Instabilities in Fluid Motion

Susan Friedlander and Victor Yudovich

—"The diversity of problems exhibited in hydrodynamics alone can only make one admire the inexhaustibility of fascinating problems that occur in nature."

—Birkhoff, Bellman, and Lin, from Hydrodynamic Instability*, 1962.*

Our everyday life is full of examples of fluid motion, from the drama of tornadoes and hurricanes, the turbulent cascading flows in rivers, and the breaking of massive ocean waves to the undramatic familiarity of stirring a cup of tea. Inspired by observation, scientists have sought for centuries to understand and predict fluid behavior. Over two hundred years ago Euler formulated his celebrated equations that give the fundamental mathematical description of fluid motion. It is interesting to note that other areas of physics such as the theory of heat or the motion of waves advanced rapidly in the early nineteenth century because linear models were in good agreement with experiments and the mathematical tool of Fourier series was successfully developed to attack such linear equations. While almost all the partial differential equations in mathematical physics are nonlinear, a number such as the heat equation and the wave equation become linear in commonly occurring physical situations (e.g., the heat equation when the coefficient of thermal conductivity is independent of the temperature itself). In contrast, the process of understanding fluid behavior was much slower because of the crucial nonlinearity in the Euler equations.

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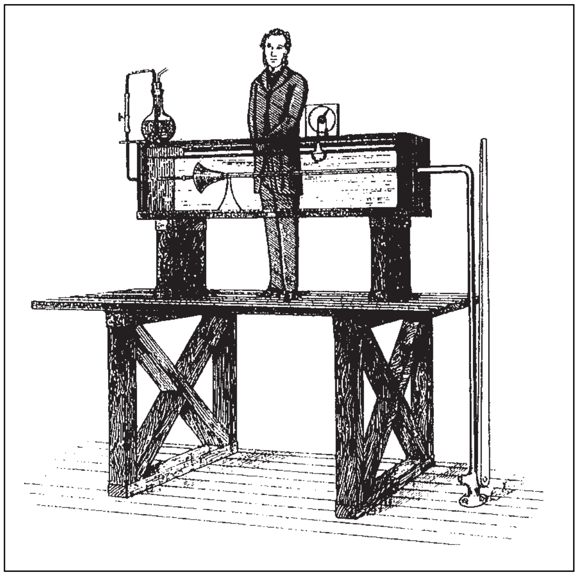
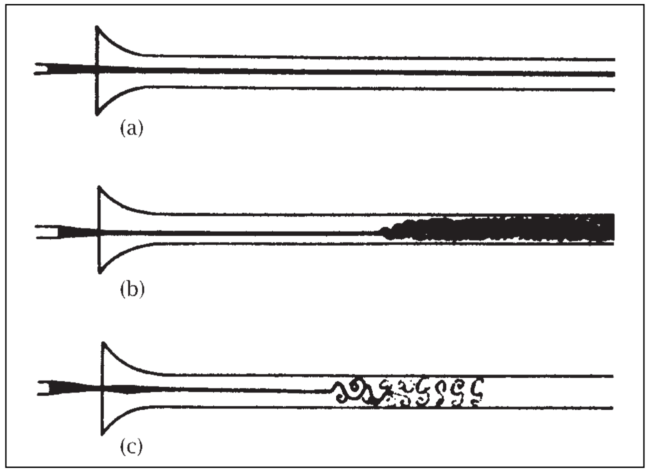


Figure 1. Diagram of the experimental apparatus used by Reynolds. After Reynolds (1883), see [1].

Only the simplest flows in which all fluid particles are moving along straight lines or around circles could be described by explicit solutions of the hydrodynamic equations. The theoretical solutions named *Poiseuille flow* in a circular pipe and *rotating Couette flow* between concentric circular cylinders were found in the mid-nineteenth century. However, by the end of the nineteenth century a striking experimental fact became clear: these flows were realizable in experiments only when they were sufficiently slow. Reynolds [1] gave a beautiful description that we reproduce below of an experiment showing that as the speed of water in a pipe is increased, the flow completely changes its nature from simple Poiseuille flow to a completely different regime that is very irregular in time and space. Reynolds called this *turbulence* and pointed out that the transition from the simple flow to the chaotic flow was caused by the phenomenon of instability.

The issue of the stability or instability of a fluid flow became one of the most basic problems in fluid dynamics and was examined experimentally and mathematically by such giants of science as Helmholtz, Kelvin, Rayleigh, and Reynolds. It presents an important example of a physical problem that can be partially addressed through sophisticated mathematics and where the answers have direct physical interpretations: stable flows are robust under inevitable disturbances in the environment, while unstable flows may break up, sometimes rapidly. Reynolds describes a simple experiment that illustrates a progression from stable to unstable to turbulent fluid behavior:

> The …experiments were made on three tubes…. The diameters of these were nearly 1 inch, $\frac{1}{2}$ inch, and $\frac{1}{4}$ inch. They were all…fitted with trumpet mouthpieces, so that the water might enter without disturbance. The water was drawn through the tubes out of a large glass tank, in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.
>
> The general results were as follows:
>
> (1) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube.
>
> (2) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuousity.
>
> (3) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the color band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water. Any increase in the velocity caused the point of breakdown to approach the trumpet, but with no velocities that were tried did it reach this. On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies.



Source: Reynolds's paper, see [1].

Figure 2. Reynolds's sketches of the flow observed in his 1883 experiment. (a) shows stable flow, (b) and (c) show unstable and turbulent flows.

As we mentioned, the stability/instability of the flow in the pipe is governed in Reynolds's experiment by increasing the velocity. In fact, the full control parameter is the dimensionless number now known as the Reynolds number,

$$R = Ud/\nu,$$

where U is the mean speed of the fluid, d is the diameter of the pipe, and ν is the viscosity of the fluid. In Reynolds's series of experiments, U was increased: alternatively one could envision decreasing the viscosity ν. The limit of vanishing viscosity (i.e., $R \to \infty$) is a very subtle one for the partial differential equations (PDE) of fluid motion, and we will return to this point later.

