Model Theory

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Model theory is a branch of mathematical logic. It has been considered as a subject in its own right since the 1950s. I will try to convey something of the model-theoretic view of the world of mathematical objects. The use of the word "model" in model theory is somewhat different from (and even opposed to) usage in ordinary language. In the latter, a model is a copy or ideal representation of some real object or phenomena. A model in the sense of model theory, on the other hand, *is* supposed to be the real thing. It is a mathematical structure, and that which it is a model of (a set of axioms, say) plays the role of the idealization.

Abraham Robinson was fond of the expression "metamathematics" as a description of mathematical logic. His dream, which he realized in many ways, was to show how metamathematical arguments and considerations can produce new mathematical results. Logic began as the delineation and classification of (logically) valid forms of reasoning. One might not expect this science of tautology to have much to say about the world. However, for reasons people do not fully understand, it does have something to say about the *mathematical* world.

Logic is, in the popular imagination, often associated with undecidability, paradoxes, and pathology. This is of course true in many ways. In fact, possibly the deepest results of twentieth-century logic, such as the Gödel incompleteness theorems, the independence of the continuum hypothesis from the standard axioms of set theory, and the negative solution to Hilbert's Tenth Problem, have this character. However, from almost the beginning of modern logic there have been related, but in a sense opposite, trends, often within the developing area of model theory. In the 1920s Tarski gave his decision procedure for elementary Euclidean geometry. A fundamental result, the "compactness theorem" for first-order logic, was noticed by Gödel, Mal'cev, and Tarski in the 1930s and was, for example, used by Mal'cev to obtain local theorems in group theory. Abraham Robinson in the 1950s introduced differentially closed fields and discovered nonstandard analysis. The eve of the modern era saw James Ax's decidability for the theory of finite fields and Ax-Kochen's asymptotic solution to a problem of E. Artin. The modern era has seen, often for unexpected reasons and sometimes because of the internal development of the subject, that model theory is in a position to discern amazing patterns and analogies in tame mathematics (those areas of mathematics not subject to Gödel undecidability and incompleteness) and in the process to obtain new results. This is what I want to discuss.

I will have to introduce some technical material, outlining the basic notions and objects of model theory as we see them now. In the process I will refer back to and explain something of the older results mentioned above, but my aim is rather to describe current work and prospects for the future. I will be limiting my description and analysis to a few "main trends" in model theory. As such I will either not touch on or give scant attention to many interesting fields of research, such as finite model

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theory (definability in finite structures, which is related to both database theory and computational complexity), universal algebra, nonstandard analysis and the model theory of infinite-dimensional spaces, infinitary logic, generalized quantifiers, “abstract” model theory, and models of arithmetic and set theory.

For further reading on model theory in the spirit of this article, I recommend starting with the volume [4]. A classic text on model theory is [2]. An elegant and very accessible text on o-minimality is [3]. For more on the connections with diophantine-geometric issues, I recommend [1] and [5].

Structures and Definability

The key notion of model theory, which I think cannot be avoided even though there are many attempts in popular expositions to get around it, is the notion of “truth in a structure”. A *structure* M here is simply a set X, say, equipped with a distinguished family of functions from X^n to X (various n) and a distinguished family of subsets of X^n (various n). Here X^n is the Cartesian product $X \times \cdots \times X$, n times. We shall assume that the diagonal $\{(x,x) : x \in X\}$ is among the distinguished sets even if it is not explicitly mentioned. There is not much to say here except that whenever mathematicians focus on a certain object X (or even a category), they are typically interested not in all subsets of X, $X \times X$, etc., but certain ones. For example, if X is an algebraic variety identified with its points in some algebraically closed field, the algebraic geometer will be interested in the algebraic subvarieties of X, $X \times X$, etc., among which of course *is* the diagonal. So the important thing in looking at a structure is not only the underlying set X but also the family of functions and sets with which it is equipped. The integers considered as an additive group (i.e., with addition the only distinguished function) is a very different structure from the integers considered as a ring.

A slightly more general and mathematically quite natural notion is that of a *many-sorted* structure. In the place of a single underlying set or universe X, we have an indexed family $(X_i)_i$ of universes. The distinguished relations and functions live on or go between various Cartesian products of the X_i. Everything we say below, including the notion of truth of first-order formulas in a structure and the compactness theorem, adapts unproblematically to the many-sorted setting.

Nevertheless, let us work with a single underlying set X. If f_i and A_j are the distinguished functions and sets for a structure M, we write M as $(X, f_i, A_i)_{i,j}$. The additive group of reals becomes $(\mathbb{R}, +, 0)$ in this notation. Here $+$ is the addition function from $\mathbb{R}^2$ to $\mathbb{R}$, and 0 refers either to the constant function from $\mathbb{R}^0$ to $\mathbb{R}$ or to the singleton set $\{0\}$. But we are getting ahead of ourselves. We want to be able to distinguish clearly between labels and the objects to which they refer.

