Free Boundary Problems in Science and Technology

Avner Friedman

Free boundary problems deal with solving partial differential equations (PDEs) in a domain, a part of whose boundary is unknown in advance; that portion of the boundary is called a *free boundary*. In addition to the standard boundary conditions that are needed in order to solve the PDEs, an additional condition must be imposed at the free boundary. One then seeks to determine both the free boundary and the solution of the differential equations. The theory of free boundaries has seen great progress in the last thirty years; for the state of the field up to 1982 we refer to [3] and [10].

Two types of problems stimulated this progress. The first one is the "obstacle problem", which, in the time-independent case, can be stated as a variational problem. The second one is the Stefan problem, describing the process of melting and solidification. These two problems will be briefly reviewed in the first two sections. The main objective of the present article, however, is to describe several more recent free boundary problems, selected from an increasingly large number of such problems that arise in science and technology and that require the development of new mathematical ideas and analytical techniques.

Since the aim of this article is not to provide an exhaustive review of the subject of free boundary problems, but rather to brief the general mathematical public on a few of the current research directions, we had to drastically curtail the list of references and the full historical allocation of credit.

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Variational Problems

The Dirichlet problem seeks to find a function $u(x)$ satisfying

$$\begin{aligned} -\Delta u &= f(x) \quad \text{in a bounded domain } D \text{ in } \mathbb{R}^m, \\ u &= g(x) \quad \text{on the boundary } \partial D. \end{aligned}$$

For smooth f, ∂D, and g, the solution can be obtained as the unique solution of the following "variational problem": Find u in K minimizing the functional

$$J(v) = \int_D \left(\tfrac{1}{2} |\nabla v|^2 - vf \right) dx, \quad v \in K,$$

where K is the class of all functions in $H^1(D)$ with $v = g$ on ∂D; here $H^1(D)$ denotes the space of functions that belong to $L^2(D)$ together with their first derivatives.

Following the centuries-old "variational principle", one can consider small perturbations of the minimizer u and derive the so-called Euler-Lagrange equation $-\Delta u = f$. Boundary value problems that can be written as Euler-Lagrange equations for some functionals are called *variational problems*.

Let φ be a smooth function in $\overline{D}$ with $\varphi \le g$ on ∂D, and denote by K_φ the class of functions v in K with $v \ge \varphi$ a.e. in D. Then the minimizer u of $J(v)$ over the class K_φ is unique, and, proceeding as in the derivation of the Euler-Lagrange equation, one finds that

$$\begin{aligned} -\Delta u &= f(x) \quad \text{a.e. in } \Omega \equiv \{u > \varphi\} \cap D, \\ -\Delta u &\ge f(x) \quad \text{a.e. in } D \setminus \Omega. \end{aligned}$$

The above problem is called the *obstacle problem* for the following reason: Consider a membrane $x_{m+1} = u(x)$ held fixed above ∂D at $x_{m+1} = g(x)$, which is being forced downward by pressure $f(x)$ while at the same time being constrained to remain above a solid obstacle with surface $x_{m+1} = \varphi(x)$. Then $u(x)$ is the minimizer of $J(v)$ for $v \in K_\varphi$.

Figure 1 is a schematic of the obstacle problem in the one-dimensional case. It is known ([3] and [10]) that the second derivatives of $u(x)$ are bounded, so that Ω is an open set and

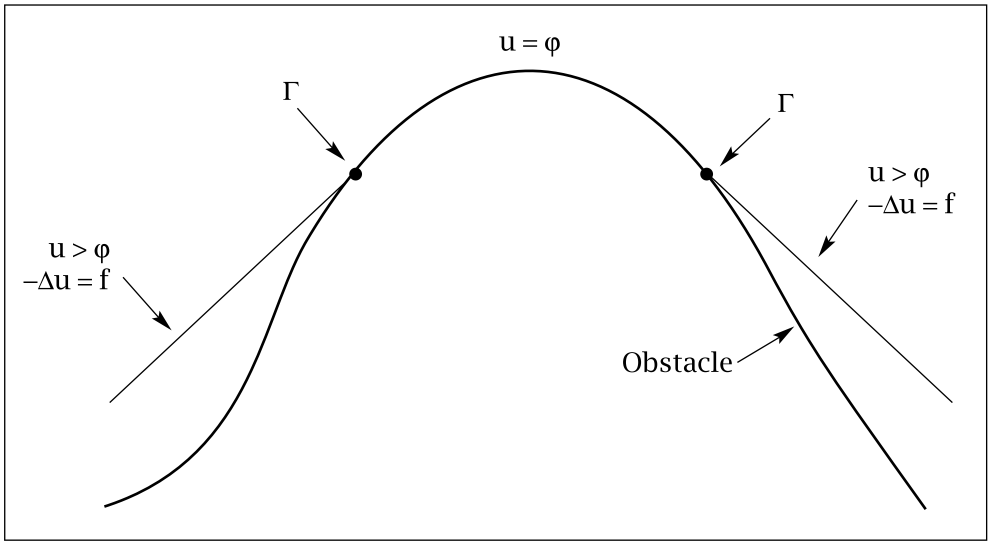

Figure 1. The obstacle problem.

