Pointwise Fourier Inversion in Several Variables

Mark A. Pinsky

Fourier analysis is widely acknowledged as one of the major driving forces behind the development of modern real analysis. The theories of Lebesgue integration and subsequent operator theory find many applications and motivations in classical Fourier analysis. Consequently, it has become widely believed in many circles that Fourier analysis is impossible without all of the tools of contemporary real analysis.

In this report we outline some recent developments in Fourier analysis which can be discussed using only elementary calculus of several variables. As in many areas of mathematics, the key to getting nice theorems is a solid base of good examples, which we now describe in context.

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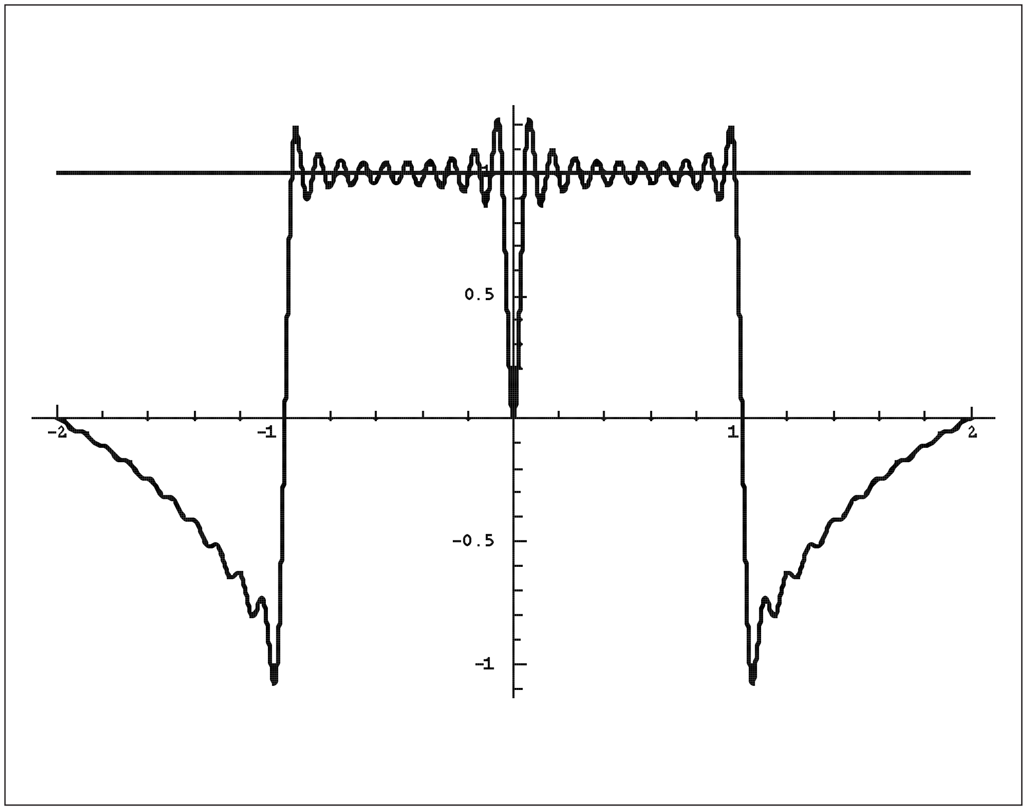
First, every student of elementary Fourier analysis is familiar with the convergence and oscillation properties of the Fourier series of a piecewise smooth function $f(x)$ defined on the interval $(-\pi, \pi)$. Three basic facts are that:

1. The partial sum $f_M(x)$ of the Fourier series converges for each x to the average of the left and right limits of the function:
$$\lim_M f_M(x) = \frac{f(x+0)+f(x-0)}{2}.$$
2. On any subinterval $[a, b]$ for which the function is smooth in an open set containing the subinterval, we have uniform convergence:
$$\lim_M \max_{a\le x\le b} |f_M(x) - f(x)| = 0.$$
3. On any subinterval containing a single jump discontinuity x_0 of the function, Gibbs' phenomenon occurs: for small $\delta > 0$
$$\lim_M (\max_{|x_0-x|\le\delta} f_M(x) - \min_{|x_0-x|\le\delta} f_M(x)) = |f(x_0+0) - f(x_0-0)|G,$$
where the constant
$$G = \frac{2}{\pi}\int_0^\pi (\frac{\sin x}{x})dx,$$
is approximately given by the sixteen-term expansion 1.17897974447216727.

