint differential operators, these Notices, 9 (1962), 119]. (Received November 6, 1963.)


Let \( X \) be a locally compact Hausdorff space and \( \mathcal{M}(X) \) denote the space of finite Baire measures on \( X \). Let \( \mathcal{F} \) be a linear space of (not necessarily bounded) real-valued Baire functions defined on \( X \) that contains the constant function and let \( \mu \in \mathcal{M}(X) \) be such that \( \int_X |f|d\mu < \infty \) for each \( f \in \mathcal{F} \). Set \( \mathcal{E}_\mu = \{ v/v \in \mathcal{M}(X) \text{ and } \int fvd\mu = \int fvd\mu' \in \mathcal{F} \} \). Theorem 1. \( \mathcal{F} \) is dense in \( L_1(\mu) \) iff \( \mu \) is an extreme point of \( \mathcal{E}_\mu \). Theorem 2. If \( \int_X f^2d\mu < \infty \) for every \( f \in \mathcal{F} \) and \( \mathcal{F} \) is also a vector lattice, then \( \mathcal{F} \) is dense in \( L_2(\mu) \) iff \( \mu \) is an extreme point of \( \mathcal{E}_\mu \). Corollary. If \( \mathcal{F} \) is a subalgebra of bounded real-valued Baire functions on \( X \) that contains the constant functions, then \( \mathcal{F} \) is dense in \( L_2(\mu) \) iff \( \mu \) is an extreme point of \( \mathcal{E}_\mu \). Remark. There exist extreme points in \( \mathcal{E}_\mu \) because it can be shown that \( \mathcal{E}_\mu \) is contained in the \( \omega^* \)-closed \([\mathcal{M}(X) = C_0(X)] \) convex hull of its extreme points. (Received November 6, 1963.)

64T-56. G. E. BAXTER and J. T. JOICHI, 400 Ford Hall, University of Minnesota, Minneapolis, Minnesota. On functions that commute with full functions.

Let \( I = [0, 1] \). A continuous mapping \( f \) of \( I \) onto \( I \) is full if \( I \) can be partitioned into a finite number of subintervals \( J_i \) such that \( f \) maps each \( J_i \) homeomorphically onto \( I \). A full function \( f \) is a hat function if the \( J_i \) are of equal length and \( f \) maps each \( J_i \) linearly onto \( I \). Theorem. Let \( f \) and \( g \) be continuous mappings of \( I \) into \( I \) such that \( f(g(x)) = g(f(x)) \) for each \( x \) in \( I \). (1) If \( f \) is a hat function with \( n \geq 2 \) branches, then \( g \) is either a constant function or a hat function. (2) If \( f \) is a full function with \( n \geq 2 \) branches and \( g \) has only finitely many maxima and minima, then \( g \) is a full function. An example shows that the finite number of maxima and minima condition in (2) is necessary. (Received November 7, 1963.)
Let $\mathcal{A}$ denote the lattice and let a binary relation $N$ "normality" be introduced into $\mathcal{A}$ in such a way that the Zassenhaus theorem holds for normal chains. Let $a, b \in \mathcal{A}$, $a \geq b$. Let $\mathcal{F}$ denote the set of all finite normal chains between $a$ and $b$, that is, $A \subset \mathcal{F} \iff A = \{a_0 \uparrow \uparrow b\}$ and $a_1^n a_{i+1}$ (i = 0, 1, ..., r - 1). The number $r$ is called the length of the chain $A$ and one writes $\lambda(A)$. Let $A, B \subset \mathcal{F}$; $A \circ B$ is defined as the Zassenhaus refinement of $A$ by $B$; that is $A \circ B$ is the set of elements $a_{i+1} \lor (a_i \land b_j)$ where $A = \{a_i\}_{i=0}^r$, $B = \{b_i\}_{i=0}^r$. According to the Zassenhaus theorem $A \circ B \subset \mathcal{F}$; thus $\mathcal{F}$ becomes a groupoid (in general not associative). Let $A, B \subset \mathcal{F}$. The chains $A, B$ will be called low simply similar and one writes $A \sim B$ if $\lambda(A) = \lambda(B) = n$ and there exists a permutation $\Phi$ such that $a_i/a_{i+1} \sim b_{\Phi(i)}$ ($i = 0, 1, 2, ..., n - 1$). It has been proved that two given normal chains $A$ and $B$ are invariant under the Zassenhaus refinement if and only if they are low simply similar: $A \circ B \iff \lambda(A) = \lambda(B)$. The pair $X, Y$ will be called a Schreier refinement of $A, B \subset \mathcal{F}$ if: (1) $X \sim Y$, (2) $A \subset X, B \subset Y$. The minimal property of the Zassenhaus refinement has been proved. It is the Schreier refinement of minimal length, which means: $A, B, X, Y \subset \mathcal{F}, X \sim Y, A \subset X, B \subset Y \Rightarrow \lambda(A \circ B) \leq \lambda(X)$. (Received November 8, 1963.)


The total rigid motion group $G_3^*$ of the Minkowski relativistic plane $M_2$ contains three essential parameters. Any map $T$ of $G_3^*$ is of the form: $Z$, or $Z = az + b$, $a \overline{a} = 1$, in which all quantities are hyperbolic complex numbers, Some elementary properties, invariant under $G_3^*$, of a triangle $T$, are developed. These are based on the Cauchy-Schwarz inequality, for which the sign may or may not be reversed depending on the position of the two vectors. For a modular or pseudomodular triangle $T$, the sides obey $b + c > a$, $c + a > b$, $a + b < c$. However no such inequalities exist for an indefinite triangle $T$. In all possibilities, the sum of the angles of a triangle $T$ are zero. Right angles possess measure zero. For a modular, pseudomodular or indefinite triangle $T$, the laws of hyperbolic sines and cosines are developed. Also for a right triangle $T$, the relations between a positive angle $A$ and the hyperbolic functions are established. The classical area formulas for a general triangle $T$ are obtained. These results are applied to an isotropic triangle $T$, in which one or two sides may be zero. Finally, various theorems are proved concerning the congruent and similar triangles $T$ of $M_2$. (Received November 8, 1963.)

