# Variations on the de Rham Complex

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## Introduction

This article is about symmetry and differential geometry. It is about symmetry because there will be symmetry groups. It is about differential geometry because these groups will be acting on manifolds. There are three basic examples, all of them three-spheres. The three-sphere is usually regarded as the vectors in $\mathbb{R}^4$ of unit length. We shall refer to this as the *round sphere*. It is, of course, extremely symmetrical: the rotation group SO(4) acts as isometries. However, this is not one of the three examples!

**A. The projective three-sphere.** This is the three-sphere regarded as the space of rays emanating from the origin in $\mathbb{R}^4$. Its symmetry group is taken to be $\mathrm{SL}(4,\mathbb{R})$, the $4\times 4$ real matrices of unit determinant. This group acts on $\mathbb{R}^4$ by multiplication on column vectors; hence it acts on the three-sphere.

**B. The conformal three-sphere.** This is the three-sphere viewed as the space of lines through the origin in $\mathbb{R}^5$ lying on the cone

$$(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2-(x^5)^2=0. \tag{1}$$

The group SO(4, 1) is the identity connected component of the group of linear transformations of $\mathbb{R}^5$ preserving the quadratic form that is the left-hand side of (1). It preserves the cone and hence acts on the three-sphere.

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**C. The contact three-sphere.** Again, the three-sphere is viewed as the space of rays emanating from the origin in $\mathbb{R}^4$. However, the symmetry group is taken to be $\mathrm{Sp}(2,\mathbb{R})$, the subgroup of $\mathrm{SL}(4,\mathbb{R})$ preserving the standard nondegenerate skew bilinear form.

It is not a coincidence that the symmetry groups in these four examples are from the classical A, B, C series of simple Lie algebras. In fact, these examples are of the form $G/P$ where $G$ is a classical group and $P$ is a so-called *parabolic* subgroup. The idea is to set up differential geometry on these spaces in a way that is invariant under the symmetry group. As often happens when symmetries are combined with calculus, the result is best expressed as a construction in the realm of Lie algebras. In this case we obtain a construction known as the *Bernstein-Gelfand-Gelfand* complex.

The differential operators arising in this way are well known in various contexts. For example, there is a case on the projective three-sphere that is due to Calabi, but it also arises in applied mathematics concerning the theory of elasticity.

Roughly speaking, Klein *defined* a geometry as a space with a group acting on it transitively, and Cartan took this as the *flat model* of a differential geometry. Thus, Euclidean space is the flat model of Riemannian geometry. This scheme works very well for $G/P$. The resulting *parabolic* differential geometries include projective, conformal, and CR geometry.

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like to thank Douglas Arnold for explaining linear elasticity to me. Finally, I would like to thank Tony Knapp for his invaluable help in the preparation of this article and, in particular, for steering me away from early attempts that started with linear elasticity. Though this is a good example, it is consuming of space and notation. Instead, this example has been consigned to a separate article [6].

## An Outline

Some notation is inevitable, but the full extent that would be necessary to make precise everything in this article is severe. Some notation comes from differential geometry and some from representation theory. By restricting to the simplest examples, we can keep the differential geometry to a minimum, but, even so, explicit formulae are relegated to the final section. The representation theory is less easily banished, but only hints to its full glory will be given here.

As the title suggests, the idea is to take the de Rham complex, play with it, and see what happens. The prototype de Rham complex comes from applied mathematics with the familiar differential operators of gradient, curl, and divergence in three dimensions. Since the main examples in this article will all be in three dimensions, this is a good place to start. So, perhaps the zeroth variation is to create the de Rham complex in three dimensions. This really comes down to distinguishing between vector fields, one-forms, and two-forms; also between functions and three-forms. Then the de Rham complex makes sense on a smooth three-manifold.

Once this is done the first variation is a vector-valued de Rham complex. In $\mathbb{R}^3$ such a variation is trivial, but on a manifold such as the round three-sphere this variation leads to the notion of curvature.

The next variation arises by trying to eliminate this curvature. On the round three-sphere the differential operators that give rise to curvature can be modified so that the curvature no longer is visible. At first sight this seems to have strayed from the de Rham complex, but it turns out to be quite close. The precise link gives rise to many more differential operators on the three-sphere invariant under the symmetry groups of Examples A, B, and C. There is a surprise though—the operators are no longer first order in general. Apart from this the story is not too different from the invariance of gradient, curl, and divergence under Euclidean motions.

This is where representation theory enters, because we are looking for operators invariant under symmetry groups. The next surprise and the final variation is that these differential operators persist even when the symmetry group is removed. This final variation is much like the first. It gives rise to various differential geometries in the same way that Euclidean space gives rise to Riemannian differential geometry. This set of variations sheds new light on the Bernstein-Gelfand-Gelfand complex. From now on, Bernstein-Gelfand-Gelfand will be abbreviated to BGG.

## The de Rham Complex

There are at least a dozen different ways of setting up the de Rham complex. This poses a significant problem in exposition of this complex, since several viewpoints are necessary for a good understanding. So the following treatment is just one of many, and, even so, the details are omitted. They can be found in [8]. Those already familiar with the de Rham complex should feel free to skip this section.

