A Sideways Look at Hilbert's Twenty-three Problems of 1900

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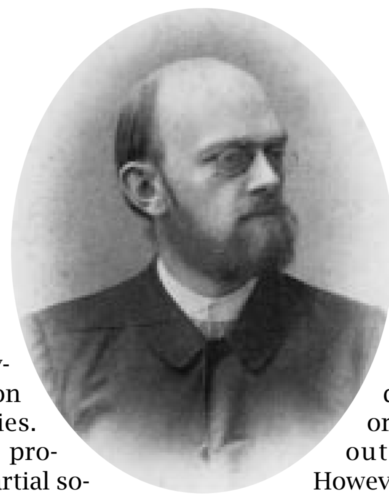

David Hilbert, circa 1900.

As the nineteenth century drew to its close, David Hilbert (1862–1943), then regarded as a leading mathematician of his generation, presented a list of twenty-three problems, which he urged upon the attention of his contemporaries. They have entered the folklore of professional mathematicians; even a partial solution of one of them has given its author(s) much prestige. Two compendia have reviewed progress to the date of their publication: [1] in the former Soviet Union, where study of the problems has been a speciality, and [4] in the United States. In addition, individual problems have been examined in various other books and special articles. Now, at the centenary of the lecture, it is opportune to compare the range of Hilbert's problems against the panoply then evident in mathematics.

Circumstances and Publications

First, some details of the preparation and publication of the list are appropriate. The motivation was the Second International Congress of Mathematicians, held in Paris early in August 1900, which Hilbert was invited to address. He seems to have thought of the topic by December 1899, for he sought then the opinion of his close friend Hermann Minkowski (1864–1909) [17, pp. 118–120], and again in March of another ally, Adolf Hurwitz (1859–1919).[1] But apparently he delayed writing the paper until May or June, so that the lecture was left out of the Congress programme. However, by mid-July he must have sent it for publication by the Göttingen Academy of Sciences, of which he was a member, for Minkowski was then reading the proofs [17, pp. 126–130]; very likely no refereeing had occurred.

Hilbert spoke in the Sorbonne on the morning of 8 August 1900, not in a plenary lecture but in the section of the Congress on bibliography and history; he proposed "the future problems of mathematics," working from a French translation of his text that was distributed to the members of the audience. A summary of it soon appeared in the recently founded Swiss journal *L'Enseignement Mathématique* (Hilbert 1900a)[2]; the original seems not to have been published. For reasons of time he described there only ten problems. The full story was soon out with the Göttingen Academy (1900b); next year it was published again, with three additions, in the *Archiv der Mathematik und Physik* (1901a). This second-ranking research journal is a somewhat surprising location: maybe its editors persuaded him to the reprint in order to raise its

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The author thanks particularly Karen Parshall, Helmut Rohlfing, and David Zitarelli for their help in the preparation of this article.

[1] *Hilbert to Hurwitz, 29 March 1900 (Göttingen University Archives, Mathematical Archive 76, letter 275).*

[2] *This notation refers to an item that is cited in the sidebar.*

prestige as they launched a new series of volumes with the new century.

The preamble and ten of the problems in these versions received an anonymous free and condensed translation, which was published in the *Revue Générale des Sciences Pures et Appliquées* (Hilbert 1901b). The *Archiv* version was translated in full into French for the Congress proceedings by the French mathematician and former diplomat Léonce Laugel, who added a few footnotes of his own.[3] His translation appeared both there and as a separate undated pamphlet under the title *Mathematical Problems* (Hilbert 1902a). Then an English translation of the version was prepared for the *Bulletin of the American Mathematical Society* by Mary Newson (1869–1959) (Hilbert 1902b), completing an initiative taken by H. S. White (1861–1943).[4]

All these manifestations were listed in the reviewing journal of the time, the *Jahrbuch über die Fortschritte der Mathematik*, and the Göttingen version was reviewed. The reviewer was Georg Wallenberg (1864–1924), no less but no more; a teacher at the Technical University in Berlin and co-editor of the *Jahrbuch*: he summarised the general preamble that launched the paper and then copied the titles that Hilbert had given to the problems [22]. Sadly he left out the Fifth, Eleventh, and Fourteenth Problems, so that readers of the *Jahrbuch* learnt about Hilbert's twenty problems!

