The Quadratic Family as a Qualitatively Solvable Model of Chaos

Mikhail Lyubich

In the last quarter of the twentieth century the *real quadratic family*

$$f_c : \mathbb{R} \to \mathbb{R}, \quad f_c : x \mapsto x^2 + c \quad (c \in \mathbb{R})$$

was recognized as a very interesting and representative model of chaotic dynamics. It contains regular and stochastic maps intertwined in an intricate manner. It also has remarkable universality properties. Complexification of this family leads to a beautiful interplay between real and complex dynamics, supplying us with powerful geometric tools. By now we have a complete picture of the basic dynamical features in this family, which will be described below.

To put the quadratic family into perspective, we begin in the first two sections with a brief discussion of the philosophy and conceptual background of the field of "chaotic dynamics". Then we pass to the main themes of this article: the Basic Dichotomy, which can be regarded as a complete qualitative description of dynamics in the real quadratic family; an outline of ideas from holomorphic dynamics that yield one half of the final picture; and renormalization theory, which completes the picture.

Philosophy of "Chaotic Dynamics"

Three-Body Problem

According to V. Arnol'd,[1] Newton's fundamental discovery was that "It is useful to solve differential equations." Unfortunately, at the same time it is hard to solve differential equations: Two centuries after the successful explanation of Kepler's laws of motion of the Earth around the Sun (neglecting the gravitational field of other planets), all attempts to integrate analytically the three-body problem (the motion of the Earth around the Sun taking into account the gravitation of, say, Jupiter) had failed. An attempt to solve this problem led Poincaré to a radical change of viewpoint: Instead of finding *explicit* analytic solutions of differential equations, one can try to describe *qualitative* behavior of these solutions.

The simplest kind of motion is "no motion", represented by stationary solutions. A little more complicated are periodic oscillations represented by closed orbits. Next are solutions that are forward or backward asymptotic to the stationary or periodic ones. The next ones would be the solutions that are asymptotically stationary or periodic in both forward and backward times ("homoclinic" trajectories). Poincaré first believed that homoclinic solutions in the three-body problem fill some smooth submanifold in the phase space. Discovery that this might be wrong[2] led Poincaré to a picture described by him in the following famous words:

> One is struck by the complexity of this figure that I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three body problem and of all the problems of dynamics in general.

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[1] *In the first paragraph of his book* Geometric Methods in the Theory of Ordinary Differential Equations.

[2] *This was one of the most productive mistakes in the history of science; see the book* Poincaré and the Three Body Problem *by J. Barrow-Green for a very interesting account of this story.*

(See Figure 1.) Basic features of this complexity are the presence of many *recurrent* (i.e., returning arbitrarily close to the initial position) trajectories, their instability, and the co-existence of many qualitatively different types of behavior. Attempts to understand phenomena of this kind led to creation of the "theory of chaos".

Goals

Instead of dealing with continuous trajectories of a differential equation, the theory of dynamical systems often prefers to consider their discrete analogues $\{f^n z\}_{n=0}^{\infty}$, where f^n are the n-fold iterates of a single map $f : M \to M$ of the phase space M. A simple way to pass from a continuous flow $F^t : M \to M$, $t \geq 0$, to a discrete system is to specialize t to the multiples of some real r and to take $f = F^r$. Another natural and useful way is to take the "first return map" to some *Poincaré section* (i.e., a transversal to the flow) if such a map exists. Of course, there are many other sources of dynamical systems with discrete time.

When dealing with "chaotic" dynamical systems (depending on some parameters), there is little chance to describe, even qualitatively, all trajectories of every single system in the class. So one should try to look for typical phenomena within this class. Poincaré's Recurrence Theorem and Boltzmann's Ergodic Hypothesis originated this approach.

This immediately raises a question: What should be considered typical? Since the beginning of the twentieth century there have been two competing approaches to this issue, from the measure-theoretic (or rather, probabilistic) viewpoint and from the viewpoint of Baire category. In his address to the International Congress of Mathematicians in Amsterdam in 1954, Kolmogorov compared these two approaches:

> Approach from the categorical side is interesting more as a tool of proving existence results…, while an approach from the measure-theoretic side seems to be physically reasonable and natural…."[3]

This viewpoint is generally accepted nowadays. Though dynamics is a highly nonhomogeneous field with many different branches and viewpoints, there seems to be general agreement on its main destination:

Main Goal of Dynamics: Study asymptotic behavior of *almost all* orbits for *almost any* parameter value in *representative* finite-parameter families of dynamical systems.

[3] *What Kolmogorov announced in his address was the biggest breakthrough in dynamical systems theory since Poincaré: persistence of most (in the measure-theoretic sense) invariant tori under a perturbation of an integrable system of classical mechanics. The theory that grew out of this discovery is now called KAM theory after Kolmogorov, Arnol'd, and Moser.*

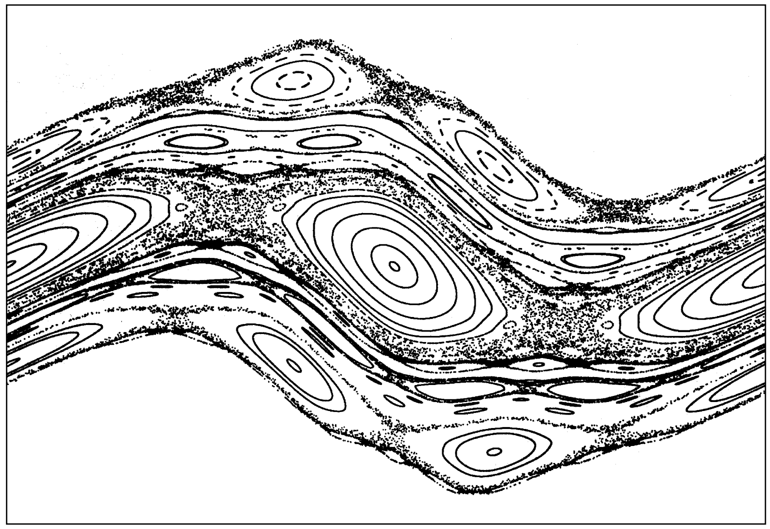

Figure 1. This is the kind of picture Poincaré would have seen if he had had access to the public-domain program dstool described in *Notices* 39 (1992), 303–309. Such phase portraits are typical for Poincaré return maps of nonintegrable systems of classical mechanics with two degrees of freedom. This particular picture is produced by iterating a model map $(x, y) \mapsto (x + y - \epsilon \sin(2\pi x),\ y - \epsilon \sin(2\pi x))$ **with** $\epsilon = 0.154\ldots$.

This formulation raises several questions: "Almost all" with respect to which measure? In what terms can the asymptotic behavior of orbits be described? What families of dynamical systems are "representative"?

