Dynkin Diagrams and the Representation Theory of Algebras

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The Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$ and the associated extended Dynkin diagrams $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ (also called Euclidean diagrams) appear in many different parts of mathematics. Often their occurrence can be traced back to fairly elementary considerations, and the first part of this paper is devoted to such a discussion. In the second part we discuss how and why these diagrams have come up in the representation theory of finite-dimensional algebras and related topics. They have occurred in many major results, so this area illustrates well the importance of the diagrams. We stress the connections with the elementary considerations in the first part, as well as the important role the Dynkin and extended Dynkin diagrams have played in establishing new and interesting relationships with other fields of mathematics.

Dynkin Diagrams

We start by stating some simple problems whose solutions are given by a collection of graphs and where the answers turn out to be the same even though the problems look quite different. In this connection we list the graphs known as Dynkin and extended Dynkin diagrams.

Additive and Subadditive Functions

A graph $\Sigma$ consists of vertices together with edges between vertices. Assume that $\Sigma$ is finite and connected and with no loops, that is, no edge $\emptyset$. We denote the set of vertices by $\Sigma_0 = \{1, \ldots, n\}$. Denote by $\mathbb{N}$ the positive integers, and let $a: \Sigma_0 \rightarrow \mathbb{N}$ be a function. Writing $a(i) = a_i$, we say that $a$ is subadditive if for each $i$ we have $2a_i \geq \sum_j a_j$, where the sum runs over all edges connected with the vertex $i$. If we have equality for all $i$, then $a$ is said to be additive. We consider graphs equipped with functions with values in the positive integers. We write the values of the functions at the vertices and use the following examples for illustration.

(1) $\begin{array}{ccc} a_1 & \rightarrow & a_2 \\ \downarrow & & \downarrow \\ 1 & \rightarrow & 1 \end{array}$

(2) $\begin{array}{ccc} 1 & \rightarrow & 1 \\ \downarrow & & \downarrow \\ 2 & \rightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \rightarrow & 1 \end{array}$

We ask the following questions.

- Which graphs $\Sigma$ admit an additive function?
- Which graphs $\Sigma$ admit a subadditive function?

We now consider our examples from this point of view.

(1) If $a_1 = a_2 = 1$, we get a subadditive function which is not additive. There cannot be an additive function, since $2a_1 = a_2$ and $2a_2 = a_1$ is impossible.
(2) We see that the function listed is additive. Note that the summation is over different edges, so that the relevant equation is \( 2 \cdot 1 = 1 + 1 \).

(3) There is no subadditive function, since \( 2a_1 \geq 3a_2 \) and \( 2a_2 \geq 3a_1 \) is impossible.

(4) It is easy to see that the function listed is additive.

**Quadratic Forms**

We now define a quadratic form \( q_\Sigma \) associated with a finite connected graph \( \Sigma \) without loops, with vertices \( \Sigma_0 = \{1, \ldots, n\} \). Denote by \( \mathbb{R} \) the real numbers and by \( x = (x_1, \ldots, x_n) \) a vector in \( \mathbb{R}^n \).

Define the quadratic form \( q_\Sigma: \mathbb{R}^n \rightarrow \mathbb{R} \) by

\[
q_\Sigma(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i \neq j} a_{ij} x_i x_j,
\]

where the last summation is over all edges. We consider the following basic questions.

* When is \( q_\Sigma \) positive definite, that is, \( q_\Sigma(x) > 0 \) for \( x \neq 0 \)?

* When is \( q_\Sigma \) positive semidefinite, that is, \( q_\Sigma(x) \geq 0 \) for all \( x \)?

Again, we investigate the graphs in the four examples, at the same time illustrating the definition.

1. \( \begin{array}{c}
1 \\
\end{array} \mapsto \begin{array}{c}
2
\end{array} \) We have \( q_\Sigma(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2 = (x_1 - x_2)^2 + 3x_2^2 > 0 \) for \( (x_1, x_2) \neq (0, 0) \). Hence \( q_\Sigma \) is positive definite.

2. \( \begin{array}{c}
1 \\
\end{array} \mapsto \begin{array}{c}
2
\end{array} \) We have \( q_\Sigma(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 x_2 = (x_1 - x_2)^2 \geq 0 \) for all \( (x_1, x_2) \). This shows that \( q_\Sigma \) is positive semidefinite, but it is not positive definite since \( q_\Sigma(1, 1) = 0 \).

3. \( \begin{array}{c}
1 \\
\end{array} \mapsto \begin{array}{c}
2
\end{array} \) We have \( q_\Sigma(x_1, x_2) = x_1^2 + x_2^2 - 3x_1 x_2 \). Since \( q_\Sigma(1, 1) = -1 \) and \( q_\Sigma(-1, -1) = 1 \), it follows that \( q_\Sigma \) is not positive semidefinite.

**Dynkin and Extended Dynkin Diagrams**

Looking at the answers to the two questions for the four examples in the previous sections, we see that the existence of an additive function corresponds to the associated quadratic form being positive semidefinite but not positive definite and that the existence of a subadditive nonadditive function corresponds to the quadratic form being positive definite. Surprisingly, this connection holds in general, and here the Dynkin and extended Dynkin diagrams, which are listed in Figure 1, appear.

**Theorem 1.1** Let \( \Sigma \) be a finite connected graph with no loops.

(a) The following are equivalent.

(i) There is a subadditive nonadditive function for \( \Sigma \).

(ii) The quadratic form \( q_\Sigma \) is positive definite.

(iii) \( \Sigma \) is a Dynkin diagram.

(b) The following are equivalent.

(i) There is an additive function for \( \Sigma \).

(ii) The quadratic form \( q_\Sigma \) is positive semidefinite but not positive definite.

(iii) \( \Sigma \) is an extended Dynkin diagram.

There are two families of the Dynkin and extended Dynkin diagrams and three exceptional ones (see Figure 1). The total diagram represents the extended Dynkin diagram including the encircled vertex, and the corresponding Dynkin diagram is obtained by dropping the encircled vertex and associated edges. The index denotes the number of vertices of the Dynkin diagram. We have equipped the graphs with functions, which are additive for the extended Dynkin diagrams and subadditive nonadditive for the Dynkin diagrams. Any vertex with value 1 can be chosen as encircled vertex.

