Topology, Algebra, Analysis—Relations and Missing Links

Beno Eckmann

This is largely, but not entirely, a historical survey. It puts various matters together that are usually considered in separate contexts. Moreover, it leads to open, probably quite difficult problems and has analogues in contemporary mathematics. There are three parts.

The **first part** is about a certain range of classical results in algebraic topology concerning continuous real functions and maps, vector fields, etc., that can be stated in a very simple way: just replace "continuous" by "linear". They thus seem to be reduced to problems of algebra, essentially linear algebra, where the solution is relatively easy. The proofs of all these statements, however, do not use such a reduction principle. They are beautiful and in general quite difficult, using elaborate ideas and techniques of topology such as cohomology operations, spectral sequences, K-theory, and so on.

In these cases the absence of a *reduction principle* from continuity to linearity is thus a missing link between two areas. Of course, such a link is not necessary since the results are proved. Still, it might be interesting to have, at least in special cases, a direct reduction to linearity which could throw new light onto old and new mathematics.

We begin, in that first part, with a very elementary example. It contains in a nutshell the problem to be discussed later for analogous but more complex phenomena.

Beno Eckmann is professor emeritus of mathematics and retired founding director of the Institute for Mathematical Research at Eidgenössische Technische Hochschule (ETH) in Zürich.

The **second part** is about the homotopy of the unitary groups. It leads to a different linearization phenomenon stating that maps of a sphere into a unitary group are homotopic in the infinite unitary group to a linear map (in the situation of the first part it is, in general, not true that any continuous solution is *homotopic* to a linear one). Again, no direct proof is known—by differential geometry, by approximation, or by optimization. This linearization result is due to the close relation between the classical Hurwitz-Radon matrices and Bott periodicity. Since the latter is the source of topological K-theory and general homology theories, a direct approach would establish a link between an important old matrix problem and refined methods of algebraic topology.

The **third part** is a short outlook concerning topics of contemporary interest. It is about recent trends in homology theories for spaces with infinite fundamental group G using Hilbert space methods (Hilbert-G-modules, ℓ_2-homology, and ℓ_2-Betti numbers). The presentation, necessarily a little technical, turns around the von Neumann algebra of G considered just as an algebra, forgetting the analysis behind it. We first recall the (weak) Bass conjecture (1976), which has become a theorem for several classes of groups but is still open in general, and give it a simple von Neumann algebra formulation. Application to algebraic topology and group homology again lead, in a very special example, to linearity.

We probably all agree that eventually reducing a difficult problem to a "nice" situation is at the heart of mathematics. What I present here is a modest attempt to list topics where a direct link to linearity or to easy algebra probably exists but is not known. If found, it might, even in different contexts, have some general significance. But in most other areas, nice situations certainly are of a quite different nature; there is no need to list examples.

Vector Fields and Vector Functions

Two preliminary remarks seem appropriate.

1) The results in the first part are all about certain positive integers (dimensions, number of vector fields, etc.). They state that, apart from those values of these integers where (multi-)linear solutions exist, there are no continuous solutions. This negative statement can be turned into a positive one, mainly about the existence of zeros; see Proposition 3. Moreover, in some of those cases where a linear solution exists, there are also other continuous solutions of a different nature (not homotopic to linear ones).

2) We, of course, do not suggest that the respective topological results should have been proved from the outset by reduction to linearity. On the contrary, these problems were a source of stimulation for developing interesting tools that still are very important, e.g., in modern homotopy theory. The linearization phenomenon emerged only a posteriori after the topological theorems had been established.

Tangent Vector Field on a Sphere

A *tangent unit vector field* v on $S^{n-1} \subset \mathbb{R}^n$ (given in coordinates $\xi_1, \dots, \xi_n$ of $\mathbb{R}^n$ by $\sum \xi_i^2 = 1$) is a function that attaches to $x \in S^{n-1}$ a vector $v(x)$ satisfying

$$\langle v(x), x \rangle = 0, \tag{1}$$

$$|v(x)|^2 = \langle v(x), v(x) \rangle = 1 \tag{2}$$

for all $x \in S^{n-1}$. Here $\langle\ ,\ \rangle$ is the standard scalar product in $\mathbb{R}^n$.

Linearity implies that (2) is equivalent to

$$|v(x)|^2 = \langle x, x \rangle \quad \text{for all } x \in \mathbb{R}^n. \tag{2'}$$

If n is even, $v(x) = (-\xi_2, \xi_1, -\xi_4, \xi_3, \dots)$ is such a field, *linear* in x. What about linear fields if n is odd? Let

$$v_i = \sum a_{ik} \xi_k$$

be the components of $v(x)$. Then (1) means $\sum a_{ik} \xi_i \xi_k = 0$ for all x; i.e., the matrix a_{ik} is skew-symmetric. If n is odd, its determinant is 0, and thus there is an $x \in S^{n-1}$ with $v(x) = 0$, in contradiction to (2).