Given a structure M, say, let us introduce labels or names for the distinguished functions and sets. This choice of labels gives rise to a *vocabulary* or *language* for the structure M. In fact, the usual way of introducing structures is to start with a vocabulary $L = (f_i, R_j)_{i,j}$ consisting of function labels and set labels of specified *arities* (i.e., the integers n mentioned above) and then to define an L-structure M to be a set X equipped with distinguished functions f_i^M and sets R_j^M corresponding to these labels. One often says, for example, that f^M is the *interpretation* of the label f in the structure M. Different structures can have the same vocabulary.

For example, suppose our language L consists of a single binary function label f. We can form an L-structure M whose underlying set is the integers $\mathbb{Z}$ such that the interpretation f^M of f is addition of integers. Another L-structure N is obtained by again taking $\mathbb{Z}$ as the underlying set but defining f^M to be multiplication of integers.

For another example, suppose the language L' consists of a single set label R of arity 2. We can form an L'-structure P whose underlying structure is the integers $\mathbb{Z}$ such that the interpretation R^P of R is the relation set for $<$, i.e., the subset of $(x, y) \in \mathbb{Z}^2$ with $x < y$.

As a matter of convention, $=$ is always the label for the equality relation on a structure (the diagonal in $X \times X$).

In any case, from the labels for the distinguished functions and sets in a structure M, together with symbols for the usual logical operations “and” ($\wedge$), “or” ($\vee$), “implies” ($\rightarrow$), “not” ($\neg$), “there exists” ($\exists$), “for all” ($\forall$), as well as an infinite supply of “variables” $(x, y, \ldots)$, we can build up *first-order* expressions that can be interpreted as making statements about the structure M and/or about elements of that structure. The first-order aspect is that the variables range over *elements* of the underlying set X (rather than subsets). It should also be emphasized that a first-order formula is a *finite* string of symbols.

What about truth? Suppose f is a label for a unary function. Consider the first-order expression σ:

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y).$$

In words this expresses that whenever $f(x) = f(y)$, then $x = y$. The expression σ is true or false in M according to whether or not the function f^M from X to X is one-one. This σ is an example of a *first-order sentence*. On the other hand, the expression $\phi(x)$:

$$\exists y (f(y) = x)$$

has a “free” or “unquantified” variable x. Whether it is true or false in M depends on what x is. The expression $\phi(x)$ will be true of an element $a \in X$ if and only if a is in the range of f^M. We say that

$\phi(x)$ is an example of a *first-order formula with free variable* x. Traditional notation is:

> $M \models \sigma$ for "σ is true in M". In ordinary language "M is a model of σ". If Σ is a possibly infinite set of sentences, we say $M \models \Sigma$, M is a model of Σ, if it is a model of each $\sigma \in \Sigma$.

Regarding the formula $\phi(x)$ above, we write $M \models \phi(a)$ for "$\phi(x)$ is true of a in M".

It is important to understand not only the power of first-order expressibility but also its limitations. Consider a structure M of the form $(\mathbb{R}, f, <)$, where f is a function from $\mathbb{R}$ to itself and $<$ refers to the relation subset for "less than" in $\mathbb{R}^2$. Let $a \in \mathbb{R}$. Then there is a first-order formula $\psi(x)$ such that for any $a \in \mathbb{R}$, $M \models \psi(a)$ if and only if f is continuous at a. Specifically $\psi(x)$ informally is:

> For all y_1 and y_2 such that $y_1 < f(x) < y_2$, there are z_1 and z_2 with $z_1 < x < z_2$ such that for all w, if $y_1 < w < y_2$, then $z_1 < f(w) < z_2$.

On the other hand, there is no first-order sentence σ (or even set Σ of first-order sentences) expressing that an abelian group $(G, +, 0)$ is a torsion group. One wants to express that for every x, either $x = 0$ or $x + x = 0$ or $x + x + x = 0$ or …. An infinite disjunction appears inside the universal quantifier, so this is not a first-order sentence. A rigorous proof of nonexpressibility could be based on the compactness theorem discussed in the next section.

A key feature of introducing labels is that we can compare in various ways different structures that have the same labels (vocabulary). The notion of isomorphism is clear, but we also have the a priori weaker notion, elementary equivalence. Two structures M and N with the same vocabulary are said to be *elementarily equivalent* if the first-order sentences true in M are precisely the ones true in N. The elementary or first-order theory of a structure M is by definition the set of first-order sentences true in M. This is exactly what we call a *complete theory*.

Classical decidability results can now be stated and made sense of. Tarski (1951) proved that the elementary theory of the real field $(\mathbb{R}, +, \cdot, 0, 1)$ is decidable. That is, there is an effective procedure for deciding whether or not a given first-order sentence σ in the appropriate vocabulary is true in the structure. For example, σ may express that a given finite set of polynomial equations in indeterminates $X_1, \ldots, X_n$ over $\mathbb{Z}$ has a solution in $\mathbb{R}^n$. But there are more complicated examples: for example, whether the set of solutions in $\mathbb{R}^n$ of such a system has N connected components can also be decided, for nontrivial reasons.