In the hundred years that followed Reynolds's seminal investigation of fluid instabilities, there developed an enormous body of literature on the subject. This includes not only papers in mathematical journals but also a large body of work in the engineering, physics, astrophysics, oceanographic, and meteorological literature. It is beyond the scope of this article to even touch on this extensive bibliography, but the interested reader may find much background material in the substantive texts of Lin (1955), Chandrasekhar (1961), Joseph (1976), Drazin and Reid (1981), Swinney and Gollub (1981), who have contributed much to various aspects of the theory of fluid stability through their papers and books. An extensive bibliography can be found in the recent volume on hydrodynamics and nonlinear instabilities edited by Godrèche and Manneville [2]. As well as different aspects of the physics of the problem, different mathematical tools are used to tackle the equations governing fluid stability. For example, an elegant mathematical treatment of nonlinear stability is

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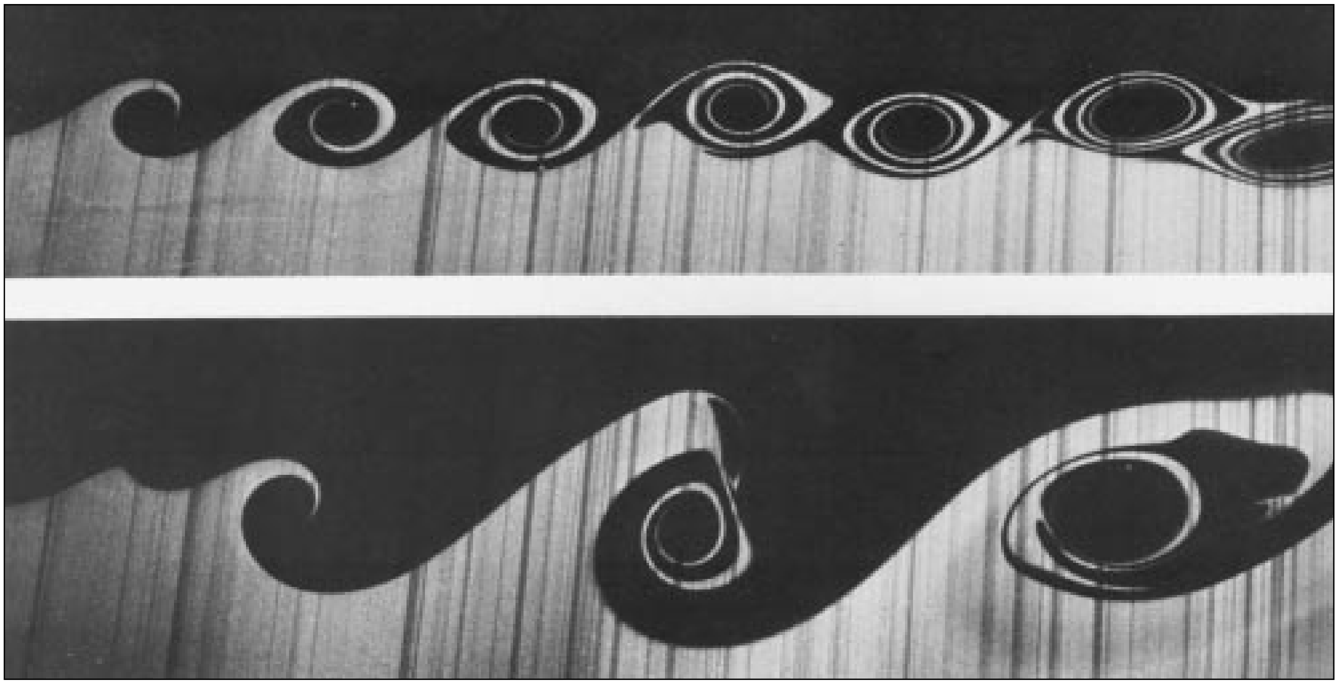


Figure 3. Kelvin-Helmholtz instability of superposed streams. The upper stream of water is moving to the right faster than the lower one, which contains dye that fluoresces under illumination by a vertical sheet of laser light. The faster stream is perturbed sinusoidally at the most unstable frequency in the upper photograph, and at half that frequency in the lower one so that the motion locks into the subharmonic.

given by Arnold (1966), which is applicable to certain two-dimensional inviscid fluid motions. An extensive discussion of the power of "energy theory" as applied to nonlinear stability of fluid flows is given by Galdi and Padula (1990).

Partly because there exist few known solutions to the nonlinear partial differential equations that govern fluid motion, it is very difficult to obtain general results and give detailed analysis of fluid instabilities for arbitrary flows. Much work has concentrated on a relatively small number of rather special fluid configurations, and even in these cases open questions remain. For example, the Reynolds experiment of 1883 is not yet fully explained by current theory. Although there is no rigorous proof of the stability of Poiseuille flow in a circular pipe, analytical and numerical evidence to date for such a flow suggests (theoretical) stability for all Reynolds numbers. As we have described, experiments show instability for sufficiently large Reynolds numbers. The resolution of this paradox appears to be the instability of such flows with respect to small but finite disturbances combined with their stability to infinitesimal disturbances. One can conjecture that the area of attraction in phase space is contracting to one point (the Poiseuille flow itself) as $R \to \infty$. However, the proof of this conjecture is one of the challenges of hydrodynamic stability theory.

Two Examples of Instabilities

To illustrate the type of physics involved, we will give a brief description of two types of instabilities involving specific, rather simple basic flow patterns, namely, particles moving in straight lines or circles.

Example 1. Instability in $\mathbb{R}^2$: Kelvin–Helmholtz Instability.

Consider two streams of *inviscid* (i.e., frictionless), incompressible fluids flowing in parallel one above the other with different velocities, say, one positive and one negative. At the interface where there is a discontinuity in the velocity, the *vorticity* (or curl of the velocity) is nonzero. The vorticity is a delta-function-like distribution that can be modeled by a vortex sheet. Consider a small sinusoidal disturbance of this sheet. Vorticity is an important concept in fluid dynamics, and for two-dimensional flows such as the one we are considering, it is conserved under the motion of the fluid particles. This conservation property for an inviscid fluid implies that the vorticity in parts of the sheet displaced upwards (or downwards) induces a velocity in the positive (or negative) direction. At the undisturbed points of the sine wave, the vorticity induces a rotational velocity about such points that amplifies the sine wave; i.e., the disturbance grows in magnitude.