This "Gibbs constant" has been computed to a high degree of accuracy; a thorough and interesting survey of the literature on Gibbs' phenomenon is contained in the beautiful article by Hewitt and Hewitt [8].

Now, if one looks at higher dimensional analogues of Fourier series or the Fourier transform, new Gibbs-like phenomena occur which are beyond the classical theory of Fourier series. These phenomena exhibit oscillation of Fourier inversion at a smooth point—even a point of local constancy—of the function, resulting from global rather than local discontinuities. We will see below that the obvious counterparts of the

Mark A. Pinsky is professor of mathematics at Northwestern University in Evanston, Illinois. His e-mail address is: `pinsky@math.nwu.edu`.



Eigenfunction expansion (1) with twenty terms.

basic facts 1), 2) are false in dimensions three or more.

Let us first look at eigenfunction expansions of radial functions (functions depending only on $r = |x|$) in $\mathbf{R}^3$ for the Laplacian. For radial functions, the Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r},$$

so the ordinary differential equation for eigenfunctions is

$$f'' + \frac{2}{r}f' + \lambda f = 0.$$

The solutions which vanish at $r = 1$ can be chosen to be

$$\phi_k(r) = \frac{\sin k\pi r}{r}, \quad k = 1, 2, \ldots.$$

These functions are orthogonal with respect to the weighted measure $r^2 dr$. If we now expand the indicator function of the unit ball

$$f(r) = \begin{cases} 1, & \text{if } 0 < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

in a generalized Fourier series with respect to this basis, we obtain the identity

$$(1) \qquad 1 = 2\sum_{k=1}^{\infty} \frac{\sin k\pi r}{k\pi r} \qquad (0 < r < 1).$$

This is another way of writing the ordinary Fourier series of the function $f(r) = r$ on the interval (0,1):

$$(2) \qquad r = 2\sum_{k=1}^{\infty} \frac{\sin k\pi r}{k\pi} \qquad (0 \le r < 1).$$

However, at $r = 0$ there is a difference between (1) and (2). The right side of (2) converges for $r = 0$, but the right side of (1) diverges. This 3-dimensional example clearly displays both the presence of Gibbs' phenomenon and the *lack* of pointwise convergence, through bounded oscillation of the partial sums. We now turn to the general situation of the Fourier transform on $\mathbf{R}^n$.

The Fourier transform of an integrable function f on $\mathbf{R}^n$ is defined as a continuous superposition of characters:

$$\hat{f}(\xi) \doteq (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-i\langle \xi, x\rangle} f(x)dx.$$

The character $x \to e^{-i\langle \xi, x\rangle}$ is an eigenfunction of the Laplace operator $\Delta \doteq \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$; thus $\Delta e^{-i\langle \xi, x\rangle} = -|\xi|^2 e^{-i\langle \xi, x\rangle}$. The spherical partial sum is defined as the integral over a ball of radius M: $f_M(x) \doteq \int_{|\xi| \le M} e^{i\langle \xi, x\rangle} \hat{f}(\xi)d\xi$. If in addition f is smooth and rapidly decreasing at infinity, then classical arguments show that $\lim f_M(x) = f(x)$, i.e., Fourier inversion holds everywhere. More generally, if all partial derivatives through order $n + 1$ exist as integrable functions, it is also classical that we have Fourier inversion everywhere, in the sense of absolute convergence. However, Fourier inversion fails if f is only *piecewise* smooth and rapidly decreasing. The desired example is the same as that above: the indicator function of a ball of radius a: $f(x) = 1_{[0,a]}(|x|)$ where $a > 0$ is arbitrary and the dimension $n = 3$. Explicit computation shows that when $M \to \infty$

$$\lim f_M(x) = 0 \quad (|x| > a),$$
$$\lim f_M(x) = 1 \quad (0 < |x| < a),$$
$$\lim f_M(x) = \frac{1}{2} \quad (|x| = a),$$
$$\liminf f_M(0) = 1 - 2/\pi,$$
$$\limsup f_M(0) = 1 + 2/\pi.$$