$$(1)\qquad \begin{aligned} -\Delta u &= f(x) \quad \text{in } \Omega, \\ u &= g \quad\quad \text{on } \partial\Omega \cap \partial D. \end{aligned}$$

Furthermore, $u = \varphi$ and $\nabla u = \nabla \varphi$ on all of $D \setminus \Omega$, and, in particular,

$$(2)\qquad u = \varphi \quad \text{on } \Gamma,$$

$$(3)\qquad \frac{\partial u}{\partial n} = \frac{\partial \varphi}{\partial n} \quad \text{on } \Gamma,$$

where Γ is the part of the boundary of the set $\{u > \varphi\}$ that lies in D. Since Γ may not have unique normals, we should actually replace (3) by $\nabla u = \nabla \varphi$ on Γ.

The set Γ is not given in advance. We could have, in fact, posed the variational problem in the following way: Find a domain Ω (or a free boundary Γ) such that u solves the Dirichlet problem (1)–(2) in Ω, the additional Neumann boundary condition (3) at the free boundary, and

$$u > \varphi \text{ in } \Omega, \quad -\Delta \varphi \geq f \text{ in } D \setminus \Omega.$$

The variational approach described above can readily be extended to other functionals. For example, if Q is positive and χ_A denotes the characteristic function of A, let us minimize the functional

$$J_0(v) = \int_D \left(\tfrac{1}{2} |\nabla v|^2 + Q^2(x) \chi_{\{v>0\}} \right) dx$$

over the same class K as above (assuming $g \geq 0$). Then we find that any minimizer satisfies the equation

$$(4)\qquad \Delta u = 0 \quad \text{in } \Omega \equiv \{u > 0\} \cap D$$

and the boundary conditions

$$(5)\qquad u = 0 \quad \text{on } \Gamma,$$

$$(6)\qquad \frac{\partial u}{\partial n} = Q \quad \text{on } \Gamma,$$

where Γ, the free boundary, is the part of the boundary of the set $\{u > 0\}$ lying in D and n is the inward normal.

The stream function of an ideal fluid in two dimensions satisfies (4). It satisfies (5), since Γ is a streamline, and by Bernoulli's law

$$\tfrac{1}{2} |\nabla u|^2 + gy = \text{const.},$$

$$g = \text{gravity constant}, \quad y = \text{height},$$

also (6) is satisfied with $Q = \text{const.} - 2gy$.

How regular is the free boundary? For the obstacle problem, Γ may in general have singularities. However, if the free boundary Γ is Lipschitz continuous, then Γ will be as smooth as the data [2]; in particular, Γ will be C^∞ or analytic if the same is true of f and φ. For details see L. Caffarelli, 1977, and D. Kinderlehrer and L. Nirenberg, 1977. On the other hand, for the minimizers of $J_0(v)$ over K the free boundary in the two-dimensional case is always analytic; see H. Alt and Caffarelli, 1982.

The minimization problem for $J_0(v)$ has been used as the starting point in the study of a two-dimensional jet of fluid emerging from a nozzle. This approach has been further extended to axially symmetric jets, to compressible fluids, and to two-fluid problems; see Alt, Caffarelli, and A. Friedman, 1982–1985; Friedman, 1986; and Friedman and Y. Liu, 1995.

The Stefan Problem

Consider the problem of melting and solidification, denote by T_1 the water temperature and by T_2 the ice temperature, and let Γ be the ice/water interface. Then

$$(7)\qquad \begin{aligned} &\frac{\partial T_i}{\partial t} = \nabla(k_i \nabla T_i) \quad \text{in } \Omega_i \\ &(i = 1 \text{ for water}, \ i = 2 \text{ for ice}), \end{aligned}$$

where k_i are constants,

$$(8)\qquad T_1 = T_2 \quad \text{on } \Gamma,$$

and

$$k_1 \frac{\partial T_1}{\partial n} - k_2 \frac{\partial T_2}{\partial n} = V_n \quad \text{on } \Gamma. \tag{9}$$

In (9) V_n is the velocity of the free boundary Γ in the direction of the normal n. If Γ is given by $\Psi(x,t) = 0$, then (9) can be written in the form

$$k_1 \nabla_x T_1 \cdot \nabla_x \Psi - k_2 \nabla_x T_2 \cdot \nabla_x \Psi = \Psi_t.$$

The system (7)–(9) supplemented by initial conditions at $t = 0$ and boundary conditions on the fixed boundary (for all time) is called the *two-phase Stefan problem*; if $T_2 \equiv 0$ (or $T_1 \equiv 0$), then it is called the *one-phase Stefan problem*. Condition (9) is the balance-of-energy equation.