It is observed that the sheet develops vortex rolls and may eventually break up in a turbulent fashion. The basic mechanism of this instability will be affected by physical forces that may be present such as gravity, surface tension, or frictional effects of viscosity. Such forces may enhance (i.e., destabilize) or damp (i.e., stabilize) the Kelvin–Helmholtz instability. For example, suppose there is a bottomheavy density distribution in the two layers. Then for instability to occur the velocity difference in the two layers must be sufficiently strong to overcome the stabilizing effects of gravity. We remark that Kelvin–Helmholtz type instabilities may often occur in nature, and one such manifestation familiar to all travelers by plane is so-called "clear air turbulence".

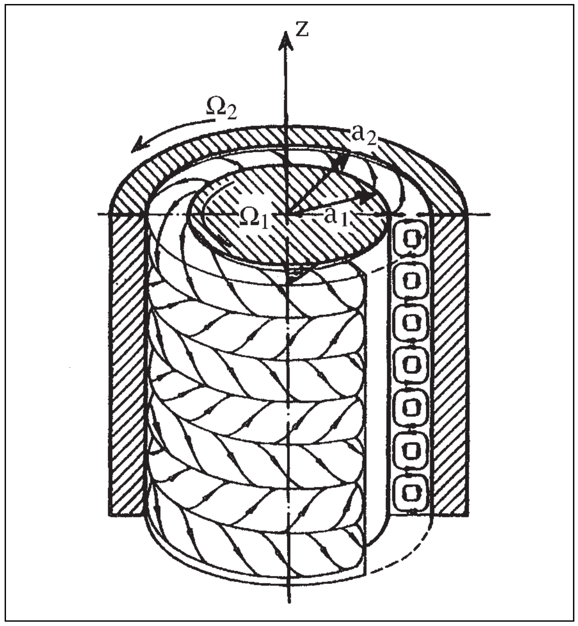
2. An example of instability in $\mathbb{R}^3$: Centrifugal Taylor–Couette Instabilities.

Consider the motion of a fluid between two concentric cylinders rotating in the same direction with different angular velocities. For steady motion in the absence of other forces, the centrifugal force at any radius must be balanced by the pressure gradient. Consider a small radial displacement of a ring of fluid. Conservation of angular momentum implies a change in the angular velocity, which now may or may not be sufficient to offset the centrifugal force. If so, the displaced ring moves back to its original position and a wave is set up. If not, the ring moves away from its original position and the disturbance is enhanced (i.e., there is instability). From such an argument a necessary and sufficient condition for instability to axisymmetric disturbances can be deduced in terms of the radial derivative of the angular velocity of the fluid. Experiments with a viscous fluid in which the angular velocity of the inner cylinder is gradually increased show that above a certain critical inner rotation rate the flow becomes unstable and one sees a pattern of small counterrotating Taylor vortices superimposed on the basic flow. There follows a hierarchy of successive instabilities: azimuthal travelling waves, twisting regimes, quasiperiodic regimes, and so on, which lead step by step to developed turbulence. The mathematical challenge after understanding the primary instability is that of studying bifurcations and detecting the secondary, tertiary, etc., regimes. Details concerning bifurcation theory and fluid behavior can be found in the book of Chossat and Iooss [3].

The Euler Equations for the Motion of an Ideal Fluid

We now turn to the mathematical description of fluid motion. The equations that describe the most fundamental behavior of a fluid were derived by Euler in 1755. They are the equations of conservation of momentum and conservation of mass of a fluid that is incompressible, has constant density, and is inviscid. Such a fluid is sometimes called a "perfect" or an "ideal" fluid. Let $\mathbf{q}(\mathbf{x}, t)$ denote the velocity vector and $P(\mathbf{x}, t)$ denote the scalar pressure field. The Euler equations are the following system of nonlinear PDEs:

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = -\nabla P \tag{1}$$



Source, Schlichting, "Boundary Layer Theory", 1979, McGraw-Hill, New York.

Figure 4. Taylor-Couette experimental set-up. The flow regime shown here corresponds to toroidal Taylor vortices. After Schlichting (1979).

$$\nabla \cdot \mathbf{q} = 0. \tag{2}$$

Equation (1) is the mathematical formulation of Newton's second law of motion written for an element of an ideal fluid. Equation (2) represents conservation of mass for an incompressible fluid. Heuristic derivations of these equations can be found in many texts, including those of Batchelor (1967) and a very readable introduction to fluid dynamics by Acheson (1990). From a physical point of view we are most interested in the case in which $\mathbf{q}$ is a time-dependent vector field that takes its values in $\mathbb{R}^3$; however, it also makes some sense to consider motions restricted to a plane. In order to predict the motion of the fluid at time t in terms of the initial configuration, it is necessary to solve the system (1)–(2) with given initial condition

$$\mathbf{q}(\mathbf{x}, 0) = \mathbf{q}_0(\mathbf{x}) \tag{3}$$

and appropriate boundary conditions. The latter include such physically reasonable conditions as (a) rigid boundaries with no flow perpendicular to the boundary or; (b) flow on a manifold without a boundary such as a torus, where $\mathbf{q}(\mathbf{x}, t)$ and $P(\mathbf{x}, t)$ are spatially periodic; (c) flow in an infinite domain with a condition of decay at infinity to give finite energy. In order to determine the motion of an actual fluid particle given the velocity vector $\mathbf{q}(\mathbf{x}, t)$, it is necessary to solve the Cauchy problem

