Aperiodic Dynamical Systems

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Since the work of Poincaré it has been known that on every smooth closed manifold M of Euler characteristic zero there is a nonsingular smooth vector field. The Euler characteristic equals the alternating sum of the Betti numbers of M, and it is zero for all closed odd-dimensional manifolds by a duality argument. The existence of a nonsingular vector field implies that there is a fixed-point-free map homotopic to the identity. Then by the Lefschetz fixed point formula, the Euler characteristic is zero. Even-dimensional spheres have Euler characteristic two. Thus no even-dimensional sphere admits a nonsingular vector field, which in dimension two is popularly known as the "hairy-ball theorem". It is easy to find specific examples of nonsingular vector fields on the 3-dimensional sphere S^3. One of the most readily recognized is a vector field tangent to the fibers of the Hopf fibration, which decomposes S^3 into copies of S^1. In Figure 1, S^3 is the union of two solid tori $D^2 \times S^1$ with the boundaries identified, and each of the tori can be foliated by circles compatibly on the common boundary of the tori.

Integrating a C^1 vector field gives a *dynamical system*, or a *flow*, on a closed manifold M, i.e., an $\mathbb{R}$-action, where $\mathbb{R}$ is the additive group of the reals. Equivalently, a dynamical system on M is a map $\Phi : \mathbb{R} \times M \to M$ satisfying the following conditions:

1. $\Phi(0, p) = p$,
2. $\Phi(t + s, p) = \Phi(s, \Phi(t, p))$.

The map Φ is in general of the same degree of differentiability as the vector field, although locally on each trajectory the differentiability increases by one. The image of each fiber $\mathbb{R} \times \{p\}$ is called a *trajectory* or an *orbit*. The parameter t is often interpreted as time, and the above two conditions emerge as very natural. The first condition states that the flow starts from the identity on M at time $t = 0$, and the second condition, easily following

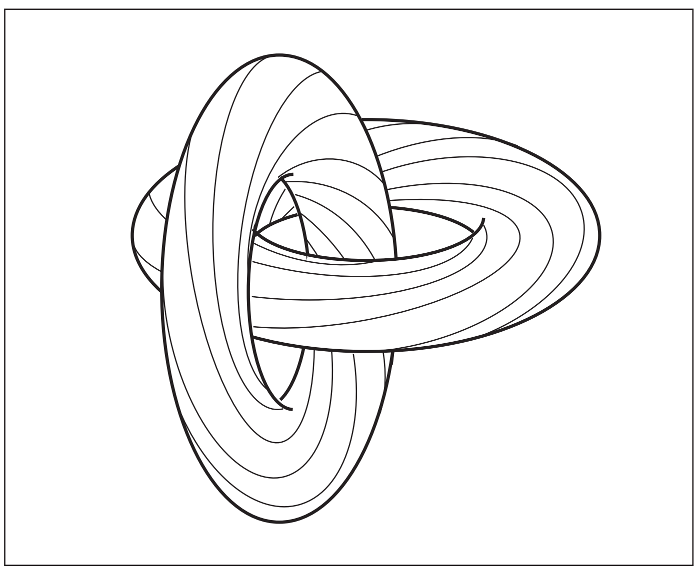

Figure 1. Hopf fibration.

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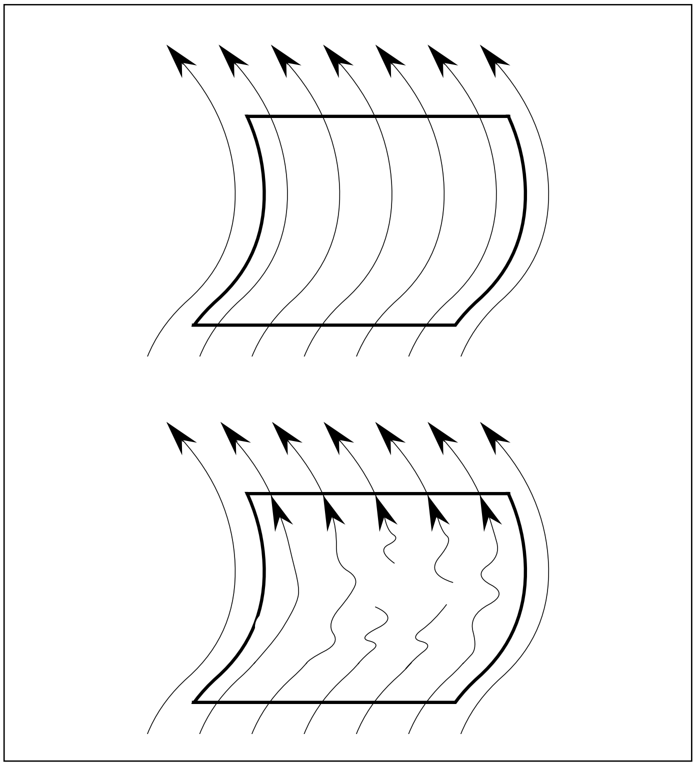