It is often convenient to describe a finite graph \( \Sigma \) with no loops and vertex set \( \Sigma_0 = \{1, \ldots, n\} \) in terms of an associated matrix \( C = (c_{ij})_{i,j \in \Sigma_0} \), called the Cartan matrix. By definition \( c_{ii} = 2 \) for \( 1 \leq i \leq n \), and \( -c_{ij} = -c_{ji} \) is the number of edges between \( i \) and \( j \). So associated with the graph \( \begin{array}{c}
1 \\
\end{array} \rightarrow \begin{array}{c}
2
\end{array} \) is the matrix \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

One naturally wonders if there is a direct connection between existence of additive or subadditive functions for a graph \( \Sigma \) and properties of the associated quadratic form. We give an indication of such a connection.
Dynkin diagrams:  

\[ A_n \]

\[ \tilde{A}_n \]

\[ D_n \]

\[ \tilde{D}_n \]

\[ E_6 \]

\[ \tilde{E}_6 \]

\[ E_7 \]

\[ \tilde{E}_7 \]

\[ E_8 \]

\[ \tilde{E}_8 \]

Figure 1. The Dynkin and extended Dynkin diagrams.

Let \( \Sigma \) be a finite connected graph with no loops, vertex set \( \Sigma_0 = \{1, \ldots, n\} \) and additive function \( a: \Sigma_0 \to \mathbb{N} \). For each edge \( e \) between \( i \) and \( j \) and \( \mathbf{x} = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \), define \( q_e(\mathbf{x}) = \frac{1}{2a_ia_j}(a_jx_i - a_ix_j)^2 \). The coefficient of \( x_ix_j \) in \( q_e \) is \(-2a_ia_j/a_ia_j = -1\), and the constant \( x_i^2 \) is \( a_i \). Since \( a \) is additive, the coefficient of \( x_j^2 \) in \( \Sigma e q_e \), where the sum is over all edges in \( \Sigma \), is \( \frac{1}{2a_ia_j}(\sum_{i \neq j} a_j) = 1 \). The last sum is taken over all edges with \( i \) as end vertex. We then see that \( q_\Sigma = \Sigmamq_q \), so that \( q_\Sigma \) is positive semidefinite. Since \( q_\Sigma(a_1, \ldots, a_n) = 0 \), it is not positive definite.

A finite graph \( \Sigma \) with single edges and an integer \( p \geq 3 \) or \( p = \infty \) attached to each edge is called a Coxeter diagram. When \( p = 3 \) is attached to an edge, the convention is to drop the corresponding number in the Coxeter diagram. If \( p \) is attached to the edge \( i \rightarrow j \), define \( t_{ij} = 2\cos \frac{\pi}{p} \) if \( p \geq 3 \) is an integer and \( t_{ij} = 2 \) if \( p = \infty \). Now define for a Coxeter diagram \( \Sigma \) with vertex set \( \Sigma_0 = \{1, \ldots, n\} \) the quadratic form \( q_\Sigma: \mathbb{R}^n \to \mathbb{R} \) by \( q_\Sigma(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2 - \sum_{i \neq j} t_{ij}x_ix_j \), where the sum runs over all edges. For example, if \( \Sigma \) is the Coxeter diagram \( \cdots \rightarrow \), then \( q_\Sigma(x_1, x_2) = x_1^2 + x_2^2 - 2(\cos \frac{\pi}{3})x_1x_2 = x_1^2 + x_2^2 - \sqrt{3}x_1x_2 \). Note that when \( p = 3 \), we have \( 2\cos \frac{\pi}{3} = 1 \), so we have the old definition when all associated numbers are 3. In addition to \( A_n, D_n, E_6, E_7, \) and \( E_8 \), the Coxeter diagrams

\[ B_n: \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet , \]

\[ F_4: \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet , \]

\[ H_3: \bullet \rightarrow \bullet \rightarrow \bullet , \]

\[ H_4: \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet . \]
and

$$L_2(m): \rightarrow \leftarrow$$

also give positive definite quadratic forms [20].

**Weyl Groups**

As before let $\Sigma$ be a finite connected graph without loops, with vertices $\Sigma_0 = \{1, \ldots, n\}$. Associated with $\Sigma$ is a group of linear transformations $W$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ called the Weyl group of $\Sigma$. It is generated by reflections $\sigma_1, \ldots, \sigma_n$, defined as follows. For $x = (x_1, \ldots, x_n)$ we have

$$\sigma_i(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$

where $y_j = x_j$ for $j \neq i$ and $y_i = - x_i + \sum_{j \neq i} x_j$, where the sum runs over all edges having $i$ as an end vertex.

For example, if $\Sigma$ is the graph $1 \rightarrow 2 \rightarrow 3$, we have

$$\sigma_1(x_1, x_2, x_3) = (-x_1 + x_2, x_2, x_3)$$

$$\sigma_2(x_1, x_2, x_3) = (x_1, -x_2 + x_1 + x_3, x_3)$$

$$\sigma_3(x_1, x_2, x_3) = (x_1, x_2, -x_3 + x_2).$$

Note that if $q_\Sigma$ is the quadratic form associated with $\Sigma$ and $B_\Sigma$ is the corresponding symmetric bilinear form, given by

$$B_\Sigma(x, y) = q_\Sigma(\frac{1}{2}(x + y)) - q_\Sigma(\frac{1}{2}(x - y)),$$

then $\sigma_i(x) = x - 2B(f_i, x)f_i$. Here $f_i$ denotes the $i$th coordinate vector.

There is the following connection with Dynkin diagrams.

**Theorem 1.2** Let $\Sigma$ be a finite connected graph without loops and $W_\Sigma$ the associated Weyl group. Then $\Sigma$ is a Dynkin diagram if and only if $W_\Sigma$ is a finite group.

For a Dynkin diagram $\Sigma$ with $n$ vertices the elements of $\mathbb{R}^n$ of the form $w(f_i)$ for $w \in W$ are the roots. It is known that for each root $x$ either $x$ or $-x$ is positive, where $x$ is said to be positive if $x_i \geq 0$ for all $i$ and $x_i > 0$ for at least one $i$. So in our example the positive roots are $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)$, and the negative ones are the opposite ones.