$$\dot{\mathbf{x}} = \mathbf{q}(\mathbf{x}, t)$$

with

$$\mathbf{x}\Big|_{t=0} = \mathbf{a}$$

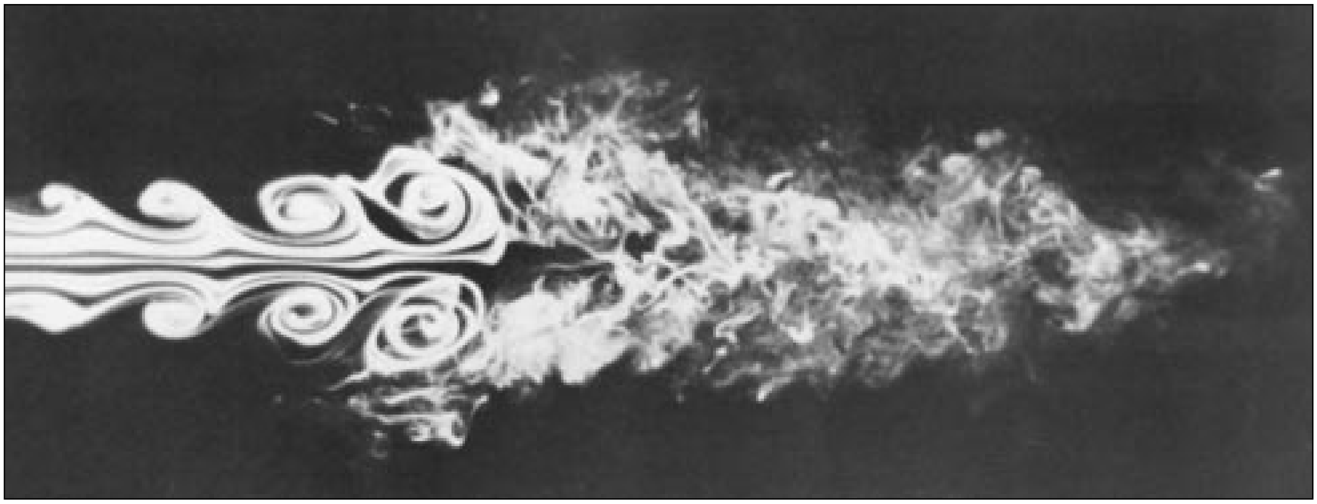


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Figure 5. Instability of an axisymmetric jet. A laminar stream of air flows from a circular tube at Reynolds number 10,000 and is made visible by a smoke wire. The edge of the jet develops axisymmetric oscillations, rolls up into vortex rings, and then abruptly becomes turbulent.

where **a** denotes the initial position of the fluid particle. This system may give rise to another type of instability known as Lagrangian turbulence, which is the irregular, chaotic motion of the particles themselves.

The initial-boundary-value problem for the Euler equations is surprisingly difficult. Perhaps it is one of the most challenging of all problems in PDE that arises directly from physics. Even the basic questions of uniqueness and existence for all time of the solutions in three dimensions remain open. More precisely, it is possible to prove uniqueness for smooth solutions to the three-dimensional Euler equations. However, to date, there is no proof of global existence, even of generalized solutions, to these equations. It is a great challenge to answer the following question: Does an initially smooth velocity field continue to be smooth for all time as it evolves under the Euler equations? In two dimensions the answer is yes. But in three dimensions the question remains highly controversial. Some mathematicians and physicists believe that singularities will arise from an initially smooth velocity field in a finite time. But no unambiguous examples to support this contention have been found, and no theorems have been proved to rule out the possibility of such a singularity. This question is important in understanding how flows become turbulent.

All real fluids are at least very weakly viscous. Viscosity is necessary to generate flows, and its influence is very complicated. In particular, not only is it able to smooth and stabilize fluid motions but it also sometimes actually destroys and destabilizes. Incorporation of the effects of viscosity (or friction) leads to versions of (1)–(2) known as the Navier-Stokes equations. Since friction is a fact of nature, it could be argued that only the Navier-Stokes equations are physically relevant. But there is much to learn from the Euler equations.

Loosely speaking, all possible fluid flows can be classified into three kinds. The first are slow or very viscous flows. Usually these flows go to a single globally stable stationary flow (i.e., the attractor consists of a single stationary point). The second is the class of flows with moderately large Reynolds numbers (small, but not very small viscosity). It is in this class that such phenomena can be found as transitions, successive bifurcations connected with loss of stability, stochastic oscillations, and growth of the dimension of hydrodynamical attractors (regimes of the Taylor-Couette instability give examples of this behavior). The third class is developed turbulence. Figure 5 illustrates turbulence developing at Reynolds number 10,000. While most fluid dynamicists believe that turbulence obeys some universal laws and admits a more exact description, as yet such laws are unknown. It is likely that the core of any such theory lies in the asymptotics of the limit of vanishing viscosity, and the first step towards constructing such an asymptotic theory is the study of inviscid fluids governed by the Euler equations. It is an interesting, and somewhat surprising, observation that zero viscosity limit results are often also applicable and consistent with experiments for flows of the second type, i.e., moderate viscosity.

The limit of vanishing viscosity, which is the appropriate limit in many physical problems, is well known to be a subtle, singular limit. There are many open questions connected with this limit; for example, it is not known how regular or irregular the limiting inviscid flows are. This is why it is necessary to examine all possible ideal flows in the hope that this knowledge will illuminate some general laws. Constantin [4] argues in connection with turbulence, which requires large gradients in the limit of vanishing viscosity, that the possible development of singularities in finite time in solutions to the Euler equations rather than the Navier-

Stokes equations is the physically relevant problem.

The mathematical challenges raised by the Euler equations are different and sometimes more difficult than those connected with the Navier–Stokes equations. In some general sense the Navier-Stokes equations are parabolic. For certain transitions in the Navier-Stokes equations, center-manifold techniques permit the phase space to be treated as finite dimensional, although the dimension of the attractor for realistic values of the Reynolds number is very large compared with the Avogadro number in molecular physics. However, the phase space dynamics of the Euler equation is truly infinite dimensional. The Euler equations are outside the conventional classification of PDE theory, and they manifest a unique individuality. As Arnold (1966) and Ebin and Marsden (1971) demonstrated, the Euler equations are a particular case of equations of geodesics on an infinite dimensional Lie group.

There are many interesting ways in which viscosity plays a role in the stability or instability of fluid motion, and there are a number of beautiful results concerning viscous instability theory. However, in order to keep this short article focused, we will discuss only instabilities in the Euler equations for the motion of an inviscid, incompressible fluid, namely, equations (1)–(2). In the context of the Euler equations Arnold [5] observed that "there appear to be an infinitely great number of unstable configurations." Recent progress in the study of fluid instabilities bears out this observation. We will describe certain of these recent results and put them in the context of some of the tools used to study instabilities.