64T-59. B. W. BOEHM, The RAND Corporation, 1700 Main Street, Santa Monica, California. Functions whose best rational Tchebycheff approximations are polynomials.

A continuous function $f$ on $[-1,1]$ is said to be of class $P(n,m)$ if the best approximation to $f$ by a rational function of degree $\leq n$ in the numerator and $\leq m$ in the denominator is a polynomial, Theorem. A function $f$ belongs to $P(n,n)$ for all $n$ if and only if $f$ is a constant plus a multiple of a Tchebycheff polynomial, Theorem. If, for some fixed $m$, $f$ belongs to $P(n,m)$ for all $n$, then the
degrees of the successive polynomials of best approximation to $f$ differ by at least $m + 1$. **Theorem.** For fixed $m$, there exist nonpolynomial functions $f$ belonging to $P(n,m)$ for all $n$. For fixed $c > 1$, there exist nonpolynomial functions belonging to $P([cn],n)$ for all $n$. (Received November 12, 1963.)

64T-60. B. W. BOEHM, The RAND Corporation, 1700 Main Street, Santa Monica, California. **Convergence of best rational Tchebycheff approximations.**

The following theorems concern the approximation of continuous functions by reciprocals of functions from a linear class. **Theorem.** Let $X$ be a compact topological space, and let $\{g_1, g_2, \ldots\}$ be fundamental in $C[X]$. The elements of $C[X]$ which are uniformly approximable to any degree of accuracy by functions of the form $1/\sum_1^n c_i g_i$ are precisely those functions which do not change sign at any point of $X$. **Corollary (Walsh).** Every nonnegative function $f \in C[a,b]$ is uniformly approximable to any degree of accuracy by reciprocals of polynomials. Furthermore, the exponents in such polynomials may be restricted to a sequence $\{p_i\}$ such that $\sum p_i = \infty$. **Theorem.** Let $\{g_0, g_1, \ldots\}$ be a Markoff system on $X$, so that each "generalized polynomial of degree $n$," $P(a,n) = E_0^n g_i$ can have no more than $n$ zeros on $X$. The functions of $C[X]$ which are uniformly approximate to any degree of accuracy by $P(a,n)/P(b,m)$, with $n$ fixed, are precisely those functions $f$ which change sign at $j \leq n$ points $x_i \in X$, and are such that $f(x)/\pi(x - x_i)$ is continuous on $X$. (Received November 12, 1963.)

64T-61. B. W. BOEHM, The RAND Corporation, 1700 Main Street, Santa Monica, California. **The degree of convergence of best rational Tchebycheff approximations.**

For a fixed $f \in C[-1,1]$, let $E(n,m)$ denote the distance (with respect to the Tchebycheff norm) from $f$ to the set of rational functions having degree $\leq n$ in the numerator and degree $\leq m$ in the denominator. A study of $E(n,m)$ has revealed a number of results, of which the following are representative. **Theorem (of Jackson type).** If $|f(x)| > \rho > 0$ on $[-1,1]$, and if $\rho > 0$, then for all sufficiently large $m$, $E(0,m) < (1 + \varepsilon) \|f\| \omega(\pi/m + 1)/\rho^2$, where $\omega$ is the modulus of continuity of $f$. **Theorem (of Bernstein type).** Given a sequence $\{A_i\}$ of real numbers decreasing to zero and a sequence of positive integers satisfying $m_{i+1} \geq 3m_i$, there exists an $f \in C[-1,1]$ such that $E(0,m_i) = A_i$ for all $i$. Similar results hold for $E(n,m)$ when $n$ is fixed. **Theorem.** Given a sequence $A_n \downarrow 0$, there exists an $f \in C[-1,1]$ such that $E(n,n) = A_n$ for infinitely many $n$. (Received November 12, 1963.)


Suppose $[a,b]$ is a number interval. **Definition.** If $R$ is a real-valued function of sub-intervals of $[a,b]$, then the statement that $R$ is refinement-unbounded means that if $K$ is a positive number, then there is a subdivision $D$ of $[a,b]$ such that if $I$ is an interval of a refinement of $D$, then $R(I) > K$. **Theorem.** If each of $g$ and $m$ is a real-valued nondecreasing function on $[a,b]$, then the following two statements are equivalent: (1) For each real-valued, refinement-unbounded function $R$ of subintervals of $[a,b]$, the Hellinger type integral $\int_{[a,b]} \min \{dg,R(I)dm\}$ exists and is $g(b) - g(a)$; and (2) $g$ is absolutely continuous with respect to $m$. (Received November 12, 1963.)
64T-63. WITOLD BOGDANOWICZ and TAD KRAUZE, Georgetown University, Washington 7, D. C. On a functional-integral equation.

Let \( R \), \( R^+ \) denote the set of all reals and the set of all nonnegative reals respectively. Let \( u(x), v(y) \) be real-valued functions defined on \( R \) and \( R^+ \) respectively. Let the function \( u \) be continuous for \( x = 0 \). Then the functions \( u, v \) satisfy the functional equation \( v(\int_R |x|^n u(x) dx) = \int_R u(x) dx (n > 1) \) for all measures \( \mu \) such that \( \mu(R) = 1 \) and \( \int_R x^d \mu = 0 \) if and only if they are of the form \( u(x) = a + bx + c|x|^n \) and \( v(y) = a + cy \) where \( a, b, c \) are any constants. The proof of the theorem is very elementary and makes use only of the notion of limit. (Received November 12, 1963.)