Let $\Sigma$ be a Dynkin diagram. The linear transformations $\sigma_i$ have the property that $\sigma_i^2 = 1$ for each vertex $i$. Also, it is easy to see from the definition that for $i \neq j$ we have $(\sigma_i \sigma_j)^2 = 1$ if and only if there is no edge between $i$ and $j$ and that $(\sigma_i \sigma_j)^3 = 1$ if there is an edge. It turns out that the group $W_\Sigma$ is determined by the generators $\sigma_i$ with the above relations. A group generated by a finite number of elements $s_1, \ldots, s_n$ with relations $s_i^2 = 1$ for $i = 1, 2, \ldots, n$ is called a Coxeter group. It is determined by the associated Coxeter graph, which has vertices $\{1, \ldots, n\}$. There is an edge between $i$ and $j$ if and only if $(s_i s_j)^2 \neq 1$. Then this edge is labelled by $\infty$ if $(s_i s_j)^t \neq 1$ for all $t$, and by $m_{ij}$ if $(s_i s_j)^{m_{ij}} = 1$ and $m_{ij}$ is minimal with this property. (If $m_{ij} = 3$, the usual convention is to drop the number 3.) In particular, we see that when $\Sigma$ is Dynkin, the group $W_\Sigma$ has $\Sigma$ as associated Coxeter diagram.

**Origins**

Two different lines of research led to the study of the Weyl groups. On one hand there are the five regular polyhedra (Platonic solids) from ancient times. A study of their rotation groups was started in the last century. Also, finite and, more generally, discrete groups generated by reflections with respect to hyperplanes in $\mathbb{R}^n$ were studied, first for $n$ equal to 2 or 3. The symmetry groups of the regular polyhedra are examples of finite groups generated by reflections. Associated with a discrete reflection group is a so-called fundamental region, bounded by the reflecting hyperplanes. For the classification of such reflection groups Coxeter associated a (Coxeter) graph with the fundamental region (or alternatively with the reflection group as explained in the preceding section). We illustrate the basic ideas with easy examples in $\mathbb{R}^2$.

Consider two lines $l_1$ and $l_2$ in the plane intersecting at an angle $\frac{\pi}{p}$, where $p \geq 3$ is an integer. We have marked a fundamental region by stripes, and $f_1$ and $f_2$ are unit vectors perpendicular to the lines $l_1$ and $l_2$.

The associated Coxeter diagram has two vertices, corresponding to the two lines $l_1$ and $l_2$, and there is an edge between the vertices expressing that the lines are not perpendicular. Corresponding to the angle being $\frac{\pi}{p}$, we label the edge by $p$, to get \( \rightarrow \) (where one writes \( \rightarrow \rightleftarrows \) when $p = 3$). Associated with the two lines $l_1$ and $l_2$ is the group generated by the reflections $\sigma_1$ and $\sigma_2$ in these lines. Note
that \( \sigma_1 \sigma_2 \) is rotation by \( \frac{2\pi}{p} \) around 0, so that \( (\sigma_1 \sigma_2)^p = 1 \) (and this relation determines the group). When \( p = 3 \), we have the symmetric group \( S_3 \). A positive definite quadratic form is defined directly from the geometry as follows. Let \( x_1 f_1 + x_2 f_2 \) with \( x_1 \) and \( x_2 \) in \( \mathbb{R} \) be an element in \( \mathbb{R}^2 \) and \( (, ) \) the standard bilinear form. Since \( (f_1, f_2) = -\cos \frac{\pi}{3} = \langle f_2, f_1 \rangle \), we have \( (x_1 f_1 + x_2 f_2, x_1 f_1 + x_2 f_2) = x_1^2 + x_2^2 - (2\cos \frac{\pi}{3})x_1 x_2 \), in particular \( x_1^2 + x_2^2 - x_1 x_2 \) if \( p = 3 \). This is the form we associated directly to the Coxeter diagram \( \bullet - \bullet - \bullet \) after Theorem 1.1, and the classification of finite reflection groups is in terms of the Coxeter graphs with positive definite quadratic form listed above.

The extended Dynkin diagrams (in addition to some further Coxeter diagrams) occur in the classification of discrete reflection groups which are not finite. As an example, consider three lines in the plane where any two intersect at an angle \( \frac{\pi}{3} \), and mark by stripes a fundamental region.

\[
\begin{align*}
\end{align*}
\]

The same principle gives the graph \( \triangle \). Since the three unit vectors perpendicular to the lines are not a basis for \( \mathbb{R}^2 \), we now get a quadratic form \( x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3 \), which is positive semidefinite but not positive definite. The connection with additive functions also occurred in this setting (see [9]).

The classification of complex simple Lie algebras started in the last century. These Lie algebras are classified in terms of a finite set of elements, called roots, of the dual vector space \( H^* \) of some subalgebra \( H \) of the simple Lie algebra \( L \). There is a subset of simple roots, which form a basis for \( H^* \). Also, there is a natural symmetric nondegenerate bilinear form on the simple Lie algebra \( L \), equivalently a positive definite quadratic form, which induces a bilinear and a quadratic form on \( H^* \).

As a concrete example we have the Lie algebra \( sl(3, \mathbb{C}) \) whose elements are the \( 3 \times 3 \) matrices over \( \mathbb{C} \) where the sum of the diagonal elements is zero. Here \( H \) can be chosen to be the subalgebra of \( L \) spanned over \( \mathbb{C} \) by the elements \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \). The simple roots are \( \alpha: H \to \mathbb{C} \) where \( \alpha(h_1 + sh_2) = -2s + s \) and \( \beta: H \to \mathbb{C} \) where \( \beta(h_1 + sh_2) = r - 2s \), for \( r \) and \( s \) in \( \mathbb{C} \). For the induced bilinear form \( (, ) \) on \( H^* \) we have \( (\alpha,\alpha) = \frac{1}{2} = (\beta,\beta) \) and \( (\alpha,\beta) = (\beta,\alpha) = -\frac{1}{6} \). "Normalizing", we get the matrix

\[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

which is the associated Cartan matrix and which corresponds to the Dynkin diagram \( \bullet - \bullet - \bullet \). In this case there are six roots: \( \alpha, \beta, \alpha - \beta, -\alpha, -\beta, -\alpha - \beta \). We consider the real vector space spanned by \( \alpha \) and \( \beta \) and transform to \( \mathbb{R}^2 \) with the standard bilinear form. We then have \( (\alpha, \beta) = ||\alpha|| ||\beta|| \cos \theta \), where \( \theta \) is the angle between \( \alpha \) and \( \beta \), so that \( \cos \theta = -\frac{1}{2} \). Hence we have the following picture

\[
\begin{align*}
\end{align*}
\]

