Symplectic Structures—A New Approach to Geometry

Dusa McDuff

Introduction

Symplectic geometry is the geometry of a closed skew-symmetric form. It turns out to be very different from the Riemannian geometry with which we are familiar. One important difference is that, although all its concepts are initially expressed in the smooth category (for example, in terms of differential forms), in some intrinsic way they do not involve derivatives. Thus symplectic geometry is essentially topological in nature. Indeed, one often talks about symplectic topology. Another important feature is that it is a 2-dimensional geometry that measures the *area* of complex curves instead of the *length* of real curves.

The classical geometry over the complex numbers is Kähler geometry, the geometry of a complex manifold with a compatible Riemannian metric. This is a very rich geometry with a detailed local structure. In contrast, symplectic geometry is flabby, though as should become clear, not completely flabby—there are interesting elements of global structure. The comparison can be roughly stated as follows:

$$\left\{\begin{matrix}\text{Kähler}\\ \text{rich detail}\end{matrix}\right\} \quad \text{versus} \quad \left\{\begin{matrix}\text{symplectic}\\ \text{flabby, global}\end{matrix}\right\}.$$

In this article I will try to give an idea of symplectic geometry by comparing it with Kähler geometry. I will do this in three areas:

- Embeddings of round balls
- Structure of 4-manifolds
- Properties of automorphisms

Basic Notions

Let M^{2n} be a smooth *closed* manifold, that is, a compact smooth manifold without boundary. A symplectic structure ω on M is a closed ($d\omega = 0$), nondegenerate ($\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$) smooth 2-form. The nondegeneracy condition is equivalent to the fact that ω induces an isomorphism

$$\begin{array}{ccc} T_xM & \xrightarrow{\cong} & T_x^*M \\ X & \mapsto & \iota_X\omega = \omega(X,\cdot) \\ \text{vector fields} & & \text{1-forms.} \end{array}$$

Basic Example. The form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ on Euclidean space $\mathbf{R}^{2n}$. In this case, the above isomorphism is given explicitly by the formulae

$$\begin{array}{ccc} X = \frac{\partial}{\partial x_j} & \mapsto & \iota_X\omega_0 = dy_j \\ \frac{\partial}{\partial y_j} & \mapsto & -dx_j. \end{array}$$

Thus, if we identify both the tangent space $T_x\mathbf{R}^{2n}$ and the cotangent space $T_x^*\mathbf{R}^{2n}$ with $\mathbf{R}^{2n}$ in the usual way, viz:

$$\frac{\partial}{\partial x_j} \equiv e_{2j-1} \equiv dx_j, \quad \frac{\partial}{\partial y_j} \equiv e_{2j} \equiv dy_j,$$

this isomorphism is a rotation through a quarter turn.

Every symplectic structure ω determines a volume form $\omega^n/n!$, that is, a nonvanishing top-di-

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mensional form that integrates to give a volume. In two dimensions, of course, ω is simply an area form. In higher dimensions it was suspected long ago that a symplectic structure is much richer than a volume form, but there was no hard evidence of this until the early 1980s, with Eliashberg's work on symplectic rigidity, the Conley–Zehnder proof of the Arnold conjecture for the torus, and Gromov's proof of the nonsqueezing theorem. We will discuss some of this below. For a much more detailed treatment of these questions and many further references the reader can consult [MS].

Here is the first main theorem on symplectic structures.

Theorem 1 [Darboux]. Every symplectic form is locally diffeomorphic to the above form ω_0.

Thus locally all symplectic forms are the same. In other words, all symplectic invariants are global in nature. It has turned out that, apart from obvious invariants such as the de Rham cohomology class $[\omega] \in H^2(M, \mathbf{R})$ of the symplectic form, it is hard to get one's hands on these global invariants, which is why symplectic geometry has taken so long to be developed. Another important fact that goes along with the local uniqueness of symplectic structures (one cannot exactly call it a consequence) is that a symplectic structure has a rich group of automorphisms. We discuss this further below.

Symplectic structures have two main aspects: the geometric and the dynamic. We start with the geometric, the connection with Riemannian and Kähler geometry.

The Geometric Aspect

There is a contractible family of Riemannian metrics on M associated to ω which are constructed via ω-compatible almost complex structures J. Here J is an automorphism

$$J : TM \to TM, \qquad J^2 = -\mathrm{Id}$$

that turns TM into a complex vector bundle. The compatibility conditions are:

$$\omega(x, y) = \omega(Jx, Jy),$$
$$\text{and } \omega(x, Jx) > 0 \quad \text{for all } x \neq 0.$$

They imply that the bilinear form

$$g_J : \quad g_J(x, y) = \omega(x, Jy)$$

is a Riemannian metric. For each ω the set of such J is nonempty and contractible.

Examples

• The standard almost complex structure J_0 on $\mathbf{R}^{2n}$ given by

$$J_0\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J_0\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

is compatible with ω_0.

• The almost complex structure J induced by the complex structure on a Kähler manifold.

There is an important difference between Kähler manifolds and symplectic manifolds. A Kähler manifold M has a fixed complex structure built into its points; M is made from pieces of complex Euclidean space $\mathbf{C}^n$ that are patched by holomorphic maps. One adds a metric g to this complex manifold and then defines the symplectic form ω_J by setting

$$\omega_J(x, y) = g(Jx, y).$$

(For this to work, g must be compatible with J in a rather strong sense: J has to be parallel with respect to the Levi-Civita connection in order for ω_J to be closed. Not all complex manifolds can be given a Kähler structure.)