If we take $(x^1, x^2, x^3)$ as coördinates on $\mathbb{R}^3$ and write $\partial/\partial x^k = \nabla_k$ for $k = 1, 2, 3$, then the gradient $v$ of a smooth function $u$ is given by

$$v = \operatorname{grad} u = \nabla u = (\nabla_1 u, \nabla_2 u, \nabla_3 u). \tag{2}$$

There is often no harm in regarding the result as a vector field, but this turns out not to be completely natural. Let us postpone for the moment finding a proper home for $\nabla u$ and simply denote the result by its three components $v_k = \nabla_k u$. This notation can cause some discomfort. Some might prefer a *set* of components $\{v_k\}$ or perhaps an *array* $(v_k)$. But, in addition to being evidently cumbersome, none of these notations is really satisfactory for what eventually turns out to be a *tensor field*. So let us just stick with $v_k$ as a *list* of three functions. We would now like to take the curl of $v_k$ whether or not $v_k$ is a gradient. The expression

$$w_{jk} = \nabla_j v_k - \nabla_k v_j \tag{3}$$

is skew in its two indices, namely, $w_{jk} = -w_{kj}$. In three dimensions $w_{jk}$ can be identified with a vector field

$$w_{23}\frac{\partial}{\partial x^1} + w_{31}\frac{\partial}{\partial x^2} + w_{12}\frac{\partial}{\partial x^3},$$

and (3) becomes $w = \operatorname{curl} v$. But (3) makes sense on $\mathbb{R}^n$, in which case $w_{jk}$ has $n(n-1)/2$ independent components. Furthermore, this process may be continued, the next operator being

$$w_{jk} \longmapsto \nabla_i w_{jk} - \nabla_j w_{ik} + \nabla_k w_{ij}. \tag{4}$$

The result is skew in its three indices, so has $n(n-1)(n-2)/6$ components. Three dimensions is special in that (4) can be identified as taking the divergence. The formulae (2), (3), (4) may be continued: if $w_{jk\cdots lm}$ has $p$ indices, then

$$w_{jk\cdots lm} \longmapsto \nabla_i w_{jk\cdots lm} - \nabla_j w_{ik\cdots lm} + \cdots + (-1)^p \nabla_m w_{ijk\cdots l}. \tag{5}$$

We still need a proper home for $w_{jk\cdots lm}$, but already it is clear that if $\Lambda^p$ denotes its eventual home, then (5) will define a sequence of differential operators on $\mathbb{R}^n$ terminating at $\Lambda^n$:

$$\Lambda^0 \xrightarrow{\nabla} \Lambda^1 \xrightarrow{\nabla} \Lambda^2 \xrightarrow{\nabla} \Lambda^3 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1} \xrightarrow{\nabla} \Lambda^n. \tag{6}$$

As in three dimensions, this sequence is a *complex*: the composition of any two consecutive operators is zero. This comes down to the fact that partial derivatives commute: $\nabla_j \nabla_k = \nabla_k \nabla_j$. Once $\Lambda^p$ is defined, (6) is the de Rham complex.

The definition of $\Lambda^p$ is extremely natural when (6) is set up on a smooth manifold. But to define $\Lambda^p$ and carefully establish (6) borders on the tedious. As far as an abstract manifold is concerned, specified by local coördinates and transition functions, the consistency of the definitions boils down to the chain rule. More intuitive is the story for submanifolds of Euclidean space, the round sphere $S^n \subset \mathbb{R}^{n+1}$ being typical. At each point $x \in S^n$ the tangent vectors form a vector space that we will denote by $T_x S^n$. Though $T_x S^n$ is $n$-dimensional, it has no canonical basis and so should not be regarded simply as $\mathbb{R}^n$. Better is to consider the disjoint union

$$TS^n = \coprod T_x S^n \tag{7}$$

as a *vector bundle* on $S^n$. Roughly speaking, a vector bundle $E$ on a smooth manifold $M$ is the smooth assignment of a vector space $E_x$ to each point $x \in M$. If $M$ is connected, then dim $E_x$ is necessarily constant and is called the *rank* of $E$. The *tangent bundle* $TM$ of $M$ generalises (7) and is the prototypical vector bundle. More information may be found in [8]. A *section* $\sigma$ of a vector bundle $E$ is a smooth mapping $\sigma : M \to E$ with $\sigma(x) \in E_x$ for all $x \in M$. For the tangent bundle a section is a vector field and formalises the intuitive notion of the smooth assignment of a tangent vector to each point of the manifold. For any vector bundle $E$ there is the dual vector bundle $E^*$, with $E^*_x$ being the dual of $E_x$ for each $x \in M$. The *cotangent bundle* $T^*M$ is the dual of the tangent bundle. A section of $T^*M$ is traditionally known as a *one-form* and, in local coördinates, is usually written

$$v_1\, dx^1 + v_2\, dx^2 + \cdots + v_n\, dx^n.$$

Here $\{dx^k\}$ are locally defined sections of $T^*M$ determined by the local coördinates: at any point they give a basis of the cotangent space dual to that given by $\{\partial/\partial x^k\}$ in the tangent space at the same point.

The punchline is that if $u$ is a smooth function on $M$, then $\nabla u$ should be regarded as a one-form. There are several ways of seeing this, the most naïve of which is to check that an arbitrary smooth change of coördinates $x^j \mapsto y^k$ gives rise to a change of basis

$$dy^k = \sum_j \frac{\partial y^k}{\partial x^j} dx^j \tag{8}$$

for the cotangent spaces. The chain rule then implies that $\nabla u = \sum_k \nabla_k u\, dx^k$ is coördinate-free.

More generally, the vector bundle $\wedge^p T^*M$ associates the $p$-th exterior product $\wedge^p (T_x M)^*$ to $x \in M$, and its sections are called $p$-forms. When a particular smooth manifold is understood, let us write $\Lambda^p$ for the $p$-forms. A zero-form is simply a smooth function. The de Rham complex (6) now makes good sense on any smooth manifold. In this context $\nabla$ is called the *exterior derivative*. In summary, the exterior derivative makes sense of (5) independent of choice of local coördinates.

On any smooth manifold (6) is a complex. Locally, more is true. For any open subset $U \subseteq M$ let us write $\Gamma(U, \Lambda^p)$ for the $p$-forms on $U$.

***Theorem 1.*** *If $B \subseteq M$ is a ball in some local coördinate patch, then*

$$0 \to \mathbb{R} \to \Gamma(B, \Lambda^0) \xrightarrow{\nabla} \Gamma(B, \Lambda^1) \xrightarrow{\nabla} \Gamma(B, \Lambda^2) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Gamma(B, \Lambda^n) \to 0$$

*is exact, i.e. the kernel of each mapping is the image of the one that precedes it.*

This result is often called the Poincaré Lemma.

