Random Waves and Localization

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This essay aims to give a brief and informal survey of recent work on Anderson localization. In this subject the problems are easy to state, and most of them are open.

Throughout history mathematicians have had considerable experience of waves: the electromagnetic waves with which we see the world, the sound waves with which we hear language and music, and the electron waves that glue the world together. However, it was only in 1747 that d'Alembert wrote down the first wave equation, and it took a good part of a century for mathematicians to understand its solutions.

This development brought an understanding of wave motion in a space that is empty or at least homogeneous in some sense. It is natural to ask about wave motion in a space that is not empty, where small random inhomogeneities can scatter the waves. Even today relatively little is known about this situation.

The mathematical context of the description of wave motion is that of conservative linear wave equations. In most elementary accounts waves propagate in empty space, coasting along until they hit an obstacle or inhomogeneity located in some bounded region. At this point they scatter and take some other direction of propagation in the empty space. This is not always a realistic model of the true situation. For instance, the sound wave or the light wave may propagate in an atmosphere in which the local propagation speed varies from point to point. The corresponding classical wave equation is

$$\frac{\partial^2 w}{\partial t^2} = c(x)^2 \frac{\partial^2 w}{\partial x^2}, \tag{1}$$

where $c(x)$ is the irregularly varying local propagation speed. Or the matter wave may propagate through a material that is not a perfect crystal, that is, a material whose electric potential is not periodic. This is described by the Schrödinger equation

$$i\frac{\partial w}{\partial t} = -\frac{\partial^2 w}{\partial x^2} + v(x)w, \tag{2}$$

where $v(x)$ is a potential function with irregular wiggles. In such situations there are inhomogeneities throughout space, and the scattering process never ceases. For these cases a random conservative linear wave equation often gives an appropriate description. We shall always assume that the coefficients in the equation satisfy some sort of local independence condition, so that the medium in which the waves propagate is truly random.

Until relatively recently it seemed clear that the mathematical solution of such equations describes diffusive motion. The wave travels for a while in free motion, but occasionally scatters and takes on another direction of travel. In the long run all memory of the initial direction of travel is lost: the wave performs a kind of ran-

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dom walk and hence manages eventually to diffuse over long distances. Wave diffusion is encountered in everyday experience, since heat conducts through insulating solids by randomly scattered elastic waves [1].

In 1958 Anderson [2] conjectured that in some situations there is an entirely different behavior: the waves do not propagate at all. The scattering of the inhomogeneities is so strong that the net effect is to form standing waves that simply slosh back and forth in a bounded region of space. This phenomenon is called "Anderson localization" or "localization".

There is a reason that a physicist made this discovery while studying the motion of electrons. In macroscopic situations the localization effect is often masked by friction, and the appropriate mathematical description is by a dissipative wave equation rather than by a conservative wave equation. At low temperatures electron waves experience little friction, so the purely conservative effect of repeated scattering is observable.

Anderson wanted to understand matter waves propagating in a disordered medium in three-dimensional space. It is natural to ask what happens when the wave depends on only one space dimension. In 1977 Goldsheid, Molchanov, and Pastur [3] proved that in this case the wave is always localized, even if the inhomogeneities are weak and widely separated. This is an astonishing fact; it says that though the wave may set out to propagate through a long distance, the net effect is always complete back-scattering that produces a standing wave.

This one-dimensional situation is now relatively well understood; there are excellent books that treat the subject [4, 5, 6, 7]. The major open problems are in wave propagation in more than one dimension. The landmark result is that for a discrete version of the wave equation there is localization in all dimensions for sufficiently strong random inhomogeneities. How about for random disturbances that are weak and sparse? The conjecture is that when the dimension of space is greater than two, wave diffusion is possible. Many practical calculations are based on this assumption. In these calculations one goes to great effort to calculate a diffusion constant, which would not have much point if the answer were zero. On the other hand, it is widely believed that when the dimension of space is two, there is localization even for weak disorder.

What are the implications of the varying behavior in different dimensions of space? The localization of electron waves in one dimension (and possibly in two dimensions) may indicate that conductivity in a wire with random impurities is an essentially three-dimensional effect. This is not a notion that is immediately intuitive, so it is worth going more deeply into the mathematics.