Conceptual Background

Hyperbolicity

A central dynamical idea that was developed in the 1960s and 1970s (by Smale, Anosov, Sinai, and many others) was the idea of hyperbolicity. Roughly speaking, it means that over a recurrent part of the phase space there exist two transverse invariant foliations, stable and unstable, that are (respectively) uniformly exponentially contracted and expanded by the dynamics. This implies that all recurrent trajectories are exponentially unstable, either in forward or in backward time. Despite this instability, the global dynamics of uniformly hyperbolic systems turned out to be tractable. At the same time, it turned out that these systems are quite scarce, not dense in the space of dynamical systems on a given manifold (except for the one-dimensional case, as we will see below). For instance, Newhouse (1979) discovered that infinitely many attracting cycles can coexist for a locally generic (in the Baire sense) set of dynamical systems on a compact manifold M (while uniformly hyperbolic systems on M can have only finitely many attracting cycles).

This development indicated that the purely topological viewpoint should be replaced by a measure-theoretic one and stimulated a search for

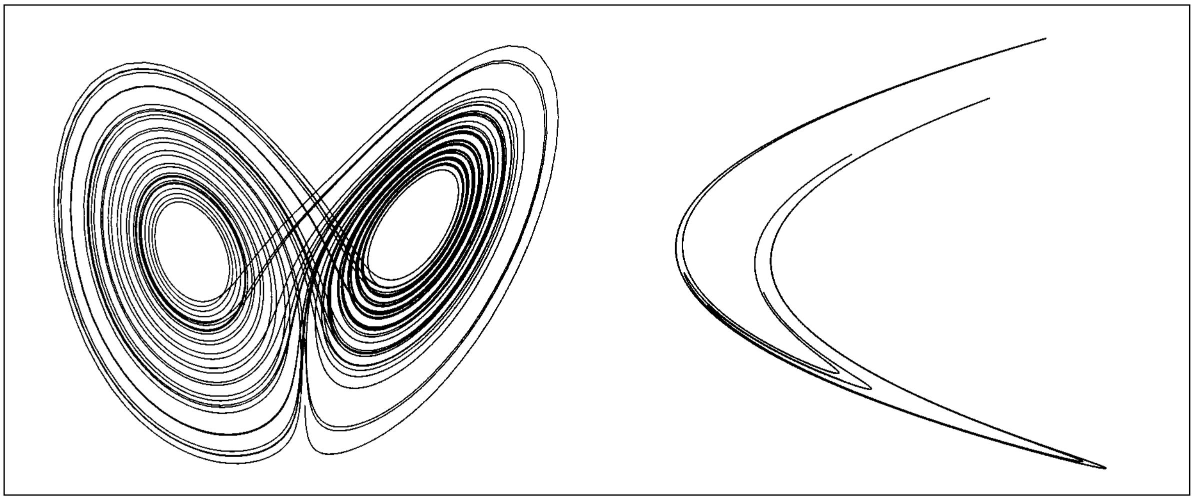

Figure 2. Lorenz butterfly and Hénon swallow.

some less restrictive notions of hyperbolicity. Oseledets-Pesin theory (mid 1970s) developed a very general form of "hyperbolicity" of an invariant measure μ. Roughly speaking, this kind of hyperbolicity means existence of transverse stable and unstable manifolds for μ-typical points, which are exponentially contracted and expanded by the dynamics but perhaps in a *nonuniform* way. It in particular yields positivity of the leading *characteristic exponent*

$$\chi(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\| > 0 \tag{1}$$

of almost all orbits of μ, i.e., exponential instability of μ-almost all trajectories.

But a chaotic dynamical system usually has a wealth of invariant measures. What would be the best choice?

SRB Measures

Consider a dynamical system generated by a smooth map $f : M \to M$ on a smooth Riemannian manifold M. This manifold is endowed with the Riemannian measure λ, which is of course physically most natural. According to the above discussion, we wish to describe the asymptotic behavior of λ-almost all orbits $\{f^n x\}_{n=0}^{\infty}$. The idea is that this asymptotic behavior should be governed by some invariant measure. Assume that there exists a compactly supported invariant measure μ such that:

- For a set of initial points $x \in M$ of positive Riemannian measure, the Birkhoff averages weakly converge[4] to μ:

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \to \mu \quad \text{as} \quad n \to \infty,$$

where δ_y is the delta measure supported at the point $y \in M$;

- μ has positive entropy.[5]

Then μ is called an *SRB measure* of f, after Sinai, Ruelle, and Bowen, who proved that a uniformly hyperbolic dynamical system on a compact space has finitely many such measures that jointly govern—together with finitely many attracting cycles—the behavior of λ-almost all orbits.

The first property in the definition of SRB measure is tied to the Birkhoff Ergodic Theorem. If an invariant probability measure μ is absolutely continuous with respect to λ (such a measure will be abbreviated as "a.c.i.m.") and if μ is *ergodic* in the sense that all measurable invariant sets have μ-measure 0 or 1, then the theorem implies that the first property is satisfied.

If the first property in the above definition is satisfied for λ-a.e. $x \in M$, then μ will be called a *global* SRB measure. A map f with a global SRB measure will be called *stochastic*.

Low-Dimensional Phenomenon

The next question is, which families of dynamical systems are "representative"? A democratic answer would be "the most popular." A natural science answer would be "those that model important phenomena in nature." A purely mathematical answer would be either "simplest nontrivial families" or "generic families." In the course of the article we will see that all these approaches actually point in the same direction.

One of the most stimulating discoveries in contemporary dynamics was the discovery of the "Lorenz attractor" (1963). It appeared in an innocent-looking system of three ordinary differential equations approximating equations of gas dynamics. Computer experiments showed that for some parameter values the trajectories of this system converge to an attractor with an intricate structure. It demonstrated that high-dimensional

[4] *The space of measures is regarded as the dual of the space of compactly supported continuous functions.*

[5] *Entropy measures how chaotic the invariant measure is. The notion of SRB measure has not been canonized yet, and the reader can find in the literature slightly different versions of it.*

phase spaces are not needed in order to encounter chaos.

The next step was made by Hénon (1976), who suggested a very simple discrete two-dimensional model

$$(x, y) \mapsto (x^2 + c + by, x), \quad (x, y) \in \mathbb{R}^2,$$

for a Poincaré section of the Lorenz attractor.[6] Again, computer experiments showed that for some parameter values (e.g., $c = -1.4$, $b = 0.3$) the orbits of this system converge to a "strange attractor". Much later, it was proven by Benedicks, Carleson, Lai-Sang Young, and Viana (1990s) that there is a positive measure set of parameters (c, b) with a tiny b for which the attractor indeed exists and supports a global SRB measure.

If we now let $b \to 0$, the Hénon family will degenerate to the one-dimensional quadratic family $f_c : x \mapsto x^2 + c$. These reductions suggested that the quadratic family can give some clues to a qualitative understanding of the nature of turbulence. At the same time R. May (1976) suggested the quadratic family as a model for population dynamics, and this work ignited a great interest in this family.[7]

At this point we have to stop in our search for the simplest model for chaotic dynamics: the quadratic family cannot be simplified any further. Still, it turns out that this family exhibits extremely rich properties and sends important messages to the bigger world.

Real Quadratic Family

Works of Sharkovsky (mid-1960s); Milnor, Thurston, and Misiurewicz (mid-1970s); and other people demonstrated how rich one-dimensional dynamical systems can be from the topological and combinatorial points of view. Soon afterwards people started to explore the measure-theoretic picture.