## Type A: The Projective Three-Sphere

Suppose we start with a vector field

$$u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2} + \cdots + u^n \frac{\partial}{\partial x^n}$$

on $\mathbb{R}^n$ instead of a function. We could take each of its coefficients and build $n$ separate de Rham complexes

$$u^m \mapsto \nabla_k u^m, \quad v_k{}^m \mapsto \nabla_j v_k{}^m - \nabla_k v_j{}^m, \quad \text{and so on}$$

for $m = 1, 2, \ldots, n$. However, the coefficients should not really be separated. Instead, we should view the result as a vector-valued de Rham complex. To this end, on a smooth manifold $M$ let us write $\Lambda^p(T)$ for the sections of $\wedge^p T^*M \otimes TM$, the vector bundle that associates to each $x \in M$, the vector space $\wedge^p (T_x M)^* \otimes T_x M$. Then, in the special case that $M = \mathbb{R}^n$, we obtain a complex

$$\Lambda^0(T) \xrightarrow{\nabla} \Lambda^1(T) \xrightarrow{\nabla} \Lambda^2(T) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1}(T) \xrightarrow{\nabla} \Lambda^n(T) \tag{9}$$

and a vector-valued version of Theorem 1. In setting up Riemannian differential geometry, one of the first tasks is to find a suitable analogue of (9). This is the first variation on the de Rham sequence alluded to in the "Outline". It is more difficult than the de Rham complex because the differential operators lose some invariance. We will return to this point in the final section.

There is another significant variation, even when $M = \mathbb{R}^n$. It arises because $\Lambda^1(T)$ naturally decomposes. In fact, for any $n$-dimensional real vector space $V$ we have

$$V^* \otimes V = \mathrm{End}(V) = \mathrm{End}_\circ(V) \oplus \mathbb{R},$$

where $\mathrm{End}_\circ(V)$ denotes the trace-free endomorphisms of $V$. Explicitly,

$$A = \underbrace{A - \left(\frac{1}{n}\text{ trace } A\right)\mathrm{Id}}_{\text{trace-free}} + \left(\frac{1}{n}\text{ trace } A\right)\mathrm{Id}$$

for $A \in \mathrm{End}(V)$. Applying this decomposition with $V = T_xM$, let us write

$$\Lambda^1(T) = \Lambda^1_\circ(T) \oplus \Lambda^0 \tag{10}$$

for the corresponding point-by-point decomposition of $\Lambda^1(T)$. Then it is natural to decompose $\nabla_k u^m$ and, in particular, to consider

$$u^m \mapsto \nabla_k u^m - \left(\frac{1}{n}\sum_p \nabla_p u^p\right)\delta_k{}^m \tag{11}$$

with image in $\Lambda^1_\circ(T)$. Here $\delta_k{}^m$ is the Kronecker delta:

$$\delta_k{}^m = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m. \end{cases}$$

Representation theory is beginning to show itself. The splitting (10) is best viewed as the canonical decomposition of $T^*_xM \otimes T_xM$ into invariant subspaces under the general linear group of $T_xM$. There is a similar canonical decomposition

$$\Lambda^2(T) = \Lambda^2_\circ(T) \oplus \Lambda^1 \tag{12}$$

where

$$w_{jk}{}^m \in \Lambda^2_\circ(T) \iff \sum_p w_{pk}{}^p = 0.$$

Let us now suppose $n \geq 3$, since otherwise $\Lambda^2_\circ(T)$ vanishes. We obtain a natural differential operator $\nabla : \Lambda^1_\circ(T) \to \Lambda^2_\circ(T)$ given explicitly by

$$v_k{}^m \mapsto \nabla_j v_k{}^m - \nabla_k v_j{}^m + \frac{1}{n-1}\sum_p \left(\nabla_p v_j{}^p \delta_k{}^m - \nabla_p v_k{}^p \delta_j{}^m\right). \tag{13}$$

Now it is easily verified that the composition

$$\Lambda^0(T) \xrightarrow{\nabla} \Lambda^1_\circ(T) \xrightarrow{\nabla} \Lambda^2_\circ(T)$$

vanishes. In fact, on a ball it is exact. This suggests that the above sequence might be the start of a complex very much like the de Rham complex. This is true, but it is quite surprising that on $\mathbb{R}^3$ the next and final differential operator is *second* order:

$$w_{jk}{}^m \xmapsto{\nabla^{(2)}} \sum_m \nabla_m(\nabla_i w_{jk}{}^m - \nabla_j w_{ik}{}^m + \nabla_k w_{ij}{}^m). \tag{14}$$

In the case of $M = \mathbb{R}^3$ we obtain a complex

$$\Lambda^0(T) \xrightarrow{\nabla} \Lambda^1_\circ(T) \xrightarrow{\nabla} \Lambda^2_\circ(T) \xrightarrow{\nabla^{(2)}} \Lambda^3 \tag{15}$$

that resembles the de Rham complex and is exact on a ball. It is a particular BGG complex. The main aim of this article is to explain how complexes such as this arise. Already the formulae (11), (13), (14) are quite complicated. Fortunately such explicit formulae can be avoided with a more advanced viewpoint. We shall soon see that BGG complexes arise quite naturally on the spaces given in the three examples at the beginning of this article. Explicit formulae are not needed for the general theory but can always be obtained by writing the differential operators in suitable local coördinates.