64T-64. WITOLD BOGDANOWICZ, Georgetown University, Washington 7, D. C. Existence and uniqueness of asymptotically almost periodic and almost periodic solutions of nonlinear operational-differential equations.

Let \( Y \) be a complex Banach space and \( I \) either the additive group \( R \) of reals or the additive semigroup \( R^+ \) of nonnegative reals. Let \( C \) be the space of all continuous bounded functions from \( I \) into \( Y \) with the uniform topology. Let \( P \) be the set of all functions \( g \) from \( C \) such that the set \( \{ g(x) : x \in I \} \) is precompact in \( C(g(x) = g(x + x)) \). If \( I = R \), the space \( P \) is the space of almost periodic functions. If \( I = R^+ \), the space \( P \) is the space of asymptotically almost periodic functions. Let \( A = \sum A_k (d/dx)^k \) be a closed, linear, differential operator from a linear subset of \( P \) into \( P \). Let \( f(t, y) \in P \) for every \( y \) in \( Y \) and let \( f \) satisfy Lipschitz's condition with respect to \( y \) with a constant \( L_1 \). Let there exist for every \( g \in P \) a unique solution \( y \in P \) of the equation \( Ay = g \). Then there exists a constant \( C \) depending only on \( A \) such that if \( L < C \) then the differential equation \( Ay = f(t, y) \) has a unique solution \( y \in P \). If the function \( f(x, y) \) is uniformly almost periodic respectively asymptotically almost periodic in \( x \) with respect to \( y \), and is uniformly continuous in \( y \) with respect to \( x \), and maps the set \( I \times Y \) into a compact part of \( Y \), then the nonlinear differential equation has a solution \( y \in P \). The proofs make use of contraction mapping and Schauder fixed point theorems. (Received November 12, 1963.)

64T-65. WITOLD BOGDANOWICZ, Georgetown University, Washington 7, D. C. On a nonlinear hyperbolic system of differential equations.

Let \( J_1, J_2 \) be two continuous curves in the rectangle \( I_1 \times I_2 \), \( I_1 = [0, t_1], I_2 = [0, t_2] \), with the equations \( x_2 = J_1(x_1) \) for \( x_1 \) in \( I_1 \) and \( x_1 = J_2(x_2) \) for \( x_2 \) in \( I_2 \). Let the only point of intersection of the curves be the origin \( O \). Let \( L \) be the space of functions continuous on \( I_1 \times I_2 \times R \times R \times R \) satisfying Lipschitz's condition in the last three variables, let \( C_k \) be the space of functions continuous on \( J_k \), and \( R \) be the space of reals. Let \( A_k \in C_k \). The system of equations \( u_{12} = f(x_1, x_2, u_1, u_2) \) in \( I_1 \times I_2 \), \( u_1 + a_1 u_2 = g_1 \) on \( J_1 \), \( u_2 + a_2 u_1 = g_2 \) on \( J_2 \), \( u(0) = h \) has, for every \( f \in L \), \( g_k \in C_k \), \( h \in R \), a unique solution iff \( |a_1(0) a_2(0)| < 1 \). The proof that the condition is sufficient makes use of (1) an isomorphism of the space \( C \times C_1 \times C_2 \times R \), where \( C \) is the space of continuous functions on \( I_1 \times I_2 \), and the space of all functions \( u \) continuous together with the derivatives \( u_1, u_2, u_{12} \) on the rectangle, and (2) contraction mapping theorem. If the system always has a unique solution, then an operator in the space of convergent sequences must have an inverse which exists iff \( |a_1(0) a_2(0)| < 1 \). Thus the condition is necessary. (Received November 12, 1963.)
Let \( Y \) be a complex Banach space and \( R \) the space of reals. Let \( A \) be a linear continuous operator from \( Y \) into itself whose spectrum does not intersect the imaginary axes on the complex plane. If \( f(x,y) \) is uniformly continuous from \( R \times Y \) into \( Y \), has a period \( T \) in \( x \), and maps the space \( R \times Y \) into a compact set, then the differential equation \( y' + Ay = f(x,y) \) has a solution with period \( T \).

Let \( S \) be a sphere in \( Y \) with the center at the point \( 0 \), and let \( f \) map \( R \times S \) into a compact part of \( Y \) and be uniformly continuous. There exists a positive constant \( C \) such that if \( \|f(x,y)\| \leq C \) for \( x \in R \) and \( y \in S \) and \( f \) is periodic in \( x \) with period \( T \), then there exists a periodic solution of the differential equation with the period \( T \). The proof makes use of Schauder fixed point theorem and some construction in the ring of linear continuous operators from the space \( Y \) into itself. (Received November 12, 1963.)

Suppose that for a denumerable state space Markov chain one has for any two states \( x, y \) that the series \( \sum_{n=0}^{\infty} (p^n_{xx} - p^n_{yx}) \) converges. If \( N_n(A) \) is the occupation time of a set \( A \) by time \( n \), then for any finite \( A \), nonnegative integer \( k \), and states \( x, y \) one has that \( \lim_{n \to \infty} \sum_{n=0}^{R} \sum_{y \in S} (y) = 0 \) exists and this limit is explicity found. Under much more restrictive conditions one may show that the ratios \( \lim_{n \to \infty} \frac{P_x(N_n(A) = k)}{P_y(N_n\{y\}) = 0} \) also have a limit. Previously it was shown (H. Kesten and F. Spitzer) that ratios (2) have limits for the partial sums of independent, identically distributed integer lattice valued random vectors. Ratio theorems of the above type are also investigated for other functionals such as \( Y_n(A) \), the time of last visit to \( A \), \( Z_n(A) \), the time of next visit to \( A \), etc. (Received November 13, 1963.)