In one dimension the one-phase Stefan problem is the following: Find a curve $x = s(t)$ and a function $u(x,t) \geq 0$ such that

$$\begin{aligned} u_t &= u_{xx} \quad &&\text{if } 0 < x < s(t) \text{ and } t > 0, \\ u(0,t) &= g(t) &&\text{if } t > 0, \\ u(x,0) &= h(x) &&\text{if } 0 < x < s(0), \end{aligned}$$

and

$$u(s(t),t) = 0, \quad -u_x(s(t),t) = \frac{ds}{dt}, \quad t > 0.$$

Here $h > 0$ and $g > 0$.

Alternatively, the condition $u(0,t) = g(t)$ can be replaced by $-u_x(0,t) = k(t)$ for $t > 0$, and we assume $k(t) \geq 0$.

In either case this problem has a unique solution, and $s(t)$ is C^∞ with $\dot{s}(t) > 0$ for $t > 0$; see [3] and the references given there.

In higher dimensions the Stefan problem has a unique "weak solution" in the sense that the function

$$T = \begin{cases} T_1 & \text{if } T_1 \geq 0 \\ T_2 & \text{if } T_2 \leq 0 \end{cases}$$

and its gradient $\nabla_x T$ belong to H^1 for all $t > 0$, and T is continuous; see Caffarelli and Friedman, 1979; and Caffarelli and L. Evans, 1983. However, the free boundary Γ may in general develop singularities. For the one-phase problem, if Γ has the form

$$x_m = k(x_1, \ldots, x_{m-1}, t) \text{ in a neighborhood } N,$$

where k is Lipschitz continuous, then Γ is C^∞ in the same neighborhood; see Caffarelli [2], and Kinderlehrer and Nirenberg, 1977.

Image Development in Electrophotography

The photocopying process consists of several steps. The main ones are: (i) the formation of an electric image of the document on the surface of the moving photoconductor drum; (ii) the development of a visible image formed by electrically charged toner particles, some of which get deposited just above the electric image; (iii) the transfer of the toner deposition from the surface of the photoconductor drum onto paper. We consider here a mathematical model for step (ii). For simplicity we consider a two-dimensional model and concentrate on an electric image of just one black interval. We take the development zone to be a rectangle

$$R = \{-a < x < a,\ -c < y < b\},$$

the surface of the photoconductor to be $y = 0$ (with toner particles coming from above $y = 0$), and the electric image to have intensity σ at $\{y = 0,\ -d < x < d\}$ and zero elsewhere on $y = 0$ $(d < a)$. Then the electric potential $-\varphi$ satisfies

$$-\Delta\varphi = \begin{cases} \rho & \text{in } D(t) \\ 0 & \text{in } R \setminus D(t), \end{cases} \tag{10}$$

where $D(t)$ is the region occupied by the toner at time t; ρ is the toner density (assumed constant); and

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= 0 \quad &&\text{on } x = \pm a, \\ \varphi &= -M &&\text{on } y = b,\ M > 0, \\ \varphi &= 0 &&\text{on } y = -c. \end{aligned} \tag{11}$$

The potential difference at the boundaries $y = b$ and $y = -c$ is maintained in order to guide the charged toner particles downward. The electric image is accounted for by the jump condition

$$\frac{\partial\varphi}{\partial y}(x, 0+) - \frac{\partial\varphi}{\partial y}(x, 0-) = -\sigma, \quad -d < x < d. \tag{12}$$

We anticipate the toner region $D(t)$ to have the form

$$D(t) = \{(x,y) \mid 0 < y < f(x,t),\ -\gamma(t) < x < \gamma(t)\}, \tag{13}$$

where

$$\Gamma = \{(x,y) \mid y = f(x,t),\ -\gamma(t) < x < \gamma(t)\} \tag{14}$$

is the free boundary. Initially there is no toner in the development zone; i.e.,

$$f(x,0) = 0, \quad -d < x < d. \tag{15}$$

Across Γ, φ and $\nabla_x\varphi$ are continuous, so we need just one additional condition on the free boundary in order to complete the formulation of the problem. This condition comes from the fact that the rate of increase of the free boundary in the normal direction, V_n, is proportional to the force $\nabla\varphi \cdot n$ of the electric field at the boundary; here n denotes the normal pointing into $R \setminus D(t)$. Thus

$$V_n = -\frac{\partial\varphi}{\partial n}. \tag{16}$$

This condition is similar to the Stefan condition.

Theorem 1 [7]. *The toner problem (10)–(16) has a unique solution with $C^{2+\alpha}$ free boundary for some time interval* $0 \leq t < T$.