The particular example (15) is obtained in this way from a complex on the projective three-sphere $S^3$. Here $S^3$ is regarded as the space of rays emanating from the origin in $\mathbb{R}^4$ as suggested in the introduction. Certainly we have the de Rham complex on $S^3$. On the projective three-sphere, however, we also have the $\mathbb{R}^4$-valued de Rham sequence

$$\Lambda^0(\mathbb{R}^4) \xrightarrow{\nabla} \Lambda^1(\mathbb{R}^4) \xrightarrow{\nabla} \Lambda^2(\mathbb{R}^4) \xrightarrow{\nabla} \Lambda^3(\mathbb{R}^4), \tag{16}$$

where $\Lambda^p(\mathbb{R}^4)$ denotes the sections of $\wedge^p T^*S^3 \otimes \mathbb{R}^4$ over some open subset of $S^3$ (which is understood). We shall derive (15) from (16). The point is that $\mathrm{SL}(4,\mathbb{R})$ is acting, not only on $S^3$, but also on (16). More specifically, a vector-bundle $E$ on $S^3$ is said to be *homogeneous* under $\mathrm{SL}(4,\mathbb{R})$ if for every $g \in \mathrm{SL}(4,\mathbb{R})$ and $x \in S^3$ there is a linear transformation

$$\rho(g) : E_x \to E_{gx} \text{ so that } \rho(gh) = \rho(g)\rho(h). \tag{17}$$

In particular, the trivial vector bundle $S^3 \times \mathbb{R}^4$, associating the same vector space $\mathbb{R}^4$ to every point of $S^3$, may be viewed as a nontrivial homogeneous vector bundle by decreeing that $\mathrm{SL}(4,\mathbb{R})$ act in its usual way by left multiplication on column vectors. There is a natural action of $\mathrm{SL}(4,\mathbb{R})$ on $TS^3$ simply obtained by differentiating the action of $\mathrm{SL}(4,\mathbb{R})$ on $S^3$. This induces an action on $T^*S^3$ and more generally on $\wedge^p T^*S^3$. It is by combining all these natural actions that (16) should be viewed. Moreover, since the exterior derivative respects coördinate changes, the $\mathbb{R}^4$-valued exterior derivative in (16) respects the action of $\mathrm{SL}(4,\mathbb{R})$.

Let us take as basepoint $b \in S^3$ the positive $x^1$-axis. Its stabilizer is the subgroup

$$P = \left\{ \begin{pmatrix} \lambda & * & * & * \\ 0 & & & \\ 0 & & A & \\ 0 & & & \end{pmatrix} \text{ s.t. } \begin{array}{l} \lambda > 0 \\ \lambda \det A = 1 \end{array} \right\}$$

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\Lambda^0(-1) & & \Lambda^1(-1) & & \Lambda^2(-1) & & \Lambda^3(-1)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\Lambda^0(\mathbb{R}^4) & \xrightarrow{\nabla} & \Lambda^1(\mathbb{R}^4) & \xrightarrow{\nabla} & \Lambda^2(\mathbb{R}^4) & \xrightarrow{\nabla} & \Lambda^3(\mathbb{R}^4)\,.\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\Lambda^0(T)(-1) & & \Lambda^1_\circ(T)(-1)\oplus\Lambda^0(-1) & & \Lambda^2_\circ(T)(-1)\oplus\Lambda^1(-1) & & \Lambda^2(-1)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & & 0 & & 0 & & 0
\end{array}$$

**Figure 1.**

of $\mathrm{SL}(4,\mathbb{R})$. If we take $x = b$ and $g, h \in P$, then (17) says that $\rho$ is a *representation* of $P$, i.e., a homomorphism of $P$ into a general linear group. Conversely, it is easy to reconstruct the entire homogeneous vector bundle on $S^3$ from such a representation. Intuitively, the only obstruction to defining $\rho$ arbitrarily on $\mathrm{SL}(4,\mathbb{R})$ in (17) is what happens for group elements that fix $b$, and the representation property on $P$ handles them.

***Theorem 2.*** *There is a one-one correspondence between* $\mathrm{SL}(4,\mathbb{R})$ *-homogeneous vector bundles on* $S^3$ *and finite-dimensional representations of* $P$.

The standard action of $P$ on $\mathbb{R}^4$ by left multiplication on column vectors is responsible for the trivial $\mathbb{R}^4$-bundle on $S^3$. But this representation decomposes: the $x^1$-axis is a one-dimensional invariant subspace. The natural representation on the quotient $\mathbb{R}^4/x^1$-axis is three-dimensional and irreducible. We obtain, in accordance with Theorem 2, a corresponding sequence of bundles on $S^3$ and bundle maps

$$0 \to L \to S^3 \times \mathbb{R}^4 \to Q \to 0, \tag{18}$$

where $L$ and $Q$ are homogeneous vector bundles of ranks 1 and 3 respectively. The sections of $L$ can be described more concretely. A function on $\mathbb{R}^4 \setminus \{0\}$ is said to be *homogeneous* of degree $d$ if and only if

$$f(\lambda x^1, \lambda x^2, \lambda x^3, \lambda x^4) = \lambda^d f(x^1, x^2, x^3, x^4)$$
$$\text{for } \lambda > 0.$$

Such a function is determined by its restriction to the unit sphere. Let us write $\Gamma(S^3, \Lambda^0(d))$ for the smooth functions on the three-sphere extended to be homogeneous of degree $d$ on $\mathbb{R}^4 \setminus \{0\}$. It is important to include $d$ in the notation because the action of $\mathrm{SL}(4,\mathbb{R})$ sees the homogeneity. A careful unravelling of definitions shows that sections of $L$ can be identified with functions homogeneous of degree $-1$, thus with the space $\Lambda^0(-1)$ (over some open subset of $S^3$). Similarly, $Q_x$ can be identified as $T_xS^3 \otimes L_x$ for all $x \in S^3$. Therefore, let us write $\Lambda^0(T)(-1)$ for the sections of $Q$ (over some open subset of $S^3$). Then (18) induces an exact sequence on sections

$$0 \to \Lambda^0(-1) \xrightarrow{\iota} \Lambda^0(\mathbb{R}^4) \to \Lambda^0(T)(-1) \to 0 \tag{19}$$

known as the *Euler* sequence. The other terms in (16) decompose similarly:

$$0 \to \Lambda^p(-1) \to \Lambda^p(\mathbb{R}^4) \to \Lambda^p(T)(-1) \to 0$$

and in case $p = 1, 2$ even further. In fact, we have already seen these further splittings in equations (10) and (12). We obtain

$$0 \to \Lambda^1(-1) \to \Lambda^1(\mathbb{R}^4) \to \Lambda^1_\circ(T)(-1) \oplus \Lambda^0(-1) \to 0$$

and

$$0 \to \Lambda^2(-1) \to \Lambda^2(\mathbb{R}^4) \to \Lambda^2_\circ(T)(-1) \oplus \Lambda^1(-1) \to 0.$$