Mathematical Definitions of Stability/Instability

Instability occurs when there is some disturbance of the external or internal forces acting on the fluid. In practice it is very hard to suppress all possible small disturbances, and, loosely speaking, the question of stability or instability is the question of whether there exist disturbances that "grow" as time progresses. Mathematically the most important concept of a stable steady state has its origin in Lyapunov stability theory, which was first developed for ordinary differential equations (i.e., for systems with a finite number of degrees of freedom) and then generalized to infinite-dimensional dynamical systems. The steady state described by a velocity field $\mathbf{U}_0(\mathbf{x})$ is called *Lyapunov stable* if every state $\mathbf{U}(\mathbf{x}, t)$ "close" to $\mathbf{U}_0(\mathbf{x})$ at $t = 0$ stays close for all $t > 0$. In mathematical terms "close" is defined by considering metrics in a normed space X. While in finite-dimensional systems the choice of the norm is not significant because all Banach norms are equivalent, in infinite-dimensional systems like a fluid configuration this choice is crucial, as was emphasized by Yudovich in 1974.

The following general definition of stability is given in the book of Yudovich [6]. Consider a dynamical system whose phase space is a Banach space X. Examples for X are the space C of bounded continuous functions, the usual L^p space, and the Sobolev space $W^{k,p}$ of functions having (weak) derivatives through order k in L^p. We assume that if the initial value $\mathbf{u}_0 \in X$ is given, the future evolution $\mathbf{u}(t)$, $t > 0$, of the system is uniquely defined. This means there is a mapping M^t sending the initial value $\mathbf{u}_0$ into a space M of vector functions with values in X:

$$\mathbf{u}(t) = M^t \mathbf{u}_0.$$

Let us assume that zero is a steady state, i.e., $M^t(0) = 0$. (We note that it is always sufficient to consider the stability of the zero state because we can pass to this from disturbances about an arbitrary steady state.) The fixed point $\mathbf{u}_0 = 0$ is called (X, M) *stable* if the mapping M^t is continuous at the point zero. For example, if the space M is $C\big([0,\infty), X\big)$, then (X, M) stability, as defined above, gives classical Lyapunov stability in the Banach space X. If M is the subspace of vector functions in $C\big([0,\infty), X\big)$ that go to zero as $t \to \infty$, we obtain the definition of asymptotic stability in the Banach space X.

To illustrate the dependence of stability on the choice of norm, let us consider the following simple example given in [6]. We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = x\,\frac{\partial u}{\partial x}$$

with initial conditions

$$u(0) = \phi(x).$$

The unique solution for an arbitrary smooth function $\phi(x)$ is

$$u(x,t) = \phi(xe^t).$$

A simple calculation leads to the formula

$$\left\| \frac{\partial^k u(\cdot, t)}{\partial x^k} \right\|_{L^p(\mathbb{R})} = e^{(k-p^{-1})t}\, \|\phi^{(k)}\|_{L^p(\mathbb{R})}.$$

Hence we see that this linear equation is:

- asymptotically stable in $L^p(\mathbb{R})$ for $1 \le p < \infty$ and that disturbances decay exponentially as $t \to \infty$;
- Lyapunov stable, but not asymptotically stable in the Banach spaces $L^\infty(\mathbb{R})$, $C(\mathbb{R})$ and $W^{1,1}(\mathbb{R})$;
- exponentially *unstable* in any space $W^{k,p}(\mathbb{R})$ with $k > 1, p \ge 1$ or $k = 1, p > 1$.

The Linearized Euler Equations

We now return to the question of stability or instability of a specific system, namely, the Euler equations (1)–(2) for an ideal fluid. Since we are interested in the behavior of fluid motion close to a given steady flow $\mathbf{U}_0(\mathbf{x})$, it is informative to write the flow as the sum of $\mathbf{U}_0$ and a perturbation $\mathbf{v}(\mathbf{x}, t)$:

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x}, t) \tag{4}$$

$$P(\mathbf{x}, t) = P_0(\mathbf{x}) + p(\mathbf{x}, t). \tag{5}$$

We assume that the time-dependent perturbations $\mathbf{v}$ and p are "small" with respect to some metric. From the Euler equations (1) and (2), the steady fields must satisfy:

$$(\mathbf{U}_0 \cdot \nabla)\mathbf{U}_0 = -\nabla P_0, \quad \nabla \cdot \mathbf{U}_0 = 0. \tag{6}$$

We note that although there are many classes of vector fields $\mathbf{U}_0(\mathbf{x})$ that satisfy (6), not every vector field is an equilibrium fluid flow. Neglecting terms in (1) that are quadratic in $\mathbf{v}(\mathbf{x}, t)$ gives an approximate *linear* system of PDEs for the evolution of the perturbation velocity

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{U}_0 \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla) \cdot \mathbf{U}_0 - \nabla p \equiv L\mathbf{v} \tag{7}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{8}$$

with initial condition $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ and appropriate boundary conditions. Equations (7)–(8) are the linearized Euler equations for infinitesimal disturbances, and issues of linear stability concern the nature of the spectrum of the operator L for a given steady velocity $\mathbf{U}_0(\mathbf{x})$. From the point of view of spectral theory for PDEs, the operator L is very nonstandard, since it is nonselfadjoint and nonelliptic. For most flows $\mathbf{U}_0$, none of the standard theorems from spectral theory of PDE and ODE can be invoked to study the spectrum[1] of the operator L associated with $\mathbf{U}_0$. Rather, the problem falls in the context of general operator and semigroup theory, which provides only a framework for proving specific theorems.

Because of the nonstandard nature of the operator L, the spectrum $\operatorname{Spec} L$ is generically the union of a continuous part, where $\mathbf{f}(\mathbf{x})$ may be a generalized function, and a set of discrete eigenvalues, where $\mathbf{f}(\mathbf{x})$ is an eigenfunction. If the intersection of $\operatorname{Spec} L$ with the right half of the complex plane is nonempty, then the linearized Euler equation has a solution that grows exponentially with time. The steady flow $\mathbf{U}_0(\mathbf{x})$ is then called linearly unstable. We note that the structure of $\operatorname{Spec} L$ will depend on the Banach space of functions in which we seek solutions to (7)–(8). An interested reader may find refinements and further mathematical definitions of stability and instability in some of the referenced texts.