Let \( \epsilon \) denote the set of all nonnegative integers, \( \Lambda \) the collection of all isols and \( \Lambda_R \) the collection of all regressive isols. For \( f(x) \) a recursive function, let \( F(X) \) denote the canonical extension of \( f \) to \( \Lambda \). (For particular definitions see: J.C.E. Dekker, *Infinite series of isols*, Proc. Sympos. Pure Math. pp. 77-96, Amer. Math. Soc. (1962) and J. Myhill, *Recursive equivalence types and combinatorial functions*, Proceedings of the symposium on logic, methodology and philosophy of science, Stanford, (1962), 46-55.) A function \( f \) from \( \epsilon \) into \( \epsilon \) is called monotone increasing if \( m < n \) implies \( f(m) \leq f(n) \). A function \( f \) from \( \epsilon \) into \( \epsilon \) is called eventually monotone increasing if, for some nonnegative integer \( k \), the function \( g(n) = f(n + k) \) is monotone increasing. Theorem. Let \( f \) be a recursive function. Then \( F \) maps \( \Lambda_R \) into \( \Lambda_R \) if and only if \( f \) is eventually monotone increasing. (Received November 13, 1963.)

S. S. Holland, Jr. (Trans. Amer. Math. Soc. 108, 68) writes $P \perp Q$ for elements $P$, $Q$ of an orthomodular lattice if (i) $P = (P \land Q^\perp) \lor (P \land Q)$. Call $P$ orthogonal to $Q$ in the sense of walls and floors if (ii) $Q \perp P \land (P \land Q)^\perp$. Proposition: Conditions (i), (ii), and the symmetric condition (iii) $P \land (P \land Q) \lor (P \land Q)^\perp$, are mutually equivalent in an orthomodular lattice, for any pair of elements $P$, $Q$. Proposition: Bounded self-adjoint projections on a Hilbert space commute ($PQ = QP$) iff they satisfy any (hence all) of the above conditions. (Received November 13, 1963.)

64T-73. B. W. BOEHM, University of California, Los Angeles, California. Existence of best rational Tchebycheff approximations.

Let $X$ be a compact perfect topological space. The set of functions $g_1, \ldots, g_n, h_1, \ldots, h_m \in C[X]$ is said to be of class L on $X$ if, for any vector $\gamma = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ in real $m + n$-space $\mathbb{R}^{n+m}$ with $b_i$ not all zero, the function $\sum_{i=1}^{m} a_i h_i$ is different from zero on a set $Y_{\gamma}$ dense in $X$, and for any $x_0 \in X - Y$, the limit $r_{\gamma}(x_0) = \lim (x \in Y, x \to x_0) \sum a_i g_i(x)/\sum b_i h_i(x)$ exists on the extended real line. For $x \in Y$, define $r_{\gamma}(x) = \sum a_i g_i(x)/\sum b_i h_i(x)$; the resulting $r_{\gamma}$ is called a rational function, defined on $X$. Theorem. If the set $g_1, \ldots, g_n, h_1, \ldots, h_m$ is of class L on $X$, and if $f \in C[X]$, then there exists a rational function $r^*$ of best approximation to $f$ in the Tchebycheff sense, i.e. satisfying $\sup(x \in X) |f(x) - r^*(x)| = \inf (\gamma \in R^{n+m}) \sup (x \in X) |f(x) - r_{\gamma}(x)|$. Remarks. Any closed set $C \subset \mathbb{R}^{n+m}$ containing a vector $\gamma$ such that the $b_i$ are not all zero can be substituted for $R^{n+m}$ in the theorem. Similar results hold for existence of best Tchebycheff approximations of the form $C_{\gamma} f_i + \sum a_i g_i/\sum b_i h_i$ and of the form $(\sum C_{\gamma} f_i)(\sum a_i g_i)$. (Received November 14, 1963.)


A linear polygenic partial differential equation of first order is one of the form $Aw_z + Bw_z + Cw_z + Dw_z = E + Fw + G\bar{w}$, where $w = \phi(x, y) + i\psi(x, y)$ is a polygenic function of at least class $C^3$ and the $A$, $B$, $C$, $D$, $E$, $F$, and $G$ are polygenic functions of at least class $C^2$ in a certain region $R$ of the $z$-plane. A theory of real generalized Cauchy-Riemann equations for the above partial differential equation is obtained. For example, $w$ is monogenic at a point $z$ if and only if $Aw_z + Dw_z = 0$, where $|A| \neq |D|$. Similarly, $w$ is reverse monogenic at a point $z$ if and only if $Bw_z + Dw_z = 0$, where $|B| \neq |C|$. In the Minkowski plane, $w$ is monogenic at a point $z$ if and only if $A(w_z - \bar{w}_z) + B(w_z + \bar{w}_z) = 0$, where $\bar{A} - \bar{A} \neq 0$. Also, $w$ is reverse monogenic at such a point $z$ if and only if $A(w_z + \bar{w}_z) + B(w_z - \bar{w}_z) = 0$, where $\bar{A} + \bar{A} \neq 0$. Finally, in the Laguerre inversive plane, $w$ is monogenic at a point $z$ if and only if $(A + B)(w_z + \bar{w}_z) + B(w_z - \bar{w}_z) = 0$, where $2B + A\bar{B} + \bar{A} \neq 0$. It is reverse monogenic at such a point $z$ if and only if $(C + D)(w_z + \bar{w}_z) + D(w_z - \bar{w}_z) = 0$, where $2DD + \bar{C} \bar{D} + \bar{C} \bar{D} \neq 0$. (Received October 31, 1963.)