Similarly, when $p = 3$, we can rewrite the last term $\Lambda^3(T)(-1)$ as $\Lambda^2(-1)$ to obtain

$$0 \to \Lambda^3(-1) \to \Lambda^3(\mathbb{R}^4) \to \Lambda^2(-1) \to 0.$$

In summary, each of the terms in (16) decomposes under the action of $\mathrm{SL}(4,\mathbb{R})$, and we can assemble the result into the diagram of Figure 1. This is entirely automatic from the action of $\mathrm{SL}(4,\mathbb{R})$ on the $\mathbb{R}^4$-valued de Rham complex (16). Now observe that $\Lambda^0(-1)$, $\Lambda^1(-1)$, and $\Lambda^2(-1)$ appear twice in this diagram. Not only that, but there are mappings between them:

$$\Lambda^p(-1) \to \Lambda^p(\mathbb{R}^4) \xrightarrow{\nabla} \Lambda^{p+1}(\mathbb{R}^4) \to \Lambda^p(-1).$$

These mappings all turn out to be the identity! A diagram chase yields the complex

$$\Lambda^0(T)(-1) \xrightarrow{\nabla} \Lambda^1_\circ(T)(-1) \xrightarrow{\nabla} \Lambda^2_\circ(T)(-1) \xrightarrow{\nabla^{(2)}} \Lambda^3(-1) \tag{20}$$

as a replacement for (16). When the many details of this sketch are completed, several conclusions emerge. In particular, the intricacies of the diagram chase yield differential operators with orders as shown by superscripts on $\nabla$. And it really is a replacement for (16): since (16) is a complex, so is (20). Not only that, but the $\mathbb{R}^4$-valued version of Theorem 1 gives

***Theorem 3.*** *If* $B \subset S^3$ *is a ball in some local coördinate patch, then*

$$0 \to \mathbb{R}^4 \to \Gamma(B, \Lambda^0(T)(-1)) \xrightarrow{\nabla} \Gamma(B, \Lambda^1_\circ(T)(-1)) \xrightarrow{\nabla} \Gamma(B, \Lambda^2_\circ(T)(-1)) \xrightarrow{\nabla^{(2)}} \Gamma(B, \Lambda^3(-1)) \to 0$$

*is exact.*

Furthermore, $\mathrm{SL}(4,\mathbb{R})$ acts on (20), and the three differential operators in (20) are invariant under this action. Finally, (20) agrees with (15) when viewed in suitable local coördinates. Specifically, this happens if we identify a hemisphere of $S^3$ with $\mathbb{R}^3$ by central projection

$$\mathbb{R}^3 \ni (x^1, x^2, x^3) \longmapsto \frac{(1, x^1, x^2, x^3)}{\sqrt{1 + (x^1)^2 + (x^2)^2 + (x^3)^2}} \in S^3. \tag{21}$$

We will come back to this point later.

To recapitulate, the main feature is that when the $\mathbb{R}^4$-valued de Rham sequence (16) is viewed under the action of $SL(4, \mathbb{R})$, there are some automatic cancellations that result in (20). It is then a matter of calculation in suitable local coördinates to obtain (15).

Representation theory provides these automatic cancellations. Without going into any details, the point is that invariant differential operators are equivalent to homomorphisms of so-called *induced modules*, constructed from the Lie algebras of $SL(4, \mathbb{R})$ and its subgroup $P$. So, the $\mathbb{R}^4$-valued de Rham complex (16) is equivalent to a complex of induced modules, and the required cancellations are immediate from a theorem of Harish-Chandra concerning the action of the centre of the universal enveloping algebra of the Lie algebra of $SL(4, \mathbb{R})$.

This construction works starting with *any* finite-dimensional irreducible representation of $SL(4, \mathbb{R})$. The result is a BGG complex, and this method of constructing such complexes is essentially due to G. J. Zuckerman. As another example, the representation $\wedge^2\mathbb{R}^4$ gives rise to a complex on the projective three-sphere whose differential operators, when viewed in the local coördinates (21), are as follows:

$$\begin{array}{lll} u_m & \longmapsto & \nabla_l u_m + \nabla_m u_l \\ v_{lm} & \longmapsto & \nabla_j \nabla_k v_{lm} - \nabla_l \nabla_k v_{jm} \\ & & \quad - \nabla_j \nabla_m v_{lk} + \nabla_l \nabla_m v_{jk} \\ w_{jklm} & \longmapsto & \nabla_i w_{jklm} - \nabla_j w_{iklm} + \nabla_l w_{ikjm}. \end{array}$$

These operators arise also in applied mathematics. The middle operator, which is second order, relates strain and stress in the linear theory of elasticity [6]. As a complex on $S^3$, however, this example (constructed in this way) is due to Calabi [3].

Further examples are postponed pending a better notation. But already we have encountered the key point, namely, that the many pleasing properties of the BGG complex may be seen as inherited from the de Rham complex.

### Type B: The Conformal Three-Sphere

The story for the conformal three-sphere is much the same as for the projective three-sphere, except that the simplest example past the de Rham complex is a little more complicated. These complications arise from the counterpart to the Euler sequence (19). If we view (19) as a filtration of $\Lambda^0(\mathbb{R}^4)$, then the corresponding filtration on the conformal three-sphere is one stage longer, having four terms rather than three.