On the other hand, a symplectic manifold *first* has the form ω, and *then* there is a family of J imposed at the tangent space level (not on the points). Note that the only intrinsic measurements that one can make on a symplectic manifold are 2-dimensional; i.e., if S is a little piece of 2-dimensional surface, then one can measure

$$\int_S \omega = \mathrm{area}_\omega\, S.$$

It was the great insight of Gromov to realize that in symplectic geometry the correct replacement for geodesics are *J-holomorphic curves*. These are maps $u : (\Sigma, j) \to (M, J)$ of a Riemann surface Σ into M that satisfy the generalized Cauchy–Riemann equation:

$$du \circ j = J \circ du.$$

(Here j is the complex structure on the Riemann surface.) In fact, the image $u(\Sigma)$ is a minimal surface in M when it is given the metric g_J, so the analogy with geodesics is not far-fetched. There is a very nice theory of these curves—one application is mentioned below—and they occur as an essential ingredient in many symplectic constructions, for example, in Floer theory.

In his 1998 Gibbs lecture Witten discussed two "deformations" of classical physics, one to quantum theory and the other to string theory. I would like to propose that in some sense the passage from Riemannian (or Kähler) to symplectic geometry is analogous to these deformations. Symplectic geometry was of course first explored because of the fact that the classical equations of motion can be put in Hamiltonian form and that symplectic properties can be exploited to solve these equations in certain important cases. Therefore, because symplectic structures are built into the classical theory, they are very important in the new deformed theories. In "classical" symplectic geometry very little was understood about global topological

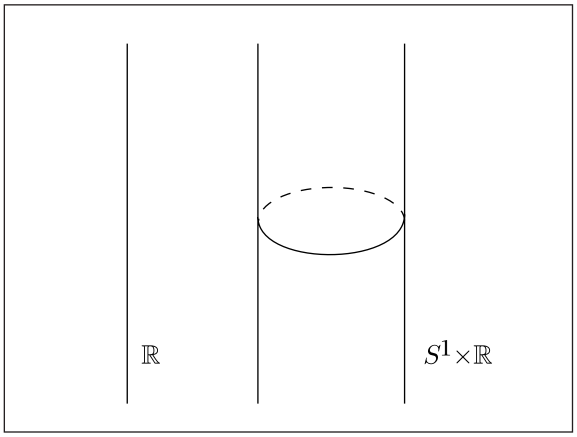

Figure 1. The classical time line is the real line R. This is complexified in string theory to $S^1 \times \mathbf{R}$.

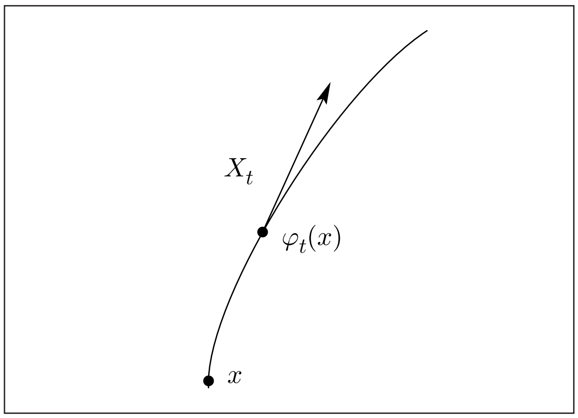

Figure 2. The path ϕ_t has flow lines $\{\phi_t(x)\}_{t\in[0,1]}$ tangent to the vector field X_t at $\phi_t(x)$. It is Hamiltonian if $\iota(X_t)\omega = dH_t$ for all t.

properties of symplectomorphisms. Now, in both of Witten's deformations, new structures are being found that relate in some way to the new global symplectic geometry that is concurrently being developed.

I shall not say anything here about the problem of quantization (though recently Fedosov and Kontsevich have achieved great success with this question), but now briefly discuss the interconnection with string theory. Witten pointed out that one basic consequence of replacing the points that make up the configuration spaces of classical physics by strings (which in the closed case are just circles) is that the time line of a point—usually identified with the real line $\mathbf{R}$—is replaced by the time line of a string, which is a cylinder $S^1 \times \mathbf{R}$; see Figure 1. This cylinder can be identified with the quotient $\mathbf{C}/\mathbf{Z}$ of the complex plane $\mathbf{C}$ by a translation and so has a natural complex structure. Thus the passage to string theory involves replacing $\mathbf{R}$ by $\mathbf{C}$ (or $\mathbf{C}/\mathbf{Z}$), and so going from a geometry in which 1-dimensional objects such as geodesics are of paramount importance to one in which 2-dimensional objects such as J-holomorphic curves are the crucial elements. It is no accident that some of the new ideas that have come into mathematics from physics (such as quantum cohomology and mirror symmetry) involve J-holomorphic curves in an essential way.