The associated Weyl group \( W \) is generated by the reflections \( \sigma_\alpha \) and \( \sigma_\beta \) in the lines perpendicular to the vectors \( \alpha \) and \( \beta \) (given by \( \sigma_\alpha(\xi) = \xi - 2(\xi,\alpha)\alpha \) and \( \sigma_\beta(\xi) = \xi - 2(\xi,\beta)\beta \)). The elements of \( W \) permute the roots. We have \( (\sigma_\alpha \sigma_\beta)^3 = 1 \), and \( W \) is isomorphic to the symmetric group \( S_3 \) in this case.

The Dynkin diagrams \( A_n, D_n, E_6, E_7, E_8 \) correspond to the case where all roots have equal length. Further diagrams \( B_n, C_n, F_4, G_2 \) occur in general. (See [6, 9, 19] for more details on the material in this section.)

**Representation Theory of Algebras**

When the Dynkin or extended Dynkin diagrams appear in a classification theorem in mathematics, there are some immediate interesting questions. Is there a naturally defined quadratic form or some (sub)additive function explaining the occurrence? Is there a direct relationship with other theorems where the same diagrams occur? If so, can new proofs be given, taking advantage of the connection? Can results already developed in another field be applied to get new information?

Within the representation theory of finite-dimensional algebras, and the related area of Cohen-Macaulay modules over certain commutative rings, the Dynkin and extended Dynkin diagrams have appeared in many important theorems. We discuss some of the main occurrences, emphasizing the above point of view.
Background on Representation Theory of Algebras

We give some background on the representation theory of finite-dimensional algebras, which we always assume to be associative with unit.

First we give typical examples of the algebras we are talking about. Let $k$ throughout be an algebraically closed field. Some first examples of (finite-dimensional $k$-algebras) are group algebras $kG$ where $G$ is a finite group, and factor rings of polynomial rings over $k$, such as $k[x]/(x^2)$ and $k[x, y]/(x, y)^2$. Also, there are various subrings of the ring of $n \times n$ matrices over $k$ such as $(k_{0})_{0}$; other examples come from the representation theory of Lie algebras.

An interesting class of examples is defined in terms of quivers. A quiver $\Gamma$ is an oriented graph, i.e., a set of vertices together with a set of arrows between vertices. We assume here that $\Gamma$ has only a finite number of vertices and arrows. A path in $\Gamma$ is either the "trivial" path $e_{i}$ associated with a vertex $i$ or a sequence $\alpha_{n} \cdots \alpha_{1}$ of arrows with $n \geq 1$ where if $n > 1$, $\alpha_{n}$ ends at the vertex where $\alpha_{n+1}$ starts for $i = 1, \ldots , n - 1$. A nontrivial path $\alpha_{n}, \ldots , \alpha_{1}$ is an oriented cycle if $\alpha_{n}$ ends at the vertex where $\alpha_{1}$ starts. As a vector space over $k$, the path algebra $k\Gamma$ has a basis consisting of the paths in $\Gamma$. Then $k\Gamma$ is finite-dimensional over $k$ if and only if $\Gamma$ has no oriented cycles. Multiplication of basis elements is given by composition of paths when possible and is defined to be $0$ otherwise. For example, if $\Gamma$ is the quiver

\[ \begin{array}{ccc}
1 & \rightarrow & 2 \\
\alpha & \rightarrow & \beta \\
\end{array} \]

then $\{e_{1}, e_{2}, e_{3}, \alpha, \beta, \beta \alpha\}$ is a $k$-basis for $k\Gamma$. And for the multiplication we have, for example, $\beta \cdot \alpha = \beta \alpha$, $\alpha \cdot \beta = 0$, $e_{2} \cdot \alpha = \alpha$, $\alpha \cdot e_{2} = 0$, $e_{1} \cdot \alpha = 0$, $\alpha \cdot e_{1} = \alpha$, $\alpha \cdot \beta \alpha = 0$, and $\beta \alpha \cdot \alpha = 0$.

In the representation theory of algebras we are interested in studying the modules over the algebra, in particular those which are finitely generated. For the group algebras $kG$ this is the same as investigating the representations of $G$ over $k$. Amongst the modules, we are interested in their building blocks up to direct sums, that is, the indecomposable modules. For a finite-dimensional $k$-algebra $\Lambda$ a nonzero $\Lambda$-module $M$ is indecomposable if whenever $M = L \oplus N$, a direct sum, then $L$ or $N$ is zero. In particular, the simple modules, that is, the modules having no submodules except zero and the whole module, are indecomposable. For any finitely generated $\Lambda$-module $M$ we have $M = M_{1} \oplus \cdots \oplus M_{n}$, where the $M_{i}$ are indecomposable, and such a decomposition is unique up to isomorphism and order of summands. Basic questions are centered around trying to understand the module theory. Much attention has been devoted to questions of when we have finite or tame representation type. $\Lambda$ is said to have finite representation type, or finite type for short, if there is only a finite number of indecomposable (finitely generated) $\Lambda$-modules up to isomorphism. Amongst the finite-dimensional $k$-algebras of infinite type there is the important class of tame algebras. The definition of tame is rather technical, but loosely speaking, this is a class of algebras where an explicit classification of the indecomposable modules is possible. The Kronecker algebra $k(\bullet \rightarrow \bullet \rightarrow \bullet)$ is a typical example of a tame algebra whose classification of indecomposable modules was carried out by Kronecker around 1890.