A function on $\mathbb{R}^5 \setminus \{0\}$ is said to be *homogeneous* of degree $d$ if and only if

$$f(\lambda x^1, \lambda x^2, \lambda x^3, \lambda x^4, \lambda x^5) = |\lambda|^d f(x^1, x^2, x^3, x^4, x^5) \quad \text{for } \lambda \neq 0.$$

The lines in (1) can be identified with the three-sphere by intersecting with the hyperplane $\{x^5 = 1\}$. As before, a homogeneous function is determined by its restriction to the three-sphere. This notion of homogeneity is different from that on the projective sphere, and so we will write $\Lambda^0[d]$ to distinguish these local functions as seen by $SO(4, 1)$. As with (19) there is an inclusion

$$0 \to \Lambda^0[-1] \xrightarrow{\iota} \Lambda^0(\mathbb{R}^5) \tag{22}$$

but also a projection

$$\Lambda^0(\mathbb{R}^5) \xrightarrow{\pi} \Lambda^0[1] \to 0\,, \tag{23}$$

and now the vector fields on the three-sphere are identified by

$$\frac{\ker \pi}{\operatorname{im} \iota} = \Lambda^0(T)[-1]. \tag{24}$$

In other words, (22), (23), and (24) constitute a filtration of $\Lambda^0(\mathbb{R}^5)$ that is one stage longer than the filtration of $\Lambda^0(\mathbb{R}^4)$ given in (19). In fact, this makes little difference to our previous reasoning. If we start with the standard representation of $SO(4, 1)$ on $\mathbb{R}^5$, then we encounter a series of cancellations leaving a complex on the three-sphere whose differential operators are invariant under the action of $SO(4, 1)$.

We do not yet have the notation to describe the vector bundles on $S^3$ that appear in this complex. But we can write the differential operators in local coördinates. In this case, the best choice of local coördinates is obtained by stereographic projection. The reason is that the group $SO(4, 1)$ is acting on $S^3$ by conformal transformations, and stereographic projection is a conformal mapping. We will come back to this point later. Here are the differential operators in these local coördinates:

$$\begin{array}{lll} u & \longmapsto & \nabla_k \nabla_l u - \left(\frac{1}{3} \sum_p \nabla_p \nabla_p u\right) \delta_{kl} \\ v_{kl} & \longmapsto & \nabla_j v_{kl} - \nabla_k v_{jl} \\ & & \quad + \frac{1}{2} \sum_p \left(\nabla_p v_{jp} \delta_{kl} - \nabla_p v_{kp} \delta_{jl}\right) \\ w_{jkl} & \longmapsto & \sum_l \nabla_l \left(\nabla_i w_{jkl} - \nabla_j w_{ikl} + \nabla_k w_{ijl}\right). \end{array}$$

### Type C: The Contact Three-Sphere

By its definition the contact three-sphere is like the projective three-sphere but is equipped with some extra structure derived from the nondegenerate skew form $\omega$ on $\mathbb{R}^4$. This form makes itself felt in combination with the Euler sequence (19). There is a projection

$$\begin{array}{ccccc} \Lambda^0(\mathbb{R}^4) & \xrightarrow{\pi} & \Lambda^0(1) & \to & 0 \\ \cup & & \cup & & \\ X(x) & \longmapsto & \omega(x, X(x)) & & \end{array}$$

parallel to (23) and therefore, from (19), a vector subbundle $D \subset TS^3$ so that

$$\frac{\ker \pi}{\operatorname{im} \iota} = \Lambda^0(D)[-1]$$

parallel to (24). This is the natural filtering of $\Lambda^0(\mathbb{R}^4)$ automatically arising from the action of $\mathrm{Sp}(2, \mathbb{R})$.

A new feature of this example is that the de Rham complex itself reduces. The usual reasoning applied to the trivial representation of $\mathrm{Sp}(2, \mathbb{R})$ gives

$$\Lambda^0 \xrightarrow{\nabla} \Lambda^0(D^*) \xrightarrow{\nabla^{(2)}} \Lambda^3(D) \xrightarrow{\nabla} \Lambda^3 \tag{25}$$

as a replacement for the de Rham complex. This is the simplest of the BGG complexes on the contact three-sphere. Even in this simplest of examples, one of the differential operators is second order.

## Representation Theory

The spaces we have been considering in our various examples are of the form $G/P$, where $G$ is a simple Lie group and $P$ is a parabolic subgroup. This is a general setting in which BGG complexes may be constructed. With our examples in mind, however, it is not necessary to explore the general definitions to get a reasonable idea of what is going on. All phenomena of interest occur in the examples.

Theorem 2 is valid for any homogeneous space whatsoever. In particular, the BGG complex consists of *irreducible* homogeneous vector bundles, corresponding to *irreducible* representations of $P$. The differential operators between the bundles are invariant under the action of $G$.

There is a notation that records the symmetry group $G$, the stabiliser group $P$, and an irreducible representation of $P$. It starts from the Dynkin diagram of the complexified Lie algebra of the group $G$. We do not propose to define Dynkin diagrams in general, but will be content with saying what they are for our three examples. For $G = \mathrm{SL}(4, \mathbb{R})$, the diagram has three nodes connected by single lines:

$$\bullet\!-\!\bullet\!-\!\bullet.$$

For $\mathrm{SO}(4, 1)$ the diagram has two nodes connected by a double line pointing to the right:

$$\bullet\!\Rightarrow\!\bullet.$$

For $\mathrm{Sp}(2, \mathbb{R})$, the diagram again has two nodes connected by a double line, but we shall view the arrow as pointing to the left.