Table 1 shows the twenty-three problems by short description of their subject matter; where possible I have quoted Hilbert. A full survey of the relevant branches of mathematics is far beyond the scope of this article; indeed, it would require a formidable but worthwhile monograph. Instead, I shall point to some general and particular features of the problems, elaborating on the information in the middle and last columns of the table. I shall refer almost entirely to the full version, noting the three additions. For reasons of space, references are confined almost entirely to literature of the time; many articles in [12] partly fill the historical gaps.

Problems: Range and Definition

Range

The few pages of preamble appraised problems in general and the development of mathematical knowledge as Hilbert saw it; near the end he expressed his optimism with a slogan that he would repeat in later life: "*for in mathematics there is no ignorabimus!*" (Hilbert 1902b, p. 7, italics restored). The modernistic flavour of the problems lay not only in their unresolved status but also in the high status given to axiomatisation in solving or even forming several of them.

Several main branches of mathematics were impressively covered or at least exemplified by problems: number theory and higher and abstract algebra (Hilbert's two main research specialities up to that time), most of real- and complex-variable analysis, and the still emerging branch of topology. Geometry was more patchily handled; in particular, the achievements of the Italian geometers largely eluded him. Apparently untalented in languages, he had trouble reading even technical Italian.

Among problems directly inspired by Hilbert's own work, the Fourteenth Problem grew out of his proofs in the early 1890s that systems of algebraic invariants always possess finite bases. However, he forgot to cite Hurwitz's recent contribution [14]; he apologised to his friend in November 1900 and added a paragraph to the *Archiv* version.[5]

Some problems were handled with great perspicuity. In particular, in the Fifth Problem on the theory of Sophus Lie (1842–1899) of continuous groups of transformations, not only did he pose a specific problem invoking the differentiability of the pertaining functions, but also a broader one about weakening that property. The latter is still far from a general answer; indeed, the pertinent articles in [1] and [4] suggest that the distinction between the two problems is not well recognised.

Hilbert grouped together some problems of similar content. In particular, he pointedly placed as the First Problem questions in the set theory of Georg Cantor (1845–1918), which was just then gaining general acceptance among mathematicians after a somewhat difficult development [7]; then as the Second Problem he proposed an issue in the foundations of mathematics that he was soon to enrich as his "proof theory". Some other bunching of problems can be seen in the table. However, it might have been tighter: the gap between the Eleventh and the Seventeenth on quadratic forms is hard to grasp, and maybe also that between the Nineteenth and the Twenty-third on the calculus of variations.

Definition

From now on, my look becomes rather more sideways. To begin with, Hilbert often proposed a list of problem areas rather than individual ones: for example, those on Cantor and on Lie each form pairs. But he seems not to have thought carefully about the notion of problem as such. Without degenerating into language-games philosophy, one can valuably press distinctions between a problem as such and a research programme, a foundational

[3] *In 1901 a booklet containing the conference timetable and related details was published [9]; this information appeared again in the front matter of the proceedings [10].*

[4] *White to Hilbert, 28 April and 12 May 1902 (Göttingen University Archives,* Nachlass *Hilbert, sec. I, letters 432/3–4). I gather that this collection does not contain any manuscript versions of the lecture or full versions.*

[5] *Hilbert to Hurwitz, 21 November 1900 (as in footnote 2, letter 278). The addition is the second paragraph of the Fourteenth Problem.*

Table 1. Hilbert Problems.

Hilbert Problem (*Archiv* Paper)	Hilbert Problem (Lecture)	Apparent Number Of Problems	In Hilbert *Revue* Paper	Problems/Topics
1	1	2	Yes	Set theory: continuum hypothesis; well-ordering principle
2	2	0?1?	Yes	"Consistency of arithmetic axioms"
3		1	No	Equality of volumes of two tetrahedra of equal base area and height
4		1	No	Shortest line between two points
5		2	No	Lie groups and differentiability of its functions
6	3	0?2?	Yes	"Mathematical treatment of the axioms of physics"
7		1 group	Yes	"Irrationality and transcendence of certain numbers" (e.g., $e^{i\pi z}$, α^{β})
8	4	2	Yes	"Prime number problems": Riemann hypothesis; distribution of primes
9		1	No	General reciprocity law in algebraic number theory
10		1	No	"Decidability of solvability of Diophantine equations"
11		1	No	"Quadratic forms with arbitrary algebraic number coefficients"
12	5	2	No	Generalising theory of field extensions to arbitrary rational domains
13	6	1	No	"Impossibility of solving the general quintic"
14		1	No	Invariants and covariants of rational "function systems"
15		0?1?	No	Rigorisation of enumerative geometry
16	7	2	Yes	Topology of curves; maximal number of limit cycles
17		1	No	Reduction of quadratic forms to sums of squares
18		2	Yes	Filling space with congruent polyhedra; functions definable from differential equations
19	8	1	Yes	Analytic solution of problems in the calculus of variations
20		1	No	General solution of Dirichlet's problem
21	9	1	Yes	Monodromy groups over differential equations
22	10	1	Yes	Relationships between automorphic functions
23		1 group	No	Solubility of problems in calculus of variations, with one or several functions and integrals