Fourier inversion fails at the center of the sphere. This is somewhat unexpected from the viewpoint of one and two dimensions, where the corresponding limit exists at the center. When we pass to higher dimensions, the situation becomes worse, since it can be shown by direct computation that the spherical partial sums are *unbounded* at $x = 0$ in case $n > 3$.

How to pass from these basic examples to a general theory? The key is to note that in the above example we have the expected behavior away from the origin—familiar from one-dimensional Fourier analysis—where Fourier inversion holds everywhere for an arbitrary piecewise smooth function. With respect to the origin the function is less smooth than elsewhere, in terms of the behavior of the *spherical average centered at* x defined as the surface integral $r \to \bar{f}_x(r) := \frac{1}{r^{n-1}\omega_{n-1}} \int_{|y-x|=r} f(y)dS_y$, where the normalization ensures that the average of $f \equiv 1$ is $\bar{f}_x \equiv 1$. Clearly $r \to \bar{f}_x(r)$ is discontinuous in case $x = 0$, whereas a direct computation using surface integrals shows that for $x \neq 0$ the corresponding function is differentiable of order $[(n-2)/2]$.

This is used to study Fourier inversion by first interchanging the orders of n-dimensional integrations to rewrite the spherical partial sum as a one-dimensional integral $f_M(x) = \int_0^\infty D_n^M(r)\bar{f}_x(r)r^{n-1}\,dr$, where D_n^M is the n-dimensional *Dirichlet kernel*, obtained by integrating the character $e^{-i\langle\xi,x\rangle}$ over the ball $|\xi| \le M$. Various identities are then used to express this in terms of the $n-2$ dimensional Dirichlet kernel as $D_n^M = \frac{-1}{2\pi r}\frac{d}{dr}D_{n-2}^M$. Repeated integration-by-parts reduces the original spherical partial sum to a *convergent* spherical partial sum in one or two dimensions (depending on whether n is odd or even) together with appropriate boundary terms; the number of necessary integrations is exactly $1 + [(n-3)/2)]$. The original spherical partial sum $f_M(x)$ converges precisely when the boundary terms converge, which is equivalent to the condition that the spherical average is smooth of degree $[(n-3)/2]$. This is summarized in a theorem which gives necessary and sufficient conditions for Fourier inversion at a preassigned point.

Consequently, it has become widely believed in many circles that Fourier analysis is impossible without all of the tools of contemporary real analysis.

Theorem [3]. *Let f be an integrable function and $x \in \mathbf{R}^n$ such that the spherical average centered at x is piecewise smooth with compact support. Then $\lim f_M(x)$ exists if and only if this spherical average is differentiable of class* $[(n-3)/2]$.

One obtains a more succinct statement by defining the *smoothness index* $j(f;x)$ as the least integer $j \ge -1$ so that the spherical average centered at x is differentiable of degree j. Thus

FOURIER INVERSION at $x \in \mathbf{R}^n$ IF AND ONLY IF $j(f;x) \ge [\frac{n-3}{2}]$.

This result gives a sharp criterion for Fourier inversion at a preassigned point. The criterion depends on the global behavior of f, as expressed through the spherical average centered at x. The smoothness condition is automatically satisfied in case $n = 1, 2$. In case $n = 3, 4$ the condition requires continuity; in dimensions $n = 5, 6$ we require one derivative, etc., etc. When the smoothness condition holds, the limit is the limiting average on small spheres $\bar{f}_x(0+)$. In case the smoothness condition is violated it is shown that the spherical partial sum oscillates as $M^\nu, M \to \infty$ where $\nu = \frac{n-5}{2} - j(f;x) \ge 0$.