The parabolic subgroup $P$ is specified by telling what nodes do not contribute to it, and we mark a cross through these nodes to indicate $P$. Finally, the finite-dimensional irreducible representations of $P$ are parametrised by their highest weights, which may be regarded as a system of real numbers, one for each node, with the numbers that are attached to the uncrossed nodes being nonnegative integers. There are some subtleties with the notation that need not be discussed here; details may be found in [2]. Here are examples of this notation for the projective three-sphere:

$$\overset{d}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = \Lambda^0(d) \qquad \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} = \Lambda^0(T) \qquad \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} = \Lambda^1.$$

Here are some examples on the conformal three-sphere:

$$\overset{d}{\times}\!\Rightarrow\!\overset{0}{\bullet} = \Lambda^0[d] \qquad \overset{0}{\times}\!\Rightarrow\!\overset{2}{\bullet} = \Lambda^0(T) \qquad \overset{-2}{\times}\!\Rightarrow\!\overset{2}{\bullet} = \Lambda^1.$$

On the contact three-sphere, vector fields are reducible. Instead:

$$\overset{d}{\times}\!\Leftarrow\!\overset{0}{\bullet} = \Lambda^0(d) \qquad \overset{0}{\times}\!\Leftarrow\!\overset{1}{\bullet} = \Lambda^0(D) \qquad \overset{-4}{\times}\!\Leftarrow\!\overset{0}{\bullet} = \Lambda^3.$$

The de Rham complex on the projective three-sphere is

$$\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}.$$

The complex (20) becomes

$$\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet} \xrightarrow{\nabla^{(2)}} \overset{-5}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

and the general BGG complex is as in Figure 2 for nonnegative integers $a, b, c$. It is constructed from the de Rham complex and the irreducible representation of $\mathrm{SL}(4, \mathbb{R})$ with highest weight given by $\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}$. The Calabi complex comes from $\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$. There is a general counterpart to Theorem 3 with exact sequence

$$0 \to \overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet} \to \Gamma(B, \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \longrightarrow \Gamma(B, \overset{-a-2}{\times}\!-\!\overset{a+b+1}{\bullet}\!-\!\overset{c}{\bullet}) \longrightarrow \cdots$$

for $B \subset S^3$ a ball.

The general BGG complex on the conformal three-sphere is indicated in Figure 3, and on the contact three-sphere the general BGG complex is as in Figure 4. The particular cases discussed earlier were $a = 1, b = 0$ on the conformal three-sphere, and $a = b = 0$ on the contact three-sphere.

## Consequences

To reiterate the theme of this article, the BGG complex is analogous to, and may be derived from, the de Rham complex. This derivation has several consequences. We have already seen in Theorem 3 a counterpart to the Poincaré Lemma (Theorem 1), and there is a similar counterpart for the general BGG complex. There are at least two other properties of the de Rham complex that one hopes will carry over to make the BGG complex a true replacement.

The first concerns the Helmholtz decomposition. Classically, this says that a smooth vector field on $\mathbb{R}^n$ may be written as the sum of a gradient $\nabla f$ and a field with vanishing divergence. This decomposition is unique if we insist that $f$ tends to

0 at infinity. On a smooth compact manifold, there are similar decompositions of the smooth $p$-forms known collectively as Hodge theory. The full extent of these decompositions for the BGG complex is not clear, but certainly there are analogues of the classical Helmholtz decomposition (for example, Theorem 2.6.2 of [10]).

The second concerns the de Rham Theorem that simplicial cohomology agrees with de Rham cohomology on a smooth manifold. This is very much related to the piecewise linear approximation of smooth differential forms and, in the applied mathematical setting, to finite element schemes. There are only glimpses of a corresponding theory for the BGG complex.

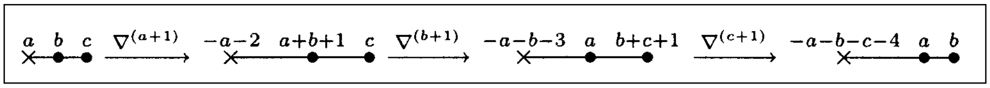

**Figure 2.**

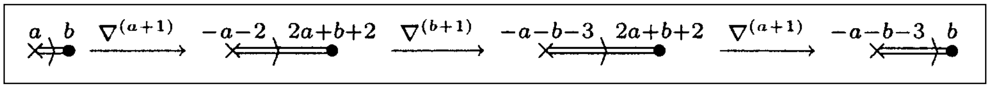

**Figure 3.**

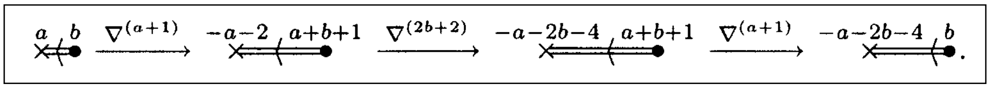

**Figure 4.**

## Parabolic Differential Geometry

Up until now we have been concerned with highly symmetrical manifolds, namely, those of the form $G/P$ for $P$ a parabolic subgroup of a simple Lie group $G$. It turns out that much of what we have constructed persists in a considerably more general setting. The term *parabolic geometries* was coined by Graham [7] following C. Fefferman's usage *parabolic invariant theory* in the CR case. At that stage it was clear that there were many examples. However, the final definition of a parabolic geometry corresponding to any $G/P$ is due to Čap and Schichl [4]. Here we will consider just a few examples. Some differential geometry will be assumed.

Consider the conformal four-sphere, built from SO(5, 1) acting on $\mathbb{R}^6$ in a fashion similar to the action of SO(4, 1) on $\mathbb{R}^5$ in Example B. As is often the case when the orthogonal group is involved, the even-dimensional spheres behave differently from the odd-dimensional. This difference is already apparent in the de Rham complex. It so happens that the homogeneous vector bundle $\wedge^2 T^* S^4$ splits into two homogeneous subbundles, each of rank 3. The two-forms split accordingly, and the de Rham complex reads

$$\begin{array}{ccccc} & & \Lambda^2_+ & & \\ & \nearrow & & \searrow & \\ \Lambda^0 \to \Lambda^1 & & \oplus & & \Lambda^3 \to \Lambda^4. \\ & \searrow & & \nearrow & \\ & & \Lambda^2_- & & \end{array} \tag{26}$$

The general BGG complex on the conformal four-sphere has the same shape. The Hodge $\star$-operator on an oriented Riemannian four-manifold sends two-forms to two-forms and has $\star^2 = 1$. Hence, the two-forms split

$$\begin{aligned} \Lambda^2 &= \Lambda^2_+ \oplus \Lambda^2_- \\ &= \{\nu \mid \star\,\nu = \nu\} \oplus \{\nu \mid \star\,\nu = -\nu\} \end{aligned}$$

into self-dual and anti-self-dual types. Moreover, the $\star$-operator is *conformally invariant* in this middle dimension, i.e., is unchanged if the metric is scaled by an arbitrary positive smooth function. So, a conformal structure (a metric defined only up to scale) is all that is needed to have (26). Conformal differential geometry is the parabolic differential geometry associated to Example B and to this example, which corresponds to classical type D.