examination, and an algorithm. For example, the Twenty-third Problem seeks the "Further development [Weiterführung] of the methods of the calculus of variations." But then why not urge the same for every branch of mathematics? (This branch was so selected because he had recently been drawn to it by the Twentieth Problem on proving the Dirichlet principle, a major issue in potential theory; it is overly present in the list as a whole.[6]) The same query could be made also about the Second ("consistency of … axioms"), the Sixth ("treatment of axioms"), and the Fifteenth ("rigorisation").

Numerical mathematics ought to have gained a problem or two, especially as it contains many in the proper sense of the term. The Thirteenth Problem on solving the general septic equation was laid out in terms of nomography, a graphical method of handling functional relationships for numerical purposes, but it actually concerned the (im)possibility of reducing functions of several variables to functions of functions of fewer variables.

Missing from the list are two of the most spectacular problems of the time. One is Fermat's Last Theorem of number theory: that

$$\text{(1)} \qquad \text{if } xyz \neq 0, \ \text{then } x^n + y^n = z^n$$

has no solutions in positive integers if $n > 2$. Maybe it could be squeezed in as a Diophantine equation under the Tenth Problem if the variable n is tolerated, but no such mention was made. The other is the three-body problem in dynamics, especially as posed and examined by Henri Poincaré (1854–1912) in 1889–90 and so formally falling under the Sixth on mechanics. Yet in both the lecture and the full versions they were explicitly mentioned as problems in the preamble but omitted from the lists. So are there twenty-five problems in all?

The Place of Applied Mathematics

The three-body problem should have been recalled in the elaboration of the Sixth Problem; but this raises the issue of applied mathematics in general, which needs separate consideration. In his preamble Hilbert stated that "the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena" (Hilbert 1902b, p. 3). Yet applications were poorly treated in the list: while the Twentieth on the Dirichlet problem was relevant, only the Sixth *explicitly* related to applications, and in unsatisfactory ways over and above not being a proper problem anyway. While he stated "physics" in the title of the Sixth Problem, most of the references then given were to mechanics (perhaps prompted by Minkowski, who had studied physics): the difference between physics and mechanics was elided, although it had been a major theme for the whole nineteenth century.

For physics itself, in the second paragraph Hilbert mentioned the role of probability theory—quite rightly in view of the current development of gas theory and statistical mechanics—but he passed over electromagnetism and the interpretation of Maxwell's equations, long a *major* research area in mathematics. In particular, in the spring of 1900 J. J. Larmor (1857–1942) had published a substantial survey of current knowledge in his *Aether and Matter* [16]. His subtitle admirably conveyed the aim: "A development of the dynamical relations of the aether to material systems on the basis of the atomic constitution of matter including a discussion of the influence of the Earth's motion on optical phenomena": that is, physics, but with mathematics *centrally* involved. A variety of problems emerge from the book concerning elastic properties required of the supposed aether, modes of its excitation, means of approximation to the contraction equations, and the mystery of Maxwell's displacement current.

Presumably Hilbert had not read Larmor's new book; but he must have at least seen the recent survey of electromagnetism by his Göttingen colleague Emil Wiechert (1861–1928), for it had been prepared for the unveiling in Göttingen on 17 June 1899 of a statue to Gauss and Wilhelm Weber [23], an occasion for which Hilbert himself had expounded upon the foundations of geometry (Hilbert 1899). But some months later, when thinking out his Paris address, the subject passed him by: in his lecture, he just proposed

> To establish the systems of axioms of the calculus of probabilities, of rational mechanics and of the different branches of physics, then to found upon these axioms the rigorous study of these sciences

with no elaboration at all (1900a, p. 352). Further, both here and in the full version he never mentioned probability theory again, thus omitting most of its uses and problems, which had also been well surveyed recently [6].