Proposition 1. A linear tangent unit vector field on S^{n-1} exists if and only if $n-1$ is odd.

We now ask the same question for continuous vector fields. These are, of course, more interesting from the viewpoint of geometry and analysis. Given such a field on S^{n-1}, we consider the great circle determined by x and $v(x)$ and move the point x, in the $v(x)$-direction, to its antipode $-x$, for all $x \in S^{n-1}$. This is a homotopy (a continuous deformation) between the identity map and the antipodal map of S^{n-1}.

At this point the homological concept of *degree* comes into play. Its value is 1 for the identity and $(-1)^n$ for the antipodal map. Homotopic maps have the same degree, so we get $1 = (-1)^n$; i.e., n is even.

Proposition 2. A continuous tangent unit vector field on S^{n-1} exists if and only if $n-1$ is odd.

The crucial point is now to express these facts in a different way. We write P_n for the problem: Does there exist on S^{n-1} a tangent unit vector field?

Theorem L ("Linearization"). *If P_n has a continuous solution, then it also has a linear solution.*

Theorem L is proved "indirectly" by using a method from algebraic topology. There is no harm in doing so. By a "direct" proof we would mean a procedure replacing a continuous field by a linear one (for example, through a variational principle where in the space of all continuous fields there would be an extremal expected to be linear). Such a direct proof would reduce the topological problem to very elementary linear algebra.

In the more complicated situations to be described below, a direct reduction to a transparent algebraic argument, though not necessary, might be even more interesting.

Remarks. 1) The negative statement "There is, for even $n-1$, no continuous tangent unit vector field on S^{n-1}" can be turned into a nontrivial existence statement for zeros, as follows.

Proposition 3. Let $f_i(\xi_1, \dots, \xi_n)$, $i = 1, \dots, n$, be continuous functions satisfying

$$\sum \xi_i f_i(\xi_1, \dots, \xi_n) = 0$$

for all x with $|x| = 1$. If n is odd, then the f_i have a common zero.

Otherwise the f_i could be normalized so as to be the components of a unit tangent vector field on S^{n-1}. If the f_i are polynomials, one has an algebraic statement for which no algebraic proof seems to be known. For the complex analogue, however, there is an algebraic proof by van der Waerden (1954).

2) The above proof of Proposition 2 is by elementary algebraic topology. For differentiable vector fields there are other classical proofs, using analysis or geometry; they are all based on some version of the concept of degree. A very different analytic proof, however, is due to Milnor (1978). None of these proofs is by reduction to linearity.

Vector Functions of Two (or More) Variables, Multiplications in $\mathbb{R}^n$

The well-known vector cross-product $x \times y$ in $\mathbb{R}^3$ is a function of two vectors that is bilinear and fulfills

$$\langle x \times y, x\rangle = \langle x \times y, y\rangle = 0, \tag{3}$$

$$|x \times y|^2 = |x|^2|y|^2 - \langle x, y\rangle^2. \tag{4}$$

For which n does such a bilinear vector product exist in $\mathbb{R}^n$?

We assume that it exists in $\mathbb{R}^n$ and imbed $\mathbb{R}^n$ in $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$. We write $X \in \mathbb{R}^{n+1}$ as $X = \xi + x$, $\xi \in \mathbb{R}$, $x \in \mathbb{R}^n$, and similarly $Y = \eta + y \in \mathbb{R}^{n+1}$, and put

$$X \cdot Y = \xi\eta - \langle x, y\rangle + \xi y + \eta x + x \times y. \tag{5}$$

Then $1 + 0 \in \mathbb{R}^{n+1}$ is a two-sided identity for that product, and an easy computation using (3) and (4), namely,

$$\begin{aligned} |X \cdot Y|^2 &= \xi^2\eta^2 + \langle x, y\rangle^2 - 2\xi\eta\langle x, y\rangle + \xi^2|y|^2 \\ &\quad + \eta^2|x|^2 + 2\xi\eta\langle x, y\rangle + |x|^2|y|^2 - \langle x, y\rangle^2 \\ &= \xi^2\eta^2 + \xi^2|y|^2 + \eta^2|x|^2 + |x|^2|y|^2, \end{aligned}$$

yields

$$|X \cdot Y|^2 = |X|^2|Y|^2\ . \tag{6}$$

The product (5) turns $\mathbb{R}^{n+1}$ into an "algebra"; the commutative and associative laws are not required. It fulfills, however, the *norm product rule* (6).

We consider for a moment such algebras in $\mathbb{R}^n$ with norm product rule. The product $X \cdot Y$ can easily be modified so as to contain a two-sided identity whose existence we will assume in the following and denote 1. A bilinear product can be given by the multiplication table of a basis of $\mathbb{R}^n$; it is convenient to have 1 as a basis element.