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64T-75. LEONARD CARLITZ, Duke University, Durham, North Carolina. Recurrences for the Bernoulli and Euler polynomials, II.

In an earlier paper with the same title (to appear in Journal für die reine und angewandte Mathematik) the writer proved that the Bernoulli number \( B_n(t) \), where \( t \) is a positive integer, cannot satisfy a recurrence of the type \( \sum_{r=0}^{k} A_r(t)(n) B_n(t-r) = A(t)(n) \) for all \( n > N \), where the coefficients are polynomials in \( n \) with integral coefficients and \( k \) is independent of \( n \). A similar result was obtained for the Euler numbers. The method of proof was arithmetical and hence could not be applied for arbitrary \( t \). In the present paper results of this kind are obtained for arbitrary \( t \neq 0, -1, -2, \ldots \).

The restriction on \( t \) is indeed necessary. (Received November 15, 1963.)

64T-76. LEONARD CARLITZ, Duke University, Durham, North Carolina. The distribution of irreducible polynomials in several indeterminates.

Let \( f_k(m) \) denote the total number of normalized polynomials in \( m \) indeterminates with coefficients in \( \text{GF}(q) \) and let \( \psi_k(m) \) denote the number of normalized irreducible polynomials. In a recent paper (Illinois Journal of Mathematics, vol. 7 (1963), 371-375) the writer has proved that \( \psi_k(m) \sim f_k(m) \) when \( k > 1 \) but fixed and \( m \to \infty \). In the present paper we consider a more refined classification of irreducibles; for simplicity we suppose that the number of indeterminates is two.

By the degree of a polynomial \( A(x,y) \) will be meant the pair \( (m,n) \), where \( m \) is the degree in \( x \) and \( n \) the degree in \( y \). Now let \( f(m,n) \) denote the number of normalized polynomials of degree \( (m,n) \) and \( \psi(m,n) \) the number of normalized irreducibles. Then we show that, for fixed \( m \), \( \psi(m,n)/f(m,n) \to 1 - q^{-m} \) as \( n \to \infty \); more precisely \( \psi(m,n) = (1 - q^{-m})f(m,n) + O(mq^{mn}) \), where the constant in the \( O \)-term depends on \( q \) and \( m \). (Received November 15, 1963.)

64T-77. LEONARD CARLITZ, Duke University, Durham, North Carolina. Generalized Dedekind sums.

Put \( s(h,k; x,y) = \sum_{\mu \equiv 0 \pmod{k}} ((h(\mu+y)/k + x))/((((\mu+y)/k))) \), where \( ((x)) = x - \lfloor x \rfloor - \frac{1}{2} \) or \( 0 \) according as \( x \) is not or is an integer. At the 1963 Number Theory Conference at Boulder, Colorado, Rademacher proved a generalized reciprocity theorem for \( s(h,k; x,y) \). In the present paper this result is extended to the sum \( s_p(h,k; x,y) = \sum_{\mu \equiv 0 \pmod{k}} B_p((h(\mu+y)/k + x)B_1(((\mu+y)/k)) \), where \( B_p(x) \) denotes the Bernoulli function. (Received November 15, 1963.)

64T-78. LEONARD CARLITZ, Duke University, Durham, North Carolina. An extension of the congruences of Bauer and Lubelski.

Let \( p \) denote an odd prime. Results of the following sort are obtained. \( \prod_{r=0}^{p-1} (x + r) \equiv (B_p(x))p^{n-1} + 2^{-1}p^n(x^p - x)p^{n-1} \pmod{p^{n+1}} \) (\( p > 3, n > 1 \)), \( \prod_{r=0}^{p^n-1} (x + rp) \equiv x^{p^{n-1}} - 2^{-1} p^n x^{p^n-1} \pmod{p^{n+1}} \) (\( p > 3, n > 1 \)); more generally \( \prod_{r=0}^{p^n-1} F(x + rp) \equiv (F(x))p^{n-1} - 2^{-1} p^n (F(x))p^{n-1} \pmod{p^{n+1}} \) (\( p > 3, n > 1 \)), where \( F(x) \) is an arbitrary polynomial with integral coefficients.

(Received November 15, 1963.)
64T-79. ALBERT SADE, 86, Cours de la République, Pertuis, Vaucluse, France. Le groupe d'anti-autotopie.

Un groupeoidé, \( G = \mathcal{E}( ) \), a un groupe d'anti-autotopie, \( \mathcal{J} \), si les isotopies qui le projettent sur lui-même et sur son conjoint forment un groupe. Soit \( \mathcal{K} \) le groupe d'autotopie de \( G \), le normalisateur de \( \mathcal{K} \) dans \( \mathcal{E}( ) \), son centralisateur, \((a, \beta, \gamma)\) une anti-autotopie de \( G \). Alors,

(i) \( \exists \mathcal{J} \Rightarrow B \in \mathcal{K} \land B^2 \in \mathcal{K} \Rightarrow \mathcal{K} \) self-conjoint \( \land B^2 \in \mathcal{K} \),