Figure 2. An insertion.

from the integration, is the additive property of time.

A compact orbit is also called *periodic*, and it is either a fixed point or a simple closed curve, a closed orbit. If the orbit is not compact, then it is a one-to-one image of $\mathbb{R}$ and its topological closure is the union of orbits; i.e., it is *invariant*. A dynamical system, respectively a vector field, with no periodic orbits is called *aperiodic*. For example, the so-called "irrational flow" on the n-torus $S^1 \times \cdots \times S^1$, $n \geq 2$, has every orbit dense in the torus, and thus it is aperiodic. A nonempty compact invariant set is *minimal* if it contains no nonempty compact invariant proper subset. A minimal set always exists: it may be a single point, a simple closed curve, or quite a large set. A dynamical system is *minimal* if the only minimal set is the whole underlying space.

The orbits of the vector field tangent to the fibers of the Hopf fibration are the fibers themselves, and all are periodic. A small tilt of the vectors can easily change the vector field so that there is only one compact orbit. In 1950 H. Seifert proved that if $\mathcal{V}$ is a C^1 vector field on S^3 and the angles between the vectors of $\mathcal{V}$ and the circles of the Hopf fibration are sufficiently small, then there is at least one periodic orbit.

Seifert's theorem produced a natural question as to whether every dynamical system on S^3 must possess a compact orbit. An affirmative answer to this question became known as the Seifert Conjecture, and it was believed until a truly beautiful counterexample given in 1974 by P. A. Schweitzer [9]. Schweitzer's construction of an aperiodic dynamical system on S^3 is based on the existence of an aperiodic C^1 vector field on the torus $S^1 \times S^1$ that is not minimal, known as a Denjoy vector field. There are two minimal sets in Schweitzer's vector field on S^3: two copies of the Denjoy set, each embedded in a C^∞ punctured torus $(S^1 \times S^1) - D^2$. It is known that such a vector field cannot be C^2. However, a clever modification by J. Harrison [2] in 1988, not requiring that the minimal sets be embedded in a smooth surface, yields a $C^{2+\delta}$ counterexample to the Seifert Conjecture. The method used by Harrison puts a natural restriction on the differentiability of her example—it cannot be C^3. Neither Schweitzer's nor Harrison's examples resolve a stronger conjecture, the Modified Seifert Conjecture stated in [9] and [10], asserting that: Every dynamical system on S^3 has a minimal set of dimension[1] one or zero. The dimension of the Denjoy minimal set equals one.

A different approach, explored in [7], launched another series of aperiodic examples. As in the examples of Schweitzer and Harrison, a basic building element is a *plug*. In a rather intuitive description, a C^r plug $\mathcal{P}$, $r \leq \infty$, is a nonsingular vector field on the Cartesian product of an $(n-1)$-dimensional compact connected manifold F and the interval $I = [0, 1]$. It is assumed that $F \times I$ can be embedded in $\mathbb{R}^n$ so that all I-fibers $\{f\} \times I$ are straight segments and are parallel. Thus when $n = 3$, F can be any orientable compact surface with nonempty boundary. It is also required that

1. In a neighborhood of the boundary $\partial(F \times I)$, $\mathcal{P}$ is tangent to the fiber I.
2. The interior of $F \times I$ can be identified with a chart U in an n-manifold furnished with a nonsingular vector field $\mathcal{V}$, to replace $\mathcal{V}$ on U with a vector field conjugate to $\mathcal{P}$, in such a way that the resulting vector field $\mathcal{W}$ on M satisfies the following two conditions:

   a. (Matched ends) If a segment A of a trajectory of $\mathcal{W}$ is a subset of U and joins two points on the boundary of U, then A replaces a segment of a trajectory of $\mathcal{V}$.

   b. (Trapped orbit) At least one trajectory of $\mathcal{W}$ enters U at some point and does not leave U at a later time.