Regular and Stochastic Behavior

First, let us restrict ourselves to the parameter values $c \in [-2, 1/4]$, for which the quadratic map f_c has a nontrivial invariant interval I_c.[8] The parameter interval and the dynamical intervals are all endowed with the Lebesgue measure λ. For the above parameter values two types of measure-theoretic behavior can be described:

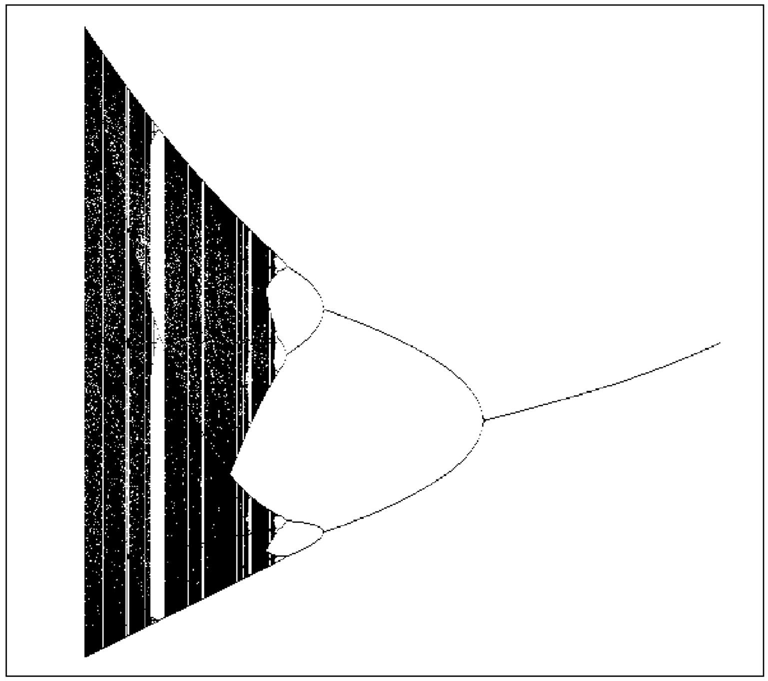

Figure 3. Real quadratic family as a model of chaos. This picture presents how the limit set of the orbit $\{f_c^n(0)\}_{n=0}^{\infty}$ bifurcates as the parameter c changes from $1/4$ on the right to -2 on the left. Two types of regimes are intertwined in an intricate way. The gaps correspond to the regular regimes. The black regions correspond to the stochastic regimes (though of course there are infinitely many narrow invisible gaps therein). In the beginning (on the right) one sees the cascade of doubling bifurcations. This picture became symbolic for one-dimensional dynamics.

Regular maps. There exists an *open* set $\mathcal{R}$ of parameter values c, which are called *regular* values, such that f_c has an attracting cycle $\bar{\alpha}_c = \{f^k \alpha_c\}_{k=0}^{p-1}$ (as an example, for $c \in (-3/4, 1/4)$, f_c has an attracting fixed point). Consider the attracting basin of this cycle:

$$D(\bar{\alpha}_c) = \{x \in I_c : f_c^n x \to \bar{\alpha}_c\}.$$

It turns out that it has full Lebesgue measure in I_c (Guckenheimer, 1979) and, moreover, that f_c is uniformly exponentially repelling on the complement to the basin. The last property shows that regular maps are uniformly hyperbolic in the sense of Smale.

Stochastic maps. There exists a positive measure set S of parameter values c, which are called *stochastic* values, such that the map f_c has an absolutely continuous (with respect to λ) invariant measure μ (a.c.i.m.). This result was proven in works of Jakobson (1981) and Benedicks and Carleson (1985).[9] It turns out that in this setting any

[6] *An actual Poincaré section of the Lorenz attractor looks quite different.*

[7] *Another motivation for studying one-dimensional dynamics came from KAM theory through the works of Arnol'd and Herman on circle dynamics.*

[8] *For $c > 1/4$, all real orbits of f_c escape to ∞. For $c \le 1/4$, both fixed points $\alpha_c \le \beta_c$ of f_c are real. Moreover, for $c \in [-2, 1/4]$, $c \ge -\beta_c$ and the interval $I_c \equiv [-\beta_c, \beta_c]$ is f_c-invariant. For $c < -2$, the critical point 0 escapes to ∞. One can show that in this case all points except a zero measure Cantor set escape to ∞ as well.*

[9] *There had been many previous works constructing an a.c.i.m. for particular parameter values. One of the earliest was the work of Bunimovich (1970) concerned with the sine family, which already contained such key ideas as relation to the Renyi expanding map, recovery from the contraction near the critical point due to expansion near a repellor, and distortion bounds.*

a.c.i.m. is automatically a global SRB measure (Blokh and Lyubich, 1991). Moreover, the characteristic exponent (1) is positive, so that stochastic maps are nonuniformly hyperbolic in the sense of Oseledets and Pesin.

The absolutely continuous invariant measure of a stochastic map f_c is supported on a cycle of intervals with disjoint interiors,

$$J_0 \underset{f}{\to} J_1 \underset{f}{\to} \dots \underset{f}{\to} J_{p-1} \underset{f}{\to} J_0, \tag{2}$$

with the convention that 0 is in J_0. If one selects a λ-typical point $x \in I_c$, then its orbit eventually lands in this cycle and its Birkhoff averages will converge to μ.

Thus both regular and stochastic[10] phenomena cannot be neglected in the real quadratic family. The immediate question is whether these two phenomena exhaust everything observable.

Basic Dichotomy *[L4]. For almost every $c \in [-2, 1/4]$, the quadratic map $f_c : x \mapsto x^2 + c$ is either regular or stochastic.*

This result can be regarded as a complete qualitative description of the nature of chaos in the real quadratic family, in the sense of the Main Goal of Dynamics stated earlier.

We will see that what is hidden behind this brief statement is intricate small-scale properties of the "Julia sets" and the "Mandelbrot set", "Mostow-like rigidity phenomena", as well as a general version of the "Feigenbaum Universality Conjecture".

Renormalization

Renormalization is a central concept in contemporary dynamics. The idea is to study the small-scale structure of a class of dynamical systems by means of a renormalization operator R acting on the systems in this class. This operator is constructed as a rescaled return map, where the specific definition depends essentially on the class of systems.

The quadratic family is naturally embedded in the space $\mathcal{U}$ of C^2-smooth unimodal interval maps[11] with nondegenerate critical point, considered up to rescaling of their intervals of definition. In this setting the definition of renormalization is very simple: a unimodal map f is called *renormalizable* if it has a cycle of intervals like (2). Then the restriction of f^p to the central interval J_0 is again a unimodal map. If p is the smallest period of periodic intervals of f, then the map

$$Rf = f^p | J_0 \tag{3}$$

considered up to rescaling of J_0 is called the *renormalization* of f. This map can possibly be renormalizable itself. In this case f is called twice renormalizable, and we can consider its second renormalization $R^2 f$, and so on.