When studying representations, each finite-dimensional $k$-algebra $\Lambda$, where $k$ is an algebraically closed field, can be replaced by the factor algebra $k\Gamma/I$ of a path algebra $k\Gamma$ modulo an ideal $I$ contained in $J^{2}$, where $J$ is the ideal in $k\Gamma$ generated by the arrows in $\Gamma$. It then turns out that the quiver $\Gamma$ is uniquely determined by the $k$-algebra $\Lambda$. A useful point of view on the module theory of $k\Gamma$ is to associate with a module over $k\Gamma$ a so-called representation of the quiver $\Gamma$. This means that (for a quiver $\Gamma$) there is a (finite-dimensional) vector space $V(i)$ associated with the vertex $i$, and if $\alpha$ is an arrow from $i$ to $j$, there is a linear transformation $f_{\alpha}: V(i) \rightarrow V(j)$. For example, let $\Gamma$ be the quiver

Then a representation of $\Gamma$ over $k$ is given by four vector spaces $V(1), V(2), V(3),$ and $V(4)$, together with linear transformations $f_{\alpha}: V(1) \rightarrow V(2), f_{\beta}: V(1) \rightarrow V(3),$ and $f_{y}: V(1) \rightarrow V(4)$, associated with the arrows $\alpha, \beta$, and $y$. Writing the vector space $V(i)$ at the vertex $i$ and the linear transformation corresponding to an arrow next to the arrow, we get that

is such a representation of the quiver $\Gamma$. (See [5] for details.)
Path Algebras of Finite Representation Type

One of the first main theorems in the modern phase of representation theory of finite-dimensional algebras, dating back to the early seventies, was Gabriel's classification of the path algebras of finite (representation) type in terms of Dynkin diagrams. As we are going to discuss, this appearance of Dynkin diagrams strongly influenced the development of representation theory.

Theorem 2.1 Let \( \Gamma \) be a finite connected quiver without oriented cycles.

(a) The path algebra \( k\Gamma \) over the field \( k \) is of finite type if and only if the underlying graph \(|\Gamma|\) of \( \Gamma \) is a Dynkin diagram.

(b) Assume that \(|\Gamma|\) is a Dynkin diagram with vertices \( \{1,\ldots,n\} \), and let \( V \) be a representation of \( \Gamma \) over \( k \). Then the assignment

\[
V \rightarrow [V] = (\dim_k V(1), \ldots, \dim_k V(n)) \in \mathbb{R}^n
\]

sending a representation to its dimension vector provides a one-one correspondence between indecomposable representations of \( \Gamma \) (or indecomposable \( k\Gamma \)-modules) and positive roots for \(|\Gamma|\).

As an example, consider the quiver \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \) with underlying graph \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \). The positive roots are listed just before Theorem 1.2. The corresponding indecomposable representations of \( \Gamma \) are \( k \rightarrow 0 \rightarrow 0 \), \( 0 \rightarrow k \rightarrow 0 \), \( 0 \rightarrow 0 \rightarrow k \), \( k \rightarrow 0 \rightarrow 0 \), \( 0 \rightarrow k \rightarrow 0 \), and \( k \rightarrow k \rightarrow k \).

Since the answer to the question of finite type was given in terms of Dynkin diagrams and there was hence a connection with the quadratic form \( q_{|\Gamma|} \) being positive definite, Tits was inspired to give a direct argument, using geometry, for the fact that if \( k\Gamma \) is of finite type, then \( q_{|\Gamma|} \) is positive definite (see [8]). This suggested a closer connection with the Weyl group being finite. And indeed shortly thereafter Bernstein-Gelfand-Ponomarev gave a new and elegant proof of Gabriel's theorem based on such connections [8]. They proved finite type in the Dynkin case by establishing a direct one-one correspondence between indecomposable modules and positive roots for the associated Dynkin diagram.

Also, quivers \( \Gamma \) where \(|\Gamma|\) is extended Dynkin are important in representation theory, and they correspond to the tame algebras which are not of finite type. The diagrams \( B_n, C_n, F_4, \) and \( G_2 \) appear for finite type when the field \( k \) is not algebraically closed [11].

In addition to the immediate offsprings of Gabriel's theorem, there have been some further long-range influences. Quadratic forms still play an important role in representation theory, and the central theory of tilting which compares the module theory of an algebra with the module theory of certain endomorphism algebras, has its origin in this work. Also, for path algebras \( k\Gamma \) of infinite representation type the dimension vectors of the indecomposable representations are the roots (real and imaginary) for certain infinite-dimensional Lie algebras [21]. In recent years there has been work by Ringel, Riedtmann, Schofield, and others on constructing part of the Lie algebra starting with the path algebra. Also, there are, via the Dynkin diagrams, connections with \( C^* \)-algebras [17, 13].

Self-injective Algebras of Finite Type

In this section we try to give a rough idea of how and why the Dynkin diagrams appear in the classification of another class of finite-dimensional \( k \)-algebras of finite type, called self-injective algebras.

Let \( \Lambda \) be a finite-dimensional \( k \)-algebra. Recall that a sequence \( 0 \rightarrow A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \rightarrow 0 \) of \( \Lambda \)-modules and \( \Lambda \)-homomorphisms is exact if \( f \) is one-one, \( g \) is onto, and the kernel of \( g \) is equal to the image of \( f \). This means that the module \( C \) is isomorphic to the factor module \( B/f(A) \), also as a vector space over \( k \). Hence we have the equality \( \dim_k B = \dim_k A + \dim_k C \).

As an example, assume now that there are exactly four indecomposable \( \Lambda \)-modules—\( V_1, V_2, V_3, V_4 \)—up to isomorphism and that \( V_4 \) is isomorphic to \( \Lambda \). Assume further that we have maps \( f_i: V_i \rightarrow V_{i+1} \) and \( g_i: V_{i+1} \rightarrow V_i \) for \( i = 1, 2, 3 \) giving rise to the exact sequences

\[
0 \rightarrow V_1 \overset{f_1}{\rightarrow} V_2 \overset{g_1}{\rightarrow} V_1 \rightarrow 0
\]

\[
0 \rightarrow V_2 \overset{(f_2)}{\rightarrow} V_3 \oplus V_1 \overset{(g_2, f_1)}{\rightarrow} V_2 \rightarrow 0
\]

\[
0 \rightarrow V_3 \overset{(f_3)}{\rightarrow} V_4 \oplus V_2 \overset{(g_3, f_2)}{\rightarrow} V_3 \rightarrow 0
\]

We have the following quiver, where the vertex \( f \) corresponds to the module \( V_f \) and the arrows come from the six maps \( f_i, g_i \) for \( i = 1, 2, 3 \):

\[
1 \overset{f_1}{\rightarrow} 2 \overset{f_2}{\rightarrow} 3 \overset{f_3}{\rightarrow} 4
\]
Replacing each pair of arrows \( \bullet \rightarrow \bullet \bullet \rightarrow \bullet \) by an edge \( \bullet \rightarrow \bullet \), we get the graph \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \). If we write \( a_1 = \dim_k V_1 \), the above three exact sequences imply \( 2a_1 = a_2, 2a_2 = a_1 + a_3 \), and \( 2a_3 = a_2 + a_4 > a_2 \). Hence we get a subadditive nonadditive function \( a_1 \rightarrow a_2 \rightarrow a_3 \). This way we obtain a Dynkin diagram, here \( A_3 \), associated with \( \Lambda \).