(ii) \( \exists \mathcal{J} \Rightarrow B \text{ pair } \land (I \in \mathcal{J} \Rightarrow (I \to I^{-1}(p^{12})) \)

est un anti-automorphisme de \( \mathcal{J} \), (iii) si \( R \) est isotope de \( G \), \( R = G(\xi, \eta, \zeta) \), \( \exists \mathcal{J} \Rightarrow R \Rightarrow U = (a, \xi^{-1}, \eta^{-1}, \zeta^{-1}) \in \mathcal{J} \land U^2 \in \mathcal{K} \), (iv) \( \mathcal{J} \subseteq \mathcal{J} \Rightarrow \mathcal{J} = \mathcal{J} \Rightarrow v = (\xi^{-1}, \eta^{-1}, \zeta^{-1}, 1) \in \mathcal{J} \land VBV = B 

(v) \( \mathcal{J} \Rightarrow \mathcal{K} = \{U, B\} \); \( (a, \beta, c) \in \mathcal{K} \), \( \mathcal{H}(a, b, c) \to (b, a, c) \)

De plus, si \( S = G(a, b, c) \), tous les \( S \) ont des groupes d'autotopie isomorphes et les anti-autotopies de \( S \) sont dans \( \mathcal{J} \); enfin \( \mathcal{J}^{12} = \mathcal{J} \Rightarrow \mathcal{K} \Rightarrow \mathcal{K} \Rightarrow \mathcal{J} \). (vi) Pour qu'un groupeoidé, \( G \), ait un groupe partiel d'anti-autotopie, il faut et il suffit qu'il existe une anti-autotopie \( I \) de \( G \), transformant en lui-même un sous-groupe \( \mathcal{K} \) de \( \mathcal{K} \) contenant \( I^2 \), \( I^2 \in \mathcal{K} \). (Received November 18, 1963.)

64T-80. J. A. WOLF, University of California, Berkeley 4, California. Translation-invariant function algebras on compact groups.

Let \( X \) be a compact topological group. It is shown that every closed translation-invariant (stable under right and left translations) subalgebra of \( C(X) \) is self adjoint, if and only if, every representation of degree 1 of \( X \) has finite image. If \( X \) is connected, an equivalent condition is that \( X \) be an inverse limit of semi-simple Lie groups; if \( X \) is connected and Lie, an equivalent condition is that \( X \) be semi-simple. Various classification results then follow, via the Stone-Weierstrass Theorem, for the closed translation-invariant subalgebras of \( C(X) \) which contain the constants. (Received November 18, 1963.)

64T-81. J. A. WOLF, University of California, Berkeley 4, California. On the classification of hermitian symmetric spaces.

A quick classification is given for the hermitian symmetric spaces. It is based on the observation that the compact irreducible hermitian symmetric spaces are just the spaces \( G/K \) where (1) \( G \) is connected compact simple centerless Lie group, (2) \( K \) is the analytic subgroup for a subalgebra \( \mathcal{H} \) of the Lie algebra \( \mathfrak{g} \) of \( G \), (3) \( \mathcal{H} \) contains a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \), (4) there is a system \( \Psi \) of simple \( \mathfrak{h} \)-roots of \( \mathcal{H} \) and a root \( \psi \in \Psi \) of multiplicity one in the greatest root, and (5) \( \mathcal{H} = \mathcal{H} + \{\mathfrak{h} \cap \sum_{\alpha \in A} \mathfrak{a}_\alpha \} \), where \( A \) is the set of all roots not involving \( \psi \) and \( \mathfrak{a}_\alpha \) is the (complex) root space for the root \( \alpha \). (Received November 18, 1963.)

64T-82. R. E. JAFFA, University of California, Berkeley 4, California. Extremal quasi doubly stochastic matrices. Preliminary report.

Let \( \mu \) be a probability measure in \( \{1, 2, \ldots, n\} \) such that for all \( i, \mu(i) > 0 \). Let \( \beta_{ij} = \mu(i)/\mu(j) \), \( \mu \) is said to be primitive if there do not exist \( S_1, S_2 \subseteq \{1, 2, \ldots, n\} \) such that \( \mu(S_1) = \mu(S_2) \). A real non-negative \( n \times n \) matrix \( (\gamma_{ij}) \) is said to be \( \mu \)-quasi doubly stochastic if for all \( i, \sum_j \gamma_{ij} = 1 \) and for all \( j, \sum_i \beta_{ij} \gamma_{ij} = 1 \). Let \( \Gamma \) be the class of \( \mu \)-quasi d.s. matrices and let \( E_1 \) be the class of
extremal members of $\Gamma$. A criterion for extremality in $\Gamma$ is established. From this, an algorithm is obtained for computing the non-zero entries of a member of $E_\Gamma$ from their configuration. This yields a method of exhibiting $E_\Gamma$ with the aid of a computer. It is then shown that if $(\gamma_{ij}) \subseteq E_\Gamma$, $\gamma_{ij}$ can be written $\sum_{j=1}^{n} \delta_{ij} \rho_{ij}$ where $\delta_{ij} = 1, 0, \text{ or } -1$. For primitive measures and for some non-primitive measures including the equal-weights measure, certain relationships existing among the $\delta_{ij}$ yield a description of the members of $E_\Gamma$ generalizing that of the equal weights case. Révész, as quoted by Mirsky, has conjectured that $\text{card } E_\Gamma \leq n!$. However, it is shown that $n! \leq \text{card } E_\Gamma \leq (n!)^\delta$, instances being exhibited where the left hand inequality is strict. Finally, partial extensions of the extremality criterion are obtained in the case of an arbitrary atomistic measure space. (Received November 18, 1963.)

64T-83. G. J. RIEGER, Bernt-Notke-Weg 26, 8000 Munich, West Germany. On the number of integers the form $p_1^n + p_2^n$ and below a positive bound.