It is remarkable that the above considerations can be transferred with little change to other non-Euclidean spaces on which Fourier analysis is customarily performed. The simplest of these is the standard sphere $\mathbf{S}^n$ with the *spherical harmonics* Y_m, which form an orthonormal basis of eigenfunctions for the standard Laplace operator restricted to the sphere. If we are given an integrable function f on $\mathbf{S}^n$, we may form the Fourier coefficients $\hat{f}_m \doteq \int f Y_m$ and the Fourier series $\sum \hat{f}_m Y_m(x)$ where we sum in the order of the increasing eigenvalues of the spherical Laplace operator. Then we show in [3] that, if f is piecewise smooth with respect to $x \in \mathbf{S}^n$, the series converges at x if and only if the spherical average centered at x, defined here for $0 < r < \pi$, is smooth of degree at least $[(n-3)/2]$.

The sphere is the canonical model of a simply connected complete Riemannian manifold of constant positive sectional curvature. When we pass to the space of negative curvature, there is also a Fourier transform theory on the standard hyperbolic space $\mathbf{H}^n$, developed by Helgason, Strichartz, and others. In this setting one can make sense of the spherical average centered at $x \in \mathbf{H}^n$, the Fourier transform, and the corresponding spherical partial sum. It is then rigorously true [3] that we have pointwise convergence at $x \in \mathbf{H}^n$ if and only if the spherical average centered at x is smooth of degree at least [(n − 3)/2]. The previous result holds with no change whatsoever.

One might conclude from these observations that the theory would be identical in the case of multiple Fourier series, which give the spectral resolution of the Laplace operator on the standard torus $\mathbf{T}^n = (-\pi, \pi)^n$. However, there are complications here, due to the lack of rotational symmetry. More precisely, the Euclidean group of rotations and translations in the first three cases ($\mathbf{R}^n$, $\mathbf{S}^n$, $\mathbf{H}^n$) allows us to reduce all of the convergence questions to problems of one-dimensional analysis, since the spherical partial sum $f_M(x)$ can always be expressed as an integral over the real line of the appropriate "Dirichlet kernel" applied to the spherical average, as a function of the radius [6].

On the torus the Dirichlet kernel is no longer a radial function, and more delicate methods must be used. At this point we have obtained in [3] a sharp result in case the function to be expanded is itself *radial*, meaning that $f(x) = F(|x|)$ where F is a piecewise smooth function on $(0, a)$ for some $a \leq \pi$ and f is defined to be zero for points of the torus outside the ball of radius a. The Fourier series of an arbitrary integrable function on the torus is written

$$f(x) \sim \sum_{m \in \mathbf{Z}^n} a_m e^{i\langle m,x\rangle}.$$

One quickly shows that the Fourier coefficients of a radial function are also radial and are the restriction of a smooth function $r \to A(r)$; thus $a_m = A(|m|)$ for $m \in \mathbf{Z}^n$. The spherical partial sum $f_M(0)$ is written in terms of *Landau's lattice point counting function* $N(r) \doteq \text{card}\{m \in \mathbf{Z}^n : |m| \leq r\}$ as a Steiltjes integral and partial integration [5]; thus