We briefly describe the strategy of the proof. We try to describe the toner region at time t, $D(t)$, as traced by a family of curves $\Gamma(\tau)$, $0 \le \tau \le t$. Thus $\Gamma(t)$ coincides with the curve defined in (14). It is convenient to write the curves $\Gamma(\tau)$ in the form

$$x = \tilde{x}(\sigma, \tau), \quad y = \tilde{y}(\sigma, \tau),$$

where σ is a parameter. If $\sigma = h(s)$ and $h(s)$ is a monotone function, then s can also serve as a parameter. One can choose a function $h(s)$ so that, if we set

$$x = x(s, \tau), \quad y = y(s, \tau)$$

when $x(s,\tau) = \tilde{x}(h(s), t)$ and $y(s,\tau) = \tilde{y}(h(s), \tau)$, then the condition (16) takes the form

$$\frac{dx}{dt} = -\varphi_x(x, y, t), \quad \frac{dy}{dt} = -\varphi_y(x, y, t). \tag{17}$$

Given a family of increasing domains $D(t)$, one constructs a family of solutions $\varphi(x, y, t)$ of (10)–(12) and then defines a new family of domains $\tilde{D}(t)$ traced by the solution curves of (17) with the initial conditions $x(s, 0) = s$ and $y(s, 0) = 0$. This process defines a mapping

$$\{D(t)\} \xrightarrow{W} \{\tilde{D}(t)\}.$$

One can show that W has a unique fixed point if T is small enough.

As an interesting by-product of the proof, we obtain a shape of $D(t)$, for small t, as shown in Figure 2. We see that the visible image of a black interval is lighter in the center and darker near the edges. This phenomenon, observed especially in older photocopy machines, is called the *edge effect*.

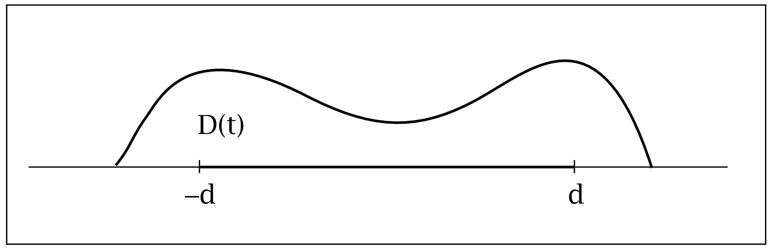

Figure 2. The toner region.

Chemical Vapor Deposition

A *wafer* is a cylindrical slab of a crystal (mostly silicon) 10–16 cm in diameter on which a semiconductor is built. One of the steps in this process is chemical vapor deposition. The wafer lies on the table in a reactor chamber, and a flux g of chemical species (plasma) is injected from the top of the chamber. The plasma diffuses in the chamber, and some of it is deposited on the wafer. The surface of the wafer has already been divided into many small regions, which are the seats of the transistors. Each transistor is of linear dimension smaller than 10^{-6}m; it has a metallic conductor in the center of its upper face and insulating silicon oxide at the boundary. Hence the physical properties of the surface of the wafer vary significantly on a very small scale.

Figure 3 shows a simplified two-dimensional geometry in which the reactor chamber is a rectangle

$$R = \{-a < x < a,\ 0 < y < b\}$$

and the wafer extends from $x = -a$ to $x = a$. The surface of the wafer

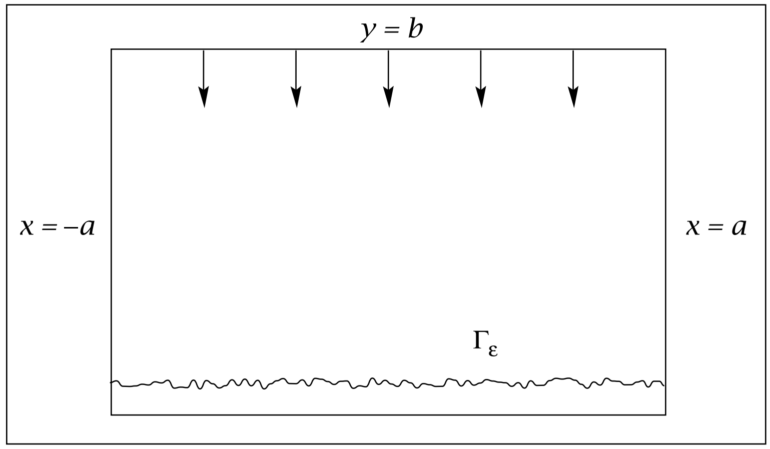

Figure 3. A domain with microscale features.

$$\Gamma_\epsilon = \{(x, y) \mid y = \epsilon f_\epsilon(x, t)\}$$

is nearly flat, but it is growing with the plasma deposition, and it is growing unevenly at a small scale ϵ (where $\epsilon = 10^{-5}a$) due to the oscillation in the physical properties of the surface of the transistors.

For simplicity we assume that the plasma concentration u diffuses according to

$$u_t - \Delta u = 0 \quad \text{in } R_\epsilon, \tag{18}$$

where

$$R_\epsilon = R \cap \{y > \epsilon f_\epsilon(x, t)\}; \tag{19}$$

we do not include chemical reactions and temperature effects. We also have the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \qquad \text{at } x = \pm a, \tag{20}$$

$$\frac{\partial u}{\partial y} = g(x, t) \quad \text{at } y = b. \tag{21}$$

On the free boundary the deposition is given by

$$\frac{\partial u}{\partial n} + p_\epsilon u = 0 \quad \text{on } \Gamma_\epsilon, \tag{22}$$

where n is the outward normal and the absorption coefficient p_ϵ is positive and fast oscillating. Finally, by conservation of mass,

$$V_n = \epsilon \frac{\partial u}{\partial n}, \tag{23}$$

where V_n is the velocity of the free boundary in the direction $-n$.