Decidability is known also for the theory of the fields $\mathbb{C}$ of complex numbers and $\mathbb{Q}_p$ of p-adic numbers. One can consider also the set of sentences true in all members of a certain class of structures. In the case of the class of all finite fields, decidability was proved by James Ax (1968). So we have decidability results for fields surrounding number theory. Number theory itself lives directly in the structure $(\mathbb{Z}, +, \cdot)$, and Gödel's work yielded undecidability of the theory of this structure. In fact, Matijasevich's negative solution of Hilbert's Tenth Problem says that there is a fixed polynomial $P(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ over $\mathbb{Z}$ such that there is no effective procedure for determining, for any given $a_1, \ldots, a_m \in \mathbb{Z}$, whether or not the equation $P(X_1, \ldots, X_n, a_1, \ldots, a_m) = 0$ has a solution in $\mathbb{Z}$. Gödel's coding methods show that all "effective mathematics" is coded in or contained in the structure $(\mathbb{Z}, +, \cdot)$.

Related to decidability issues is the question of identifying axioms for theories of specific structures. For example, it was proved by Tarski (1949) that the theory of the field $\mathbb{C}$ is axiomatized by the axioms for algebraically closed fields of characteristic 0, and thus all algebraically closed fields of characteristic 0 are elementarily equivalent. This is a rigorous formulation of the "Lefschetz principle". It may seem at first sight rather mysterious that the field $(\overline{\mathbb{Q}}, +, \cdot)$ of algebraic numbers is elementarily equivalent to $(\mathbb{C}, +, \cdot)$, as the latter contains transcendental elements. As in the issue of torsion abelian groups discussed above, the *limitations* of first-order expressibility are responsible. If one tries to write down a sentence expressing the existence of a transcendental element, one inevitably finds oneself writing $\exists x$ followed by an *infinite* conjunction.

Deeper results along these lines, but for "Henselian valued fields", were obtained by James Ax and Simon Kochen (1965, 1966) and Yuri L. Ershov (1965). A *valued field* K is a field equipped with a map v from K into $\Gamma \cup \{\infty\}$, where Γ is an ordered group, the restriction of v to the multiplicative group K^* is a homomorphism, $v(0) = \infty$, and $v(x + y) \geq min\{v(x), v(y)\}$. A valued field is *Henselian* if it satisfies "Hensel's Lemma", whose statement we do not need and consequently omit. It is convenient to think of a valued field as a "composite" of three structures—the underlying field $(K, +, \cdot)$, the value group $(\Gamma, +, <)$, and the "residue field" $(k, +, \cdot)$—together with the valuation $v : F \to \Gamma \cup \{\infty\}$ and the residue map π from the valuation ring of K to k. The Ax-Kochen-Ershov result says that if we restrict ourselves to Henselian valued fields of characteristic 0 whose residue fields are of characteristic 0, the theory of this composite structure is determined by the theory of the value group and the theory of the residue field.

Often the proofs of decidability and axiomatizability results yield information on the "definable sets" in a structure. In fact, identification of the definable sets in a structure has become a central focus. So we should explain what these definable sets are.

If a structure M has underlying set X and vocabulary L, and if an L-formula $\phi(x_1, \ldots, x_n)$ is given, the set ϕ^M defined by ϕ in M is by definition the set of $(a_1, \ldots, a_n) \in X^n$ such that ϕ is true of $(a_1, \ldots, a_n)$ in M. The *definable sets* in M are precisely the sets ϕ^M. A function f from the definable set Y to the definable set Z is said to be *definable* if its graph is. Each member of the original family of distinguished sets and functions is definable. The reader can verify that the logical connectives translate into the closure of the class of definable sets under various set-theoretic operations: for example, complementation, finite unions and intersections (in a given ambient space X^n), and finite Cartesian products. Also, the image of a definable set under a definable function is definable (using the existential quantifier). We obtain from the structure M a category $D(M)$, the category of definable sets and functions, which can also be defined syntactically from (and thus only depends on) the theory of M. There is a somewhat richer category $D'(M)$ containing, in addition to the definable sets, the fibers of definable functions. This is the category of sets definable in M *with parameters*.

Let us consider the case of the structure $(\mathbb{R}, +, \cdot, 0, 1)$. The natural ordering $x < y$ is definable by the formula $\exists z(y - x = z^2)$. Let us write $\overline{X}$ for a tuple $(X_1, \ldots, X_n)$ of indeterminates. Tarski's quantifier elimination theorem (1951) says that the definable sets are precisely finite unions of sets defined by

$$P(\overline{X}) > 0 \wedge Q_1(\overline{X}) = 0 \wedge \cdots \wedge Q_m(\overline{X}) = 0,$$

where P and the Q_j are polynomials over the rationals $\mathbb{Q}$. The sets definable with parameters are just the same thing except that the coefficients in P and the Q_j can be arbitrary in $\mathbb{R}$. These are precisely the "semialgebraic sets" in the subject of algebraic geometry over $\mathbb{R}$. The content of the theorem is that the projection of a semialgebraic set from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ is also semialgebraic. In any case, what we obtain are precisely the objects of "semialgebraic geometry". Similarly the sets definable in $(\mathbb{C}, +, \cdot, 0, 1)$ are precisely the $\mathbb{Q}$-constructible sets (Chevalley). The definable sets in the p-adic field $(\mathbb{Q}_p, +, ., 0, 1)$ are the sets quantifier-free definable after adding relations for the nth powers, as proved by Macintyre (1976).