[1]*By the* spectrum *of* L *we mean the set* $\{\sigma \in \mathbb{C}\}$ *such that* $\mathbf{v}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x})e^{\sigma t}$ *satisfies (7)–(8) for some* $\mathbf{f}(\mathbf{x}) \neq 0$.

Plane Parallel Shear Flow

One of the very simplest steady fluid flows $\mathbf{U}_0(\mathbf{x})$ is plane parallel shear flow in which the particles move in straight lines with a velocity that may vary in magnitude from level to level. In Cartesian coordinates x_1, x_2, x_3 in $\mathbb{R}^3$, we write

$$\mathbf{U}_0(\mathbf{x}) = (F(x_2), 0, 0). \tag{9}$$

We assume that we have periodic boundary conditions and that F is C^∞-smooth. In this particular example the unstable spectrum is purely discrete except for possible limit points, and the spectral problem reduces to determining the eigenvalues of an ordinary differential equation known as the Rayleigh equation for eigenfunctions $\Phi(x_2)$:

$$\begin{aligned}(F(x_2) - i\sigma)\left[\frac{d^2}{dx_2^2} - k^2\right] \Phi(x_2)\\ -F''(x_2)\,\Phi(x_2) = 0,\end{aligned} \tag{10}$$

where k is in $\mathbb{Z}$ and $\Phi(x_2)$ satisfies periodic boundary conditions at $x_2 = 0,\ 2\pi$. Construction of an "energy integral" obtained by multiplying (10) by the complex conjugate $\Phi^*(x_2)$, integration by parts, and use of the boundary conditions yield the famous Rayleigh Criterion, namely, that a sufficient condition for linear stability with respect to the energy metric is no inflection points in the function $F(x_2)$ in $[0, 2\pi]$ (i.e., plane parallel shear flow with no inflection points is an example of a flow that is stable in L^2 to infinitesimal disturbances).

Recently Belenkaya, Friedlander, and Yudovich (1999) used the method of averaging to prove that for certain profiles $F(x_2)$ where there are many inflection points, namely rapidly oscillating profiles, the converse is true and there always exist *unstable* eigenvalues. For the case of sinusoidal profiles, techniques using continued fractions were employed to construct the complete unstable spectrum. It is interesting, and in some sense remarkable, that the particular case of sinusoidal profiles leads to a *nonconstant* coefficient eigenvalue ODE (equation (10)), where it is possible to construct in explicit form the transcendental "characteristic" equation that relates the eigenvalues σ and the wave numbers. Usually this can be done only for constant coefficient situations. In this special problem a Fourier series representation for the eigenfunctions leads to a tridiagonal infinite matrix for the algebraic system satisfied by the Fourier coefficients. This system leads to two recurrence relations for the ratio of the coefficients that can be represented by infinite continued fractions. Equating the two continued fractions gives the explicit form of the characteristic equation. Classical theorems from the theory of continued fractions, whose use in fluid dynamics might be a little unexpected, can be used to prove the existence of roots to the characteristic equation with $\mathrm{Re}\,\sigma > 0$ and the convergence of the Fourier series. This was first done for the viscous problem by Meshalkin and Sinai (1961) and for equation (10) by Friedlander, Strauss, and Vishik [7].

The following simple example of Yudovich [8] can be used to illustrate the importance of the norm in which possible growth is measured. As we noted above, plane parallel flow with no inflection point in the profile is linearly stable when the growth of a perturbation is measured by the energy. Consider the following three-dimensional vector:

$$\mathbf{q}(\mathbf{x},t) = (F(x_2),\ 0,\ W(x_1 - t\ F(x_2))). \tag{11}$$

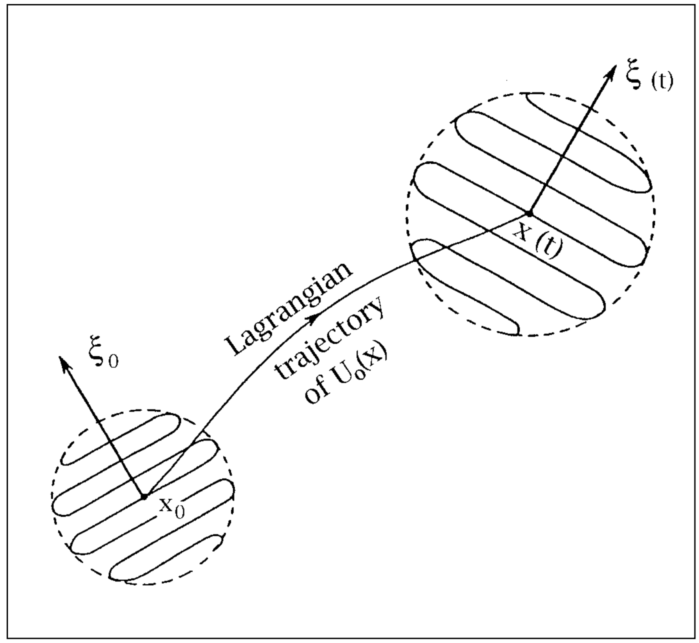
It is easy to check that this is an exact solution, with $\nabla P \equiv 0$, to the full nonlinear Euler equations (1)–(2) for any functions F and W. In particular, we can take the initial value of W to be small in any metric on the velocity and view (11) as a small perturbation of the shear flow (9). As we have mentioned, a very important concept in fluid motion is the vorticity $\omega(\mathbf{x},t) = \mathrm{curl}\,\mathbf{q}(\mathbf{x},t)$, which measures the vortex-like motions in the flow. The curl of (11) is the vector

$$\begin{aligned}\nabla \times \mathbf{q} = -(tF'(x_2)W'(x_1 - t\ F(x_2)),\\ W'(x_1 - t\ F(x_2)),\ F'(x_2)).\end{aligned} \tag{12}$$

Hence the vorticity of such a perturbation *grows* linearly with time provided only that F and W are nonconstant functions. Thus, if we measure growth in any norm that incorporates the magnitude of the vorticity, we find that no matter how small the perturbation W is initially, we have growth in time with respect to this "vorticity" metric. Hence, in this sense even a shear flow with no inflection points in the profile is (weakly) *nonlinearly unstable*. This is an instability in the full nonlinear Euler equations that is not restricted to infinitesimal disturbances. This observation is consistent with experimental evidence that shear flow in a channel with a linear profile (so-called plane Couette flow) is in fact unstable at high Reynolds numbers, despite mathematical results indicating stability. This is the same paradox that we discussed at the beginning of this article in connection with Reynolds's experiment. Again, it is likely that turbulence appears as a consequence of instability to small but *finite* disturbances that generate vorticity.