The microscale features of p_ϵ and $f_\epsilon(x,0)$ make it impossible to compute the solution by standard finite-differences or by finite-element methods. One therefore seeks to compute the "averaged" solution instead of the solution itself. This is often possible to do if the oscillations of the coefficients and of the boundary are periodic; problems of this kind are called *homogenization problems*. We shall now proceed by assuming that the microscale features have a periodic structure, namely,

$$p_\epsilon = p\left(x, \frac{x}{\epsilon}\right), \quad f_\epsilon(x,0) = F_0\left(x, \frac{x}{\epsilon}\right), \tag{24}$$

where $p(x,\xi)$ and $F_0(x,\xi)$ are periodic in ξ of period 1.

As for the control function g, we assume that it belongs to a set

$$A = \left\{ g \in L^\infty(Q) \;\middle|\; 0 \le g \le K \text{ and } \iint_Q g\,dx\,dt \le M \right\},$$

where

$$Q = Q_{T_0} = \{-a < x < a,\ 0 < t < T_0\}.$$

A question of interest is how to determine the (macroscopic) control function g so that certain features of the fast-oscillating free boundary are best achieved. As a first step we would like to approximate the free boundary by a curve

$$y = \epsilon f\left(x, \frac{x}{\epsilon}, t\right),$$

where $f(x,\xi,t)$ is periodic in ξ of period 1.

Theorem 2 [4]. *There exists a $T_0 > 0$ depending only on K such that for each $g \in A$ there exists a unique solution $(u_\epsilon, \epsilon f_\epsilon)$ of (18)–(24). Furthermore,*

$$\|u_\epsilon - u_0\|_{L^2(R_\epsilon)} \le C\epsilon$$

and

$$\left| f_\epsilon(x,t) - f\left(x, \frac{x}{\epsilon}, t\right) \right|_{L^\infty} \le C\epsilon,$$

where $f(x,\xi,t)$ is periodic in ξ of period 1 and u_0 and f satisfy the following system:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} - \Delta u_0 &= 0 && \text{in } R,\\
\frac{\partial u_0}{\partial x} &= 0 && \text{on } x = \pm a,\\
\frac{\partial u_0}{\partial y} &= g(x,t) && \text{on } y = b,\\
-\frac{\partial u_0}{\partial y} + P u_0 &= 0 && \text{on } y = 0,
\end{aligned}$$

$$\frac{\partial f}{\partial t}(x,\xi,t) = p(x,\xi) u_0(x,0,t)\left[1 + \left(\frac{\partial f}{\partial \xi}(x,\xi,t)\right)^2\right]^{1/2},$$

$$f(x,\xi,0) = F_0(x,\xi).$$

In this system $P(x,t)$ is an average of p_ϵ:

$$P(x,t) = \int_0^1 p(x,\xi)\left[1 + \left(\frac{\partial f}{\partial \xi}(x,\xi,t)\right)^2\right]^{1/2} d\xi.$$

The function u_0 satisfies an elliptic problem coupled by the boundary condition at $y = 0$ to a hyperbolic problem for $f(x,\xi,t)$. The function $f(x,\xi,t)$ is called the *homogenization* of $f_\epsilon(x,t)$.

Consider now the control problem of trying to choose the best g so that the free boundary will be as close as possible to a prescribed curve in the L^2 norm. In view of Theorem 2 it is natural to prescribe this curve in the form $\epsilon c(x, x/\epsilon, t)$, where $c(x,\xi,t)$ is periodic in ξ of period 1. Then the functional we wish to minimize is

$$J(g) = \int_0^1 \int_Q |f(x,\xi,t) - c(x,\xi,t)|^2\,dx\,dt\,d\xi, \quad g \in A.$$

Take any minimizer g_0 of $J(g)$ and any function ℓ such that $g_0 + \delta\ell$ belongs to A for all small $\delta > 0$. Then

$$J(g_0 + \delta\ell) \ge J(g_0) \quad \text{for all small } \delta > 0. \tag{25}$$

We expand the left-hand side in δ and derive an inequality on the coefficient of the linear term in δ. Next, by introducing the "adjoint problem", we construct a function $W(x,t)$ depending on g_0 but not on ℓ, such that the inequality that we derived from (25) reduces to

$$\int_Q W(x,t)\ell(x,t)\,dx\,dt \ge 0.$$

We then deduce:

Theorem 3 [5]. *There exists a constant λ such that*

$$g_0 = \begin{cases} K & \text{if } W < \lambda \\ 0 & \text{if } W > \lambda. \end{cases} \tag{26}$$

This condition is also "almost" sufficient for the optimality of g_0.