In the general case it is natural to "complete" $D(M)$ by adjoining quotient sets of the form Y/E, where Y is definable and E is a definable equivalence relation on Y. We call this completion $D^{eq}(M)$ or $D(M^{eq})$. As the latter notation suggests, this is the category of definable sets in a certain many-sorted structure M^{eq} attached to M. It is now recognized for model-theorists that to understand a given structure M is to understand the category $D^{eq}(M)$. In practice this has two aspects or phases. The first is to understand the category $D(M)$ and takes the form of identifying a manageable class of first-order formulas such that every definable set is defined by one of these formulas, as in the examples of the real, complex, and p-adic fields above. Such a procedure is called *quantifier elimination*. The second aspect, understanding quotients, is rather more complicated. In the best of all possible worlds, for any Y/E as above, there exist a set Z in $D(M)$ and a definable surjection f from Y to Z such that $f(x) = f(y)$ if and only if $E(x, y)$. If this happens (as it does for the structures $(\mathbb{C}, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$), we say that M has *elimination of imaginaries*.

Compactness

The compactness theorem says that if every finite subset of a set Σ of sentences has a model, then Σ has a model. Equivalently, if the sentence τ is a logical consequence of the set Σ of sentences, then τ is already a logical consequence of a finite subset of Σ. What was at the time considered a rather surprising implication of this was Ax's injectivity-implies-surjectivity theorem: any injective morphism from a complex algebraic variety V to itself is surjective. The proof goes as follows. Suppose for a contradiction that $f : V \to V$ is a counterexample. The variety V is defined by a finite system $\overline{P}(\bar{x}, \bar{a}) = 0$ of polynomial equations in which we exhibit the coefficients $\bar{a}$. Similarly f is defined by a system $\overline{F}(\bar{x}, \bar{a})$ of polyomials. We can now find a first-order sentence τ saying: there is $\bar{y}$ such that $\overline{F}(-, \bar{y})$ defines an injective but nonsurjective map from the set defined by $\overline{P}(\bar{x}, \bar{y})$ to itself. The sentence τ is true in $(\mathbb{C}, +, \cdot)$, hence is a consequence of the axioms for algebraically closed fields of characteristic 0. By compactness, τ is true in $(\overline{\mathbb{F}}_p, +, \cdot)$ for some (in fact almost all) primes p, where $\overline{\mathbb{F}}_p$ is the algebraic closure of the field with p elements. We get a contradiction by counting. In fact, the algebraic closure K of $\mathbb{F}_p$ is locally finite. So if V is a variety over K and f is an injective morphism over K from V to V, then for each finite subfield F of K over which V and f are defined, the restriction of f to $V(F)$ is onto. As V is the union of all such $V(F)$, $f : V(K) \to V(K)$ is onto.

A similar proof yields Lazard's theorem that an algebraic group with underlying variety affine n-space is unipotent. This general method gives, among other things, a rigorous formulation of "reduction mod p" technology.

Another implication with a similar flavor, although now using the deep Ax-Kochen-Ershov analysis of the elementary theories of Henselian valued fields mentioned above, is the asymptotic solution to a conjecture of E. Artin, as follows:

Theorem 1. *For any given d, for all sufficiently large primes p, for all $n > d^2$, every homogeneous* *polynomial over $\mathbb{Q}_p$ in n variables of degree d has a nontrivial zero in $\mathbb{Q}_p$.*

The actual conjecture was the statement above but with "for all primes" replacing "for all sufficiently large primes".

Here is a sketch of the proof of Theorem 1. It is enough to prove the theorem for the case $n = d^2 + 1$. Let τ be a first-order sentence in the language of rings expressing that every homogeneous polynomial of degree d in $d^2 + 1$ variables has a nontrivial zero. By a theorem of Lang,

(*) τ is true in the field $\mathbb{F}_p((t))$ of quotients of the ring of formal power series over the finite field $\mathbb{F}_p$, for all primes p.

Let Σ be the set of sentences in the language for valued fields consisting of

1. the axioms for Henselian valued fields of characteristic zero whose residue fields have characteristic zero;
2. for the value group, the sentences true in $(\mathbb{Z}, +, 0, <)$;
3. for the residue field, the condition "characteristic zero" (the infinite set of sentences expressing that the characteristic is different from p for each prime p), together with the theory of finite fields.

From $(*)$ together with the Ax-Kochen-Ershov result, one deduces with not much difficulty that τ is a logical consequence of Σ. By compactness, τ is a logical consequence of a finite subset Σ' of Σ. For all sufficiently large p, the p-adic valued field $\mathbb{Q}_p$ is a model of Σ', thus of τ, and the proof is complete.