The classical examples for $n = 1, 2, 4$, and 8 are:

$\mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^2 = \mathbb{C}$

$\mathbb{R}^4$ = quaternion algebra $\mathbb{H}$ (associative but not commutative)

$\mathbb{R}^8$ = "Cayley numbers" or Octonion algebra (not associative, not commutative)

We do not give the well-known multiplication tables for $\mathbb{C}$ (basis 1, i), $\mathbb{H}$ (basis 1, i, j, k), and the Octonions. We recall, however, that all these algebras fulfill the norm product rule (which implies that there are no zero-divisors).

If we write ξ_j for the components of X, η_j of Y, ζ_j of $X \cdot Y$, the norm product rule becomes

$$(\xi_1^2 + \cdots + \xi_n^2)(\eta_1^2 + \cdots + \eta_n^2) = \zeta_1^2 + \cdots + \zeta_n^2\ . \tag{7}$$

Because of the "composition of quadratic forms" given by (7), the algebras with norm product rule are also called *composition algebras*. In 1898 Hurwitz proved that such a "composition of quadratic forms" with bilinear functions ζ_j of the ξ_j and η_j, with real or complex coefficients, can exist for $n = 1, 2, 4, 8$ only.

Proposition 4. A bilinear multiplication in $\mathbb{R}^n$ with two-sided identity and with norm product rule exists if and only if $n = 1, 2, 4, 8$.

In the same spirit as before, we consider the corresponding problem for continuous multiplications. One would expect that continuity gives much more flexibility than bilinearity. However, Adams proved in 1960 that:

Theorem A. A continuous multiplication with two-sided identity and norm product rule exists only for $n = 1, 2, 4, 8$.

Thus, if we now write P_n for the continuous multiplication problem in $\mathbb{R}^n$ (with the above properties), one again has Theorem L, except that *linear* is to be replaced by *bilinear*. And a "direct" proof would reduce the proof of Adams's famous Theorem A to the very old Hurwitz argument of linear algebra.

The original proof of Adams's theorem was a real tour de force, using the whole range of methods of algebraic topology known at that time. A very simple proof became available later thanks to the development of topological K-theory and the Atiyah-Hirzebruch integrality theorems; the proof is simple, but the prerequisites are certainly not.

Here too an algebraic corollary can be mentioned for which no algebraic proof is known.

Proposition 5. $\mathbb{R}^n$ is a bilinear division algebra if and only if $n = 1, 2, 4$, or 8.

Division algebra means a product without zero-divisors (associativity and commutativity are not required). If such a product is given, it can be renormalized so as to fulfill the norm product rule, but one loses bilinearity. Theorem A then says that $n = 1, 2, 4$, or 8.

If we return to vector products of two vectors in $\mathbb{R}^n$, the earlier arguments combined with the Hurwitz Theorem (Proposition 4) yield

Proposition 6. A nontrivial bilinear vector product fulfilling (3) and (4) exists in $\mathbb{R}^3$ and in $\mathbb{R}^7$ and in no other $\mathbb{R}^n$.

Indeed, it follows that $n + 1$ must be 1, 2, 4, or 8. We have to show only that such a vector product actually exists in $\mathbb{R}^7$. We first note that for $n = 3$ the product (5) defines the usual quaternion multiplication in $\mathbb{R}^4$, where in $X = \xi + x$ the $\mathbb{R}$-multiple ξ of 1 is the "real part", x the "imaginary part". Conversely, starting from the quaternion product, one considers imaginary quaternions $x, y \in \mathbb{R}^3$ and puts

$$x \times y = x \cdot y + \langle x, y \rangle,$$

which is imaginary. Then

$$\begin{aligned}(x \cdot y) \cdot y = x \cdot (y \cdot y) &= -|y|^2 x \\ &\quad (\text{since } y \cdot y = -|y|^2 \text{ for } y \in \mathbb{R}^3) \\ &= (x \times y) \cdot y - \langle x, y \rangle y \\ &= -\langle x \times y, y \rangle + \text{imaginary terms;}\end{aligned}$$

whence $\langle x \times y, y \rangle = 0$, and similarly $\langle x \times y, x \rangle = 0$. By the norm product rule valid for the quaternions one has

$$\begin{aligned}|x \cdot y|^2 = |x|^2 \cdot |y|^2 &= |x \times y - \langle x, y \rangle|^2 \\ &= |x \times y|^2 + \langle x, y \rangle^2 \;;\end{aligned}$$

i.e., (4) holds.

Exactly the same procedure works for the Octonions in $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. Although the product is *not* associative, the "alternative" law $(X \cdot Y) \cdot Y = X \cdot (Y \cdot Y)$ holds, and only this has been used (actually the alternative law holds in any composition algebra). Thus $x \times y = x \cdot y + \langle x, y \rangle$, $x, y \in \mathbb{R}^7$, is a bilinear vector product.