In this way we can classify all quadratic maps f_c (and the corresponding parameter values $c \in [-2, 1/4]$) as *at most finitely* or *infinitely* renormalizable. Let $\mathcal{F}$ stand for the set of nonregular parameter values that are at most finitely renormalizable, while $\mathcal{I}$ stands for the set of infinitely renormalizable parameter values. It turns out that the set S of stochastic parameter values is strictly contained in $\mathcal{F}$. Thus we arrive at the following classification of the real quadratic maps:

$$[-2, 1/4] = \mathcal{R} \cup \underset{\underset{S}{\cup}}{\mathcal{F}} \cup \mathcal{I},$$

with the sets $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{I}$ disjoint by definition. With this decomposition the Basic Dichotomy amounts to the following two statements:

Theorem A. *Almost every $c \in \mathcal{F}$ is stochastic: $\lambda(\mathcal{F} \setminus S) = 0$.*

Theorem B. *Infinitely renormalizable parameter values are neglectable: $\lambda(\mathcal{I}) = 0$.*

Theorem A was proven jointly by the author with Marco Martens and Tomasz Nowicki in [L2], [MN], while Theorem B was proven by the author in [L4]. These results are formulated in purely real terms. However, the tools for their proofs came from the quite different world of holomorphic dynamics.

Puzzles, Rigidity, and Invariant Measures

Holomorphic dynamics (iteration theory of holomorphic maps in the complex plane) was founded by P. Fatou and G. Julia in the late 1910s. For a long time the subject had been isolated from the rest of mathematics and almost forgotten. The situation changed radically by the end of the century.

In the 1930s classical complex analysis was enriched with the theory of quasi-conformal maps (Grötzsch, Lavrentiev, Morrey, …). These maps found profound applications to the theory of deformations of Riemann surfaces, the theory of Fuchsian and Kleinian groups (Teichmüller, Ahlfors and Bers, …), and hyperbolic geometry (Mostow, Thurston, …). In the early 1980s D. Sullivan's insight on the intimate connection between these developments and the Fatou-Julia theory led to the revival of holomorphic dynamics. Since then, ideas of rigidity and quasi-conformal deformations have played a prominent role in the field, and the field itself has become an intrinsic part of the theory of dynamical systems, analysis, and geometry.

Julia Sets and the Mandelbrot Set

Complex quadratic maps have a much richer topological and geometric structure than their real

[10] *The above description of the behavior of typical orbits implies that a quadratic map cannot be simultaneously regular and stochastic: $\mathcal{R} \cap S = \emptyset$.*

[11] *A smooth map of an interval to itself is* unimodal *if it has a unique critical point.*

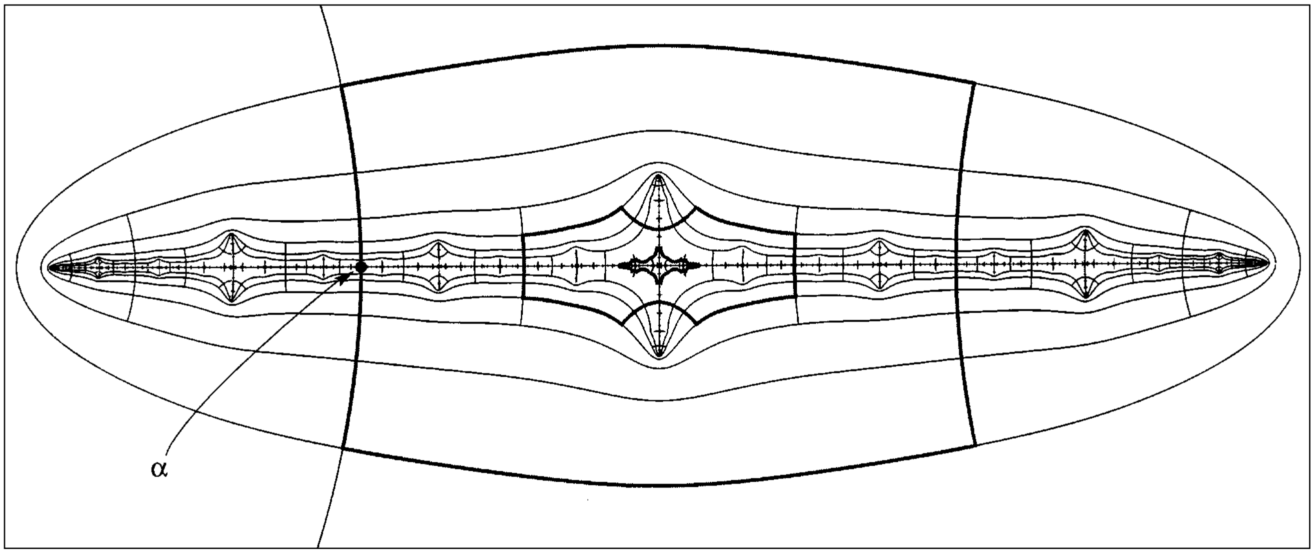

Figure 4. The Julia set of the "Fibonacci map" $f_c : z \mapsto z^2 + c$, $c = -1.870\ldots$ (see Lyubich and Milnor, 1993) with several initial levels of the puzzle. Highlighted are three principal puzzle pieces.

traces. To see this, just compare the real interval with the complex Julia sets depicted in Figure 4. Despite the intricate structure of these sets, they have a very short definition.

Near infinity all orbits of any complex quadratic map $f \equiv f_c : z \mapsto z^2 + c$, $c \in \mathbb{C}$, escape to ∞. Define the basin of infinity as the set of all escaping orbits:

$$D_f(\infty) = \{z : f^n z \to \infty \text{ as } n \to \infty\}.$$

The complementary compact set is called the *filled Julia set*, $K(f) = \mathbb{C} \setminus D_f(\infty)$, and its boundary is called the *Julia set* $J(f) = \partial K(f)$. It is classically known that for a quadratic polynomial, the Julia set and the filled Julia set are either Cantor or connected depending on whether the critical point 0 escapes to infinity or not. The parameter values c for which the Julia set $J(f_c)$ is connected form the *Mandelbrot set M* in the parameter plane (see Figure 7). For instance, for $c \in [-2, 1/4]$, the critical orbit does not escape the interval I_c, so that the Julia set $J(f_c)$ is connected. In fact, $[-2, 1/4] = M \cap \mathbb{R}$.

If the filled Julia set is connected, then by the Riemann Mapping Theorem its complement can be conformally uniformized by the complement of the unit disk,

$$(\mathbb{C} \setminus \bar{\mathbb{D}}, \infty) \to (\mathbb{C} \setminus K(f), \infty).$$

The images of radial rays and concentric circles under this Riemann map are called respectively *external rays* and *equipotentials*. They form two f-invariant foliations of $\mathbb{C} \setminus K(f)$.