The above situation is realized for \( \Lambda = k[X]/(X^3) \). The structure theorem for modules over the principal ideal domain \( k[X] \) can be used to show that the indecomposable \( \Lambda \)-modules are \( V_1 = k[X]/(X), V_2 = k[X]/(X^2), V_3 = k[X]/(X^3), \) and \( V_4 = \Lambda \).

The above sequences are the \textit{almost split sequences} for \( k[X]/(X^3) \), as introduced in joint work with Auslander in the early seventies (see [5]). That an exact sequence

\[
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\]

of \( \Lambda \)-modules is almost split means by definition the following: (i) The end terms \( A \) and \( C \) are indecomposable. (ii) The sequence does not split; that is, there is no \( \Lambda \)-homomorphism \( h:C \rightarrow B \) with \( gh = \text{id}_C \). (iii) Given any map \( s:X \rightarrow C \) with \( X \) indecomposable such that \( s \) is not an isomorphism, there is some \( t:X \rightarrow B \) such that \( gt = s \).

Let \( C \) be an indecomposable module over a finite-dimensional \( k \)-algebra \( \Lambda \) such that \( C \) is not a direct summand of \( \Lambda \). Then there is an almost split sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) unique up to isomorphism. Similarly we can start on the left with any indecomposable \( \Lambda \)-module \( A \) which is not a direct summand of the \( \Lambda \)-module \( \text{Hom}_k(\Lambda, k) \). There is also in general an associated quiver, defined as in the above example, called the \( AR \)-quiver of \( \Lambda \).

For the algebra \( k[X]/(X^4) \) we see that the set of indecomposable modules occurring on the right of almost split sequences coincides with the set of those occurring on the left. This property characterizes the class of \textit{self-injective} algebras where group algebras \( kG \) for a finite group \( G \) are also examples. For an almost split sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) write \( A = \tau C \).

In our example we had \( \tau C = C \) for all almost split sequences, but this is normally not the case, even for self-injective algebras of finite type. For this class one has \( \tau^t C = C \) for some positive integer \( t \). In this case one obtains in a similar way a Dynkin diagram. Now the vertices correspond to \( \tau \)-orbits.

Associating Dynkin diagrams with self-injective algebras of finite type goes back to [24] and was the basis for a classification theorem for this class of algebras. The idea of using dimension functions in this investigation appeared in [27,18].

**Preprojective Algebras**

Again let \( \Sigma \) be a finite connected graph with no loops and vertices \( \{1, \ldots, n\} \). We associate with \( \Sigma \) a \( k \)-algebra \( \Pi(\Sigma) \), called the \textit{preprojective} algebra of \( \Sigma \). Interesting ring theoretic properties of \( \Pi(\Sigma) \) depend on \( \Sigma \), and again the Dynkin and extended Dynkin diagrams appear in a natural way.

Associate with the graph \( \Sigma \) the quiver \( \Xi \) having the same vertices as \( \Sigma \) and where each edge \( i \rightarrow j \) in \( \Sigma \) is replaced by a pair of arrows \( i \rightarrow j \) and \( j \rightarrow i \). Then \( \Gamma \) is any quiver, without oriented cycles, with underlying graph \( \Sigma \). Consider the element \( r = \sum_{\alpha} \alpha^* \alpha - \alpha \alpha^* \) in \( k \Sigma \), where the sum runs over all arrows \( \alpha \) in \( \Gamma \). Then \( \Pi(\Sigma) \) is defined to be the factor algebra \( k \Sigma / (r) \). For example, if \( \Sigma \) is the graph \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \), then we can choose \( \Gamma \) to be the quiver \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \). Then \( \Xi \) is the quiver

\[
\bullet \quad \xrightarrow{\alpha} \quad \bullet \quad \xrightarrow{\beta} \quad \bullet \quad \xrightarrow{\alpha^*} \quad \bullet
\]

and \( r = \alpha^* \alpha - \alpha \alpha^* + \beta \beta^* - \beta^* \beta \). The algebra \( \Pi(\Sigma) \) is not necessarily finite dimensional over \( k \). In fact, there is the following result.

**Theorem 2.2** Let \( \Sigma \) be a finite connected graph without loops and \( \Pi(\Sigma) \) the associated preprojective algebra over \( k \).

(a) \( \Pi(\Sigma) \) is a finite-dimensional \( k \)-algebra (or an artin ring) if and only if \( \Sigma \) is a Dynkin diagram.

(b) \( \Pi(\Sigma) \) is a noetherian ring if and only if \( \Sigma \) is a Dynkin or an extended Dynkin diagram.

Let \( \Gamma \) be a finite connected quiver without oriented cycles and \( \Sigma = |\Gamma| \) the underlying graph. In [7] there is a construction of the preprojective algebra \( \Pi(\Sigma) \) in terms of the module theory for \( k \Gamma \).

Part (a) of Theorem 2.2 goes back to [16, 24, 12]. In [24] such ideas appeared in connection with self-injective algebras of finite type, and in [12] a more general setting is treated.