For a *continuous* vector product in $\mathbb{R}^n$, the formula (5) defines, exactly as in the bilinear case, a continuous product with two-sided identity and norm product rule in $\mathbb{R}^{n+1}$. Again using Adams's Theorem A, we get

Proposition 7. A continuous vector product in $\mathbb{R}^n$ fulfilling (3) and (4) exists if and only if $n = 3$ or 7.

Writing P_n for the existence problem of a vector product of two vectors in $\mathbb{R}^n$, one has Theorem L (with *bilinear* instead of *linear*).

An interesting corollary of Proposition 7 concerns almost-complex structures on S^n. Such a structure is given by a continuous field $J(x)$ of linear transformations of the tangent space at $x \in S^n$ with $J(x)^2 =$ (minus)identity. (On a complex-analytic manifold, multiplication of complex vector components by $\sqrt{-1}$ is such a field. But we do not assume that complex-analytic coordinates are given.)

Given the field J on S^n, we consider $x \in S^n$, a unit tangent vector $y(x)$ (i.e., two vectors $x, y \in \mathbb{R}^{n+1}$ with $|x| = |y| = 1$ and $\langle x, y \rangle = 0$), and the oriented tangent 2-plane determined by y and $J(x)y$. We choose $x \times y$ to be the unit vector orthogonal to y in that plane and corresponding to the orientation.

We then have a vector product $x \times y$ defined for $|x| = |y| = 1$ and $\langle x, y \rangle = 0$ only, but it can easily be extended to all vectors $x, y \in \mathbb{R}^{n+1}$ so as to fulfill (3) and (4) and be continuous. Therefore $n + 1$ must be 3 or 7; whence $n = 2$ or 6.

Proposition 8. S^n admits an almost-complex structure only for $n = 2$ and 6.

On S^2 such a structure exists, of course, since S^2 can be turned into the Riemann sphere. On S^6 a (linear) almost-complex structure can be derived from the Octonions in $\mathbb{R}^8$; it has been known since 1951 that it cannot come from a complex-analytic structure on S^6.

A *vector product of r vectors in* $\mathbb{R}^n$, $r < n$, is a (multilinear) vector function $v(x_1, \dots, x_r) \in \mathbb{R}^n$ that fulfills

$(1)_r$ $\langle v(x_1, .., x_r), x_j \rangle = 0, \; j = 1, ..., r;$

$(2)_r$ $|v(x_1, ... x_r)|^2 =$ determinant of the $\langle x_j, x_k \rangle$.

Condition $(2)_r$ above implies that $v \neq 0$ if and only if the r vectors x_j are linearly independent.

For $r = 2$ this is the vector product above. The case $r = 1$, vector field on a sphere, has been treated in the very beginning; a solution exists only if n is even.

For which (r, n) does there exist such a vector product? We assume $r \geq 2$ and fix an arbitrary unit vector x_r. Restricting the other variables to the $\mathbb{R}^{n-1}$ orthogonal to x_r clearly yields a vector product of $r - 1$ vectors in $\mathbb{R}^{n-1}$. Continuing with the reduction, we get a vector product of 1 vector in $\mathbb{R}^{n-r+1}$, which implies that $n - r$ must be odd. For $r \geq 2$, reducing to 2 vectors yields $n - r + 2 = 3$ or 7; i.e., $r = n - 1$ or $r = n - 5$.

For arbitrary $n > 1$ and $r = n - 1$ there is a well-known multilinear solution: Take for v the vector orthogonal to the hyperplane spanned by the vectors x_j, $j = 1, \dots, n - 1$ (if they are linearly independent), with suitable orientation and suitably normalized. In terms of the $n \times (n-1)$-matrix of the components of the x_j, the components of v are given by the $(n-1) \times (n-1)$-minors with the usual signs.

For $r = n - 5$ we know that $(2, 7)$ has a bilinear solution. What about $(3, 8)$? Here again, a multilinear vector product can be given explicitly in terms of the Octonions. A more elaborate argument, using Octonions again, shows that $(4, 9)$ does not have any multilinear solution. On the other hand, a topological method (cross-section of a Stiefel-manifold fibering, Steenrod squares; see survey [E1]) proves that even the continuous $(4, 9)$-problem does not have a solution.

In summary: *A continuous vector product of r vectors in* $\mathbb{R}^n$ *exists only in the cases* $(1, n)$ *with n even,* $(n-1, n)$, $(2, 7)$, and $(3, 8)$*, and in these cases there is a (multi-)linear solution.*

Writing $P_{(r,n)}$ for the problem, Is there a vector product of r vectors in $\mathbb{R}^n$? we have:

Theorem L. If $P_{(r,n)}$ admits a continuous solution, then for that pair (r, n) it also has a (multi-)linear solution.