Douady and Hubbard (Orsay Notes, 1984–85) suggested analyzing Julia sets by means of cutting them into pieces by external rays. An efficient cutting procedure was then explored by Yoccoz (1990, unpublished). Assume that both fixed points of $f \equiv f_c$, $c \in M$, are repelling (for real c this happens when $c \in [-2, -3/4)$). It turns out that one of these points, called α, is the landing point of more than one external ray. Cut the Julia set by the rays landing at $\alpha \equiv \alpha_c$, and truncate these rays by some equipotential. We obtain a partition of a neighborhood of the filled Julia set into several pieces. Then take pull-backs of this partition by the iterates of f. We obtain finer and finer partitions of shrinking neighborhoods of the Julia set into pieces (see Figure 4). They form a kind of jigsaw puzzle of the Julia picture, and consequently the whole procedure became known as the *Yoccoz puzzle*. If we can estimate the sizes and shapes of these puzzle pieces, we can get good control of the geometry of the Julia set.

Puzzle Geometry

The key information is contained in a *principal nest* of puzzle pieces, $V^0 \supset V^1 \supset V^2 \supset \cdots \ni 0$, which comes together with branched double coverings $g_n = f^{l_n} : V^n \to V^{n-1}$, where l_n is the *first return* time of the critical orbit to V^{n-1}. The principal nest keeps track of the recurrence of the critical orbit.

The critical value $g_n 0$ can land anywhere in V^{n-1}. It is important to distinguish two combinatorial situations, according to whether the critical value lands in the topological annulus $V^{n-1} \setminus V^n$ ("noncentral return") or immediately lands in the next puzzle piece V^n ("central return"); see Figure 5. The central return indicates a fast recurrence of the critical orbit, which makes the dynamics less tractable. Let $\{n_k\}$ denote the sequence of levels on which the return is noncentral: $g_{n(k)}(0) \in V^{n_k - 1} \setminus V^{n_k}$.

The geometry of the principal nest can be efficiently controlled by the conformal moduli[12] of the

[12] *Any topological annulus $A \subset \mathbb{C}$ whose boundary components are not singletons can be conformally uniformized by a round annulus $\{z : 1 < |z| < r\}$. Then by definition* $\operatorname{mod}(A) = \log r$. *It is the only conformal invariant of the annulus.*

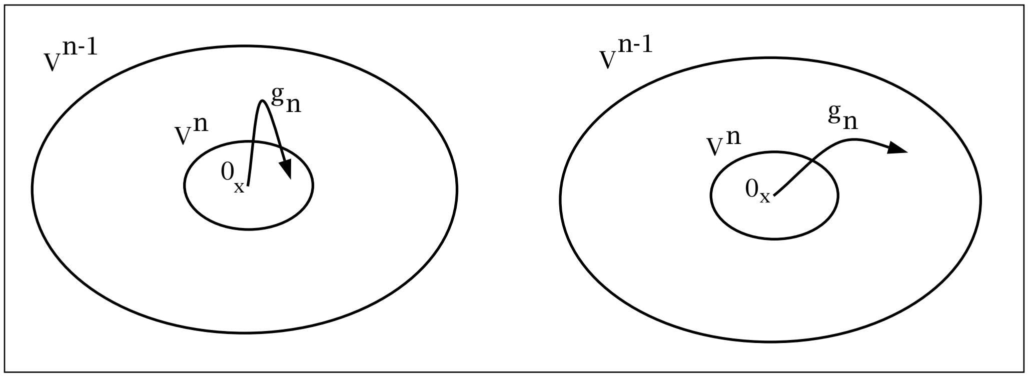

Figure 5. Central and noncentral returns.

Combinatorial Rigidity Conjecture would imply density of hyperbolic maps.

It had been observed by Sullivan that to prove the Combinatorial Rigidity Conjecture, it would be enough to show that *any two combinatorially equivalent quadratic polynomials are quasi-conformally conjugate* (which nicely fits the spirit of the proof of Mostow rigidity). That is what is proven in [L1, II] for real parameter values. Theorem 1 is what allows one to control the quasiconformal dilatation of a conjugacy between two quadratic maps.

topological annuli $V^{n-1} \setminus V^n$. A problematic scenario is when these moduli shrink to zero: then we lose geometric control of the puzzle. A very nice situation is when the moduli grow to ∞: then the double coverings $g_n : V^n \to V^{n-1}$ (which are high iterates of a quadratic map) become purely quadratic up to a small distortion. It turns out that the latter scenario actually takes place on the levels just following noncentral returns:

Theorem 1 *([L1], I). There exists a constant $C > 0$ such that*

$$\mathrm{mod}(V^{n_k} \setminus V^{n_k+1}) \geq Ck.$$

The constant C depends only on finite combinatorial data, and it is, in particular, uniform for all real parameter values $c \in [-2, -3/4)$.[13]

This result, which may sound a little technical, will find numerous applications in what follows.

Rigidity Phenomenon and Density of Hyperbolic Maps

As we have mentioned before, regular quadratic maps can be viewed as uniformly hyperbolic in the sense of Smale.

Theorem 2 *([L1], II). The set $\mathcal{R}$ of uniformly hyperbolic maps is dense in the parameter interval* $[-2, 1/4]$.

There is a deep rigidity phenomenon behind this statement intimately related to rigidity phenomena in geometry. The Mostow Rigidity Theorem tells us that two topologically equivalent compact hyperbolic manifolds of dimension > 2 must actually be isometric. It was conjectured that a similar phenomenon takes place in dynamics: *Two nonhyperbolic complex quadratic polynomials f_c and $f_{c'}$ that are combinatorially equivalent*[14] *must actually coincide.* This Combinatorial Rigidity Conjecture is equivalent to the famous MLC Conjecture, formulated by Douady and Hubbard in their Orsay Notes, asserting that the *Mandelbrot set is locally connected.* In both real and complex settings the Combinatorial Rigidity Conjecture would imply density of hyperbolic maps.

[13] *For real maps a related result was independently obtained in [GS].*

[14] *Combinatorial equivalence can be naturally defined in terms of landing properties of external rays. Topological conjugacy implies combinatorial equivalence.*

Absolutely Continuous Invariant Measures

Existence of an absolutely continuous invariant measure is related to the rate of expansion along the critical orbit, that is, the rate of growth of the derivatives $|Df^n(c)|$. It was shown by Collet and Eckmann (1983) that the a.c.i.m. does exist if the rate is exponential. This criterion was improved by Nowicki and van Strien (1988), who replaced the exponential rate with the summability condition

$$\sum |Df^n(c)|^{-1/2} < \infty. \quad (4)$$

Since the strongest contraction occurs near the critical point 0, one should expect that the rate of *expansion* along the critical point is related to the rate of *recurrence* of the critical orbit. Here is a nice result of this kind:

Theorem 3 *(Martens and Nowicki [MN]). Let $c \in [-2, -3/4)$. If all the returns to the principal nest are eventually noncentral, then f_c is stochastic.*

In fact, it is shown in [MN] that the assumption of the above theorem (together with Theorem 1) implies the summability condition (4).