Variations of this definition yield other types of plugs such as C^ω (real analytic) or PL (piecewise linear).

A dynamical system without fixed points is locally conjugate to a flow on $\mathbb{R}^n$ generated by a constant nonzero vector field. As illustrated in Figure 2, a plug can replace $\mathcal{V}_{|U}$ with a more complicated vector field $\mathcal{P}$. This procedure, called an *insertion*, can be performed in any desired category: C^r, C^ω,

[1] *The dimension is the covering dimension or, equivalently, the inductive dimension. It is different from the Hausdorff dimension, which plays a significant role in the differentiability restrictions in Harrison's example.*

or PL. An insertion alters the flow locally, but the geometry of the trajectories is changed globally. Condition 2 gives good control over the configuration of trajectories. In particular, if the plug is aperiodic, the insertion does not create new periodic trajectories.

Plugs were first defined by F. W. Wilson in [10] and used to prove that every C^∞ closed n-manifold of Euler characteristic zero admits a C^∞ dynamical system with finitely many minimal sets; each of the minimal sets is an $(n-2)$-torus $S^1 \times \cdots \times S^1$, every trajectory originates and limits on one of these tori, and the flow on each of the tori is minimal. Starting with a nonsingular vector field, he inserted copies of a plug, which he constructed, to capture every orbit so that the only minimal sets are those inside the plugs. Wilson's theorem resolves the Seifert Conjecture for spheres of dimension higher than three, but it does not settle the Modified Seifert Conjecture: all minimal sets are of codimension two. In dimension three the theorem gives the existence of vector fields on closed manifolds with finitely many periodic orbits. Wilson's construction is valid in the C^ω case, although originally it was claimed only to be valid for C^∞.

Similar methods are used in [8] to demonstrate the existence of a flow on $\mathbb{R}^3$ with uniformly bounded orbits and to answer a question posed by S. Ulam: If the diameter of the set of the iterations of a point under a continuous map of a manifold into itself is sufficiently small, does there always exist a fixed point? The time 1 map of the dynamical system on $\mathbb{R}^3$ answers Ulam's question by a counterexample.

Having only finitely many compact trajectories, one can break each one of them with an aperiodic plug matching a segment of the orbit with an orbit trapped in a plug; see Figure 3. This removes the periodicity. The first aperiodic plug and the idea of applying it to break a closed orbit are important contributions due to Schweitzer [9].

It is worth mentioning that for plug insertion, S^3 is a good representative of all orientable closed 3-manifolds. It is also the most intriguing. By the Wallace-Lickorish theorem any closed orientable 3-manifold M can be obtained from any other closed orientable 3-manifold by an integral surgery on a finite link of tori $S^1 \times D^2$. A surgery removes a solid torus $S^1 \times D^2$ from M and puts it back with a twist. The nonorientable manifolds can be treated in a similar fashion. A *twisted plug* is defined analogously to a plug with one change: Condition 1 is relaxed on the side boundary $(\partial F) \times I$ to assume only that $\mathcal{P}$ is tangent to the boundary. Unlike an inserted plug, in which the trajectories go straight up along the side boundary, a twisted plug also has orbits going around the side boundary, as in Figure 4. In [5] G. Kuperberg constructs a C^ω twisted plug and a volume-preserving C^∞

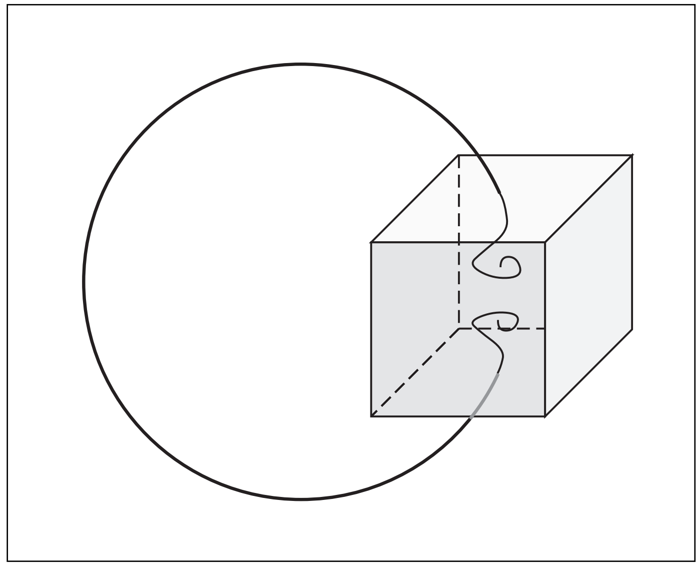

Figure 3. Breaking an orbit.