Waves Never Propagate in One Dimension

We shall assume throughout that the coefficients in the wave equation do not depend on time. In this case an arbitrary solution may be found by synthesizing the initial condition from eigenfunctions of the spatial part of the equation. The problem reduces to an eigenvalue problem for a linear differential operator.

The classical wave equation describes sound waves and light waves. The eigenvalue equation is

$$c(x)^2 \frac{d^2}{dx^2} u = -\omega^2 u. \tag{3}$$

The coefficient $c(x)$ is the local propagation speed as a function of the spatial variable x, and ω is the angular frequency. If $c(x)^2 = c^2$ is constant, then the eigenfunctions are $\sin(kx)$ and $\cos(kx)$ with eigenvalues $-\omega^2 = -(ck)^2$.

The Schrödinger equation provides the quantum mechanical description of the dynamics of nonrelativistic matter. The eigenvalue equation is

$$\left(-\frac{d^2}{dx^2} + v(x) \right) u = \lambda u. \tag{4}$$

The function $v(x)$ represents potential energy as a function of the spatial variable x. The eigenvalue λ represents the total energy. If $v(x) = v$ is constant, then the eigenfunctions are $\cos(kx)$ and $\sin(kx)$, this time with corresponding eigenvalues $\lambda = k^2 + v$.

Another way to write the eigenvalue problem for the wave equation is

$$\left(-\frac{d^2}{dx^2} - \frac{\omega^2}{c(x)^2} \right) u = 0. \tag{5}$$

This suggests that the mathematical analysis is not so different from that of the Schrödinger equation. From now on we shall focus on the example of the Schrödinger equation.

An arbitrary square-integrable function may be synthesized from sines and cosines by a Fourier integral

$$f(x) = \frac{1}{\pi} \int_0^\infty (a(k) \sin(kx) + b(k) \cos(kx))\, dk. \tag{6}$$

Since for a constant coefficient equation the eigenfunctions are sines and cosines, the solution for an arbitrary initial condition may be built up from the special solutions corresponding to eigenfunctions.

Consider the Schrödinger operator in the case when the function $v(x)$ is not constant, but is bounded and measurable. The spectral theorem for self-adjoint operators acting in Hilbert space

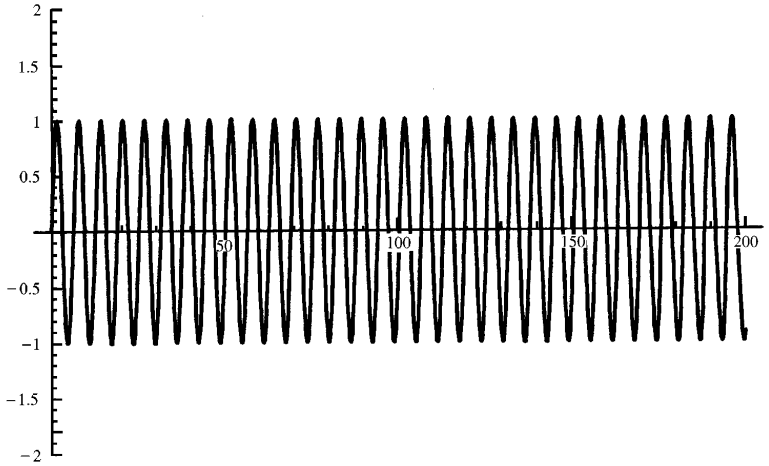

Figure 1

applies to this case. This theorem says that a self-adjoint operator is isomorphic to a multiplication operator acting on an L^2 space. The isomorphism is given by eigenvectors that in general do not belong to the Hilbert space. These eigenvectors are genuine mathematical objects, but they must be sought in a larger space [8]. For the Schrödinger operator they are eigenfunctions ψ_k satisfying

$$\left(-\frac{d^2}{dx^2} + v(x)\right)\psi_k = \lambda_k \psi_k. \tag{7}$$

An arbitrary square-integrable function may be expanded in terms of these eigenfunctions by