Parapuzzle Geometry

To prove Theorem A, we wish to show that the criterion of Martens and Nowicki is satisfied for almost all $c \in \mathcal{F}$. Let us restrict ourselves to the set $\mathcal{N} \subset [-2, -3/4]$ of *nonrenormalizable* parameter values. The heuristic argument goes as follows. Imagine a one-parameter family of return maps $g_{n,c} : T^n_c \to T^{n-1}_c$, $c \in L$, where $T^n_c = V^n_c \cap \mathbb{R}$ are real traces of the puzzle pieces. Imagine that when the parameter c runs over the interval L, the critical value $g_{n,c}(0)$ runs through T^{n-1}_c with a more or less uniform speed (see Figure 6). Then the probability that $g_{n,c}(0)$ lands at T^n_c (i.e., the probability of the central return) is comparable with $|T^n_c|/|T^{n-1}_c|$. But Theorem 1 tells us that the latter is exponentially small, provided that the previous level was not central. By the Borel-Cantelli Lemma the probability of infinitely many central returns is equal to zero.

There is one big assumption in this heuristic argument, namely, that the critical value moves with *uniform* speed through the interval T_c^{n-1} (which is also moving with c). To justify this assumption, one needs to prove uniform transversality of the two motions involved. With real methods only, this would be a desperate problem. However, one of the miracles of the complex world is that transversality can be obtained for purely topological reasons (the Argument Principle). That is how this problem is dealt with in [L2].

There is a remarkable similarity between the dynamical pictures of different quadratic maps (Julia sets) and the parameter picture of the whole quadratic family (the Mandelbrot set). As Adrien Douady put it: "You first plow in the dynamical plane and then harvest in the parameter plane." What is actually done in [L2] is the transfer of the dynamical Theorem 1 to the parameter plane. Namely, for any k, one partitions the parameter plane into "parapuzzle pieces" of level k and shows that the conformal moduli between consecutive "principal" pieces grow linearly (see Figure 7). This gives the desired geometric control of the parameter interval.

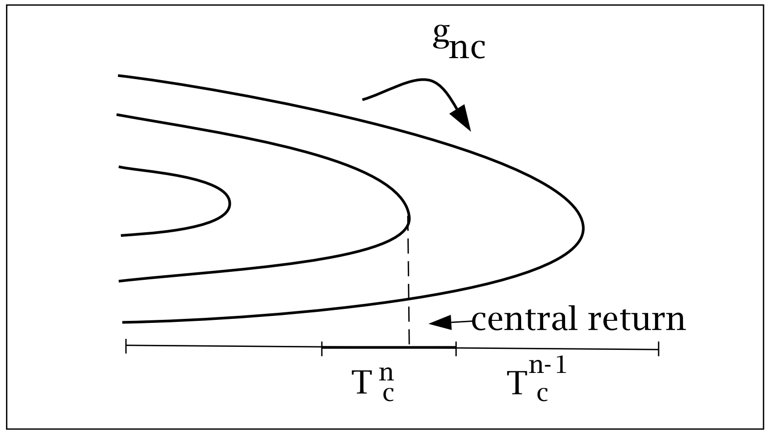

Figure 6. How the critical value $g_{n,c}(0)$ moves through the moving interval T_c^{n-1}.

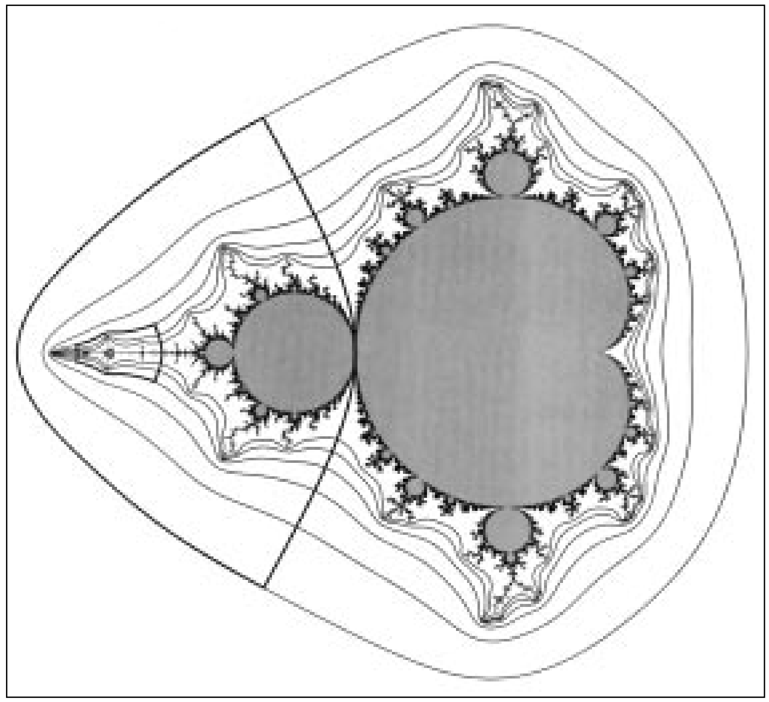
Figure 7. The Mandelbrot set with three principal parapuzzle pieces around the Fibonacci parameter value $c = -1.870\ldots$ (the pieces shrink very fast).

Universality

Discovery

In the mid-1970s a truly remarkable discovery was made by Feigenbaum and independently by Coullet and Tresser. Consider the real quadratic family $x \mapsto x^2 + c$, and let c decrease from $1/4$ to -2. In the beginning we observe an attracting fixed point, which then bifurcates into an attracting cycle of period 2, which then bifurcates into an attracting cycle of period 4, etc. (see Figure 3). Thus we observe a sequence[15] of *doubling bifurcations* c_n converging to a limit parameter value $c_* = -1.401\ldots$ called the *Feigenbaum point*. With the help of a calculator Feigenbaum observed that this convergence is exponential: $c_n - c_* \sim C\rho^{-n}$, where $\rho = 4.669\cdots$. This observation was curious, but what was really surprising is that if we take a similar family of unimodal maps, say $x \mapsto b \sin x$ on $[0, \pi]$, then we observe a similar sequence of doubling bifurcations b_n exponentially converging to a limit point b_* with the *same rate*: $|b_* - b_n| \sim C'\rho^{-n}$, where $\rho = 4.669\cdots$. In other words, the rate of convergence appears to be *universal*, independent of the particular family of unimodal maps under consideration.

Renormalization Conjecture

Motivated by the renorm-group method in statistical mechanics, Feigenbaum and Coullet-Tresser formulated a beautiful conjecture that would completely explain the above universality. Imagine an infinite-dimensional space $\mathcal{U}$ of unimodal maps, and consider the doubling renormalization operator R in this space (see (3) and Figure 8). The conjecture, now a theorem for a suitably defined $\mathcal{U}$, asserts that:

- R has a unique fixed point f_*, i.e., a unique solution of the *Feigenbaum-Cvitanović equation* $f(z) = \mu^{-1} f \circ f(\mu z)$ with an appropriate scaling factor μ.
- R is hyperbolic at this fixed point; that is, there exist two transverse R-invariant manifolds W^s and W^u through f_* such that the orbits $\{R^n f\}, f \in W^s$, exponentially converge to f_*, while the orbits $\{R^n f\}$, $f \in W^u$, are exponentially repelled from f_*.
- $\dim W^u = 1$.