Instability of More General Euler Flows

We now turn to the problem of examining Spec L in the case of more general fluid flows $\mathbf{U}_0(\mathbf{x})$. The construction of explicit eigenvalues is then a daunting task on which little progress has been made. However, it has proved possible in many cases to obtain certain information about the *continuous* spectrum. In a series of papers (for example, [9]) Friedlander and Vishik used high-frequency asymptotics and techniques of geometric optics to obtain a sufficient condition for instability with respect to growth in the energy norm. They show that a quantity Λ that can be viewed as a "fluid Lyapunov exponent" gives a lower bound on the growth rate of solutions to the linearized Euler equations. Classically a Lyapunov exponent associated with a dynamical system is the exponential growth rate of a tangent vector that is moved and stretched/contracted by the flow field of the system. The fluid Lyapunov exponent is also determined by a system of ODEs over the trajectories of the fluid velocity $\mathbf{U}_0(\mathbf{x})$.



Source: Susan Friedlander, Park City/AMS Series, vol. 5, 1999.

Figure 6. Evolution of high-frequency wavelet disturbance.

Conceptually we envision the following situation. Take a point $\mathbf{x}_0$ in the fluid. At this point we make an initial disturbance in the following form:

$$\mathbf{v}\Big|_{t=0} = \mathbf{b}_0 e^{i\boldsymbol{\xi}_0/\epsilon}, \tag{13}$$

where the amplitude $\mathbf{b}_0$ is a positive function of small support sharply peaked at $\mathbf{x}_0$. The parameter ϵ is very small, so the disturbance oscillates rapidly in space. This "wavelet" is moved by the flow to a point $\mathbf{x}(t)$ so that at a time t later the dominant wave vector is $\boldsymbol{\xi}(t)\big/\epsilon$ and the amplitude has evolved to a vector $\mathbf{b}(t)$. This is illustrated in Figure 6. The evolution with time of $\mathbf{b}(t)$ and $\boldsymbol{\xi}(t)$ along the trajectories of $\mathbf{U}_0(\mathbf{x})$ is governed by a specific coupled system of ODEs. The specific equations can be found in [9]. The fluid Lyapunov exponent Λ is defined to be

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0,\xi_0,\mathbf{b}_0\\ (\xi_0\cdot\mathbf{b}_0)=0\\ |\boldsymbol{\xi}_0|=1,|\mathbf{b}_0|=1}} \Big|\mathbf{b}(\mathbf{x}_0,\boldsymbol{\xi}_0,t)\Big|. \tag{14}$$

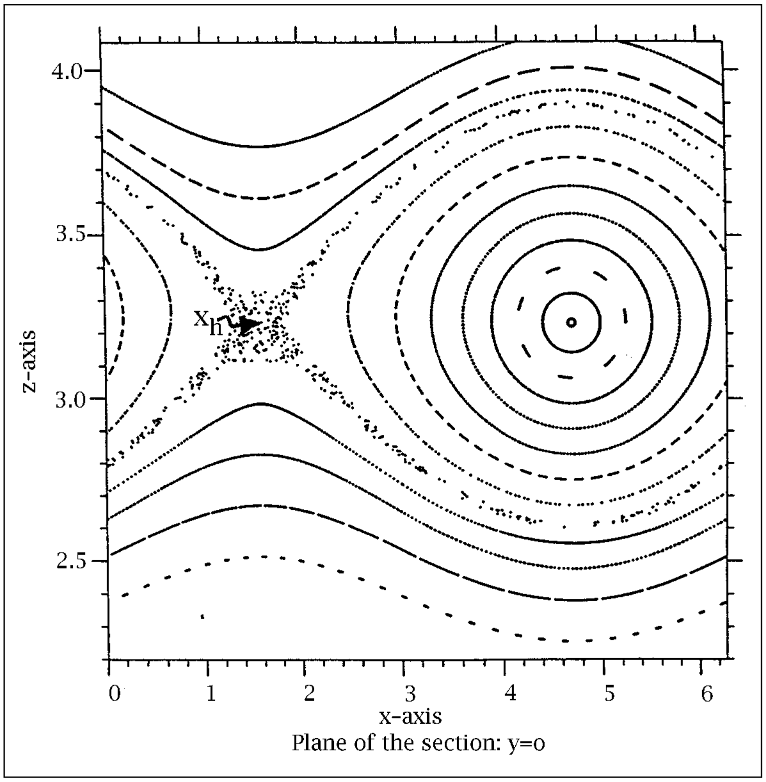


Figure 7. Detail of a Poincaré section of an ABC flow showing a hyperbolic point x_h.

The existence of this strict limit is justified by Oseledets's Ergodic Theorem. It is proved by Vishik and Friedlander (1993) that the positivity of Λ for any given steady flow $\mathbf{U}_0(\mathbf{x})$ implies the existence of a nonempty *unstable continuous* spectrum for the operator L associated with $\mathbf{U}_0$. This result is proved for the spectrum in the space L^2 for flows in arbitrary physical space dimension and with perodic or free space boundary conditions. In many particular examples of $\mathbf{U}_0$ it is possible to use the coupled system of ODE for $\boldsymbol{\xi}(t)$ and $\mathbf{b}(t)$ to calculate a lower bound on Λ and hence prove that the flow $\mathbf{U}_0$ is unstable. We note that it is possible that Λ is zero and yet the flow is still unstable because of the existence of discrete unstable eigenvalues. This is in fact the situation for plane parallel shear flow discussed earlier in this article.