$$f(x) = \int c(k)\psi_k(x)\, d\mu(k). \tag{8}$$

Here μ is a measure (on some measure space) whose existence is guaranteed by the spectral theorem. This expansion gives an isomorphism of the action of the Schrödinger operator on the functions $f(x)$ with the action of multiplication by λ_k on the coefficients $c(k)$. The solution of the time-dependent Schrödinger equation with this initial condition is

$$w(t,x) = \int e^{-it\lambda_k} c(k)\psi_k(x)\, d\mu(k). \tag{9}$$

In the case of the constant coefficient equation the measure μ is a multiple of Lebesgue measure, and $\psi_k(x)$ is $\cos(kx)$ or $\sin(kx)$. The Fourier expansion is the special case of the spectral theorem for this situation. An expansion such as this that involves functions that extend over all space is needed to describe wave propagation, since a wave that is synthesized out of such functions has no hindrance to propagating throughout the space.

One would expect that putting irregular wiggles into the coefficient would make only a small change in the qualitative behavior. This would imply that the spectral measure μ would be absolutely continuous with respect to Lebesgue measure. The localization phenomenon shows that this is not so. The expansion can take the form

$$f(x) = \sum_k c(k)\psi_k(x) \tag{10}$$

where for each k the function $\psi_k(x)$ is square-integrable as a function of x. The integral is a sum; the measure μ is counting measure. The ψ_k represent standing waves that just oscillate in place. The solution of the time-dependent equation is

$$w(t,x) = \sum_k e^{-it\lambda_k} c(k)\psi_k(x), \tag{11}$$

so if the sum can be approximated by a finite partial sum, the wave only moves over a region where finitely many of the ψ_k are significantly different from zero.

The eigenvalue equation (4) is a second-order equation

$$\frac{d^2u}{dx^2} = (v(x) - \lambda)u. \tag{12}$$

Sometimes it helps to think of it as an oscillator equation. In this interpretation, u is the displacement of an oscillator as a function of the time parameter x. The left-hand side is the acceleration, and the right-hand side is the force. If $v(x) - \lambda < 0$, we may interpret it as a spring constant, so at every moment the restoring force is proportional to the displacement. This is just an ordinary harmonic oscillator, except that the spring constant is varying in time. Thus energy is being pumped in and out of the system. What is the cumulative effect?

Perhaps surprisingly, the result of this random input and output of energy is that the input dominates and the growth of the solution is exponential. Assume that the function $v(x)$ is a random function of x. (This random process is assumed to be stationary and ergodic and non-deterministic.) Kotani [9] has shown that under this circumstance, for each λ and for each fixed initial condition for the oscillator, with probability one the growth is exponential with a strictly positive growth rate as $|x|$ tends to infinity.

Return to the picture where x is the space variable in the wave propagation problem. Kotani's result would seem to say that all eigenfunctions of the differential operator are exponentially growing in both space directions. Such functions are not allowed in the eigenfunction expansion, so this threatens a disaster for the theory.

Figure 1 shows a solution to the eigenvalue equation for a wave in empty space; it is of course a sine function that oscillates in the same

way throughout all of space. Figure 2 shows a solution to the eigenvalue equation for a wave in a random medium. The solution was obtained by integrating from left to right; one can see hints that the solution is exponentially growing.

There is a way out of the problem of exponential growth. For each fixed λ there is a random initial condition for which there is exponential decrease as $x \to +\infty$, and there is another random initial condition for which there is exponential decrease as $x \to -\infty$. (These conditions depend sensitively on the asymptotic behavior of $v(x)$ at $+\infty$ or at $-\infty$.) If the two conditions were the same, then we could fit these two solutions together and get a single solution that decays at $\pm\infty$. This would be an eigenfunction that could play a role in the discrete expansion.

This does not work in such a simple form: for each fixed value of the eigenvalue parameter λ the probability that the two conditions are the same is zero. It looks like a hopeless situation. However, the task is to find parameter values that give an acceptable eigenvalue-eigenfunction pair. An appropriate use of Fubini's theorem shows that with probability one the set of λ where the two conditions are the same is of Lebesgue measure zero. This gives a way out: the eigenvalues may belong to such a set of measure zero.