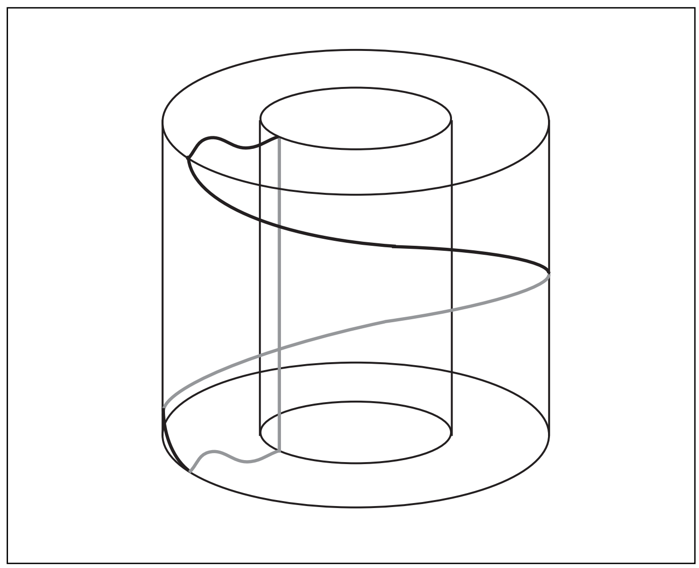

Figure 4. A twisted plug.

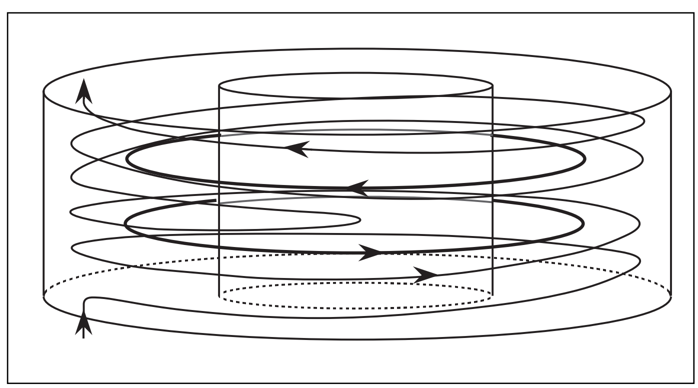

Figure 5. A plug with two closed orbits.

twisted plug, each with two periodic orbits. The plugs are used for surgery compatible with a given vector field. These methods yield an alternate proof of Wilson's theorem and similar results re-