The compactness theorem is fundamental in first-order model theory. To start with, it gives rise to "nonstandard models": any infinite structure has a proper "elementary extension"; in fact, it has elementary extensions of arbitrarily large cardinality. An *extension* N of a structure M is a structure in the same vocabulary such that for each n-ary relation symbol R, R^M is precisely $R^N \cap M^n$, etc. It is furthermore an *elementary extension* if every sentence with parameters from M that is true in M is also true in N. Nonstandard analysis, invented by Abraham Robinson, takes M to be a suitably rich structure for doing analysis; constructs a nonstandard model $^*M$; proves theorems in $^*M$ using, for example, the existence of "infinitesimals"; and then transfers the result back to the standard structure M. A related area, called "models of arithmetic", uses nonstandard models of first-order Peano arithmetic, or of weaker axiom systems, to study the strength of such axiom systems. Among the successes of this kind of work was a new "natural" Ramsey-style combinatorial fact that is true but not provable in Peano arithmetic (Paris-Harrington).

From a general model-theoretic point of view (and in many concrete examples), certain nonstandard models are the normal place to study definability. These are "universal domains" for a theory. Such an object depends on the choice of a suitable (usually uncountable) cardinal κ. We will restrict our attention to the case where κ is chosen to be the first uncountable cardinal. So given, say, a complete, countable theory T, by a *universal domain* for T we mean a model M of T such that every countable model of T elementarily embeds in M, *uniquely* up to an automorphism of M. (These are the "homogeneous-universal models" of Morley-Vaught.) In fact, the field of complex numbers (in no way a nonstandard object) is a universal domain in this sense for the theory of algebraically closed fields of characteristic 0. Sometimes by abuse of language we will say "saturated model" in place of universal domain. Saturation amounts to saying that any countable family of definable subsets of M^n, every finite subset of which has nonempty intersection, has nonempty intersection. In the context of algebraically closed fields, this provides the existence of generic points for varieties. In fact, our notion of universal domain is exactly that used by Weil in algebraic geometry (and later by Kolchin in differential algebraic geometry), although it has somewhat gone out of use, replaced by the scheme-theoretic point of view.

o-Minimality and Real Geometry

As noted above, the sets definable with parameters in the structure $(\mathbb{R}, +, \cdot, 0, 1, <)$ are precisely the semialgebraic sets. Good properties of (the class of) semialgebraic sets include these:

a. any semialgebraic set has finitely many connected components,
b. a compact semialgebraic set admits a semialgebraic triangulation,
c. any semialgebraic family of semialgebraic sets contains only finitely many sets up to semialgebraic homeomorphism.

One can ask whether similar properties hold for the sets definable in the structure $(\mathbb{R}, +, \cdot, 0, 1, <, exp) = \mathbb{R}_{exp}$, where exp is the real exponential function. Tarski was the first person to ask about the decidability of the theory of $\mathbb{R}_{exp}$. Hovanskii showed that the zero set of an "exponential" polynomial has finitely many connected components. Subsequently Alex Wilkie (1996) proved that any subset of $\mathbb{R}^n$ definable in $\mathbb{R}_{exp}$ is the projection of an exponential variety in some $\mathbb{R}^{n+m}$, namely, is *subexponential*. The content of this is a "theorem of the complement": the complement of a subexponential set is also subexponential. In any case, a consequence is that (a) holds for definable sets in $\mathbb{R}_{exp}$.

Already in the early 1980s it was realized by van den Dries (1984) and by Steinhorn and me (1984) that there is a general model-theoretic context here. What we called an *o-minimal structure* is a structure $M = (X, <, A_i)_i$, where $<$ is a dense linear ordering on X and the A_i are arbitrary subsets of various X^n satisfying: every subset of X definable with parameters in M is a finite union of points and intervals (a, b) with $a \in X \cup \{-\infty\}$ and $b \in X \cup \{+\infty\}$. The ordering gives a topology on each of $X, X \times X, \ldots$. One may want to restrict oneself to the case that X is the real line with the usual ordering. In any case, from the assumption of o-minimality, a condition on parametrically definable sets in 1-space, was deduced a host of properties of arbitrary definable sets. For example: any definable subset of $X \times \cdots \times X$ has finitely many definably connected components, definable functions are piecewise continuous, and much more. The notion and theory of o-minimality can with hindsight be considered to be a response to Grothendieck's challenge to investigate classes of sets with the tame topological properties of the semialgebraic sets.

What Wilkie actually proved in 1996 was the o-minimality of $\mathbb{R}_{exp}$. From the general theory of o-minimality one concludes that *all* the topological-geometric properties of semialgebraic sets (including (a), (b), and (c) above) pass over to definable sets in $\mathbb{R}_{exp}$. The main theme of current work is to include richer and richer classes of functions and sets in o-minimal structures on $\mathbb{R}$. It was noticed fairly early that the class of "finitely subanalytic sets" (subanalytic when viewed in projective space) is subsumed by o-minimality. Wilkie (2000) has more recently shown that the class of "Pfaffian functions" belongs to an o-minimal structure. It is hoped that the second part of Hilbert's Sixteenth Problem (uniform bounds on the number of limit cycles of planar polynomial vector fields) can be approached by finding a suitably rich o-minimal structure on $\mathbb{R}$. A problem of Hardy on asymptotics was solved by studying a certain nonstandard model of $\mathbb{R}_{exp}$, the field of LE-series (van den Dries, Macintyre, and Marker, 1994). Issues raised in this paragraph can be seen as part of the *special* theory of o-minimality. There is also a *general* theory of o-minimality, concerned with arbitrary o-minimal structures. A deep result by Peterzil and Starchenko (1998) essentially reduces this to the study of o-minimal expansions of real closed fields.