By itself this does not imply the existence of a discrete set of eigenvalues; there is also the possibility of eigenvalues described by a measure that is continuous but singular with respect to Lebesgue measure. A further analysis shows that if the randomness of $v(x)$ is sufficiently local, then this singular continuous spectrum is ruled out. The analysis establishes that the probability is one that there exists a discrete parameter k with corresponding eigenvalues λ_k that are dense in an interval. The associated eigenfunctions $\psi_k(x)$ belong to the Hilbert space and decay exponentially at $\pm\infty$. The occurrence of a dense set of eigenvalues is physically reasonable, since it says that given a frequency λ, there is a standing wave ψ_k situated at some sufficiently remote region of space with frequency λ_k arbitrarily close to λ.

Do Waves Propagate in Several Dimensions?

The Schrödinger operator in a space of higher dimension is similar to that of one dimension, except that the second derivative is replaced by the Laplace operator. It is

$$H = -\Delta + v(\mathbf{x}). \tag{13}$$

Again, if $v(\mathbf{x})$ is a bounded measurable function of $\mathbf{x}$, the spectral theorem guarantees the existence of eigenfunctions and a spectral measure.

In this case finding the expansion is considerably more difficult, since the eigenvalue equation

$$Hu = (-\Delta + v(\mathbf{x}))\, u = \lambda u \tag{14}$$

is a partial differential equation. There is no way to solve it by integrating along a line.

In the quantum mechanics of solids it is often useful to look at a "tight-binding approximation" in which there is only one electron state per site. This is described by the finite difference version of the Schrödinger equation where $\mathbf{x}$ ranges over points whose coordinates are integers. The Δ in the equation is then taken to be the discrete Laplacian corresponding to a lattice spacing of one. The eigenvalue problem with the discrete Laplace operator is

$$\begin{aligned} Hu(\mathbf{x}) &= -\sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} (u(\mathbf{y}) - u(\mathbf{x})) + v(\mathbf{x})u(\mathbf{x}) \\ &= \lambda u(\mathbf{x}). \end{aligned} \tag{15}$$

The localization phenomena are similar in this case, but there are some simplifications. This

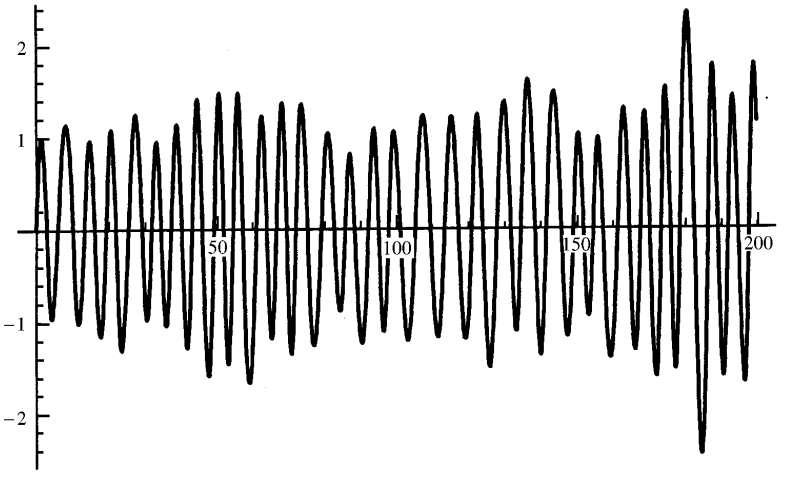

Figure 2

variant model is the subject of much of the current research on localization, and we consider it in the following.

For the special case when $v(\mathbf{x}) = v$ is constant, we can again use Fourier analysis to solve the problem. The eigenfunctions are of the form $\cos(\mathbf{k} \cdot \mathbf{x})$ or $\sin(\mathbf{k} \cdot \mathbf{x})$, and the corresponding eigenvalues for the d-dimensional problem are

$$\lambda_{\mathbf{k}} = 4 \sum_{j=1}^{d} \sin^2(k_j/2) + v \tag{16}$$

where $\mathbf{k}$ has components k_j. The effect of the discretization of space is to limit the eigenvalues to the range from ν to $4d + \nu$.