Vector Functions of One Variable, Maximal Number of Orthonormal Solutions

We return to one vector function of one variable in $\mathbb{R}^n$ (tangent vector field problem on S^{n-1}) and consider s orthonormal solutions. In other words, we have $s+1$ vector functions $v_j(x),\ j = 0, 1, \dots, s$ in $\mathbb{R}^n$, defined for $|x| = 1$ with $v_0(x) = x$, and

$$\langle v_j(x), v_k(x)\rangle = \delta_{jk},\ j, k = 0, 1, \dots, s.$$

We assume that the vectors v_j are linear functions of x and write $v_j(x) = A_j x$ where A_j is a real $n \times n$-matrix, $A_0 = E$ (unit matrix). Then

$$\langle v_j(x), v_k(x)\rangle = \langle A_j x, A_k x\rangle = \delta_{jk}\langle x, x\rangle$$

for all $x \in \mathbb{R}^n$. This implies that all A_j are orthogonal matrices, that $A_j^T A_j = E$ where A_j^T denotes the transposed matrix, and that for $j \neq k$

$$A_j^T A_k + A_k^T A_j = 0.$$

For $k = 0$ this yields $A_j^T + A_j = 0,\ j = 1, \dots, s$; whence

$$\text{(8)}\quad A_j^2 = -E, \quad A_j A_k + A_k A_j = 0 \quad \text{for } j \neq k,\ j, k = 1, \dots s.$$

In addition the matrices have to be orthogonal or, equivalently, skew-symmetric.

Such matrices, with real or complex entries, are called *Hurwitz-Radon matrices*. They were independently[1] examined around 1920 by Hurwitz and Radon; they determined for given n the maximum possible number s. If $n = \text{odd}.16^\alpha 2^\beta$, $\beta = 0, 1, 2, 3$, then

$$s_{\max} = 8\alpha + 2^\beta - 1.$$

The quantity $\rho(n) = 8\alpha + 2^\beta$ is called the Radon number of n, and the relations (8) are called the Hurwitz matrix equations.

Proposition 9. The maximum number of orthonormal tangent vector fields on S^{n-1} depending linearly on $y \in S^{n-1}$ is $\rho(n) - 1$.

Again, the continuous analogue was established by Adams (1962) in his famous

Theorem B. The maximum number of continuous orthonormal tangent vector fields on S^{n-1} is $\rho(n) - 1$.

The proof is yet more difficult and technical than the original proof of Theorem A. So far no simpler argument of algebraic topology has been found. If, by analogy to the foregoing, we write $P_{n,s}$ for the problem, Is there a system of s orthonormal tangent vector fields (s-frames) on S^{n-1}? then we again have Theorem L, this time with P_n replaced by $P_{n,s}$.

In the continuous case the same holds for linearly independent fields instead of orthonormal ones, since orthonormalization does not affect continuity.

Hurwitz-Radon Matrices and Homotopy Groups

We begin with some remarks concerning the Hurwitz-Radon matrix problem (8).

The proofs by Hurwitz and Radon were by matrix computations. A different proof, of a more conceptual nature, was given in 1942 by the author using classical representation theory applied to a certain finite group G_s (generated by symbols $A_1, \dots, A_s, \varepsilon$ with relations dictated by (8), i.e., $\varepsilon^2 = 1, A_j^2 = \varepsilon, A_j A_k = \varepsilon A_k A_j,\ j \neq k$). Instead of looking for maximal s given n, one asks for minimal n given s. Minimal n is provided by irreducible orthogonal representations with $\varepsilon \mapsto -E$. The advantage of this method is that it gives explicitly *all* solutions and shows very simply that there exist solutions with matrix entries 0, +1, and −1 only; see [E2]. We note here, for use in the next section, that the same matrix problem can, of course, be formulated for *unitary* representations and this is simpler than the orthogonal problem. In that case the minimal n is $2^{\frac{s}{2}}$ for even s and $2^{\frac{s-1}{2}}$ for odd s. All solutions are direct sums of the minimal ones. In a solution for $s+1$, omitting the last matrix A_{s+1} of course yields a solution for s. For even s, a minimal solution is obtained in this way from a minimal solution for $s+1$, since $2^{\frac{s}{2}} = 2^{\frac{(s+1)-1}{2}}$. In other words, the solutions for even s are not essential; they all come from $s+1$.

Let $A_1, \dots, A_s$ be a set of orthogonal Hurwitz-Radon matrices, i.e., a solution of (8); let $A_0 = E$; and let $\alpha_0, \alpha_1, \dots, \alpha_s$ be real numbers with $\Sigma\alpha_j^2 = 1$. From (8) it follows easily that the $n \times n$-matrix

$$\text{(9)}\qquad f(a) = \Sigma\alpha_j A_j$$

is orthogonal, and this is equivalent to (8). We write $a = (\alpha_0, \alpha_1, \dots, \alpha_s) \in S^s$ in $\mathbb{R}^{s+1}$ and consider f as a (linear) map of S^s into the orthogonal group $O(n)$. Combining with the natural imbedding of $O(n)$ into the infinite orthogonal group O (the limit of the usual inclusions $O(n) \to O(n+1)$), we get a map $F : S^s \to O$. Conversely, any linear map $F : S^s \to O$ of the form (9) (the image necessarily lies in some $O(n)$), with $A_0 = E$, is given by orthogonal Hurwitz-Radon $n \times n$-matrices. The homotopy class of F is an element of the homotopy group $\pi_s O$.