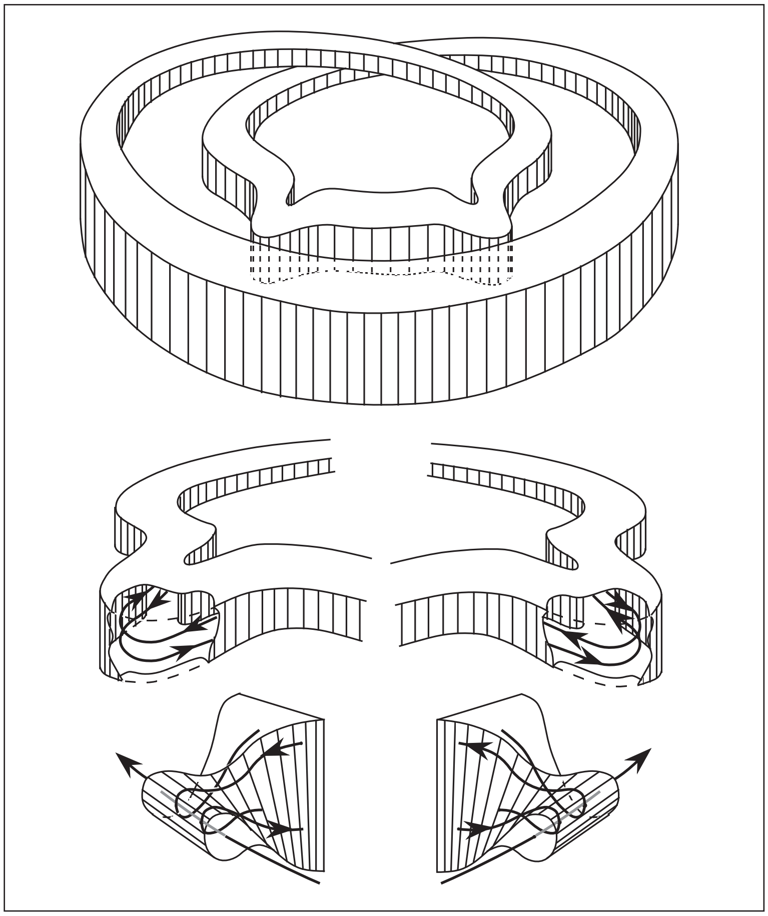

Figure 6. Preparation for a two-part self-insertion.

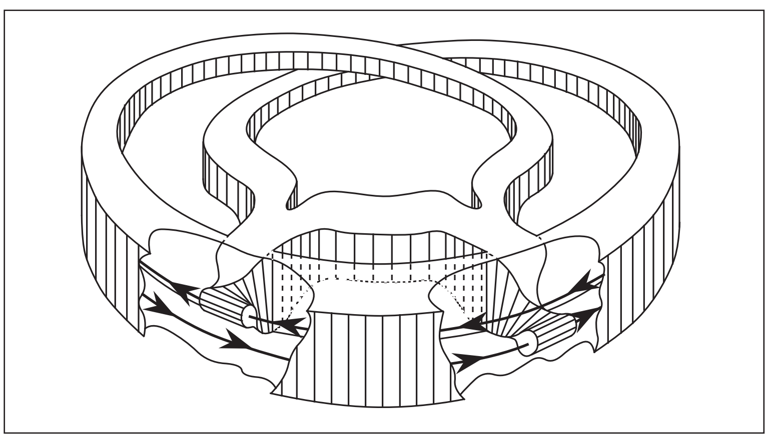

Figure 7. An aperiodic plug.

lated to measure. The author proves that every closed 3-manifold admits a volume-preserving C^∞ nonsingular vector field with at most two periodic orbits. He also constructs a volume-preserving aperiodic C^1 plug, modifying Schweitzer's example, and obtains volume-preserving counterexamples to the Seifert Conjecture. As remarked by S. Stefanov, these volume-preserving flows are ergodic. The main theorem of [5] states:

Theorem 1. *Every closed 3-manifold possesses a C^1 volume-preserving dynamical system with no compact trajectories.*

The above paper contains the only existing applications of plugs designed to alter the topological type of the manifold. Consequently, a dynamical system can be transferred by means of twisted-plug surgery from one 3-manifold onto another exporting some of its properties. For example, a minimal, volume-preserving, dynamical system on the 3-torus $S^1 \times S^1 \times S^1$ transforms to any closed orientable 3-manifold M so as to be a volume-preserving dynamical system with almost every trajectory dense; i.e., the set of dense trajectories is nonempty and open. The question might be, Which manifolds admit a minimal flow, a dynamical system with every trajectory dense? The still open Gottschalk Conjecture asserts that there is no minimal dynamical system on S^3. It is unlikely that the plug method will yield a solution to the Gottschalk Conjecture; there is always a minimal set inside the plug. However, an analogous problem for diffeomorphisms has been solved. In the early 1990s A. Fathi and M. Herman proved that there is a diffeomorphism $\tau : S^3 \to S^3$ such that for each point the set of iterations $\{p, \tau(p), \tau^2(p), \dots\}$ is dense in S^3.

Paper [8] contains a simplification of Wilson's plug, a vector field on the Cartesian product of a planar annulus and the interval. Another slight change makes the trajectories nicely layered on concentric 2-dimensional cylinders; see Figure 5. This plug with two periodic orbits is a starting element for the smooth aperiodic construction given in [7]. The trajectories entering the cylinder underneath the periodic orbit are trapped; they limit on the periodic orbit.