In the simplest random version of this model the random variables $v(\mathbf{x})$ are independent and identically distributed as $\mathbf{x}$ varies over the discrete space. This is the classical probability model where the $v(\mathbf{x})$ are coordinate functions on an infinite product space with a product probability measure $\prod_{\mathbf{x}} \rho(v_{\mathbf{x}})\, dv_{\mathbf{x}}$. In the following we use a bracket $\langle \cdot \rangle$ to denote the integral with respect to this measure; in the language of probability theory this integral is the expectation.

Fix a value of λ. In the constant coefficient case, if ν is not in the interval from $\lambda - 4d$ to λ, then λ will not be an eigenvalue. From this it is plausible that if $v(\mathbf{x})$ is so random that with high probability it is not in the interval from $\lambda - 4d$ to λ, then it is difficult for waves to propagate over long distances. There will be only occasional isolated islands where there will be $v(\mathbf{x})$ values in this range. On the other hand, these islands can be arbitrarily large. Thus there can be large regions on which an eigenfunction looks like a sine or cosine function. These large islands encourage propagation at frequency λ, but as the islands get larger, they become very sparse.

This argument suggests that there might be a renormalization argument that proves that high randomness implies localization. As one looks at the problem on larger and larger scales, the regions that are in resonance with the propagation at the given frequency occupy a smaller and smaller fraction of the space and become less and less significant.

Fröhlich and Spencer [10] made this argument rigorous. (There have been subsequent simplifications [11].) They worked with the resolvent operator $(H - \lambda)^{-1}$. The reason for this choice has to do with general issues of spectral theory. The goal is to analyze an operator $(H - \lambda)^{-1}$ with discrete eigenvalues λ_k satisfying $H\psi_k = \lambda_k \psi_k$, where the ψ_k belong to the Hilbert space. The operator $(H - \lambda)^{-1}$exists only when λ is not one of the λ_k; and when the eigenvalues are dense in an interval, it is discontinuous due to the small denominators. However, if $f = \sum_k c(k)\psi_k$ is expanded in eigenfunctions and if the coefficients satisfy $\sum_k |c(k)| < \infty$, then the Hilbert space norm

$$\|(H - \lambda)^{-1} f\|^2 = \sum_k \frac{1}{(\lambda_k - \lambda)^2} |c(k)|^2 < \infty \tag{17}$$

is finite for almost every λ [12]. This indicates that it should be possible to work with the resolvent operator in spite of its discontinuity.

For the discrete Schrödinger operator the resolvent $(H - \lambda)^{-1}$ takes a concrete form. If $f = \delta_{\mathbf{z}}$ is a source at the point $\mathbf{z}$, then

$$u(\mathbf{x}) = (H - \lambda)^{-1} \delta_{\mathbf{z}}(\mathbf{x}) \tag{18}$$

is a solution of the eigenvalue equation $Hu = \lambda u$ except at the source point $\mathbf{z}$. This is the Green's function. The advantage of calculating with this quantity is that the exponentially growing solutions are automatically ruled out, and it contains the relevant spectral information [13].

The Fröhlich-Spencer multiscale approach to estimates on the Green's function is a landmark in the understanding of localization. However, the underlying physical effect producing the localization is rather crude: strong fluctuations produce large regions where propagation at the given frequency is severely penalized. This should be contrasted with the subtle effect in one space dimension that produces localization even at very high frequencies.

Aizenman and Molchanov [14] recently found a simple way to get some of these results. The Green's function is found by solving the eigenvalue problem with a source at a given point. Write the eigenvalue equation (15) in the form

$$u(\mathbf{x}) = \frac{1}{v(\mathbf{x}) - \lambda + 2d} \sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} u(\mathbf{y}). \tag{19}$$

This says that the value of the solution at $\mathbf{x}$ is a random multiple of the average of the values at neighboring points. If we look at points where the factor in front of the average has absolute value bounded by some $\gamma < 1$, then at those points we have

$$|u(\mathbf{x})| \le \gamma \frac{1}{2d} \sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} |u(\mathbf{y})|. \tag{20}$$

The solution is subharmonic; its value at every point in this region is strictly less than its average over neighboring points. This suggests a way to prove the exponential decrease: iterate the equation and pick up a factor of $\gamma < 1$ from each iteration. There are, however, occasional points at which the factor in front is large. This is the difficulty that requires the use of the multiscale renormalization analysis.