[Remark: The matrix (9) being orthogonal shows that a solution of the Hurwitz equations (8) is

[1] *Hurwitz died in 1919. His paper appeared in 1923. Radon's work was submitted in 1922 and also was published in 1923.*

equivalent to the composition of quadratic forms, generalizing (7),

$$(\alpha_0^2+\cdots+\alpha_s^2)(\xi_1^2+,\cdots,\xi_n^2)=(\zeta_1^2+\cdots+\zeta_n^2),$$

where the ζ_i are bilinear in the α_j and ξ_i.]

One can proceed in exactly the same way with a *unitary* solution of (8). Then f and F (we use the same letters) yield a homotopy class in $\pi_s(U)$ where U is the infinite unitary group, the limit of the inclusions $U(n)\to U(n+1)$. Both cases are closely related to *Bott periodicity* (1956). Here we restrict ourselves, for simplicity, to the unitary case; the orthogonal case can be dealt with in the same way, though the details are a little more complicated.

As for the homotopy group $\pi_s U(n)$, it has been known since around 1940 that

$$\pi_s U(n)\cong\pi_s U\left(\frac{s+1}{2}\right)\quad\text{if } s \text{ is odd and } n\geq\frac{s+1}{2},$$

$$\pi_s U(n)\cong\pi_s U\left(\frac{s+2}{2}\right)\quad\text{if } s \text{ is even and } n\geq\frac{s+2}{2}.$$

The isomorphisms are given by the imbedding $U(n)\to U(n+1)$. These "stable" groups are the homotopy groups $\pi_s U$.

The Bott periodicity theorem determined the groups $\pi_s U$ completely:

$$\pi_s U=\mathbb{Z}\quad\text{if } s \text{ is odd,}$$
$$=0\quad\text{if } s \text{ is even.}$$

If s is even, any solution of (8) yields, as it must, a nullhomotopic map $f:S^s\to U(n)$, since it comes from a solution for $s+1$, so that f can be extended to a linear map $S^{s+1}\to U(n)$ and is therefore nullhomotopic (even in a *linear* way).

For odd s, however, one has the interesting result:

A minimal solution of the Hurwitz-Radon problem yields, for odd s, through $S^s\to U(n)\to U$ given by f above, a generator of $\pi_s U$.

Note that here $n=2^{\frac{s-1}{2}}$, while in the usual approach the generator, in the stable group $\pi_s U(\frac{s+1}{2})$, lies in dimension $n=\frac{s+1}{2}$. For the lowest cases $s=1$ and 3, these n are equal, and the generators are easily recognized to be identical. This is not so for $s>3$, so that a proof is needed. It makes use (see [E2]) of Bott's theorem in its full topological statement. On the algebraic side it is based on a simultaneous analysis of the solutions of the matrix problem (8) for all values of s.

What about the multiples of the generator? The group operation (addition) for two elements of $\pi_s U$ can be described in a simple way: One just places the two maps $S^s\to U(n)$ over the diagonal in $U(2n)$. One therefore can obtain all elements of $\pi_s U$, i.e., of $\pi_s U(n)$ for sufficiently high n, through unitary Hurwitz-Radon matrices. This can be expressed again as a linearization result of a different nature:

Theorem L′. Any continuous map $S^s\to U(n)$, $n\geq\frac{s+1}{2}$, or $\geq\frac{s+2}{2}$ respectively is homotopic in U to a linear map.

A direct proof of that theorem would reduce Bott periodicity to the purely algebraic discussion of Hurwitz-Radon matrices.

What Bott proved was actually more than the periodicity of homotopy groups: One considers ΩU, the space of loops in U beginning and ending in $1\in U$. The periodicity of homotopy groups $\pi_{s+2}(U)=\pi_s(U)$ for all $s\geq 0$ is essentially the same as a homotopy equivalence between $\Omega\Omega U$ and U and thus a periodicity with period 2 for all iterated loop-spaces of U. For the groups of homotopy classes of maps of arbitrary spaces (cell complexes) X into U and into the iterated loop-spaces of U, one therefore has the same periodicity. The (abelian) groups thus obtained constitute a cohomology functor called topological K-theory. This was the first example of an "extraordinary" cohomology theory, and it seems interesting that it is closely related to the unitary Hurwitz-Radon matrices.

Everything can also be said, mutatis mutandis, about the orthogonal (or the symplectic) Hurwitz-Radon matrices and the infinite orthogonal (or symplectic) group and the corresponding K-theory; here the periodicity has period 8.