Although g_0 is not known, we can use Theorem 3 to construct from an initial guess g_0 another function, g_1, by the right-hand side of (26), where λ is determined, so that

$$\int_Q g_1\,dx\,dt = M.$$

In the same way we construct from g_1 a function $W = W_1$ and use it to obtain a function g_2, etc. Numerical experiments show that starting from any g_0 and constructing the g_n inductively in a slightly different way, we obtain a sequence g_m that converges fast to a limit [5]. The slightly different way uses the functionals

$$\tilde{J}_n(g) \equiv$$
$$J(g) + \mu_n \int_Q (g_n - g)^2 \, dx \, dt$$

with $\mu_n > 0$ and $\mu_n \downarrow 0$, instead of $J(g)$, to determine g_{n+1}. It has not been proved, however, that $\{g_m\}$ is in fact convergent and that the limit, if existing, is a minimizer of $J(g)$.

Coating Flows

Photographic film is manufactured by spreading layers of "dispersions" separated by buffer layers over a film base. The buffer layers are gelatin, and the dispersion layers are gelatin mixed with silver halide microcrystals and oil droplets. The layers are initially in fluid form. A sheet of fluid emerges from a slot in a hopper and falls under the influence of gravity onto the film that is moving horizontally on a conveyer belt. After one layer dries out, another is spread. The fluid sheet is often guided by external air pressure. Figure 4 shows a stationary two-dimensional geometry with two fluid/air interfaces. Figure 5 is still a simpler geometry with just one fluid/air interface.

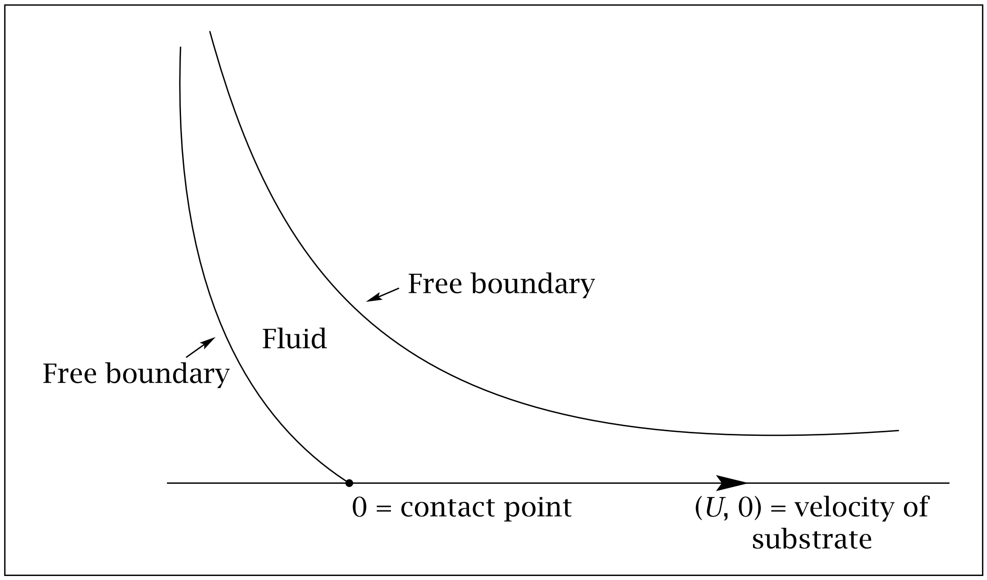

Figure 4. Free-fall coating flow.

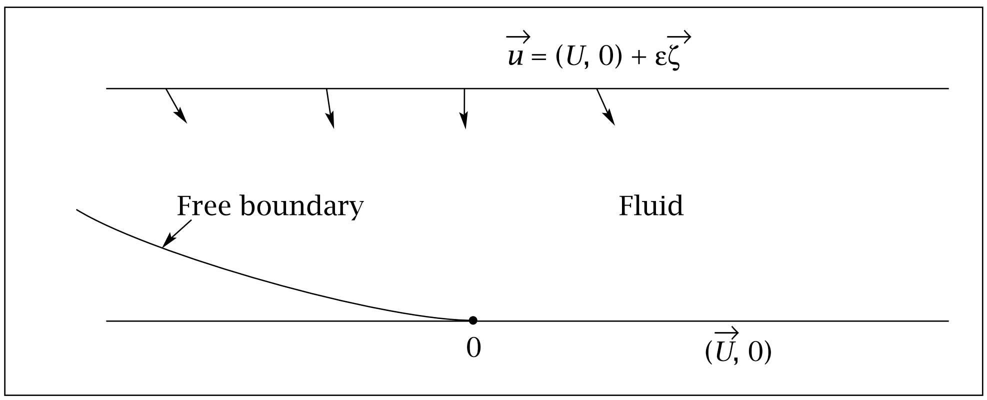

Figure 5. Coating flow in a strip.

The velocity $\vec{u}$ of the fluid at the substrate is $\vec{u} = (U, 0)$, the same as the velocity of the substrate. (This is called the *no-slip condition.*) The velocity $\vec{u}$ on the upper boundary $y = b$ of the strip in Figure 5 is pointed downward.