Analogies, model-theoretic or otherwise, between the real and p-adic fields have always been productive. There is as yet no general theory that bears the same relation to the p-adic fields that o-minimality bears to the real field. Nevertheless, the understanding of definable sets in the p-adic fields, as well as expansions by analytic functions, has led to spectacular results. I mention Jan Denef's proof (1984) of the rationality of certain Poincaré series, where the coefficients are on the face of it definable with an existential quantifier. This has been generalized by Denef and Loeser (1999) with $\mathbb{Q}_p$ replaced by the valued field $\mathbb{C}((t))$ and with rationality taken with respect to the Grothendieck ring of algebraic varieties over $\mathbb{C}$.

Interpretability and Invariants

In the article so far I have mostly described *applications* involving usually a detailed model-theoretic analysis of specific structures (axiomatizing the theory, describing definable sets), together with a sometimes ingenious use of elementary model-theoretic techniques (like compactness). The work in the previous section is a bit different in that it is informed by the model-theoretic techniques from the general theory of o-minimality. In any case, one might get the impression that model theory is simply a collection of techniques and concepts that come to life only in clever applications. In fact, there exists a subject "model theory for its own sake" that is quite developed but in some sense is still in its infancy. Moreover, there is a growing collection of applications in which the more sophisticated aspects of this theory play an increasingly important role. This section is devoted to exploring what is or could be "model theory for its own sake".

In areas of mathematics such as geometry and topology (and many others), a given class of objects, such as manifolds, algebraic varieties,…is identified, together with an appropriate notion of isomorphism ("being the same"), and tautologically a central aim of the area becomes the classification of these objects up to isomorphism. In most cases this classification will never be fully realized, but the attempt to do a classification, the techniques developed, and more importantly the connections discovered between the various classification problems in different areas are part of the process of uncovering and learning more about the mathematical world. Some of these endeavors can be seen exactly as classifying the definable sets in a given structure. For example, the classification of complex algebraic varieties up to birational isomorphism is more or less the same thing as the classification of sets definable with parameters in the single structure $(\mathbb{C}, +, \cdot)$ up to definable bijection, although on the face of it this translation is not particularly helpful.

In any case, central objects of (first-order) model theory are structures and their first-order theories, as well as possibly incomplete theories. What is the notion of sameness for structures? As these structures (and theories) may be in different vocabularies, isomorphism does not make sense. The right notion of sameness is "bi-interpretability". Let us first consider structures. Suppose M and N are structures in possibly different vocabularies. An *interpretation* of M in N is a bijection f of the underlying set of M with a definable set in N^{eq} (so possibly a quotient object), such that for every definable set Y in M, $f(Y)$ is definable in N. M and N are *bi-interpretable* if there are interpretations

f of M in N and g of N in M such that the map $f \circ g$ is definable in N and $g \circ f$ is definable in M. One can similarly define interpretations and bi-interpretability *with parameters*. As an example for any field F, the structures $(F, +, \cdot)$ and the affine plane over F considered as an incidence structure (points, lines, and incidence) are bi-interpretable (with parameters). Bi-interpretability of structures M and N translates into a syntactic (at the level of first-order formulas) bi-interpretability of the theory of M with the theory of N (a notion that the reader is free to formulate). So, tautologically, the classification of first-order structures and/or theories up to bi-interpretability is a central issue. This links up with our earlier insistence on the category of definable sets $D(T)$ of a theory T: the bi-interpretability of complete theories T_1 and T_2 roughly translates into an equivalence of categories between $D^{eq}(T_1)$ and $D^{eq}(T_2)$.

All of this is a triviality, but once expressed it explains and unifies an enormous amount of work on the pure side of model theory. In particular the issue of finding interesting and useful invariants for the bi-interpretability type of structures/theories is immediately put on the table.

Historically the development of "model theory for its own sake" focused on a certain invariant I_T. This I_T is the function that associates to a cardinal κ the number of models of T of cardinality κ, up to isomorphism; it is rather clearly an invariant of the bi-interpretability type of T. This may seem to be a rather strange invariant to consider, but it was quite natural, as set-theoretic language and concerns permeate logic. Moreover, there are many classical model-theoretic techniques for building models of theories and distinguishing them up to isomorphism. The problem of describing the possible invariants I_T was posed in the late 1960s by Saharon Shelah and solved in the early 1980s by him. The solution appears in the second edition of his book *Classification Theory and the Number of Non-isomorphic Models* (1990). He considered only countable complete theories T and the restriction of I_T to uncountable cardinals. Early work was done by Morley, who proved that if $I_T(\kappa) = 1$ for some uncountable κ, then it equals one for all uncountable κ. These theories T of Morley's, as well as their models, are called *uncountably categorical*. Shelah produced an amazing body of work, in which a number of other invariants, such as stability, superstability,…, were introduced, as well as a theory of how complicated structures are built up from simple structures (geometries) by what geometers will recognize as fibrations.