$$f_M(0) \doteq \sum_{|m| \leq M} A(|m|) = a_0 + \int_0^M A(r)dN(r)$$

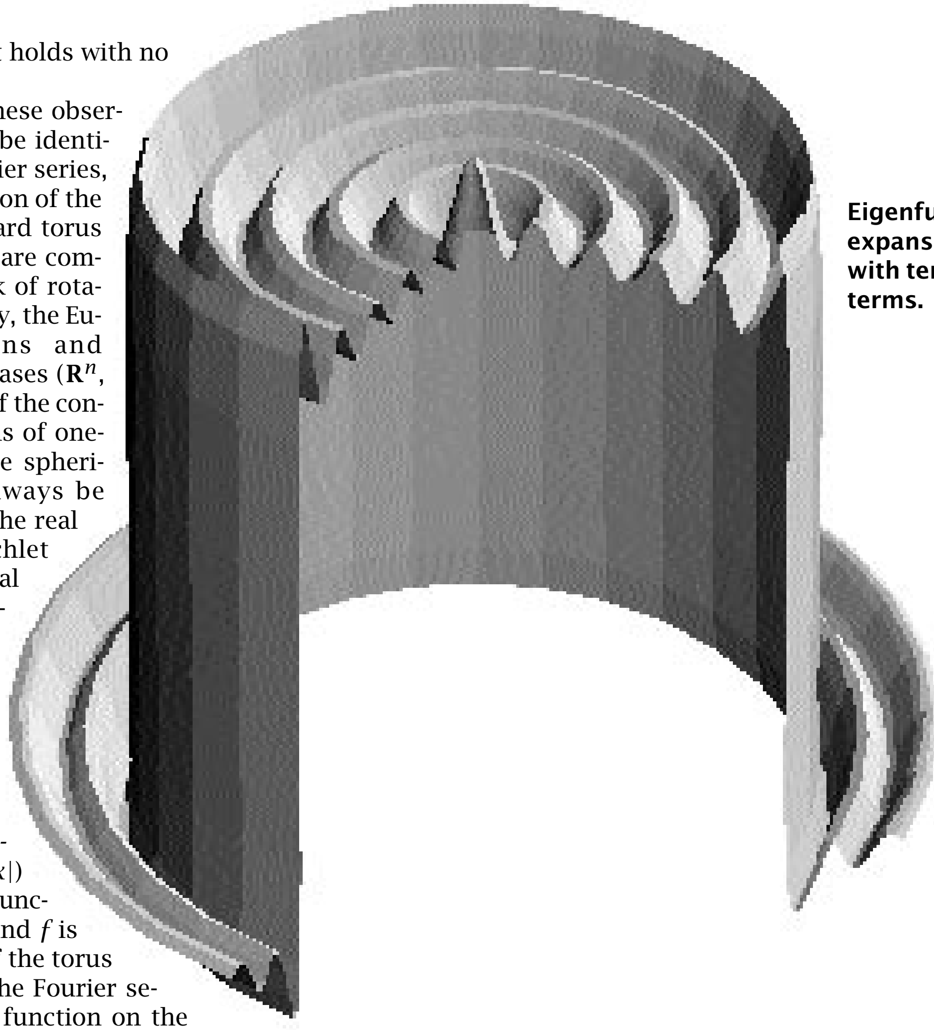
$$= A(M)N(M) - \int_0^M A'(r)N(r)dr.$$

This is compared to a spherical partial sum of the corresponding Fourier integral:

$$\int_{|x| \leq M} A(|x|)\,dx = \int_0^M A(r)d(c_n r^n)$$

$$= A(M)c_n M^n - \int_0^M A'(r)\,c_n r^n\,dr$$

$$\left(c_n \doteq \frac{\omega_{n-1}}{n}\right).$$



Eigenfunction expansion (3) with ten terms.

When we subtract these, we find an integral and a boundary term, both of which are expressed in terms of the *Landau remainder* $N(r) - c_n r^n = O(r^{n-2+\delta}), r \to \infty$, where $0 \leq \delta \leq 2/(n+1)$ if $n \geq 2$. The explicit decay of $A(r), A'(r)$, when $r \to \infty$ is then computed and used to show the convergence of both terms and thus to reduce the problem to the convergence of the corresponding Fourier integral, which has been dealt with in the previous paragraphs. Again we find that the spherical partial sum $f_M(0)$ converges if and only if F has [(n − 3)/2] derivatives [3].