The Dynamic Aspect

As mentioned above, the nondegeneracy of the symplectic form ω is equivalent to the condition that there is a bijective correspondence

$$\begin{array}{ccc} T_xM & \xrightarrow{\cong} & T_x^*M \\ X & \mapsto & \iota_X\omega = \omega(X,\cdot) \\ \text{vector fields} & & \text{1-forms.} \end{array}$$

The next important point is that the closedness of ω implies that the *symplectic* vector fields correspond precisely to the *closed* 1-forms. A vector field X is said to be *symplectic* if its flow ϕ_t^X consists of symplectomorphisms, that is, if

$$(\phi_t^X)^*\omega = \omega, \quad \text{for all } t.$$

Because

$$\frac{d}{dt}(\phi_t^X)^*\omega = (\phi_t^X)^*(\mathcal{L}_X\omega),$$

X is symplectic if and only if $\mathcal{L}_X\omega = 0$ where $\mathcal{L}_X$ denotes the Lie derivative. The calculation

$$\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega) = d(\iota_X\omega)$$

shows that X is symplectic exactly when the corresponding 1-form $\alpha = \iota_X\omega$ is closed. Since every manifold supports many closed 1-forms, the group $\mathrm{Symp}(M,\omega)$ of all symplectomorphisms is infinite-dimensional. It has a normal subgroup $\mathrm{Ham}(M,\omega)$ that corresponds to the exact 1-forms $\alpha = dH$. By definition, $\phi \in \mathrm{Ham}(M,\omega)$ if it is the endpoint of a path $\phi_t, t \in [0,1]$, starting at the identity $\phi_0 = \mathrm{id}$ that is tangent to a family of vector fields X_t for which $\iota(X_t)\omega$ is exact for all t; see Figure 2. In this case there is a time-dependent function $H_t : M \to \mathbf{R}$ (called the *generating Hamiltonian*) such that $\iota(X_t)\omega = dH_t$ for all t.

When the first Betti number $b_1 = \dim H^1(M,\mathbf{R})$ of M vanishes, $\mathrm{Ham}(M,\omega)$ is simply the identity component $\mathrm{Symp}_0(M,\omega)$ of the symplectomorphism group. In general, there is a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ham}(M,\omega) &\to \mathrm{Symp}_0(M,\omega) \\ &\to H^1(M,\mathbf{R})/\Gamma_\omega \to 0, \end{aligned}$$

where the flux group Γ_ω is a subgroup of $H^1(M,\mathbf{R})$.

Example. In the case of the torus T^2 with a symplectic form $dx \wedge dy$ of total area 1, the group Γ_ω is $H^1(M,\mathbf{Z})$. The family of rotations $R_t : (x,y) \mapsto (x+t,y)$ of the torus T^2 consists of symplectomorphisms that are not Hamiltonian. Its image under the homomorphism to $H^1(M,\mathbf{R})/\Gamma_\omega$ is the family of 1-forms $t[dy]$.

It has recently been shown [LMP 1997] that Γ_ω has rank at most b_1. One interesting question here is whether the flux group Γ_ω is always discrete. This is equivalent to asking whether the group $\mathrm{Ham}(M, \omega)$ is closed in the C^1-topology, that is, in the topology of uniform convergence of the first derivative. The group is discrete if

- the symplectic class $[\omega] \in H^2(M, \mathbf{R})$ is rational or
- if the map $\wedge[\omega]^{n-1} : H^1(M, \mathbf{R}) \to H^{2n-1}(M, \mathbf{R})$ is an isomorphism.

Because of the hard Lefschetz theorem, this last case includes all Kähler manifolds.

The group $\mathrm{Symp}(M, \omega)$ is a large and interesting group that contains a great deal of information. For example, Banyaga has shown that its structure as an abstract group uniquely determines the symplectic manifold (M, ω). In other words, if the groups $\mathrm{Symp}(M, \omega)$ and $\mathrm{Symp}(N, \sigma)$ are isomorphic as discrete groups, then there is a diffeomorphism $\phi : M \to N$ such that $\phi^*\sigma = \omega$. We will describe some other results on the group of symplectomorphisms later. Meanwhile, here is a recent result that shows that $\mathrm{Symp}(M, \omega)$ is significantly different from the group of all diffeomorphisms.

Proposition 2 [Seidel]. The natural map $\pi_0(\mathrm{Symp}(M, \omega)) \to \pi_0(\mathrm{Diff}(M))$ is not injective in many cases.

For example, the natural map is not injective if M is a $K3$ surface. To prove this, Seidel constructs a symplectic Dehn twist τ near a Lagrangian 2-sphere whose square is diffeotopic to the identity but not symplectically isotopic to the identity. There are other examples where the map $\pi_k(\mathrm{Symp}(M, \omega)) \to \pi_k(\mathrm{Diff}(M))$ is not onto (for example, when $M = S^2 \times S^2$).

Symplectic Embeddings of Balls

Gromov's Nonsqueezing Theorem

Consider a ball $B^{2n}(r)$ of radius r and a cylinder $Z(\lambda) = B^2(\lambda) \times \mathbf{R}^{2n-2}$ of radius λ in standard Euclidean space $(\mathbf{R}^{2n}, \omega)$. Here it is important that the the two coordinates (x_1, y_1) that span the disc $B^2(\lambda)$ are "symplectic", that is, $\omega_0(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) \neq 0$. The question is: when is there a symplectic embedding ϕ of the ball into the cylinder? Its answer is provided by Gromov's celebrated nonsqueezing theorem; see Figure 3.

Theorem 3 [Gromov]. There is a symplectic embedding of the ball of radius r into the cylinder of radius λ if and only if $r \le \lambda$.

The idea of the proof is very roughly the following. For each ω_0-compatible almost complex structure J the cylinder has a slicing by J-holomorphic discs of area $\pi\lambda^2$. If the ball is embedded in the cylinder, this slicing will induce a slicing of the ball; but if J is suitably compatible with the embedding, this slicing of the ball has to have some slices of ω_0-area $\ge \pi r^2$. Hence we must have $r \le \lambda$.