Von Neumann Algebra of a Group

About the Bass Conjecture

Here we consider the complex group algebra $\mathbb{C}G$ of a discrete group G. Recall that it is the complex vector space having the group elements as basis, with product given by the group multiplication of the basis elements. The identity element $1\in G$ is the identity for the algebra product. An idempotent $a\in\mathbb{C}G$ is an element fulfilling $a^2=a$. The *idempotent conjecture* says that the only idempotents in $\mathbb{C}G$ are 0 and 1, as in $\mathbb{C}$ or any division ring, provided the group G is torsion-free, i.e., has no elements of finite order $\neq 1$ (elements of finite order easily yield nontrivial idempotents of $\mathbb{C}G$). In the following we always assume G to be torsion-free.

A strong tool to deal with this problem is the "canonical" trace $\kappa(a)$, also called the Kaplansky trace of $a\in\mathbb{C}G$; it is the coefficient of $1\in G$ of a. A little more generally, let $A=(a_{ij})$ be an idempotent $(n\times n)$-matrix with entries in $\mathbb{C}G$ and $\kappa(A)=\Sigma\kappa(a_{ii})$. The image of A in $\mathbb{C}G^n$ is a finitely generated projective $\mathbb{C}G$-module P, and we write also $\kappa(P)$ for $\kappa(A)$, since κ is independent of the imbedding of P in some $\mathbb{C}G^n$.

Kaplansky Theorem. $\kappa(A)$ is a nonnegative real number, equal to 0 only if $A = 0$.

Actually $\kappa(A)$ is known to be rational, but here we will not make use of this.

We recall that the von Neumann algebra $N(G)$ of G can be defined as the algebra of all G-equivariant bounded linear operators on the Hilbert space $\ell_2 G$ of square-summable complex functions on G. A simple proof of the above theorem is obtained by imbedding $\mathbb{C}G$ into $N(G)$; then $\kappa(A)$ can be identified with the von Neumann trace of A. Although the idempotent map defined by A need not be selfadjoint, it is equivalent to a selfadjoint one (orthogonal projection). In other words, $\kappa(A) = \kappa(P)$ is the von Neumann dimension of the Hilbert-G-module $\ell_2 G \otimes_{\mathbb{C}G} P$.

Another notion of trace is given by the augmentation of Σa_{ii}. It is an integer, namely, the dimension of the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathbb{C}G} P$, which we write in short $d(P)$. The *Bass conjecture* says that these two traces are equal:

$$\kappa(P) = d(P). \tag{10}$$

This is the weak form of the conjecture, implied by the strong one, which we do not formulate here; see [B]. It has been proved for several big classes of groups G, such as linear groups, solvable groups (of finite homological dimension), hyperbolic groups, 3-manifold groups, groups of cohomological dimension 2 (over $\mathbb{Q}$). The proof of the simple equation (10) is very indirect and different for the various classes, using arithmetic methods, cyclic homology of groups, homological dimension, etc. (Bass 1976, Eckmann 1986).

Note that for an idempotent $a \in \mathbb{C}G$ the projective P is the left ideal $\mathbb{C}Ga \subset \mathbb{C}G$ and $d(P)$ is necessarily equal to 0 or 1. Thus $\kappa(a) = 0$ or $\kappa(1-a) = 0$, which proves the idempotent conjecture for the respective groups.

Projective Modules over $N(G)$

Beyond the equality (10), i.e., the weak Bass conjecture, one can say more: Through the imbedding of $\mathbb{C}G$ in $N(G)$ the projective module P becomes a finitely generated projective $N(G)$-module $N(G) \otimes_{\mathbb{C}G} P$ that turns out to be a *free* $N(G)$-module of rank equal to $d(P) = \dim_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}G} P$. This can be expressed as an isomorphism of $N(G)$-modules

$$N(G) \otimes_{\mathbb{C}G} P = N(G) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{C}G} P),$$

or more intuitively as an associativity formula

$$(N(G) \otimes_{\mathbb{C}} \mathbb{C}) \otimes_{\mathbb{C}G} P = N(G) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{C}G} P), \tag{11}$$

where everything is purely algebraic. In particular, $N(G)$ is considered just as an algebra.

For the proof, however, one uses the fact that the $N(G)$ is a finite von Neumann algebra and thus admits a center-valued trace: For the projection onto $N(G) \otimes_{\mathbb{C}G} P$ its value is precisely $\kappa(P)$.identity (here a deep result on finite conjugacy classes in G following from [B], Theorem 8.1, is used). If the weak Bass conjecture holds, it is equal to $d(P)$.identity, which is the center-valued trace of the above free module. And projective modules having the same center-valued trace are isomorphic.