An operation called a *self-insertion* similar to a plug insertion breaks the two periodic orbits with the plug itself; see Figures 6 and 7. Each of the two closed trajectories is matched to a trajectory trapped in the cylinder. The 90-degree turn of each of the inserted parts is to match the vertical vectors on the bottom to the horizontal vectors on the periodic trajectories. The entering trajectories might be broken and take a detour finitely many times, or they might continue this process indefinitely. If a roundabout orbit returns to its old path after deviating several times, it returns to itself at a later time without forming a simple closed curve, and no periodicity is added. This is guaranteed, since each detour is taken on a different outside cylinder.

The trajectories that reenter infinitely many times are those trapped inside the plug. There are in fact many such trajectories, many more than one would expect from following the path of the orbits winding on one of the two original periodic orbits. This unintended occurrence prevents the dynamical system from being volume preserving. In fact, if such a construction is C^1, then it does not preserve measure. Figure 8 illustrates this phenomenon. It shows the relation of the position of the orbits on the rectangular vertical cross-section of Figure 5 to the points of reentry on the bottom curved part. As the orbits approach the circular orbit, indicated here by a large dot, the distance between the reentering points gets smaller. For the trajectory not to be trapped, it is necessary to skip over the insertion, which would then need to be very narrow and sharp, contradicting differentiability.

The reentering trajectories bind the previously periodic (and now broken) orbits in one huge, usually 2-dimensional, minimal set. Thus the main result of [6] is the following:

Theorem 2. *There is a C^ω 3-dimensional plug with exactly one minimal set of dimension two and with no other minimal set.*

Following Schweitzer, one can insert the plug in any 3-manifold with a nonsingular vector field with finitely many closed orbits. In fact, only one plug weaving through the manifold is needed to break all periodic orbits. W. P. Thurston pointed out to the authors of [6] that because of the Morrey-Grauert theorem stating that there is only one real analytic structure on S^3, the methods give a C^ω counterexample to the Seifert Conjecture, including the stronger modified version.

The examples of Schweitzer [9] and Harrison [2] and the one just described all have natural generalizations to higher dimensions. In particular, the following holds (see [6]):

Theorem 3. *If M is a C^∞, C^ω, or* PL *closed manifold of dimension ≥ 3 admitting a fixed-point-free dynamical system in the same smoothness category, then there exists an aperiodic dynamical system on M in the same smoothness category that has exactly one minimal set of codimension one and no other minimal set.*

With the just described methods it is more difficult to obtain an aperiodic plug with one minimal set and that of dimension one. The only such example is a PL plug described in [6]. The minimal set is locally homeomorphic to a Denjoy set, but globally it has a totally different topological structure; for example, it does not embed in a 2-dimensional surface. There is also a similar PL construction in dimension three with the minimal set of dimension two obtained by breaking annuli of trajectories instead of single orbits. The last gives an interesting generalization in [6] for PL flows, i.e., directed 1-foliations, which by definition have no fixed points.

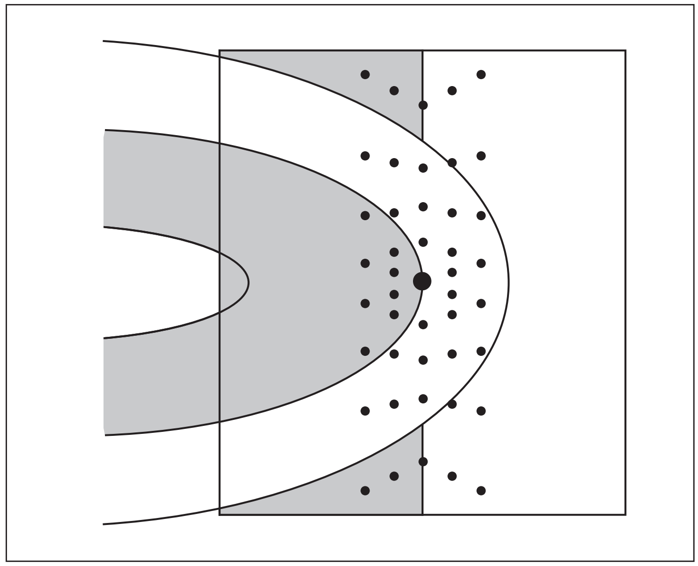

Figure 8. A section of a self-insertion.