The complex (25) of Example C persists on an arbitrary three-dimensional contact manifold $M$. By definition the tangent bundle is equipped with a 2-dimensional subbundle $D$ so that the tangent space at every point is spanned by $D$ together with the Lie bracket $[X, Y]$ for $D$-valued fields $X$ and $Y$. The same reasoning as on the flat model gives the complex (25). The *Rumin complex* extends this to general contact manifolds.

Now let us return to the projective three-sphere. On any Riemannian manifold there is a canonically defined differential operator $\nabla : \Lambda^0(T) \to \Lambda^1(T)$, namely, the Levi-Civita connection. It is the unique torsion-free connection compatible with the given metric. It induces a series of differential operators exactly as in (9). But this first variation on the de Rham complex is no longer a complex! In fact, the composition

$$\Lambda^0(T) \xrightarrow{\nabla} \Lambda^1(T) \xrightarrow{\nabla} \Lambda^2(T)$$

is a generally nonzero homomorphism of vector bundles, namely, the Riemannian curvature. As on the projective three-sphere, we can remove traces to obtain

$$\Lambda^0(T) \xrightarrow{\nabla} \Lambda^1_\circ(T) \xrightarrow{\nabla} \Lambda^2_\circ(T). \tag{27}$$

The formulae are exactly as on $\mathbb{R}^n$, namely, (11) and (13). The only difference is that now $\nabla_k$ is interpreted as the Levi-Civita connection. In fact, with Einstein's summation convention and Penrose's abstract index notation [9], the summation signs disappear and the resulting expressions are coördinate-free. The composition (27) sees only part of the Riemann curvature tensor (namely, the Weyl and the trace-free Ricci tensors). The round $n$-sphere is sufficiently symmetrical that this composition vanishes. This observation provides another natural route to the BGG complexes on spheres. Now, what about transferring the second-order operator (14) to the three-sphere? We already know this is possible by means of central projection (21), but we should be able to write the operator in terms of the Levi-Civita connection. It turns out that the formula picks up some additional *correction terms*:

$$(28) \quad w_{jk}{}^{m} \mapsto \nabla_m(\nabla_i w_{jk}{}^{m} - \nabla_j w_{ik}{}^{m} + \nabla_k w_{ij}{}^{m}) + w_{jki} - w_{ikj} + w_{ijk}$$

where the final index on $w_{jk}{}^{m}$ has been lowered using the round metric.

We are now in a position to describe these BGG operators for the corresponding parabolic geometry. Although the action of $\mathrm{SL}(4,\mathbb{R})$ on the round three-sphere does not preserve its metric, it does preserve its geodesics, namely, the great circles. In general, a *projective structure* on a smooth manifold is an equivalence class of torsion-free affine connections that have the same geodesics (considered as unparametrised curves). Details are given in [1]. This is the appropriate parabolic geometry. The notion of homogeneity on the three-sphere may be replaced by that of *projective weight* on a general smooth three-manifold so that three-forms are identified with functions of projective weight $-4$. The operators (27) are defined by (11) and (13) simply by taking any representative connection in the projective class. With an appropriate projective weight

$$\Lambda^0(T)(-1) \xrightarrow{\nabla} \Lambda^1_\circ(T)(-1) \xrightarrow{\nabla} \Lambda^2_\circ(T)(-1)$$

they are invariantly defined. Their composition, however, no longer vanishes. It gives a projectively invariant curvature that vanishes if and only if the structure is projectively equivalent to that on the sphere. The close analogy with Riemannian differential geometry is evident.

Now, what about the second-order operator (14) and its corrected form (28) on the three-sphere? A projective structure has no preferred metric with which to lower indices. Instead, there is a curvature tensor $\mathrm{P}_{ij}$ associated to a representative connection for which

$$w_{jk}{}^{m} \mapsto \nabla_m(\nabla_i w_{jk}{}^{m} - \nabla_j w_{ik}{}^{m} + \nabla_k w_{ij}{}^{m}) + \mathrm{P}_{mi} w_{jk}{}^{m} - \mathrm{P}_{mj} w_{ik}{}^{m} + \mathrm{P}_{mk} w_{ij}{}^{m}$$

is projectively invariant. On the round three-sphere $\mathrm{P}_{ij}$ coincides with the metric. See [1] for details. This addition of *curvature correction terms* to form invariant operators in a parabolic geometry is typical. Perhaps the best-known example is the Yamabe operator in conformal differential geometry. It is formed from the Laplacian in this way.

The general story is very similar. Čap, Slovák, and Souček [5] have shown that all the differential operators in the BGG complex have canonical counterparts in the corresponding parabolic geometry. Projective geometry arises from $\mathrm{SL}(n+1,\mathbb{R})$, and conformal geometry from $\mathrm{SO}(n+1,1)$ acting on the $n$-sphere. The action of $\mathrm{Sp}(n,\mathbb{R})$ on $S^{2n-1}$ gives a parabolic structure combining contact and projective. Only the Rumin complex is invariant for a contact structure alone. Many previously known constructions can be seen as special cases of this theory. Here is not the place to attempt a list.

Finally, we can return to the preferred local coördinates in Examples A and B. Stereographic projection is preferred in Example B, as it is conformal. Central projective (21) is preferred in Example A, because it takes great circles to straight lines in $\mathbb{R}^3$, and these are the geodesics of its natural projective structure.

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