Using a method of Erdős and Mahler (J. London Math. Soc. 13 (1938), 134-139) and a consequence of the theorem of Thue and Siegel, the following result is proved: Theorem. Let $F(x_1, x_2) = \sum_{j=0}^{n} a_j x_1^j x_2^{n-j}$ be an arbitrary binary form of degree $n \geq 3$ with integer coefficients, with $a_0 a_n \neq 0$, and with non-zero discriminant; denote by $B_F(u)$ the number of different $k < u$ such that $|F(x_1, x_2)| = k$ can be solved by different primes $x_1, x_2$ with $0 < x_1, x_2 < u^{1/n}$; then $\lim \inf_{u \to \infty} B_F(u) u^{-2/n} (\log u)^2 > 0$. (Received November 18, 1963.)

64T-84. TAQDIR HUSAIN, University of Ottawa, Ottawa 2, Ontario, Canada. B(\mathcal{J})-spaces and the closed graph theorem. II.

Let $E_w$ be a barrelled locally convex space, $F_u$ a Mackey space which is also a $B_\mathcal{J}(\mathcal{J})$-space (see these Notices 7 (1960), 943, Abstract 576-135), and $f$ a linear mapping of $E_w$ into $F_u$. Let $v$ be another locally convex topology on $F$, which has $\text{Cl}_v(\text{Cl}_{w^{-1}}(U))$ as a fundamental system of neighborhoods of 0 in $F$, where $U$ runs over a fundamental system of closed convex neighborhoods of 0 in $F$, and $\text{Cl}_v U$ denotes the $v$-closure of $U$. The following closed graph theorem has been proved: If the graph of $f$ is closed in $E \times F$ and if the transpose mapping $f': F'_v \to E'_w$ has the inverse which is continuous with respect to weak* topologies $\sigma(F'_v, F_v)$ and $\sigma(E'_w, E_w)$ on $F'_v$ and $E'_w$ respectively, then $f$ is continuous. A corollary when $F_u$ is quasi-barrelled follows immediately. (The paper will appear in the Mathematische Annalen.) (Received November 18, 1963.)

64T-85. HANS SAMELSON, Stanford University, Stanford, California. On Poincaré duality.

This note presents a self-contained proof of Poincaré duality for compact manifolds that uses only "elementary" properties of singular homology, but no spectral sequences or sheaf cohomology. This involves: (1) construction of a fundamental cycle; (2) construction of the dual of the diagonal (as suggested e.g. by Milnor's treatment of the differentiable case; E. H. Broen and others have been aware of the possibility of this step). Work supported by NSF grant G-20301. (Received November 19, 1963.)
On ring properties of injective hulls.

If the injective hull $\hat{R}$ of a ring $R$ with 1 is a rational extension of the right $R$ module $R_R$, it is well known that it may be turned into a ring isomorphic to $\text{Hom}_R(\hat{R}, \hat{R})$. (See for example J. Lambek, *On Utumi’s ring of quotients*, Canad. J. Math. 15 (1963), 363-370.) The author constructs examples to show that if the injective hull $\hat{R}$ of $R$ is not a rational extension of $R_R$, it may or may not be possible to give $\hat{R}$ a ring structure compatible with module multiplication. If such a ring structure is definable, then: *Theorem.* $\text{Hom}_R(\hat{R}, \hat{R})$ is an injective right $R$ module. This generalizes a result in Lambek’s paper that the theorem is true if $\hat{R}$ is a rational extension of $R$. (Received November 19, 1963.)

*64T-87.* J. B. BUTLER, JR, Division of Science, Portland State College, Portland 1, Oregon. *On the representation of the weight operator corresponding to certain ordinary differential operators of even order.* Preliminary report.

Let \( L^0 = \sum_{j=0}^{h} P_j(x)(d/dx)^j \) be an ordinary differential operator of order $h$ whose coefficients are \((\eta, \eta)\) matrices on the interval $0 \leq x < \infty$, $\eta = n = 2 \nu$, and let boundary conditions associated with $L^0$ be \( (s_{0j})u(\alpha) = 0, j = 1, \ldots, \nu \). It is assumed that $\eta$ is a bounded, positive, symmetric operator, a real parameter, and that $L = L^0 + \eta q$ determines a self-adjoint operator $H$ on $L_2, \eta, [0, \infty)$ with spectral measure $E^0(\delta)$ given by \( \langle E^0(\delta)u, v \rangle = \int_0^{\infty} \langle s_{1j}^*(s_j, n), v \rangle \eta_j d\eta_j(\delta) \) for any real interval $\gamma \leq \delta < \beta$, $j, k = 1, \ldots, \eta, \eta_j = \mathcal{P}(E^0)$. Given $u \in \mathcal{M}_0, v \in \mathcal{M}_0, \mathcal{L}_j = \delta z \to z + j$, let $T^0 \to \psi_j = (s_j, \nu)$ and $d_j^0(\mathcal{L}) = \int_0^{\nu} V_j^0 (L)_{j,k}(L)_{k,l} d\eta_j^0(\mathcal{L})$. Let $U_0 = (T^0)^{-1} ((V_{jk})^0 (L)_{j,k}(L)_{k,l})) (T^0)u$ and let $W_0(H, H^0), W_0$ denote respectively the wave and the weight operators (cf. S. T. Kuroda, Finite-dim. Perturbation and a representation of the scattering operator, *N. S. F. Tech. Rept., GP-2, Dept. of Math., Univ. of Calif., Berkeley, Calif.* (1962); I. Kay, H. E. Moses, Nuovo Cimento 3 (1966), 1978). Formally one has:

(iii) $W_0(H, H^0)u = U M_0^{-1} u$ where $M_0^{-1} u = (T^0)^{-1} ((V_{jk})^0 (L)_{j,k}(L)_{k,l}))u$; (iv) $W_0 u = M_0^{-1} M_0^{-1} u$. It is shown under conditions on $L^0$ and $q$ stated earlier (Canad. J. Math. 14 (1962), 359) that the matrices $V_{jk}$ exist and are analytic in $\eta$ so that representations (iii), (iv) are meaningful. (Received July 19, 1963.)