The archetypal example of an uncountably categorical theory is the theory of algebraically closed fields of some fixed characteristic: the uncountable models are classified simply by their cardinality, which equals the transcendence degree over the prime field. Boris Zilber invented what is now called geometric stability theory by posing the problem of classifying uncountably categorical theories up to bi-interpretability. Zilber pointed out the role and importance of the nature of *definable groups*. This observation has led to the study of simple uncountably categorical groups, motivated by the conjecture (Cherlin-Zilber) that they are precisely the simple algebraic groups over algebraically closed fields.

The issue of classification up to bi-interpretability has led also to important interactions with permutation group theory. Let us restrict our attention to countable structures M with the feature that the automorphism group $Aut(M)$ of M has finitely many orbits on n-tuples for all n. Such a structure M will be the unique countable model of its theory. $Aut(M)$ becomes a topological group when we stipulate that the stabilizers of finite tuples are basic open subsets. Then the bi-interpretability type of M is determined by $Aut(M)$ as a topological group, and one issue is when and whether this can be recovered from the abstract group $Aut(M)$.

Stability and Diophantine Geometry

A structure M is said to be *unstable* if it interprets a bipartite graph (P, Q, R) with the feature that for each n there are $a_i \in P$ and $b_i \in Q$ for $i = 1, \dots, n$ such that $R(a_i, b_j)$ if and only if $i < j$. A complete theory is unstable if some (any) model is unstable. If M is unstable (witnessed by (P, Q, R)) and saturated, then there are a_i and b_i for $i = 1, 2, \dots$ such that $R(a_i, b_j)$ if $i < j$. A structure or theory is *stable* if it is not unstable. By definition stability is an invariant of the bi-interpretability type. The connection between stability and I_T is: if T is unstable, then $I_T(\kappa) = 2^\kappa$ (the maximum possible) for all uncountable cardinals κ. So in the context of classifying the possible functions I_T, it was natural to focus on stable theories. Stability is a very strong property. There are few natural examples of stable structures: abelian groups $(G, +)$, algebraically closed and separably closed fields $(K, +, \cdot)$, differentially closed fields $(K, +, \cdot, D)$. More recently it was realized that compact complex manifolds are also stable; the structure on the compact complex manifold X consists of the analytic subvarieties of X, $X \times X$,…. On the other hand, typically the structures considered in earlier sections, such as the real field and p-adic field, are *unstable*.

Stability can be recognized also by looking at the Boolean algebra of definable subsets of a structure. Henceforth by "definable" we will mean definable possibly with parameters. Given M and an L-formula $\phi(x, \bar{y})$, we obtain the Boolean algebra of subsets of M defined by finite Boolean combinations of formulas $\phi(x, \bar{a})$, $\bar{a}$ a tuple from M. This Boolean algebra has an associated Stone space, namely the space of ultrafilters. T is stable if and only if for any countable model M of T and L-formula $\phi(x, \bar{y})$, the associated Stone space is *countable*. The issue of the complexity of Stone spaces of Boolean algebras of definable sets is pervasive in stability theory and is one route to defining various "dimensions" of definable sets. For example, given a saturated structure M and a definable set X in M, we will call X *minimal* (intuitively 1-dimensional and irreducible) if the Stone space of the Boolean algebra of all definable subsets of X has a unique nonisolated point or, equivalently, if X is infinite and any definable subset of X is finite or cofinite.

In the structure $(\mathbb{C}, +, \cdot)$ the minimal sets are the algebraic curves (up to finitely many points). In the more general structure of compact complex manifolds the minimal sets are essentially the simple compact complex manifolds.

A Cantor-Bendixon analysis of Stone spaces yields a definition of (possibly ordinal-valued or undefined) dimension for definable sets, called Morley rank or Morley dimension. Below we will refer to this as just $dim(-)$. In the context of sets defined in algebraically closed fields, this coincides with algebraic-geometric dimension. But in the case of compact complex manifolds it is normally only bounded above by the complex dimension.

Shelah's general theory explains how finite-dimensional definable sets are built up, by a finite sequence of fibrations, from minimal definable sets. The manner in which definable sets interact is also well understood. Definable sets X_1 and X_2 are said to be *orthogonal* if, up to finite Boolean combination, any definable subset of $X_1 \times X_2$ is of the form $Z_1 \times Z_2$ for Z_i a definable subset of X_i. On minimal definable sets, nonorthogonality is an equivalence relation. In fact, minimal definable sets X_1 and X_2 will be nonorthogonal if and only if there is a finite-to-finite definable relation $Y \subset X_1 \times X_2$ projecting onto both X_1 and X_2.