[15] *This sequence was first observed by Myrberg (1962).*

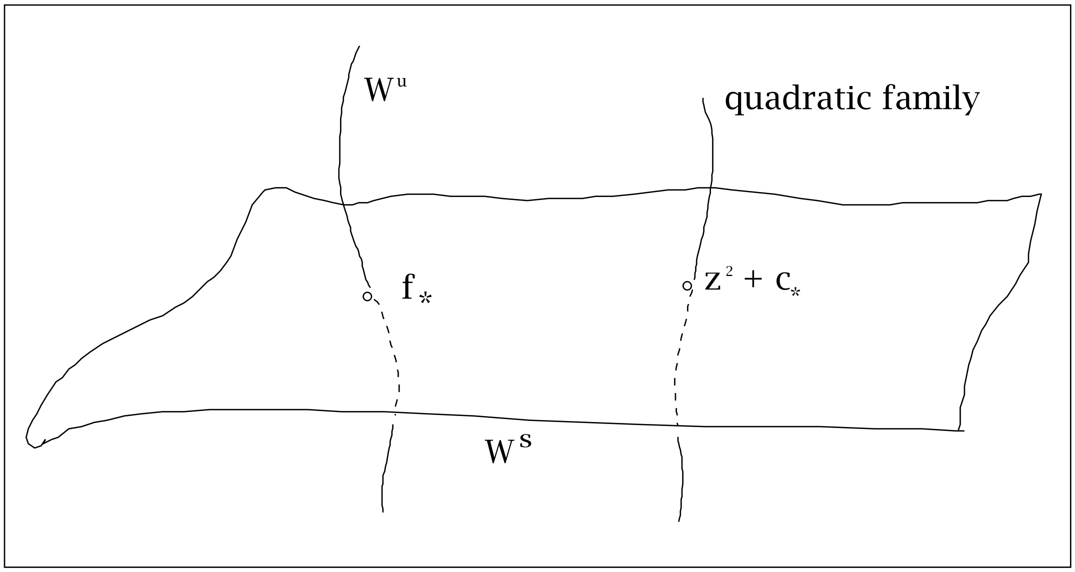

Figure 8. Hyperbolic fixed point of the renormalization operator.

• W^u transversally intersects the doubling bifurcation locus $\mathcal{B}_1$, where an attracting fixed point bifurcates into an attracting cycle of period 2.

Since the doubling bifurcations loci $\mathcal{B}_n$ of higher periods (from 2^n to 2^{n+1}) are obtained by taking preimages of $\mathcal{B}_1$ under R^n, one can readily see that any one-parameter family of unimodal maps (i.e., a curve in $\mathcal{U}$) that is transverse to W^s intersects the $\mathcal{B}_n$ at the points b_n exponentially converging to a limit point $b_* \in W^s$, where the rate of convergence, ρ, is just the *unstable eigenvalue* of $DR(f_*)$. Thus, it is independent of the particular family under consideration.

Renormalization Fixed Points

The first, computer-assisted, proof of the Renormalization Conjecture for the period doublings was given by O. Lanford in 1982. The idea was to find numerically an approximation to the solution of the Feigenbaum-Cvitanović equation and then to prove rigorously that there exists a true hyperbolic solution nearby. In this way the original conjecture was formally checked, at least locally, near the fixed point f_*.[16]

Still, the nature of the universality phenomenon remained mysterious. Also, computer-assisted proofs can conceivably handle only a few small renormalization periods, while the renormalization operator is well defined for arbitrary periods (triplings, quadruplings, etc.), not to mention arbitrary infinite strings of periods.[17] So people kept looking for a "conceptual" proof of the Renormalization Conjecture.

Such a proof was given in the works of Sullivan [S], McMullen [M], and the author [L3], consecutively dealing with different parts of the conjecture.[18] Namely, Sullivan and McMullen proved existence of the fixed point f_* and constructed its stable manifold $W^s(f_*)$, while the author proved hyperbolicity of R at the fixed point and the transversality results. The main feature of this development is that it is almost completely based on methods of holomorphic dynamics.

Let us discuss the main conceptual ingredients of this proof. Fix some combinatorial type, and consider the corresponding renormalization operator R defined by (3). Our first goal is to complexify it:

• *Quadratic-like maps*, introduced by Douady and Hubbard [DH], are complex analogues of unimodal maps. By definition a *quadratic-like map* is a double branched covering $f : U \to U'$, where $U \Subset U'$ are two topological disks. (The domain of f is not invariant!) One can define a renormalization operator acting on "renormalizable" quadratic-like maps, which extends the renormalization (3) of real maps. We will use the same notation R for the extension. It is the operator for which the Renormalization Conjecture has been proven in the above-mentioned papers.

• *Foliated structure of the space of quadratic-like maps.* Two quadratic-like maps are *hybrid equivalent* if they are conjugate by a quasi-conformal map that is conformal almost everywhere on the filled Julia set. The space of quadratic-like maps with connected Julia set is foliated by the hybrid classes. One of these classes (defined without any reference to the renormalization) will later serve as the stable manifolds of R. The quadratic family is a global transversal to this foliation [DH], [L3].

• *A priori bounds* give a bound from below for $\mathrm{mod}(U'_n \setminus U_n)$, where $U'_n \setminus U_n$ are the fundamental annuli of quadratic-like maps $R^n f : U_n \to U'_n$. They imply compactness of the orbit $\{R^n f\}_{n=0}^{\infty}$ of an infinitely renormalizable map f. This is the only part of the proof that relies on the assumption that f is real [S].

• *Teichmüller theory of Riemann surface laminations.* Sullivan's approach to the fixed point

[16] *This was perhaps the first experience with rigorous computer-assisted proofs, which nowadays have become quite widespread.*

[17] *More precisely, one should talk about* combinatorial types *of the renormalization rather than periods only. The combinatorial type is determined by the ordering of the intervals of (2) on the real line.*

[18] *See [L3] for a more detailed historical account.*

problem was to argue that the renormalization operator strictly contracts the "Teichmüller metric" in an appropriate space of conformal objects associated with quadratic-like maps [S].

• *McMullen towers* are geometric limits of rescalings of an infinitely renormalizable quadratic-like map f with *a priori* bounds. A tower is called *quasi-conformally rigid* if it does not admit nontrivial quasi-conformal deformations. McMullen has proven a result on quasi-conformal rigidity of towers, which implies that the orbit $\{R^n f\}_{n=0}^{\infty}$ converges to a unique fixed point f_* of R, [M]. This gives an alternative approach to the fixed point problem.

In fact, rigidity and universality are two ways of looking at the same phenomenon.