[Actually, more is true. The algebraic category of finitely generated projective $N(G)$-modules is equivalent to the category of finitely generated Hilbert-G-modules (and G-equivariant bounded linear operators as morphisms). And if two $N(G)$-projectives have the same center-valued trace, then the corresponding Hilbert-G-modules are isometrically G-isomorphic. The von Neumann dimension can be carried over to finitely generated projective $N(G)$-modules; a special feature is that submodules of finitely generated projective $N(G)$-modules are again projective.]

Conversely, the center-valued traces show that (11) implies the (weak) Bass conjecture for G, $\kappa(P) = d(P)$.

Theorem L′′. The associativity formula (11) is equivalent to the weak Bass conjecture $\kappa(P) = d(P)$ for all finitely generated projective $\mathbb{C}G$-modules.

For many classes of groups, $\kappa(P) = d(P)$; whence (11) is a theorem, and certainly not an easy one, with different proofs according to the respective class. Dare one ask here for a more direct approach to Theorem L′′? For which groups? In what generality?

Return to Topology: Poincaré-2-Complex

First, some very short technical remarks about Hilbert space methods in algebraic topology.

In recent years homotopy invariants of a space X (cell-complexes of finite type with infinite fundamental group G) have been introduced and applied with the help of Hilbert-G-modules: ℓ_2-homology modules (reduced, i.e., cycles modulo the closure of the boundary space) and ℓ_2-Betti numbers (their von Neumann dimension). These concepts actually go back to Atiyah (1976), but were fully developed much later.

In view of the category equivalence above, one gets a purely algebraic approach to all this: G operates as a covering transformation group on the universal covering $\widetilde{X}$ of X; the chain groups of $\widetilde{X}$ are free $\mathbb{Z}G$-modules, and tensoring them over $\mathbb{C}G$ with the algebra $N(G)$, one obtains a complex of finitely generated free $N(G)$-modules. Its homology groups are not projective in general, but finitely presented $N(G)$-modules. Their "projective part", corresponding to reduced ℓ_2-homology, yields the ℓ_2-Betti numbers, and they also yield the Novikov-Shubin invariants (Farber, Lück, 1995).

If G fulfills the Bass conjecture, then the same procedure can be applied to a *finitely dominated* space X, since the chain groups of $\tilde{X}$ are finitely generated projective $\mathbb{Z}G$-modules, and the tensor product over $\mathbb{C}G$ yields free $N(G)$-modules. This is interesting, for example, for Poincaré complexes.

A connected space is called a Poincaré-n-complex if it fulfills the classical Poincaré duality relations well known for closed n-manifolds—an approximation to the latter. We restrict attention to a very special application. It concerns the theorem (the author et al.; cf. the survey [E3]):

A Poincaré-2-complex X with infinite fundamental group G is homotopy equivalent to a closed surface of genus ≥ 1.

No finiteness assumptions are required; X is finitely dominated. An important ingredient in the proof (we mention here only the orientable case) is to show that the first ordinary Betti number $\beta_1(X)$ is ≥ 2. The methods above greatly simplify the argument, as follows.

The ℓ_2-Betti numbers $b_i(X)$ compute the Euler characteristic of the space and fulfill Poincaré duality in the manifold- or Poincaré complex-case, exactly as the ordinary Betti numbers do. The Betti number b_0 in case of an infinite group G is easily seen to be $= 0$. Thus in our situation

$$\chi(X) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1 = -b_1;$$

whence indeed $\beta_1 \geq 2$.

The Poincaré complex above is aspherical; i.e., all homotopy groups in dimensions ≥ 2 are 0. It is thus a classifying space for G, and the homology of G is the same as the homology of X. (The cohomological dimension being 2, the Bass conjecture is fulfilled; this we have already used above implicitly.) Passing to the universal cover of the surface, one can express the result in terms of the group G, yet another linearity statement:

Theorem L'''. A group whose homology fulfills Poincaré duality of dimension 2 is isomorphic to a plane motion group operating freely with compact fundamental domain on the Euclidean or hyperbolic plane.

If a group G is the fundamental group of a closed aspherical n-manifold, then its homology fulfills, of course, Poincaré duality of dimension n. Is the converse true? For dimensions $n \geq 3$ this problem is still unsolved, except for partial results in dimension 3.

References

For the first part we refer to the survey:

[E1] B. Eckmann, Continuous solutions of linear equations, *Expo. Math.* **9** (1991), 351–365, where the relevant references can be found.

Similarly for the second part:

[E2] ____, Hurwitz–Radon matrices revisited, *The Hilton Symposium* 1993 (Montreal, PQ) CRM Proceedings and Lecture Notes, vol. 6, Amer. Math. Soc., Providence, RI, 1994, pp. 23–35.

For the third part we refer to recent papers by Wolfgang Lück, Michael Farber, and the author, and to:

[B] H. Bass, Euler characteristics and characters of discrete groups, *Invent. Math.* **35** (1976), 155–196.

[E3] B. Eckmann, Poincaré duality groups of dimension two are surface groups, *Combinatorial Group Theory and Topology*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1986, pp. 35–51.