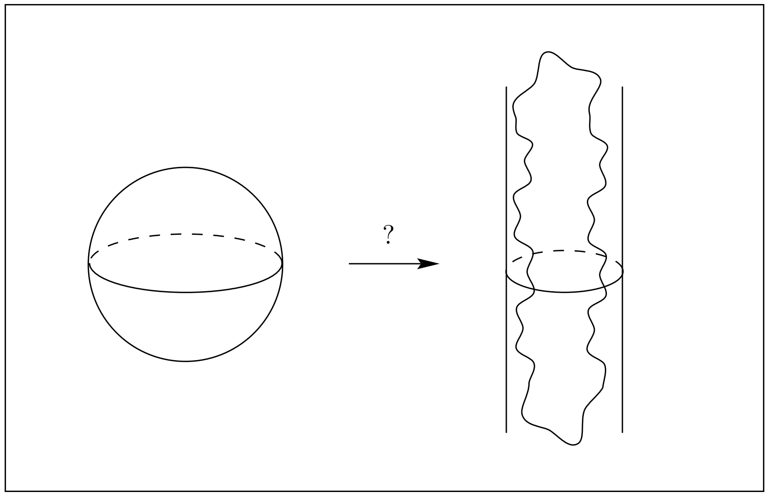

Figure 3. Trying to squeeze a ball into a thin cylinder.

This theorem underlies *all* of symplectic topology. As the following result shows, the nonsqueezing property characterizes symplectomorphisms. Darboux's theorem implies that if we want to find a criterion that characterizes general symplectomorphisms, it suffices to do this for symplectomorphisms of standard Euclidean space $(\mathbf{R}^{2n}, \omega_0)$. Define a *symplectic ball (or cylinder)* of radius r in $(\mathbf{R}^{2n}, \omega_0)$ to be the image of the standard ball (or cylinder) of radius r by a symplectic embedding. We will say that a local diffeomorphism ϕ has the nonsqueezing property if there is no symplectic ball B whose image $\phi(B)$ is contained in a symplectic cylinder with radius strictly less than that of B.

Theorem 4 [Eliashberg, Ekeland–Hofer]. If ϕ is a local diffeomorphism of $\mathbf{R}^{2n}$ such that both ϕ and its inverse ϕ^{-1} have the nonsqueezing property, then ϕ is either symplectic or antisymplectic, that is,

$$\phi^*(\omega_0) = \pm\omega_0.$$

Since the nonsqueezing condition involves only the images $\phi(B)$ of balls B, it is easy to see that it is satisfied by any uniform limit of symplectomorphisms. Hence we find:

Corollary 5 [Symplectic rigidity]. The group $\mathrm{Symp}(M, \omega)$ is closed in the group $\mathrm{Diff}(M)$ in the topology of uniform convergence on compact sets.

This is what I meant by saying in the first paragraph that symplectic geometry is intrinsically topological in nature. Not much is yet understood about symplectic geometry at this level.

Symplectic Packing

Suppose we want to embed k disjoint equal balls symplectically into a compact symplectic mani-

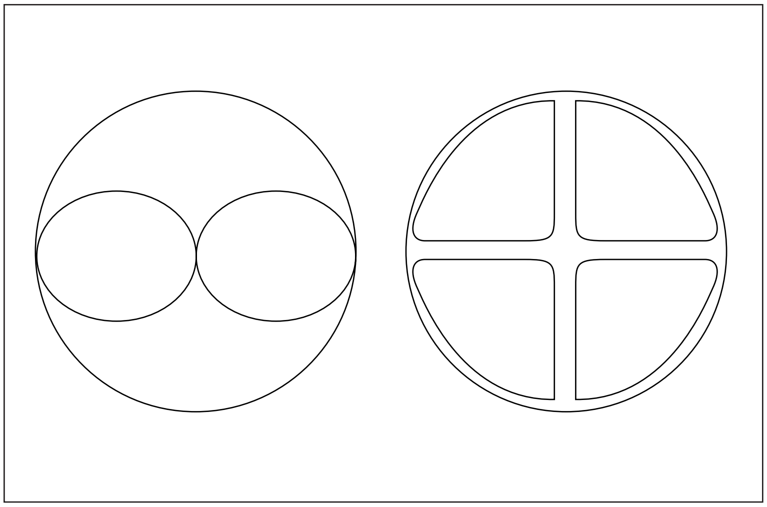

Figure 4. A 4-dimensional ball has a full packing by 4 balls, but not by 2.

fold (M, ω). What restrictions are there? One way to approach this problem is to define

$$\nu_k(M, m) = \sup \frac{\mathrm{Vol}\,(k \text{ disj. equal balls in } M)}{\mathrm{Vol}\,(M, \frac{\omega}{n!})}.$$

We say that (M, ω) has a *full packing* if $\nu_k(M, \omega) = 1$; otherwise, there are *packing obstructions*. See Figure 4.

One example that has been fully worked out is the case when M is the complex projective plane $\mathbf{C}P^2$ with the standard Fubini–Study metric. (Equivalently one could take M to be the unit ball $B^{2n}(1)$ in $\mathbf{R}^{2n}$.) In this case, results of Gromov, McDuff-Polterovich, and Biran show that $\nu_k(M, \omega)$ is as follows:

k	=	1	2	3	4	5	6	7	8	≥9
$\nu_k(M, \omega)$	=	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

The result that $\nu_k(\mathbf{C}P^2) = 1$ for all $k \geq 9$ is due to Biran [B].