Hilbert had recently given his first lecture course in mechanics (Hilbert 1898), introductory but quite wide-ranging, and from 1904 he was to examine several areas of physics in impressive detail, mostly in lecture courses, though with some publications also [5]. But the title just quoted for the Sixth Problem and the elaboration in the full version suggest that in 1900 he was not very familiar with these branches of mathematics.

Understandable Omissions

Some further omissions are worth noting in order to *defend* Hilbert. He stated no problems for three branches of mathematics that have become well

[6] *The text to the Twenty-third Problem received as an addition to the* Archiv *version the paragraph near the end citing [15].*

Hilbert References

1898. *Mechanik*, Göttingen University Archives, Nachlass Hilbert, ms. 558, unpublished lecture course.

1899. *Grundlagen der Geometrie*, 1st ed., Teubner, Leipzig; many copies are bound with [23].

1900a. Problèmes mathématiques, *l'Ens. Math.* (1) **2**, 349–355.

1900b. Mathematische Probleme, *Nachrichten Königlichen Gesellschaft Wissenschaften Göttingen, math.-physik. Klasse*, 253–297; also in [1], pp. 22–80.

1901a. Mathematische Probleme, *Arch. Math. Physik* (3) **1**, 44–63, 213–237; also in (Hilbert 1935), pp. 290–329; also in [2], pp. 247–292.

1901b. Problèmes mathématiques, *Revue Gén. Sci. Pures Appl.* **12**, 168–174.

1902a. Sur les problèmes futurs des mathématiques, in [10], pp. 58–114; translation of (Hilbert 1901a); also issued as undated repaginated pamphlet entitled *Problèmes Mathématiques*, 56 pages.

1902b. Mathematical problems, *Bull. Amer. Math. Soc.* **8**, 437–479; also in [4], pp. 1–34; translation of (Hilbert 1901).

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1898. Sur les rapports de l'analyse pure et de la physique mathématique, in F. Rudio (ed.), *Verhandlungen des I. Internationalen Mathematiker-Kongresses*, Teubner, Leipzig, pp. 81–90.

1902a. Du rôle de l'intuition et de la logique en mathématiques, in [10], pp. 115–130; also in *La Valeur de la Science*, Flammarion, Paris, 1905, ch. 1.

1902b. Review of (Hilbert 1899), *Bull. Sci. Math.* (2) **26**, 249–272; English translation in *Bull. Amer. Math. Soc.* **10** (1904), 1–23.

1904. Rapport sur les travaux de M. Hilbert, *Obshchestva Fiziko-Matematicheskago Kazan' Universiteta* (2) **14**, 11–48; expanded version of (Hilbert 1902b) for the Lobachevsky prize competition.

1909. L'avenir des mathématiques, in G. Castelnuovo (ed.), *Atti del IV Congresso Internazionale dei Matematici*, vol. 1, Accademia dei Lincei, Rome, pp. 167–192; various other printings.

established in mathematics and its higher education but that in 1900 were not on the normal mathematical scene even though the basic notions and theories were in place. These were matrix theory, mathematical statistics (apart from probability theory), and mathematical logic. Their histories are far too convoluted for even a summary account here,[7] but I cite as historical barometer the *Encyklopädie der mathematischen Wissenschaften*, a vast cataloguing of mathematical theories launched in the mid-1890s under the direction of Hilbert's Göttingen colleague Felix Klein (1848–1925) which was to be published until the mid-1930s. Here too there were no articles explicitly on matrix theory and mathematical logic and only a few on specific topics in mathematical statistics. These theories were to gain popularity, especially from the 1920s onwards, and then play roles in the solution of several of Hilbert's problems.

A more unexpected silence surrounds the application of set theory to mathematical analysis in the manner that Maurice Fréchet (1878–1973) was to call in 1906 "functional analysis", where collections of mathematical functions of given kinds were treated as sets in Cantor's sense and properties such as closure were examined. Such tasks were in the mathematical air in the 1890s, especially concerning Fourier series [21]. The historical irony is that between 1903 and 1910 Hilbert himself was to become intensively occupied with this area in connection with integral equations, which linked tightly to functional analysis (hence the notion of "Hilbert space").