We consider the problem described in Figure 5. In the fluid the velocity $\vec{u}$ and the pressure p satisfy the Navier-Stokes equation

$$\vec{u} \cdot \nabla \vec{u} + \Delta \vec{u} = \nabla p \quad \text{in } \Omega \tag{27}$$

and the incompressibility condition

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega. \tag{28}$$

Since the free boundary

$$\Gamma = \{(x, y) \mid y = f(x), \ x < 0\}$$

is a streamline,

$$\vec{u} \cdot \vec{n} = 0 \quad \text{on } \Gamma, \tag{29}$$

where $\vec{n} = (n_i)$ is the normal. We also have

$$T_{ij} n_j = \sigma \kappa n_i \quad \text{on } \Gamma, \tag{30}$$

where σ is the reciprocal of the capillary number, κ is the curvature, and

$$T_{ij} = -p\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the stress tensor.

The initial point of the free boundary, which is the common point of the free boundary and the substrate, is called the *contact point*, and the angle between Γ and the horizontal substrate at the contact point is called the *dynamic contact angle*. For good-quality coating, it is necessary to maintain uniformity of the dynamic contact angle. Experiments show that this angle depends on the substrate speed U, but there is no theory giving the precise relationship between U and the dynamic contact angle. If the dynamic contact angle is not $180°$, then the conditions $\vec{u} = (U, 0)$ and $\vec{u} \cdot \vec{n} = 0$ at 0 imply that $\vec{u}$ is discontinuous at 0. A discontinuity, however, is unacceptable physically, and thus either (i) one needs to modify the no-slip boundary condition near 0, or (ii) one should consider the dynamic contact angle as an *apparent* angle that can be measured only at some distance away from 0, whereas $f'(0) = 0$.

Experiments are not sharp enough to decide which alternative is the correct one, and there are at present several different theories based on a modification of the no-slip conditions near 0. Regarding (ii), the question arises whether the problem (27)–(30) is well posed, i.e., whether it has a unique (stable) solution with $f'(0) = 0$. An answer is given by the following result:

Theorem 4 [8]. *If the velocity $\vec{u}$ on $y = b$ is "sufficiently close" to the velocity $(U, 0)$ of the moving substrate, then the coating problem (27)–(30) shown in Figure 5 has a unique solution with free boundary satisfying the following conditions:*

$$(31)\qquad \begin{aligned} &0 < f(x) < b \quad \text{for } -\infty < x < 0,\ f(0) = 0; \\ &f(x) \sim A(-x)^{2-\rho} \quad \text{near } x = 0,\ A > 0. \end{aligned}$$

Here

$$0 < \rho < \tfrac{1}{2}, \quad \rho = \frac{1}{2\pi i} \log \frac{1 - 2i\lambda}{1 + 2i\lambda}, \quad \lambda = -\frac{U}{\sigma}.$$

The condition that $\vec{u}(x, b)$ is "sufficiently close" to the velocity $(U, 0)$ is understood in the following more precise and rather restrictive sense: $\vec{u}(x, b) = (U, 0) + \epsilon\vec{\xi}(x)$ when $\vec{\xi}(x)$ satisfies a "technical condition" and $|\epsilon|$ is sufficiently small; the "technical condition" says that, for the "linearized" problem about $\epsilon = 0$, the linearized free boundary $y = f_0(x)$ satisfies (31).

The fluid sheet emerging out of the slot presents a challenging jet problem. Even if one lets the viscosity tend to zero and the Navier-Stokes equation is replaced by the ideal fluid equation, one must maintain the surface tension term on the free boundary (due to the thinness of the sheet). In terms of the stream function ψ, the ideal fluid equation is $\Delta\psi = 0$. On the fluid/air interface ψ is constant, and Bernoulli's law has the form

$$\tfrac{1}{2}|\nabla\psi|^2 + gy = \text{const.} + \sigma\kappa,$$

where $\sigma\kappa$ as in (30) is the pressure due to surface tension. For more details, see S. Weinstein et al., 1997.

Tumor Growth

Mathematical models of tumor growth have been developed over the last twenty-five years. Here we consider a model that depends on the nutrient concentration σ and on the internal pressure p of the proliferating tissue but does not include the concentration of inhibitors, the effect of angiogenesis (that is, the sprouting of blood vessels that penetrate into the tumor), and the presence of the necrotic zone; for more details see [1] and [9].

The proliferation rate is assumed to be a linear function $\mu(\sigma - \tilde{\sigma})$, where $\tilde{\sigma}$ is a given equilibrium level and μ is a positive constant. The tumor receives nutrients through the boundary, where the concentration is a constant $\overline{\sigma}$ with $\overline{\sigma} > \tilde{\sigma}$. The diffusion equation for σ is

$$(32)\qquad c\sigma_t - \Delta\sigma + \lambda\sigma = 0 \quad \text{in } \Omega(t),$$

where $\Omega(t)$ is the evolving tumor region and λ and c are positive constants. The constant c is the quotient of two time scales, that of the diffusion (~ 1 minute) and that of tumor doubling (~ 1 day), and is therefore very small.