Recently Vishik (1996) sharpened the above result to prove, in the case of spatially periodic boundary conditions, that Λ gives exactly the maximal growth rate in the continuous spectrum of L, i.e., Λ defines the essential spectral radius of the evolution operator exp L in the space L^2.

To detect instabilities using Λ, it is sufficient to make a particular "intelligent" choice for the initial position and direction of the wavelet that is likely to give an exponentially growing amplitude $\mathbf{b}(t)$. For example, consider the situation where there exists a point $\mathbf{x}_h$ at which $\mathbf{U}_0(\mathbf{x})$ has a hyperbolic stagnation (fixed) point; i.e., $\mathbf{U}_0(\mathbf{x}_h) = 0$ and the flow lines in the neighborhood of $\mathbf{x}_h$ are hyperbolic in shape. In such a situation we choose $\mathbf{x}_0 = \mathbf{x}_h$. It is then easy to solve the system of ODEs, which become constant coefficient equations over the trajectories through the fixed point of $\mathbf{U}_0$, and hence demonstrate the existence of an exponentially growing amplitude vector $\mathbf{b}$. This is a special case of a more general result proved in Friedlander and Vishik [9] that any fluid flow with exponential stretching (i.e., positive classical Lyapunov exponent) has a positive fluid exponent Λ and hence is fluid-dynamically unstable. This confirms an observation originally made by Arnold [5] that exponential stretching is related to fluid instabilities.

One interesting specific example of a fluid velocity $\mathbf{U}_0(\mathbf{x})$ satisfying equation (6) that contains exponential stretching and hence instability is a so-called "ABC" flow, where the three components of $\mathbf{U}_0$ in Cartesian coordinates are

$$\begin{aligned} U_{0_1} &\equiv \frac{dx_1}{dt} = A\sin x_3 + C\cos x_2, \\ U_{0_2} &\equiv \frac{dx_2}{dt} = B\sin x_1 + A\cos x_3, \qquad (15) \\ U_{0_3} &\equiv \frac{dx_3}{dt} = C\sin x_2 + B\cos x_1. \end{aligned}$$

This flow is named after the three mathematicians Arnold, Beltrami, and Childress, who have contributed much to our understanding and appreciation of classes of "chaotic" flows of which (15) is an example. For nonzero values of the constants A, B, and C the system (15) is not globally integrable (Arnold [1965]), Henon [1966]). The topology of the flow lines is very complicated and can only be investigated numerically to reveal regions of chaotic behavior as illustrated, for example, by the results of Dombre et al. (1986).

Although exponential stretching is a sufficient condition for Λ to be positive, it is not necessary. There are examples of fluid flows $\mathbf{U}_0(\mathbf{x})$ that have integrals of the motion and where all the classical Lyapunov exponents are zero and yet Λ is positive [9]. Such an example is a model for a vortex ring where the flow lines wrap ergodically around the surfaces of nested tori. Under certain curvature conditions on the geometry of these flow lines, it can be shown that Λ is small but positive and hence the vortex ring is an unstable fluid flow. Experiments show instability of a vortex ring manifested by wavy perturbations on the outer shell of the ring.

Our discussion of the instability of general flows has concerned only the linearized equations (7)–(8). However, we stressed earlier in the article the importance of the *nonlinearity* of the Euler equations. Recently Friedlander, Strauss, and Vishik [7] proved that for large classes of evolution PDEs linear instability implies *nonlinear* instability under certain rather general conditions. This abstract theorem can be applied to the Euler equations to prove nonlinear instability in the Sobolev space H^s,

$s > \frac{n}{2} + 1$, of flows $\mathbf{U}_0(\mathbf{x})$ where it is known that $\operatorname{Spec} L$ contains a "gap" that allows separation of a growing solution to (7)–(8). To date we do not know enough about the detailed structure of Spec L to be sure that the gap condition always holds for any $\mathbf{U}_0(\mathbf{x})$. However, in some specific examples where the physical space dimension n is 2, it is possible to verify the gap condition. This includes plane shear flows with oscillating profiles, which are therefore nonlinearly unstable in H^s, $s > 2$.

We briefly mention that there is a considerable number of mathematicians who have studied aspects of the continuous unstable spectrum arising in the equations of fluid dynamics. Constraints of space prevent our mentioning many beautiful results, so we just give a few examples that an interested reader could pursue: the high frequency asymptotics applied to sound waves by F. G. Friedlander (1958), the pioneering work of Eckhoff (1981) on local instabilities in hyperbolic systems, the study of the continuous spectrum in ideal magnetohydrodynamics of Hameiri (1985) or Lifschitz (1989), the treatment of hydrodynamic stability "without eigenvalues" given by Trefethen et al. (1993).

Concluding Remarks

The thrust of our discussion indicates that in the absence of stabilizing forces, there is an underlying tendency to instability for almost all inviscid fluid flows. The nature of the instability will vary with the structure of the flow and the quantity that measures the growth of the disturbance. The growth rate may be fast or slow. The visible effects of the instability on the "pattern" of the fluid can be very different. The observed pattern of an unstable flow may evolve gradually, as with a sine wave that rolls up into vortices, or it may change violently, as when a laminar flow becomes abruptly turbulent. This suggests the need for a definition of a "scale" of instabilities. At present theoretical results give us just hints as to what is happening in experiments and in nature.

Acknowledgments

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About the Cover

On the cover is a photograph obtained through direct numerical simulation of "turbulence" arising in the two-dimensional Navier-Stokes equations with periodic boundary conditions and initial random distribution. The picture is a visualization of the magnitude of the vorticity field at an instant in time. Vortices correspond to elliptical regions of the flow where rotation dominates strain. The well-mixed background flow corresponds to hyperbolic regions where strain dominates rotation.

The computations were performed by Marie Farge of the Laboratoire de Météorologie Dynamique du CNRS at the École Normale Supérieure de Paris. The visualization was done in collaboration with Jean Francois Colonna of LACTAMME at the École Polytechnique. Thanks go to them for the use of their photograph.

—*S. F.*