**Rational Double Points**

We shall see how the idea of using length functions on modules in almost split sequences for finite-dimensional algebras to obtain Dynkin diagrams can be extended to some classes of commutative rings.
Associated with the rotation groups of the regular polyhedra (the polyhedral groups) are the binary polyhedral groups, from which the polyhedral groups are obtained as factors by normal subgroups of order 2. Together with cyclic and dihedral groups, the binary polyhedral groups are all the finite subgroups of the special linear group $SL(2, \mathbb{C})$ up to conjugation. Via the inclusion $G \subset SL(2, \mathbb{C})$ of a non-trivial finite group $G$ there is an action of $G$ on $\mathbb{C}^2$, inducing the quotient $\mathbb{C}^2/G$ which has a singularity only at the origin. Actually, $\mathbb{C}^2/G$ is isomorphic as a variety to the hypersurface \{(x, y, z) \in \mathbb{C}^3; f(x, y, z) = 0\} in $\mathbb{C}^3$ for some polynomial $f$. These are isolated singularities, called Kleinian singularities. Associated with each such hypersurface is a resolution graph which is known to be Dynkin. In the resolution of the singularity there is a family of curves lying above the singular point. The vertices in the graph correspond to the curves, and there are edges where the corresponding curves intersect. As a simple example (drawing the real part), we can think of

\[ \begin{align*}
&\bullet \rightarrow \bullet \\
&\bullet \rightarrow \bullet
\end{align*} \]

where there is one curve lying above the singular point, corresponding to the graph with only one vertex and no edges (see [26]).

The group $G \subset SL(2, k)$ also acts naturally on the power series ring $k[[X, Y]]$ where $k$ is an algebraically closed field of characteristic zero. We denote the corresponding invariant ring by $R = k[[X, Y]]^G$, where $r \in R$ if $g(r) = r$ for all $g \in G$. The (maximal) Cohen-Macauley modules, denoted by $CM(R)$, are the finitely generated $R$-modules which are finitely generated free as $T$-modules for a certain subring $T$ of $R$. In other words, for each module $B$ in $CM(R)$ there is some $n$ such that as $T$-module $B$ is isomorphic to $T^n$, a direct sum of $n$ copies of $T$. There is a more general existence theorem for almost split sequences which applies to this situation and was the basis for analogous theories for finite-dimensional $k$-algebras and certain classes of commutative rings [1]. The rings $R = k[[X, Y]]^G$ with $G \subset SL(2, \mathbb{C})$ have only a finite number of indecomposable modules in $CM(R)$, and for the almost split sequences the left and right terms are always isomorphic. $R$ is an indecomposable $R$-module in $CM(R)$ and is the only one which does not occur on the right, or on the left, of an almost split sequence. But a remarkable fact is that there is an exact sequence $0 \rightarrow R \xrightarrow{f} E \xrightarrow{g} R$, called the fundamental exact sequence, with properties similar to those of an almost split sequence, except for $g: E \rightarrow R$ being surjective. Also in the setting of $CM(R)$ there is a function, the rank $r$ over $T$, which has the property that $r(A) + r(C) = r(B)$ for each exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $CM(R)$. Here $r(X) = n$ when $X$ is isomorphic to $T^n$ as $T$-module. In addition, we have $2r(R) = r(E)$, and due to this extra feature we get in this situation an additive function associated with the whole $AR$-quiver. This way we get an extended Dynkin diagram by using the results in section "Dynkin and Extended Dynkin Diagrams" and first proving that there are no loops.

We illustrate with a concrete example. If we let $G = \langle g \rangle \subset SL(2, k)$ where $g = \left( \begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix} \right)$ and $\rho \neq 1$ is a third root of 1, we get $R = k[[X^3, Y, Y^3]]$ and can choose $T$ to be $k[[X^3, Y^3]]$. In this case there are three indecomposable modules: $C_1$, $C_2$, and $C_3 = R$ in $CM(R)$. The almost split sequences are

\[ \begin{align*}
0 &\rightarrow C_1 \rightarrow C_2 \oplus R \rightarrow C_1 \rightarrow 0 \\
0 &\rightarrow C_2 \rightarrow C_1 \oplus R \rightarrow C_2 \rightarrow 0 \\
0 &\rightarrow R \rightarrow C_1 \oplus C_2 \rightarrow R.
\end{align*} \]

They give rise to the $AR$-quiver

\[ \begin{align*}
&\begin{array}{c}
1 \\
3
\end{array} \\
&\begin{array}{c}
2
\end{array}
\end{align*} \]

where the vertex $i$ corresponds to the module $C_i$. We have $2r(C_1) = r(C_2) + r(R)$, $2r(C_2) = r(C_1) + r(R)$, and $2r(R) = r(C_1) + r(C_2)$. Replacing each pair of arrows \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
by an edge \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
for the extended Dynkin diagram \[ \begin{array}{c}
A_2
\end{array} \]
and a sub-additive function

\[ r(C_1) \rightarrow r(C_2) \]

for the Dynkin diagram $A_2$.

There is yet another way of associating a graph (and a quiver) with a finite group $G \subset SL(2, k)$, due to McKay. The vertices of this McKay quiver

\[ \begin{align*}
&\begin{array}{c}
1 \\
3
\end{array} \\
&\begin{array}{c}
2
\end{array}
\end{align*} \]

where the vertex $i$ corresponds to the module $C_i$. We have $2r(C_1) = r(C_2) + r(R)$, $2r(C_2) = r(C_1) + r(R)$, and $2r(R) = r(C_1) + r(C_2)$. Replacing each pair of arrows \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
by an edge \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
for the extended Dynkin diagram $\tilde{A}_2$, and a sub-additive function

\[ r(C_1) \rightarrow r(C_2) \]

for the Dynkin diagram $A_2$. 

\[ \begin{align*}
&\begin{array}{c}
1 \\
3
\end{array} \\
&\begin{array}{c}
2
\end{array}
\end{align*} \]

where the vertex $i$ corresponds to the module $C_i$. We have $2r(C_1) = r(C_2) + r(R)$, $2r(C_2) = r(C_1) + r(R)$, and $2r(R) = r(C_1) + r(C_2)$. Replacing each pair of arrows \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
by an edge \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
for the extended Dynkin diagram $\tilde{A}_2$, and a sub-additive function

\[ r(C_1) \rightarrow r(C_2) \]

for the Dynkin diagram $A_2$. 