64T-88. J. R. HATTEMER, 3852 Russell Avenue, St. Louis, Missouri 63110. *Boundary behaviour of temperatures, I.* Preliminary report.

Let parabolic limits and parabolic boundedness be defined for a function $u(X,t)$ defined on $E^+_{n+1}$ as in Abstract 63T-307 (these Notices 10 (1963)). If $u$ is a temperature, i.e., a solution to the heat equation, then: *Theorem.* $u$ has parabolic limits on a set $E \subset E_n$ if and only if

\[
\int P(X_0) t^{-n/2} [|\nabla u|^2 + t|\partial u/\partial t|^2] dX dt \text{ is finite for almost every } X_0 \in E, \text{ where } |\nabla u|^2 = \sum_{i=1}^{n} |\partial u/\partial x_i|^2.
\]

(Received October 29, 1963.)

64T-89. J. R. HATTEMER, 3852 Russell Avenue, St. Louis, Missouri 63110. *Boundary behaviour of temperatures, II.* Preliminary report.

Let $u(X,t)$ and $v(X,t)$ be respectively $k$ and $m$ dimensional vector valued solutions to the heat equation in the sense that each component is a solution. Let $|u(X,t)|^2 = \sum_{j=1}^{k} |u_j(X,t)|^2$, $|v(X,t)|^2 = \sum_{j=1}^{m} |v_j(X,t)|^2$. Let $R^2 \subset R^2$ be respectively $k$ and $m$ dimensional vector valued solutions to the heat equation in the sense that each component is a solution. Let $|u(X,t)|^2 = \sum_{j=1}^{k} |u_j(X,t)|^2$, $|v(X,t)|^2 = \sum_{j=1}^{m} |v_j(X,t)|^2$. Let
\[ \sum_{j=1}^{m} j v_j(X,t) \leq 2, \] \[ u_j, v_j \text{ being the components of } u \text{ and } v. \]

Let \( P(D) \) be a \( k \times m \) matrix, each of whose entries is a homogeneous differential polynomial of degree \( q \) in \( X \) with constant coefficients. **Theorem.**

Suppose \( \partial^{k} u / \partial^{k} = P(D) v, q = 2k, k = 1, 2, \ldots \). Then, if \( v(X,t) \) has parabolic limits on a set \( E \subset E_n \), \( u(X,t) \) has parabolic limits almost everywhere in \( E \). This theorem follows from the following results which are of interest in themselves. **Theorem.**

(a) If \( \partial^{k} u / \partial^{k} = P(D) v, q = 2k + 1, k = 0, 1, \ldots \), then, for \( a < \beta, h < k, \int \int P_{\beta,k}(X_0) t^{1-n/2} |v|^2 dX dt < \infty \) implies \( \int \int P_{\alpha,h}(X_0) t^{1-n/2} |u|^2 dX dt < \infty \). (b) If \( \partial^{k} u / \partial^{k} = P(D) v, q = 2k - 1, k = 1, 2, \ldots \), then, for \( a < \beta, h < k, \int \int P_{\beta,k}(X_0) t^{1-n/2} |v|^2 dX dt < \infty \) implies \( \int \int P_{\alpha,h}(X_0) t^{1-n/2} |u|^2 dX dt < \infty \).

**Corollary:** Let \( u \) be a one dimensional solution to the heat equation on \( E_{n+1} \). Then, the finiteness of \( \int \int P(X_0) t^{1-n/2} |\nabla u|^2 dX dt \) for \( X_0 \in E \subset E_n \) is equivalent to the finiteness of \( \int \int P(X_0) t^{1-n/2} |\partial u / \partial t|^2 dX dt \). These results are analogous to results obtained for harmonic functions by E. M. Stein [see Acta Math. 106 (1961), 137-174]. (Received October 29, 1963.)

64T-90. V. R. R. UPPULURI, Oak Ridge National Laboratory, Oak Ridge, Tennessee. On a stronger version of Wallis' formula.

By an application of a theorem in Mathematical Statistics concerning unbiased estimators, and a sequence of lower bounds in their variances, bounds are given to one form of Wallis' Formula. This is an extension of the results given by Gurland, (Amer. Math. Monthly 63 (1956)). (Received November 18, 1963.)

**ERRATA - Volume 10**


Line 9. Change "=0, then" to read "= 0, then".

Line 12. Change "0 \leq q < a(n)" to read "0 < q < a(n)"


Note that the original abstract should have shown that Mr. James Small was co-author.


Line 8. Change "\( \mathcal{O}_a \) is admissible on" to read "\( \mathcal{O}_a \) is said to be admissible on".

Line 10-11. The sentence beginning at the end of line 10 should read, "For each index \( a, \mathcal{O}_a \) is set equal to \( \{ 0 \in \mathcal{O} | X_\mu \subset P_\mu(0) \text{ for all } \mu \leq a \} \)."

Page 666, Abstract 607-17. Please replace the Abstract by the following version.

NORMAN HOSAY. The sum of a cube and a crumpled cube is \( S^3 \).

A crumpled cube is defined to be the closure of the interior of a 2-sphere in \( E^3 \). **Theorem.** The sum of a crumpled cube and a real cube formed by identifying corresponding points on the boundary of each is topologically \( S^3 \).
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