The pressure p induces velocity $\vec{q}$ of tissue cells by means of the relation $\vec{q} = \nabla p$, and, by conservation of mass, $\operatorname{div}\vec{q} = \mu(\sigma - \tilde{\sigma})$. Consequently,

$$(33)\qquad \Delta p - \mu(\sigma - \tilde{\sigma}) = 0 \quad \text{in } \Omega(t).$$

We also have the boundary conditions

$$(34)\qquad \sigma = \overline{\sigma} \quad \text{on } \partial\Omega(t),$$

$$(35)\qquad p = \gamma\kappa \quad \text{on } \partial\Omega(t),$$

where κ is the mean curvature of the free boundary $\partial\Omega$ and γ measures the cell-to-cell adhesiveness. Finally, the motion of the free boundary is determined by

$$(36)\qquad \frac{\partial p}{\partial n} = -V_n \quad \text{on } \partial\Omega(t),$$

where V_n is the velocity of the free boundary in the direction of the normal n.

For spherically symmetric initial data it was proved (Friedman and F. Reitich, 1999) that there exists a unique spherically symmetric solution

$$\sigma(r, t), \quad p(r, t), \quad \Omega(t) = \{r < R(t)\}$$

and that, as $t \to \infty$,

(i) $R(t) \to \infty$ if $\tilde{\sigma}/\overline{\sigma} > \frac{1}{3}$,

(ii) $R(t) \to R_0$ if $\tilde{\sigma}/\overline{\sigma} < \frac{1}{3}$,

provided c is sufficiently small. Here R_0 is the radius of the unique time-independent solution. The same is true in the two-dimensional case with $\frac{1}{3}$ replaced by $\frac{1}{2}$.

The above model is phenomenological and does not address the very complex processes that take place both within the tumor and in its vicinity. Such a model cannot be predictive. It is nevertheless useful if it can explain or confirm experimental results in a qualitative way. Successful examples to this effect are mentioned in H. Byrne and M. Chaplain, 1995–1999, and in references therein.

Since tumors are not spherical, it is important, in order to further validate the model (32)–(36), to exhibit solutions that are not radially symmetric. A result in this direction for time-independent solutions is given in the following theorem.

Theorem 5 [6]. *In the two-dimensional case, for any integer $\ell \geq 2$ and for any radius R_0, there is a unique $\gamma = \gamma_0$ that is a bifurcation point of a unique* *branch of analytic solutions of (32)–(35) with $c = 0$ and that has*

$$\gamma = \gamma_0 + \sum_{n\geq 2} \epsilon^n \gamma_n$$

and free boundary

$$r = R_0 + \epsilon \cos \ell\theta + \sum_{n\geq 2} \epsilon^n f_n(\theta),$$

for which the $f_n(\theta)$ are even functions of θ satisfying the orthogonality conditions

$$\int_0^{2\pi} f_n(\theta)\cos \ell\theta \, d\theta = 0, \quad n \geq 2.$$

Since $\ell \geq 2$, these solutions are not radially symmetric for small $|\epsilon|$.

The strategy of the proof goes as follows. We first transform the tumor domain Ω into the unit disc by

$$r' = r \Big/ \left[R_0 + \epsilon \cos \ell\theta + \sum_{n\geq 2} \epsilon^n f_n(\theta)\right]$$

and then write a formal power series in ϵ for σ and p. We obtain a system of differential equations, say Π_n, by collecting all the coefficients of ϵ^n. The system involves

$$(37) \qquad \frac{1}{n!} D_\epsilon^n p \Big|_{\epsilon=0}, \quad \frac{1}{n!} D_\epsilon^n \sigma \Big|_{\epsilon=0}, \quad \gamma_{n-1}, \quad f_n,$$

and γ_n. We then proceed to solve for the quantities in (37) and to estimate them inductively by $H_0 H^n$, where H_0 and H are suitable constants. This establishes convergence of the series in ϵ, for $|\epsilon| < 1/H$.

It is hoped that the result stated in Theorem 5 will stimulate further work toward developing a complete theory for the free boundary problem (32)–(36).

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About the Cover

Snow is formed when particles of water vapor from clouds freeze in the upper air, becoming crystalline flakes that fall softly to the earth. The shape of a snowflake as developed by the freezing process is governed by the distribution of temperature within the water particles and by the law of conservation of energy along the boundary. This shape is not known in advance, and its surface is a classical example of a free boundary formed by nature. The cover shows a typical snowflake, with several stems originating from the center. Each stem is a branch along which many other smaller branches have grown. Along each of these smaller branches, another sequence of yet smaller secondary branches will often develop. Mathematical models and their analysis try to capture and predict the dendritic shape of crystals, such as snowflakes, grown in solidification processes.

Solidification processes occur in industry, such as in the production of steel and in growing crystal for semiconductors. The manufacturers are interested in controlling their processes so as to make the free boundary as smooth as possible.

The graphic on the cover is taken from a poster for the 1990–91 program "Interfaces and Free Boundaries" at the Institute for Mathematics and Its Applications in Minneapolis. The organizers were R. Fosdick, M. Gurtin, W.-M. Ni, and B. Peletier.

—*A. F.*