\[ \begin{align*}
&\begin{array}{c}
1 \\
3
\end{array} \\
&\begin{array}{c}
2
\end{array}
\end{align*} \]

where the vertex $i$ corresponds to the module $C_i$. We have $2r(C_1) = r(C_2) + r(R)$, $2r(C_2) = r(C_1) + r(R)$, and $2r(R) = r(C_1) + r(C_2)$. Replacing each pair of arrows \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
by an edge \[ \begin{array}{c}
* \\
\rightarrow
\end{array} \]
for the extended Dynkin diagram $\tilde{A}_2$, and a sub-additive function

\[ r(C_1) \rightarrow r(C_2) \]

for the Dynkin diagram $A_2$.
are in one-one correspondence with the irreducible representations $V_1, \ldots, V_n$ of $G$ over $k$. The inclusion $G \subseteq SL(2, k)$ corresponds to a representation $V$, and there is an arrow from $V_i$ to $V_j$ if $V_j$ is a summand of $V \otimes_k V_i$. (Actually, the number of arrows is the multiplicity of $V_j$.) It turns out that the arrows occur in pairs $\bullet \longrightarrow \bullet$, and we obtain the McKay graph by replacing a pair of arrows $\bullet \longrightarrow \bullet$ by an edge. McKay observed that this graph is an extended Dynkin diagram [23]. If the vertex corresponding to the trivial representation $k$ is removed, it is a Dynkin diagram which coincides with the resolution graph for $R = k[X, Y]^G$.

Actually, the results for the AR-quiver were first proved through establishing an isomorphism with the corresponding McKay quiver [2]. Then a more direct approach along the lines sketched here was given, valid more generally for what is called rational double points, also when no group is involved [3]. These are two-dimensional hypersurfaces over $k$ of finite representation type for any characteristic of $k$, and they are exactly the Kleinian singularities in characteristic zero.

**Finite-Dimensional Algebras and Rational Double Points**

We have seen two ways of associating a quiver with a finite-dimensional algebra: the ordinary quiver and the AR-quiver. In each case there is a class of algebras where finite type is characterized via an associated Dynkin diagram. For the AR-quiver there are similar situations in other settings where almost split sequences exist, and we have seen how Dynkin and extended Dynkin diagrams are associated with rational double points from this point of view. A third type of occurrence was through the construction of a special type of algebra $\Pi(\Sigma)$, the preprojective algebra, from a graph $\Sigma$, where Dynkin and extended Dynkin diagrams corresponded to particularly nice classes of algebras, artinian and noetherian.

Turning things around, let us start with a Dynkin diagram $\Sigma$, with associated extended Dynkin diagram $\hat{\Sigma}$, and let $k$ be a field (which is algebraically closed and of characteristic zero). Then associated with $\Sigma$ and $\hat{\Sigma}$ are the preprojective algebras $\Pi(\Sigma)$ and $\Pi(\hat{\Sigma})$ and a rational double point $R$. Also, we can choose quivers $\Sigma'$ and $\hat{\Sigma}'$ with underlying graphs $\Sigma$ and $\hat{\Sigma}$ so that we have the path algebras $k\Sigma'$ and $k\hat{\Sigma}'$. A natural question is whether there is a useful relationship between the various objects associated with the same Dynkin diagram. Actually, in some sense $\Pi(\Sigma)$ and $\Pi(\hat{\Sigma})$ provide a link between path algebras and rational double points. We have already mentioned that $\Pi(\Sigma)$ and $\Pi(\hat{\Sigma})$ can be constructed from modules over $k\Sigma'$ and $k\hat{\Sigma}'$. On the other hand, when $R$ is a rational double point, it turns out that we have the following (see [25]).

**Theorem 2.3** Let $\Sigma$ be a Dynkin diagram, $\hat{\Sigma}$ the associated extended Dynkin diagram, and $R$ the corresponding rational double point over $k$, where $k$ is an algebraically closed field of characteristic zero. Let $M$ be the direct sum of one copy of each indecomposable module in $CM(R)$. Then we have the following:

(a) $\Pi(\hat{\Sigma}) \cong \text{End}_R(M)$.

(b) $\Pi(\Sigma) = \text{End}_R(M)$, the factor ring of $\text{End}_R(M)$ modulo the homomorphisms factoring through a free $R$-module.

Here the fact that we have a Dynkin diagram for $R$ when removing the module $R$ is reflected in the fact that $\text{End}_R(M)$ is finite dimensional over $k$ when $R$ is an isolated singularity.

It is not hard to see that the ring $R$ itself can be constructed from $\Pi(\Sigma)$, as $e \Pi(\Sigma) e$, where $e$ is the idempotent in $\Pi(\Sigma)$ corresponding to the vertex of $R$ in the quiver for $\Pi(\Sigma)$.

This connection was used in [4] to study the module theory for $\Pi(\Sigma)$ by taking advantage of known properties of the rational double point $R$ to obtain the following.

**Theorem 2.4** Let $\Sigma$ be a Dynkin diagram and $k$ a field. If $C$ is an indecomposable $\Pi(\Sigma)$-module which is not projective, then $\tau^3 C \cong C$.

This result was first proved by Ringel and Schofield using different methods. The surprising fact here is that we have periodicity for $\tau$, with low period, even though the module structure for $\Pi(\Sigma)$ may be extremely complicated.

It should also be noted that, inspired by our method, we could start with a one-dimensional ring corresponding to $R$ and get algebras related to $\Pi(\Sigma)$ where even $\tau^3 C \cong C$ for each indecomposable nonprojective module $C$ [4].

The connection between the invariant ring $R = k[X, Y]^G$ and the corresponding preprojective algebra $\Pi(\Sigma)$ has recently been further exploited in [10]. Here a family of deformations $R^\lambda$ of $R$ are studied through investigating related deformations $\Pi(\Sigma)^\lambda$ of $\Pi(\Sigma)$, where $R^\lambda$ is obtained from $\Pi(\Sigma)^\lambda$ as $R$ is obtained from $\Pi(\Sigma)$. Through this correspondence the role of the roots for the extended Dynkin diagram $\Sigma$ for the homological properties of $R^\lambda$ and $\Pi(\Sigma)^\lambda$ is emphasized. There is earlier related work by Kronheimer, Lusztig, and Nakajima.
In conclusion, we hope to have shown how the appearance of Dynkin and extended Dynkin diagrams can be traced back to basic considerations and how it has stimulated exciting developments.

References


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