Theorem 4. *Let M be a closed* PL *manifold of dimension $n \geq 3$, and let $1 \leq k \leq n-1$. A* PL*-directed 1-foliation of M can be modified in a* PL *fashion so that there are no circular leaves and there is exactly one k-dimensional minimal set and no other minimal set.*

The flexibility of the described self-insertions allows the construction of aperiodic dynamical systems with topologically diverse minimal sets. The simple condition that a trajectory reenters at a different cylinder does not impose many constraints. The self-insertions may be performed in several places and will not destroy aperiodicity provided the simple radius inequality condition of [7] or some equivalent is met. Adjusting the

flows on S^3	not volume-preserving	volume-preserving
discrete circular trajectories	C^ω (F. W. Wilson)	C^∞ (G. Kuperberg)
aperiodic, 1-dimensional minimal sets	C^1 (P. A. Schweitzer) $C^{3-\epsilon}$ (J. Harrison) PL (G. Kuperberg & K. Kuperberg)	C^1 (G. Kuperberg)
2-dimensional minimal sets	C^ω (G. Kuperberg & K. Kuperberg) PL (G. Kuperberg & K. Kuperberg)	—

Table 1. Examples of dynamical systems on S^3.

self-insertion in the direction of the radius results in topologically different flows. (In Figure 8 it would be the horizontal direction.) A complete classification of the minimal sets in these constructions, similar to the classification of the Denjoy sets or the solenoids, is unlikely. Even though examples of aperiodic dynamical systems on S^3 are plentiful, it is difficult to tell at this moment what properties can be achieved. For instance, no aperiodic C^3 dynamical system on S^3 with all minimal sets of dimension one is known.

Suppose that M is a 3-manifold with a volume-preserving dynamical system Φ with no fixed points. Then the total space of the line bundle tangent to Φ has a symplectic structure, and Φ is Hamiltonian with respect to this structure. The volume-preserving dynamical system of [5] gives in this sense a counterexample to the Hamiltonian Seifert Conjecture. However, the classical Hamiltonian Seifert Conjecture for S^3 embedded in $\mathbb{R}^4$ is still unsolved.

By far the strongest result in the direction of proving the existence of a periodic trajectory for a dynamical system on S^3 belongs to H. Hofer [3], who proved the Seifert conjecture for the so-called Reeb vector fields. The book [4] titled *Symplectic Invariants and Hamiltonian Dynamics*, by H. Hofer and E. Zehnder, is an excellent source for learning more about this field.

There are examples of aperiodic Hamiltonian systems in higher dimensions, and some of the constructions involve plugs. The nicest one is V. L. Ginzburg's example of a Hamiltonian on $\mathbb{R}^6$, a counterexample to the Hamiltonian Seifert Conjecture for S^5. In [1] Ginzburg gives a very good survey of the existing counterexamples to the Seifert Conjecture in dimension three and higher, concentrating on Hamiltonian flows.

References

[1] V. L. Ginzburg, Hamiltonian dynamical systems without periodic orbits, *Proceedings of the Berkeley-Davis-Santa Cruz-Stanford Symplectic Geometry Seminar,* Advances in Mathematics (to appear) (Eliashberg, Fuchs, Ratiu, Weinstein, eds.). LANL e-print Archive `math.DG/9811014`; `http://xxx.lanl.gov`.

[2] J. Harrison, C^2 counterexamples to the Seifert Conjecture, *Topology* **27** (1988), 249–278.

[3] H. Hofer, Pseudoholomorphic curves in symplectizations with applications to the Weinstein Conjecture in dimension three, *Invent. Math.* **114** (1993), 515–563.

[4] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser-Verlag, Basel, 1994.

[5] G. Kuperberg, A volume-preserving counterexample to the Seifert Conjecture, *Comment. Math. Helv.* **71** (1996), 70–97.

[6] G. Kuperberg and K. Kuperberg, Generalized counterexamples to the Seifert Conjecture, *Ann. of Math.* **144** (1996), 239–268.

[7] K. Kuperberg, A smooth counterexample to the Seifert Conjecture, *Ann. of Math.* **140** (1994), 723–732.

[8] K. Kuperberg and C. S. Reed, A dynamical system on R^3 with uniformly bounded trajectories and no compact trajectories, *Proc. Amer. Math. Soc.* **106** (1989), 1095–1097.

[9] P. A. Schweitzer, Counterexamples to the Seifert Conjecture and opening closed leaves of foliations, *Ann. of Math.* **100** (1974), 386–400.

[10] F. W. Wilson, On the minimal sets of non-singular vector fields, *Ann. of Math.* **84** (1966), 529–536.