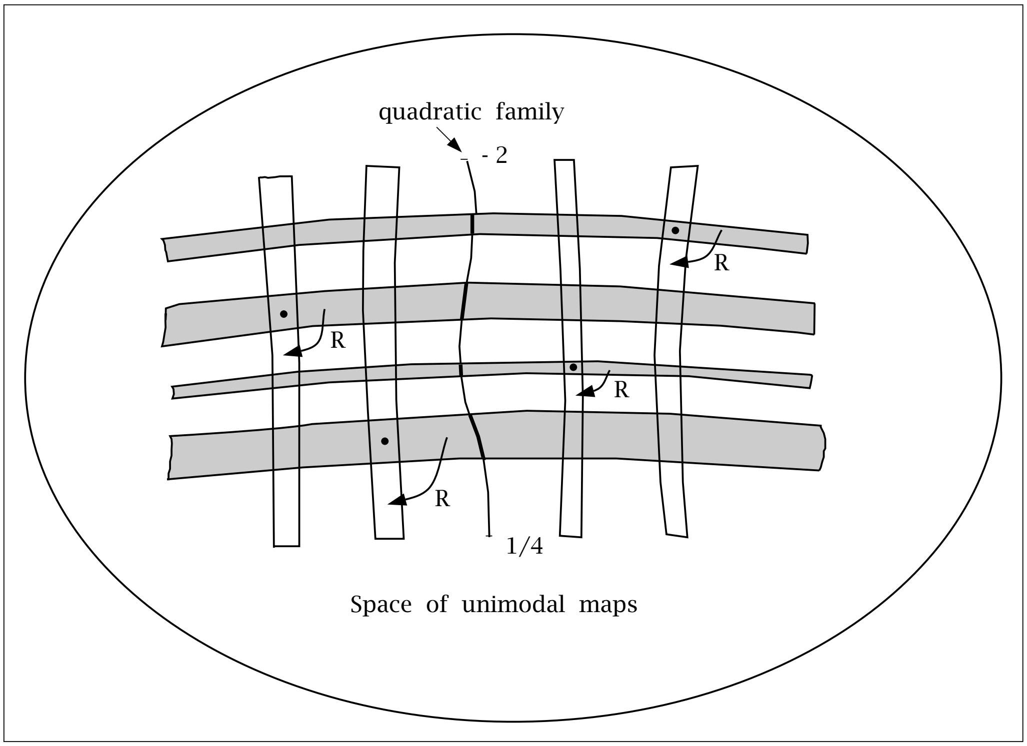

Figure 9. Full renormalization operator. Highlighted are the "renormalization windows" of renormalizable quadratic maps with the same combinatorics. Of course, there are infinitely many windows and corresponding renormalization strips. The dots indicate renormalization fixed points.

• *Combinatorial rigidity.* The rigidity phenomenon in holomorphic dynamics was discussed above. The Rigidity Theorem of [L1] can be applied to certain complex maps as well. In this form the theorem implies, in particular, that if a complex infinitely renormalizable (under R) quadratic-like map f has *a priori* bounds, then f belongs to the hybrid class $\mathcal{H}(f_*)$ of the renormalization fixed point f_*.

• *Schwarz Lemma in Banach spaces.* Using this tool, one can detect for a topological reason that R is exponentially contracting on $\mathcal{H}(f_*)$ [L3].

• *Small Orbits Theorem.* This is a tool allowing one to prove that R is transversally expanding at f_*. Namely, the lack of expansion would imply existence of a "small orbit" $\{R^n f\}_{n=0}^{\infty}$ arbitrarily close to f_* that would not belong to $\mathcal{H}(f_*)$. This would contradict the combinatorial rigidity stated above. This completes the proof of hyperbolicity of R at the fixed point f_* [L3].

• *Argument Principle.* As we have already mentioned, this phenomenon of the complex world gives one a tool to prove diverse transversality results, in particular, the last part of the Renormalization Conjecture [L3].

Full Renormalization Horseshoe

Let us now consider the renormalization operator R for all combinatorial types simultaneously. It can be regarded as defined on infinitely many "horizontal" strips of the space of real quadratic-like maps, one strip for each combinatorial type (see Figure 9). According to the previous discussion, each of these strips contains a hyperbolic fixed point. It turns out that these points are actually uniformly hyperbolic and, even more, that the renormalization operator has a global uniformly hyperbolic structure.

Theorem 4 *[L4]. The full renormalization operator R has a uniformly hyperbolic invariant subset $\mathcal{A}$ whose points represent all possible two-sided strings of combinatorial types.*

This statement encodes diverse universality properties of the bifurcation sets in one-parameter families of unimodal maps. In this way *uniform hyperbolicity* of an *infinite-dimensional* operator sheds light on the dynamics of *one-dimensional* but highly *nonhyperbolic* maps. In particular, it allows one to complete the proof of the Basic Dichotomy.

The proof of Theorem 4 requires all the machinery described above (puzzle and parapuzzle geometry, rigidity theorems, Schwarz Lemma,…) plus several extra ingredients like uniform *a priori* bounds for real infinitely renormalizable maps with arbitrary combinatorics [LS], [LY], rigidity of towers with "essentially bounded combinatorics" (related to the saddle-node bifurcation) [H], and a "Shadowing Theorem" generalizing the Small Orbits Theorem to arbitrary combinatorics [L4].

Theorem B is a simple corollary of Theorem 4. Namely, using some standard hyperbolic machinery, one shows that points $f \in \mathcal{A}$ are not density points in the unstable manifolds $W^u(f)$. Then this property is transferred to the quadratic family by means of the holonomy along the stable foliation. By the Lebesgue Density Theorem, $\lambda(\mathcal{I}) = 0$.

Perspective

Why is the quadratic family actually representative, from the mathematical viewpoint, in one-dimensional dynamics? The reason is that it is a *global transversal* to the foliation of the space of (non-regular) smooth unimodal maps into topological classes.[19] This gives a very good chance to extend the above results from the quadratic family to generic families of smooth unimodal maps. There is a beautiful interplay between holomorphic and smooth dynamics that can be efficiently exploited to achieve this goal. Active work in this direction is already under way. Let us mention the work of O. Kozlovsky extending the result on density of hyperbolic maps from the quadratic family to the space of real analytic (and hence C^r-smooth) unimodal maps, and the program carried out by W. de Melo with his collaborators on the proof of the Renormalization Conjecture for smooth unimodal maps. (A very different program based upon purely real methods has been carried out by M. Martens.) Very recently the Basic Dichotomy has been extended to arbitrary nontrivial families of real analytic unimodal maps with nondegenerate critical point (joint work of the author with Artur and Welington de Melos). These developments give good reason to believe that it will not take long to bring one-dimensional dynamics to completion.

There is a parallel program in holomorphic dynamics. Here the main problem is to prove the MLC Conjecture (local connectivity of the Mandelbrot set), which is equivalent to the Combinatorial Rigidity Conjecture. This result would imply density of hyperbolic maps in the complex quadratic family.

In two-dimensional dynamics there is an intense exploration of the dynamics in the Hénon family, both the real one mentioned above and the complex one (Hubbard, Siboni, Bedford and Smillie, ...). Unlike the one-dimensional realm, the real and complex worlds have not yet merged in dimension two. A principal reason is that *conformal* and *holomorphic* are very different concepts in higher dimensions. This is one of the most interesting problems to be addressed.

The following strong conjecture was formulated by Palis [P]:

Conjecture. *For a typical (in the sense of Kolmogorov) smooth dynamical system on a finite-dimensional manifold, there exist finitely many attracting cycles and finitely many SRB measures that govern behavior of almost all orbits.*

[19] *This statement has been made rigorous in some settings, but not yet in full generality.*

Even for diffeomorphisms in dimension two this looks like a project for the whole twenty-first century. The future will show whether the "theory of chaos" can give such a comprehensive answer to the problems of dynamics.

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