Biran has also shown that for *every* symplectic 4-manifold there is an integer N such that

$$\nu_k(M, \omega) = 1 \quad \text{for} \quad k \geq N.$$

He proves this by showing that for all $\varepsilon > 0$ there is a subset V_ε of M such that $M - V_\varepsilon$ can be identified with a disc bundle over a Riemann surface with a standard symplectic form. Then he shows how to fill this disc bundle with balls. The existence of this disc bundle uses the deep work of Donaldson mentioned below, as well as an "inflation" technique of Lalonde-McDuff that allows one to change the symplectic form so that its volume is concentrated near the submanifold.

Thus symplectic packing is basically flabby: with enough balls one can maneuver them into shapes that fill the whole space. It is not known whether the analogous problem in the Kähler category is similarly flabby. Here one considers embeddings that are suitably compatible with both the holomorphic and the symplectic structure on M so that there is a corresponding Kähler form on the blow-up.[1] It is not hard to show that the above calculations for $\nu_k(\mathbf{C}P^2)$ apply also to Kähler embeddings if $k \leq 9$. Also, one can show that the Kähler equivalent $\nu_k^{\mathrm{K}}(\mathbf{C}P^2)$ of $\nu_k(\mathbf{C}P^2)$ takes the value 1 whenever $k = d^2$. However, it is unknown if $\nu_k^{\mathrm{K}}(\mathbf{C}P^2) = 1$ for all $k > 9$. This question is related to difficult conjectures about Seshadri constants and about the structure of holomorphic curves on a generic blow-up of $\mathbf{C}P^2$. Biran has recently obtained some interesting lower bounds for the numbers $\nu_k^{\mathrm{K}}(\mathbf{C}P^2)$ that involve continued fraction expansions. However, it is as yet unknown whether the appearance of these numbers is an artifact of his construction methods or whether they reflect something intrinsic to the problem.

Symplectic 4-Manifolds

In this section we discuss some recent results on the existence of symplectic and Kähler structures on closed and connected 4-manifolds. This question is still not fully understood. The topological properties common to all manifolds with a particular geometric structure can be thought of as a large-scale global expression of this structure. Thus Donaldson's theorem that every symplectic 4-manifold has a blow-up that supports a generalized symplectic fibration is an illustration of how important fibered structures are in symplectic geometry. Fibered structures also arise when one is trying to construct the most economical embeddings of balls.

We begin with some general remarks that contrast symplectic with Kähler 4-manifolds.

• It has been known for a long time that there are non-Kähler symplectic manifolds. The first example was known to Kodaira and later rediscovered by Thurston. Here M is the nilmanifold obtained by quotienting out $\mathbf{R}^4$ by the discrete group Γ that is generated by unit translations in the first three directions together with the map

$$(x, y, s, t) \mapsto (x, x + y, s, t + 1).$$

The symplectic form $dx \wedge dy + ds \wedge dt$ descends to a form ω on M. Note that M can also be considered as made from the manifold $T^2 \times S^1 \times [0, 1]$ by identifying the point $(x, y, s, 0)$ with $(x, x + y, s, 1)$. Therefore the projection

[1] *One way of stating the conditions is as follows: one considers embeddings of a Kähler ball $(B^{2n}(r), J_0, \omega')$ into M that are simultaneously symplectic and holomorphic, where J_0 is the usual complex structure on the unit ball and ω' is a Kähler form that integrates to πr^2 over every flat J_0-holomorphic 2-disc through the origin and that restricts on the boundary to a form that is pulled back from complex projective space via the Hopf map $S^{2n-1} \to \mathbf{C}P^{n-1}$.*

$(x, y, s, t) \mapsto (s, t)$ induces a map from M onto the torus T^2 whose fiber is also a torus. The monodromy (or attaching map) of this fibration has the formula $(x, y) \mapsto (x + y, y)$. This is an area-preserving and hence symplectic map but is not holomorphic. Therefore M has no obvious Kähler structure. In fact, it is easy to see that the first cohomology group $H^1(M, \mathbf{R})$ has dimension 3. This implies that M has no Kähler structure at all because of the well-known fact that the odd Betti numbers $\dim H^{2k+1}(M, \mathbf{R})$ of every Kähler manifold must be even. Indeed, $\dim H^{2k+1}$ can be written as a sum $\sum_{p+q=2k+1} \dim H^{p,q}$, which is even when $p + q$ is odd since $\dim H^{p,q} = \dim H^{q,p}$.

• Gompf showed in 1994 that for any finitely presented group G there is a closed symplectic 4-manifold (M^4, ω) with fundamental group G. On the other hand, there are restrictions on $\pi_1(M)$ if M is Kähler. For example, the remarks above imply that if M has dimension 4, we need the rank of $H_1(M) = G/[G, G]$ to be even. (There are other more subtle restrictions as well, which are at present not very well understood.)

• Gompf–Mrowka (1993) also constructed simply connected but non-Kähler symplectic 4-manifolds using Donaldson theory. Nevertheless, some results seem to imply that symplectic 4-manifolds are very similar to Kähler ones.