Judgements: Hilbert and Poincaré

The importance of Hilbert's lecture was grasped quite soon after the Congress; for example, Laugel's translation of the full version was published in its proceedings with the plenary lectures although it had not been so delivered [10, p. 24]. But the reaction after the lecture was "a rather desultory discussion," to quote from the report on the Congress prepared by Charlotte Angas Scott (1858–1931) for the *Bulletin of the American Mathematical Society*.[8] Two comments were made. Firstly, the Italian mathematician Giuseppe Peano (1858–1932) remarked that the Second Problem on the consistency of arithmetic was already essentially solved by colleagues working on his project of mathematical logic and that the forthcoming Congress lecture by Alessandro Padoa (1868–1937) was pertinent to it [18]. Unfortunately Hilbert did not make amends in the *Archiv* version (presumably lack of Italian again), but in *L'Enseignement Mathématique* Padoa explicitly discussed this problem in one of the early publications on a Hilbert problem [19]. Secondly, the German mathematician Rudolf Mehmke (1857–1944) made a point about numerical methods that bore upon the Thirteenth Problem on resolving the quintic: it led to a new paragraph in the *Archiv* version citing [8], and Laugel elaborated further in a footnote in his translation (Hilbert 1902a, p. 92).

That was all. Maybe Hilbert's manner of delivery was partly to blame: Scott opined that the "presentation of papers is usually shockingly bad," with monotonic utterance exuding boredom; she gave no names, but hinted that eminent ones were not excluded [20, p. 77]. Two weeks after delivering his lecture, Hilbert did not mention it at all when

[7] *Some details and further historical references can be found in [12], respectively articles 6.6–6.8, 10.3–10.15, and 5.1–5.5. Hilbert's own interest in mathematical logic dates from around 1904 and in (infinite) matrix theory in connection with integral equations a little earlier. The theory of determinants was already well known by 1900.*

[8] *[20, p. 68]; see also [10, p. 21]. The five-page report for* L'Enseignement Mathématique *devoted only five lines to the lecture and none to the discussion [11]. However, the* American Mathematical Monthly *[13] recorded a good reception.*

he reported on the Congress to Hurwitz (who had not attended); indeed, he opined that "the visit was not very strong in either the quantitative or in the qualitative regard," and so may have been disappointed in general.

In this letter Hilbert also mentioned that Poincaré "was manifestly present only by duty of necessity,"[9] so maybe he did not hear the lecture. There seems to be no evidence of Poincaré's reaction to the published versions (or that of Larmor, who attended the Congress); but had he given such a survey himself there rather than muse upon "the role of intuition and logic in mathematics" (Poincaré 1902a), it would have been still broader and certainly stronger on applications. Perhaps Poincaré's (apparent) silence *is* the comment: intuition and applications please, dear colleague, not all this purist axiomatics. Never in the remaining dozen years of his life did he explicitly tackle any of the problems or mention any of them in his own survey (1909) of "the future of mathematics" (compare the title of Hilbert's lecture above) at the Third International Congress of Mathematicians in Rome in 1908. However, he praised Hilbert's work on the foundations of geometry at length, especially in (1902a) and (1904).

Hilbert seems to have conceived his lecture as a counter to the rather bland advocacy of the importance of applied mathematics made in (Poincaré 1898) at the First International Congress of Mathematicians in Zürich in 1897, but he surely swung too much the other way. The brilliance with which Hilbert focused on several specific problems (and "problems") has brought some snow-blindness to the estimation by later mathematicians of the whole collection in its historical context. His former graduate student Otto Blumenthal (1876–1944) (the first in a long sequence of students) passed a good sideways judgement many years later: "Quite few [problems] stem from the general situation of mathematics or from the problem-contexts of other researchers" [3, p. 405]. Hilbert had made a personal selection of problems, and moreover seemingly elaborated at speed and only partially grouped. In his closing remarks in the full version he stated that they "are only samples of problems," though he also claimed that they showed "how extensive is the mathematical science of today." The glamour that was to be bestowed on his selection may have distorted priorities somewhat in the development of mathematics during the twentieth century.

[9]*Hilbert to Hurwitz, 25 August 1900 (as in footnote 1, letter 277). The contexts of the translated passages read: "Der Besuch war nicht sehr stark weder in quantitativer noch in qualitativer Hinsicht," and "Poincaré war offenbar nur der Notwendigkeit gehorchend anwesend; bei den Schlussbanquet fehlte er, obwohl er präsidiren sollte" (compare [20, p. 74]). In September 1900 Hilbert reported on the Congress at the annual meeting of the* Deutsche Mathematiker-Vereinigung, *when he was elected chairman for the next twelve months (*Jahresbericht, ***7** *(1900–01), pp. 4–5, 7: no details are given).*

References

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