Among the most important and subtle aspects of geometric stability theory is the distinction between "linear" and "nonlinear" behavior of definable sets. What I call linearity is usually called "modularity" for lattice-theoretic reasons. Fix a stable saturated structure M and a finite-dimensional definable set X. We call X *linear* (modular) if there are no "large" families of definable subsets of any X^k. By a large such family we mean a definable family of definable sets Y_a for $a \in Z$ such that $dim(Y_a) = n$ for all $a \in Z$, $dim(Z) = m$, $dim(Y_a \cap Y_b) < n$ whenever $a \neq b \in Z$, and $m + n > dim(\bigcup_{a \in Z} Y_a)$.

If X happens to be minimal, to check linearity it suffices to consider definable subsets of $X \times X$. Intuitively (for X minimal), X is linear if and only if any definable family of minimal subsets of $X \times X$ is at most a 1-parameter family. Zilber conjectured a long time ago that a fundamental dichotomy holds for minimal sets in stable structures. Either X is linear or there is a definable minimal field nonorthogonal to X. This is false in general, but its truth in a certain axiomatic context (Zariski structures), as proved by Hrushovski and Zilber (1996), is of great interest for a number of examples. I will not give a definition of Zariski structure other than to say it is a minimal set in which certain definable sets are identified as being "closed" and some intersection-theoretic assumptions are made. In any case, Zilber's conjecture is true for minimal definable sets in separably closed fields, differentially closed fields, and compact complex manifolds.

Among the general theorems about modular definable sets (Hrushovski-Pillay 1987) is:

Theorem 2. *If X is a modular definable* group, *then every definable subset of X^n is, up to finite Boolean combination, a translate of a definable subgroup.*

I cannot resist pointing out how the "Mordell-Lang conjecture" over function fields in positive characteristic proved by Hrushovski (1996) is an almost immediate consequence of the modularity of certain definable groups in separably closed fields. There is a slight twist in that we will be considering sets defined by a countable collection of formulas (type-definable sets) rather than by a single formula. But there is no essential difference in the general theory.

So we consider separably closed fields K of characteristic $p > 0$ with the feature that K is finite-dimensional of dimension > 1 as a vector space over its pth powers. We work with the structure $(K, +, \cdot)$, which we assume to be saturated. Let $k = \bigcap_n K^{p^n}$. Let A be a simple abelian variety defined over K that does not descend to k (namely, is not rationally isomorphic to an abelian variety defined over k). Let $A^\sharp = \bigcap_n (p^n A(K))$ be the group of infinitely p-divisible elements of $A(K)$. Both k and $A^\sharp$ are type-definable. The validity of the Zilber dichotomy for separably closed fields yields:

Lemma 1. *$A^\sharp$ is minimal and modular.*

The point is that if not, then there is a minimal field nonorthogonal to $A^\sharp$. It has to be k. The nonorthogonality yields a rational isomorphism between A and an abelian variety defined over k, contradiction. It follows that

Lemma 2. *If A is a (not necessarily simple) abelian variety over K with k-trace 0, then $A^\sharp$ is modular.*

This yields directly the *Mordell-Lang conjecture* for abelian varieties over function fields in positive characteristic.

Theorem 3. *If A is an abelian variety over a function field K/k with k-trace 0, Γ is a finitely generated subgroup of $A(K)$, and X is a subvariety of A defined over K with $X \cap \Gamma$ Zariski-dense in X, then X is a translate of an abelian subvariety of A.*

Sketch of proof. Replacing K by its separable closure, we may assume that K and k are as in the previous paragraph; by a transfer argument we can assume $(K, +, \cdot)$ to be saturated. For each n, $p^n\Gamma$ has finite index in Γ, and so some translate of $p^n\Gamma$ meets X in a Zariski-dense set. The same is true of some translate of $p^n A(K)$. Saturation of K implies that after translation of X, $A^\sharp$ meets X in a Zariski-dense set. Modularity of $A^\sharp$ as in Lemma 2 implies, by Theorem 2, that $X \cap A^\sharp$ is a translate of a subgroup. The same is thus true of X.

It was discovered recently that the machinery of stability theory (dimension, orthogonality,...) is valid, but with a few interesting twists, in a much wider setting than stable structures—so-called "simple structures" and their theories. A great deal of energy on the pure side of model theory is currently devoted to understanding this wider class. Although the real and p-adic fields do not fall into this category, finite fields (or rather their limits, pseudofinite fields), the random graph, and sufficiently rich difference fields (fields equipped with an automorphism) *do*. Making use of definability in difference fields and the validity of the Zilber dichotomy in this context (Chatzidakis-Hrushovski 1999), Hrushovski found another proof of the "Manin-Mumford conjecture" on the intersection of the torsion points of abelian varieties A defined over *number fields* with subvarieties of A, obtaining, by virtue of model-theoretic methods, better bounds than previously known. The hard qualitative and quantitative questions concerning *rational* points of varieties over number fields remain untouched by the model-theoretic methods discussed in this section. Making progress here is a major challenge.

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