• Taubes's structure theorem (1995–96) for the Seiberg-Witten invariants of symplectic 4-manifolds shows that some important features of the Kähler case persist in the symplectic case. Using this result, Szabo and then Fintushel–Stern constructed simply connected nonsymplectic 4-manifolds with nonzero Seiberg–Witten invariants. It follows that the class of symplectic 4-manifolds is strictly larger than the class of 4-manifolds with Kähler structure and strictly smaller than the class of 4-manifolds with nonzero Seiberg-Witten invariants. It is still not understood exactly what the class of symplectic 4-manifolds is. However, as the next result shows, symplectic 4-manifolds can be considered as a kind of flabby deformation of Kähler surfaces.

• It has been known for a long time that algebraic manifolds have blowups that support Lefschetz fibrations. Since the complex structure on every Kähler surface can be slightly deformed to be algebraic, it follows that every smooth 4-manifold that has a Kähler structure also supports a Lefschetz fibration.

Donaldson has recently (1997) shown that every symplectic 4-manifold has a blowup that has the structure of a symplectic Lefschetz fibration. Philosophically this is akin to showing that every 3-manifold has a Heegaard splitting: in other words, it is a general structure theorem that as yet does not make clear all topological properties of these manifolds. In view of the importance of this result we will spend some time explaining it.

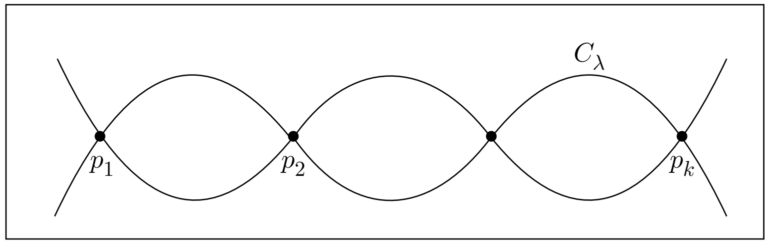

Figure 5. The pencil of subvarieties C_λ. The axis A of the pencil intersects the surface M in the points $p_1, \dots, p_k$.

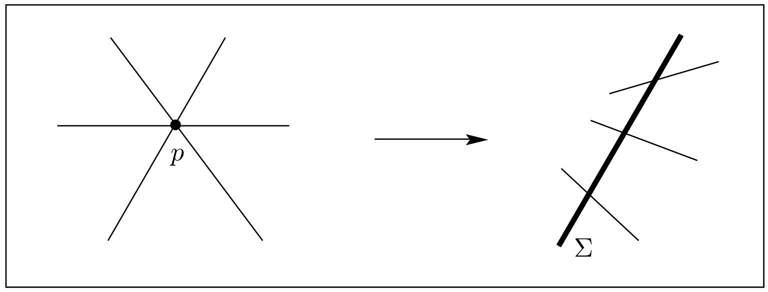

Figure 6. When the point p is blown up to the exceptional divisor Σ, the lines through p become disjoint.

Lefschetz Fibrations

Let $M \subset \mathbf{C}P^N$ be an algebraic surface. Cut M by a pencil P_λ, $\lambda \in \mathbf{C}P^1$, of hyperplanes with axis $A = \mathbf{C}P^{N-2}$. (Here P_λ is just the set of all hyperplanes through A.) This gives a family of subvarieties $C_\lambda = M \cap P_\lambda$ that all go through the set $M \cap A$; see Figure 5.

Since M has complex dimension 2 (and so real dimension 4), the set $M \cap A$ is a finite collection of points—presuming that A is generic—and the C_λ are complex curves that are nonsingular for all but a finite number of λ. Moreover, for generic A, the points in $M \cap A$ will be nonsingular on all the curves C_λ so that one can make the C_λ disjoint by blowing up these points; see Figure 6.

In this way one gets a family $\widetilde{C}_\lambda$ of disjoint curves on the blown-up manifold $\widetilde{M}$, and the map

$$\begin{array}{rccc} \tilde{f}: & \tilde{M} & \to & \mathbf{C}P^1 \\ & x \in \tilde{C}_\lambda & \mapsto & \lambda \end{array}$$

is a singular holomorphic fibration; see Figure 7.

Example. Let $C_i = \{y_i\}$ for $i = 0, 1$ be two generic conics in $\mathbf{C}P^2$. For $\lambda = [\lambda_0 : \lambda_1] \in \mathbf{C}P^1$ define

$$C_\lambda = \{\lambda_0 y_0 + \lambda_1 y_1 = 0\}.$$

This gives a family of conics, all of them nondegenerate except for three pairs of lines.

Theorem 6 [Donaldson, 1997]. Every symplectic 4-manifold M has a blowup $\widetilde{M}$ for which there is a smooth map $f : \widetilde{M} \to \mathbf{C}P^1$ such that the following holds.

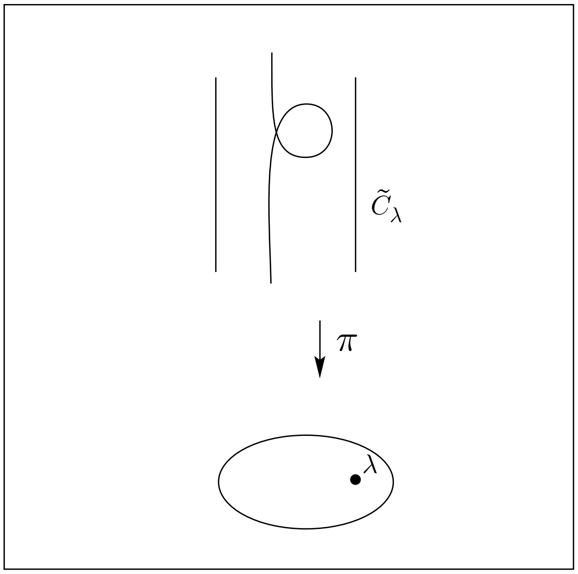

Figure 7. A Lefschetz fibration.