So how did we come to revive this area in the 1990s? This line of research was current in the 1920s and 1930s, having been abandoned in favor of other approaches [1,6,7]. The answer lies in the domain of *experimental mathematics* [2a], by which we understand *the effective use of graphical and numerical computer output to conjecture new analytical results which might not otherwise suggest themselves.* While preparing the second edition of [4], I was contacted by Alfred Gray, who had taught from the first edition and who proposed an appendix to the new edition on the use of Mathematica programming to enhance the classical text material of Fourier series and boundary-value problems. One of these graphs (showing the sum of the first twenty terms on the right side of (8) showing the sum of the first ten terms on the right side of (3)), depicted the behavior of a Fourier-Bessel series of a piecewise smooth function which displayed a strange oscillation at the origin, quite unrelated to the local behavior there. Of course these expansions can be viewed in terms of the Laplace operator, since the Bessel functions are the radial eigenfunctions of the Laplace operator which vanish at the edge of the disk—familiar from the problem of the vibrating drumhead.

Specifically, we have the series

$$1 = 2\sum_{k=1}^{\infty} \frac{J_0(x\,x_k)}{x_k J_1(x_k)} \quad -1 < x < 1, \;\; J_0(x_k) = 0 \tag{3}$$

where we use *all* of the positive zeros x_k of the Bessel function J_0. From the asymptotic behavior of $J_0(x)$ and $J_1(x), x \to \infty$, one finds that at $x = 0$ the general term of the series is asymptotic to $\frac{(-1)^{k+1}}{\sqrt{k}}$, whereas for $x \neq 0$ the general term series behaves like $(-1)^{k+1}\frac{\cos(k\pi x - \pi/4)}{k}$. Thus we obtain a *slower* rate of convergence at $x = 0$ than at other points, even though the function is smooth in a neighborhood of $x = 0$ ([9], pages 33–35).

When we pass to the corrresponding eigenfunction expansion in three dimensions, we come to the example presented at the beginning of the article, whose graph is presented above:

$$1 = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin k\pi x}{k\pi x} \quad -1 < x < 1, \quad x \neq 0.$$

At $x = 0$ the series diverges by oscillation, since the k^{th} term is $(-1)^{k+1}$, whereas at $x \neq 0$ we have a convergent one-dimensional Fourier series. If we consider the corresponding eigenfunction expansion in dimension $n > 3$, we find that the series converges for all $x \neq 0$, but at $x = 0$ the k^{th} term $\sim (-1)^{k+1}k^{(n-3)/2}, k \to \infty$, yielding a divergent series.

With these examples in hand, one confidently passes to the case of ordinary Fourier expansions in several variables, as we have described in the above paragraphs and worked out in complete detail in the reference [3].

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References

[1] S. Bochner, *Ein Konvergengensatz für Mehrvariablige Fourierische Integrale*, *Math. Z.* **34** (1931), 440–447.

[2a] A. Gray and M. Pinsky, *Computer graphics and a new Gibbs' phenomenon for Fourier-Bessel series*, *Experimental Math.* 1 (1992), 313–316.

[2b] A. Gray and M. Pinsky, *Gibbs' phenomenon for Fourier-Bessel series*, *Exposition. Math.* **11** (1993), 123–135.

[3] M. Pinsky, *Pointwise Fourier inversion and related eigenfunction expansions*, *Comm. Pure Appl. Math.* 47 (1994), 653–681.

[4] M. Pinsky, *Partial differential equations and boundary-value problems with applications* (2d ed.), McGraw-Hill, 1991; appendix on "Using Mathematica" by Alfred Gray.

[5] M. Pinsky, N. Stanton, and P. Trapa, *Fourier series of radial functions in several variables* J. Funct. Anal. **116** (1993), 111–132.

[6] E. M. Stein and G. Weiss, *Fourier analysis on Euclidean spaces*, Princeton Univ. Press, 1971.

[7] N. Wiener, *Laplacians and continuous linear functionals*, Acta Sgeged **3** (1927), 7–16.

[8] E. Hewitt and R. Hewitt, *The Gibbs-Wilbraham phenomenon: An episode in Fourier Analysis*, Arch. Hist. Exact Sci. **21** (1980), 129–160.

[9] D. Gottlieb and S. Orszag, *Numerical Analysis of Spectral Methods*, CBMS Conf. vol. 26, Soc. Ind. and Appl. Math., 1977.