There would be an easy way out if one could just take expectations of both sides of the equation and reason with these. This would give a nonrandom equation that describes regions simply in terms of their average behavior. However, if $v(\mathbf{x})$ is a continuous random variable with density $\rho(v)$, then the expectation of the prefactor $1/(v(\mathbf{x}) - \lambda + 2d)$ is an integral of the form

$$\left\langle \frac{1}{v(\mathbf{x}) - \lambda + 2d} \right\rangle = \int_{-\infty}^{\infty} \frac{1}{v - \lambda + 2d}\, \rho(v)\, dv = \infty - \infty \tag{21}$$

if $\rho(v) > 0$ near $v = \lambda - 2d$. Since the integral diverges, this simple approach is doomed.

Such a calculation would work if the singularity were less severe. The new idea in the Aizenman and Molchanov work is to take a fractional power s with $0 < s < 1$ before taking the expectation; for instance, one could take $s = 1/2$. This makes such an integral finite. The inequality for the fractional power is

$$|u(\mathbf{x})|^s \le \frac{1}{|v(\mathbf{x}) - \lambda + 2d|^s} \sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} |u(\mathbf{y})|^s. \tag{22}$$

The inequality for the expectation of the fractional power is

$$\langle |u(\mathbf{x})|^s \rangle \le \gamma \frac{1}{2d} \sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} \langle |u(\mathbf{y})|^s \rangle, \tag{23}$$

where the bracket denotes expectation with respect to the probability measure. This is a nonrandom inequality of the form

$$\phi(\mathbf{x}) \le \gamma \frac{1}{2d} \sum_{\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=1\}} \phi(\mathbf{y}) \tag{24}$$

for the unknown expectation $\phi(\mathbf{x}) = \langle |u(\mathbf{x})|^s \rangle$ at center points $\mathbf{x}$ away from the source point. For large randomness the parameter γ is less than one, and so this says $\phi(\mathbf{x})$ is a subharmonic function of $\mathbf{x}$ away from the source point.

Why go to the trouble of proving that ϕ is subharmonic? If $\mathbf{x}$ is not the source point and if the $\phi(\mathbf{y})$ at the neighboring points $\mathbf{y}$ are bounded by C, then $\phi(\mathbf{x})$ is bounded by $C\gamma$. This suggests iteration. Say that the values of ϕ have a uniform bound C_0. If $\mathbf{x}$ is n steps from the source point, then the inequality may be iterated n times without encountering a case where the center point is the source point. Each iteration lowers the bound by a factor γ. The resulting estimate $\phi(\mathbf{x}) \le C_0\gamma^n$ is the key to the proof of exponential decay of the Green's function for large disorder. Other arguments then give exponential decay of the eigenfunctions, the main ingredient in localization.

Do Waves Ever Propagate?

In one dimension there is always localization. Of course this may be a special dimension, since there are only two directions in which to propagate. In any number of dimensions large fluctuations imply localization. So is localization the general situation? If so, that would be a considerable surprise.

How about wave propagation in three-dimensional space when there are only small fluctuations of the medium? There is plenty of room to propagate around obstacles in three dimensions. Should this not imply wave diffusion instead of localization? This is for now an open question, though it should be within reach. Recently Klein [15] made progress. The discrete Schrödinger equation makes sense on a graph, if one defines the Laplacian appropriately. If the graph is a tree, then the problem simplifies, and in this situation Klein has proved that the eigenfunctions are not localized.

The open question remains: is there wave diffusion in three dimensions? The problem of two dimensions is even more intriguing; renormalization group arguments suggest that two dimensions may be like one dimension, with only localized waves. If this is true, it will be a delicate effect. After all, there seems to be plenty of room to propagate in two dimensions. Is this not enough?

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