- All but finitely many fibers of f are symplectically embedded submanifolds.
- The remaining fibers are symplectically immersed with just one double point. Moreover, a neighborhood of each of these singular fibers has a compatible complex structure.

Thus one can think of f as a complex Morse function, with singularities modelled on the most generic singularities in the holomorphic case. In particular, the monodromy around each singular fiber is given by a Dehn twist. In the complex case the singularities must satisfy subtle global compatibility conditions that are not fully understood. However, there are no such conditions in the symplectic case. If $f : M \to \mathbf{C}P^1$ is a singular fibration as above such that the fibers support a smooth family of cohomologous symplectic forms that are compatible with the local structure near the singular fibers, then there is a compatible symplectic form Ω on M provided only that there is a cohomology class $a \in H^2(M)$ that restricts on the fibers to the class of the symplectic form.

To prove this theorem, Donaldson develops an "almost holomorphic" analysis that allows him to mimic the proof for algebraic manifolds. Very recently, he completed the generalization of this argument to higher dimensions, showing that every closed symplectic manifold has a suitable blowup that supports a symplectic Lefschetz fibration; see also Auroux [Au].

Groups of Automorphisms

We come to the last of the areas in which I am contrasting symplectic with Kähler geometry. The group $\mathrm{Symp}_0(M, \omega)$ of all symplectomorphisms of M that are symplectically isotopic to the identity was introduced earlier. I will write $\mathrm{Iso}_0(M, J, \omega)$ (or simply $\mathrm{Iso}_0(M)$) for the identity component of the group of isometries of the (closed) Kähler manifold (M, J, ω) when this is provided with the corresponding metric g_J. It is well known that this is a compact Lie group (often trivial). Further, because the symplectic form ω on a Kähler manifold is harmonic with respect to the Kähler metric and because a harmonic form is unique in its cohomology class by Hodge theory, the form ω is preserved by all isometries that fix its cohomology class $[\omega]$. Hence all elements of $\mathrm{Iso}_0(M)$ preserve ω and therefore also preserve the complex structure J.

Some 4-Dimensional Examples

First of all, let me describe some cases in which these two groups are closely related. Note that they can never be equal, since $\mathrm{Symp}_0(M, \omega)$ is infinite-dimensional.

- If the complex projective plane $\mathbf{C}P^2$ is given its standard structure, $\mathrm{Iso}_0(\mathbf{C}P^2)$ is the projective unitary group PU(3), while $\mathrm{Symp}_0(\mathbf{C}P^2, \omega)$ deformation retracts to PU(3).

- Let ω^λ be the symplectic form $(1+\lambda)\sigma_0 \oplus \sigma_1$ on $S^2 \times S^2$, where $\lambda \geq 0$ and where the σ_i are area forms on S^2 of area 1, and let J_{split} be the product almost complex structure. Then $\mathrm{Iso}_0(S^2 \times S^2, J_{split}, \omega^\lambda)$ is the product $SO(3) \times SO(3)$ for all λ. On the other hand, Gromov (1985) proved that $\mathrm{Symp}_0(S^2 \times S^2, \omega^\lambda)$ deformation retracts to $SO(3) \times SO(3)$ if and only if $\lambda = 0$. Moreover, it has been shown by Abreu (1997) and Abreu-McDuff that $\mathrm{Symp}_0(S^2 \times S^2, \omega^\lambda)$ does not have the homotopy type of a compact Lie group when $\lambda > 0$. In fact, when $k - 1 < \lambda \leq k$, this group incorporates the isometry groups of the $k+1$ different complex structures $J_0 = J_{split}, J_1, \dots, J_k$ on $S^2 \times S^2$ that are compatible with the Kähler form ω^λ. Similar results are true for the blowup of $\mathbf{C}P^2$ at one point. However, nothing similar is known about most other manifolds, even one as simple as T^4.

It is obviously unreasonable to expect that the symplectomorphism group would be homotopy equivalent to the group of Kähler isometries in general. However, the next part of the discussion aims to show that some features of the Kähler case do persist in the general case.

The Group of Hamiltonian Symplectomorphisms

Let us write $\mathrm{HIso}(M)$ for the intersection of the isometry group $\mathrm{Iso}(M)$ with the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms. The Lie algebra of $\mathrm{HIso}(M)$ may then be identified with a finite-dimensional space of smooth functions H on M, normalized by the condition that the mean value $\int_M H\omega^n$ is zero. (As always, we assume that M is closed, that is, compact and without boundary.) Moreover, the exponential map is just the time one map of the corresponding flow:

$$\exp : H \mapsto \phi^H = \phi_1^H.$$

Since the exponential map is surjective when the group is compact, it follows that every element ϕ of HIso(M) is the time one map ϕ^H of a Hamiltonian function $H : M \to \mathbf{R}$. Now, every critical point of H gives rise to a fixed point of ϕ^H, since the generating vector field X_H of the flow ϕ_t^H satisfies the equation $\iota_{X_H}\omega = dH$ and so vanishes at such critical points. It follows that for every $\phi \in$ HIso(M) the number of its fixed points is at least as great as the number of critical points of a generating Hamiltonian H. Thus

$$\#\mathrm{Fix}\phi \geq \#\mathrm{Crit}H \geq \sum_i \dim H^i(M, \mathbf{R}),$$
$$\text{for all } \phi \in \mathrm{HIso}(M).$$

Arnold's famous conjecture is that the above statement remains true for every Hamiltonian symplectomorphism whose fixed points are all nondegenerate.[2] This was finally proved in 1996 for all symplectic manifolds by the combined efforts of many mathematicians, among them Floer, Hofer-Salamon, Fukaya-Ono, and Liu-Tian. Thus:

Theorem 7 [Arnold's conjecture]. If (M, ω) is any compact symplectic manifold and $\phi \in$ Ham(M) has no degenerate fixed points, then

$$\#\mathrm{Fix}\phi \geq \sum_i \dim H^i(M, \mathbf{R}).$$

Note that it is essential here that ϕ be Hamiltonian. For example, the rotation $(x, y) \mapsto (x + t, y)$ of the torus T^2 is a non-Hamiltonian symplectomorphism with no fixed points.

Hamiltonian Loops

Our final result concerns a curious and recently discovered property of Hamiltonian loops. First observe that any loop $\{\phi_t\} \in$ Diff(M) generates a homomorphism

$$\partial_\phi : H_*(M) \to H_{*+1}(M)$$

that takes a k-cycle Z in M to the $(k+1)$-cycle $S^1 \times Z \to M$ swept out by the action

$$S^1 \times M \to M : \quad (t, x) \mapsto \phi_t(x).$$

[2] *A fixed point x of ϕ is said to be nondegenerate if the graph of ϕ in $M \times M$ meets the diagonal transversally at (x, x). In the Kähler case discussed above there is no need for this hypothesis, since the fact that ϕ is an isometry implies that H is a Morse-Bott function. In particular, if it has degenerate critical points, then there are infinitely many of them. We restrict here to the nondegenerate case for the sake of simplicity: there is a corresponding conjecture in the general case that has not yet been proved for all manifolds. There are also conjectural homotopy-theoretic lower bounds for #Fixϕ that have also not yet been established.*

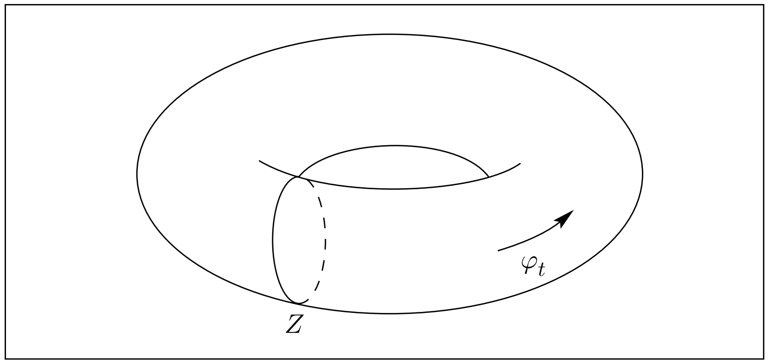

Figure 8. The cycle $\partial_\phi(Z)$.

See Figure 8. Clearly, the map ∂_ϕ depends only on the homology class of the loop $\{\phi_t\}$ in the space of continuous self-maps of M.

This map ∂_ϕ can be expressed geometrically in terms of symplectic fibrations. Given a loop ϕ_t of symplectomorphisms of M, one can construct a fibration $P_\phi \to S^2$ with fiber M by thinking of ϕ_t as a clutching function, viz:

$$\begin{array}{ccc} P_\phi & = & M \times D^+ \cup_{\phi_t} M \times D^- \\ \downarrow & & \downarrow \\ S^2 & = & D^+ \cup D^-. \end{array}$$

It is not hard to show that the loop ϕ_t is isotopic to a Hamiltonian loop exactly when there is a symplectic form Ω on P_ϕ that restricts to the form ω on each fiber M. Further, the map $\partial_\phi : H_k(M) \to H_{k+1}(M)$ is precisely the boundary map in the Wang exact sequence for the fibration $P_\phi \to S^2$.

Recent work of Lalonde-McDuff-Polterovich [LMP], which builds on ideas of Seidel, has shown that the map ∂_ϕ vanishes identically on rational homology when ϕ is a Hamiltonian loop. Thus we have the following result.

Proposition 8 [LMP]. If (P_ϕ, Ω) is a symplectic manifold that fibers over S^2 with symplectic fiber (M, ω), then there is a vector space isomorphism

$$H^*(P_\phi, \mathbf{Q}) \cong H^*(M, \mathbf{Q}) \otimes H^*(S^2, \mathbf{Q}).$$

This result generalizes in the Kähler case. Let us say that a fibration $M \to P \to B$ with the property that $H^*(P_\phi, \mathbf{Q})$ is additively isomorphic to $H^*(M, \mathbf{Q}) \otimes H^*(S^2, \mathbf{Q})$ is *cohomologically split*. Then Deligne showed that every holomorphic submersion from a Kähler manifold P to a base manifold B is cohomologically split. It is not yet known whether a similar result holds in the symplectic case, although there is a good notion of Hamiltonian fibration that generalizes the idea of a holomorphic submersion. (This is explained in the new edition of [MS] as well as in forthcoming work by [LMP].) The fact that at least some of these results on fibrations carry over to the symplectic case is yet another indication both of the naturality of fibered structures in symplectic geometry and of the special nature of Hamiltonian symplectomorphisms.

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