The Evaluation of Integrals: A Personal Story

Victor H. Moll

The fall semester of 1992 was a promising one for me. I had just returned from an extended sabbatical at the University of Utah, I was going to be considered for tenure, and I would be teaching the beginning graduate course in analysis. This article tells the story of how this promise was fulfilled, but in very unexpected ways.

One of the analysis students (George Boros), older than the rest and for many years a part-time instructor in the area, had finally decided to pursue a Ph.D. in mathematics. He was well known in New Orleans mathematical circles as "the person who can compute any integral." Having spent my graduate-student years at Courant Institute, I was comfortably unaware of integrals and did not think that this could be serious mathematics. I was wrong.

At the end of the academic year this student asked me to be his adviser. I agreed but cautioned: "George, nobody is going to give you a doctorate in mathematics for computing integrals." His response was that he was aware of this and if I was willing to suggest a general topic for the qualifiers, he would accept my choice. At that time Henry McKean and I were in the process of writing the book [6], so I suggested that George read the manuscript and that his qualifiers be related to elliptic curves. After a successful exam it came time for a thesis problem, and when I started to suggest some possibilities, he interrupted me with: "I have my own problems." This was a surprise. It was even more of a surprise to discover that new things can still be said today about the mundane subject of integration of rational functions of a single variable and that this subject has connections with branches of contemporary mathematics as diverse as combinatorics, special functions, elliptic curves, and dynamical systems.

Victor H. Moll is professor of mathematics at Tulane University. *He was supported in part by NSF Grant DMS-0070567.* *His e-mail address is* vhm@math.tulane.edu.

A Formula for the Quartic

In order to satisfy my curiosity, George told me that

$$\int_0^\infty \frac{dx}{(x^4+6x^2+1)^3} = \frac{219\pi}{2048\sqrt{2}}.$$

My response was clear: "George, there are symbolic languages that can do these things; you should not waste your time." Indeed, Mathematica yields the answer

$$\frac{3\pi}{8192}\left(-31\sqrt{6-4\sqrt{2}}+42\sqrt{3-2\sqrt{2}} + 42\sqrt{3+2\sqrt{2}}+31\sqrt{6+4\sqrt{2}}\right),$$

and George's evaluation can be obtained by using the FullSimplify command. Similarly,

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(x^4+6x^2+1)^{51}} = \frac{3^2\cdot 13^2\cdot 17\cdot 53\cdot 59\cdot 61\cdot 67\cdot N_1\cdot N_2\cdot N_3\cdot N_4}{2^{249}\sqrt{2}},$$

where N_1, N_2, N_3, and N_4 are the prime numbers

91297, 1518533, 44368952933,

and

10220677829087302935117744959039145564109.

The problem of integration of rational functions $R(x) = P(x)/Q(x)$ was considered by J. Bernoulli in the eighteenth century. He completed the original attempt by Leibniz to find a general partial fraction decomposition of $R(x)$. The main difficulty associated with this procedure is to obtain a complete factorization over the real numbers of the denominator $Q(x)$. Once this is known, the partial fraction decomposition of $R(x)$ can be computed. The fact is that the primitive of a rational function is always elementary: it consists of a *rational part* (a new rational function) and a *transcendental part* (the logarithm of a second rational function). In his classic monograph [5], G. H. Hardy states: "The solution of the problem (of definite integration) in the case of rational functions may therefore be said to be complete; for the difficulty with regard to the explicit solution of algebraical equations is one not of inadequate knowledge but of proved impossibility." He goes on to add: "It appears from the preceding paragraphs that we can always find the rational part of the integral, and can find the complete integral if we can find the roots of $Q(x) = 0$."

But knowing that a problem admits a solution in principle is not the same as being able to compute the solution. The symbolic evaluations of integrals may take considerable time. The second example above took around seventeen minutes to compute. The Mathematica manual states that "definite integrals that involve no singularities are mostly done by taking limits of the indefinite integrals. Many other definite integrals are done using Marichev-Adamchik Mellin transform methods. The results are often initially expressed in terms of the Meijer G functions, which are converted into hypergeometric functions using Slater's Theorem and then simplified." Thus it is not entirely clear what Mathematica is doing to compute these integrals. Details can be found in [1]. I became intrigued about George's methods, which were based upon the following result.

Theorem. *Let $a > -1$ and let m be a natural number. Then the integral*

$$N(a;m) := \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

is given by

$$\frac{\pi\, 2^{-m-3/2}}{(a+1)^{m+1/2}} \sum_{j=0}^{m} \binom{2m+1}{2j} (a+1)^j \times \sum_{k=0}^{m-j} \binom{m-j}{k} \binom{2k+2j}{k+j} 2^{-3(k+j)} (a-1)^k.$$

The proof is elementary and employs Wallis's integral formula

$$\int_0^{\pi/2} \cos^{2n}\varphi\, d\varphi = \binom{2n}{n} \pi/2^{2n+1}. \qquad (1)$$

The reader is invited to compare the expression given in the theorem with the expression obtained by residues.

The structure of $N(a;m)$ now became clear. In particular

$$P_m(a) := \frac{2^{m+3/2}(a+1)^{m+1/2}}{\pi} N(a;m)$$

is a polynomial in a of degree m. The theorem implies that the coefficient $d_l(m)$ of the term a^l in the polynomial $P_m(a)$ is a triply indexed sum of expressions that are products of binomial coefficients, powers of $1/2$, and plus or minus signs. The first few polynomials are

$$\begin{aligned}
P_0(a) &= 1,\\
P_1(a) &= \frac{1}{2}(2a+3),\\
P_2(a) &= \frac{3}{8}(4a^2+10a+7),\\
P_3(a) &= \frac{1}{16}(40a^3+140a^2+172a+77),\\
P_4(a) &= \frac{5}{128}(112a^4+504a^3+876a^2+708a+231).
\end{aligned}$$

The fear of *reinventing the wheel* now appeared. It was quite possible that the polynomials $P_m(a)$ were well known. We[1] wondered if there is a simple expression for the polynomials $P_m(a)$ and whether all their coefficients are positive. Colleagues told us that the $P_m(a)$ must be expressible in terms of hypergeometric functions, but an initial search in standard integral tables did not find our quartic integral. Responding to an inquiry about the coefficients $d_l(m)$, Doron Zeilberger replied: "…the triple sum that you have does not seem to have a closed form in both m and l. For a fixed $m - l = p$, it does, but as p gets bigger, the 'closed form' gets uglier." Encouraged by this, we searched for a proof that $d_l(m) > 0$.

Ramanujan, Double Square Root, and Positivity

The proof of positivity appeared from a most unexpected place. It turns out that there is a connection between the Taylor series of $h(c) := \sqrt{a+\sqrt{1+c}}$ at $c = 0$ and the polynomial $P_m(a)$. This and a theorem of Ramanujan yield a simple formula for the coefficients $d_l(m)$.

The evaluation of the quartic integral described in the previous section gives, in particular,

$$\int_0^\infty \frac{dx}{bx^4+2ax^2+1} = \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{a+\sqrt{b}}}.$$

While playing around with the parameters, we noticed that the derivatives of $h(c)$ at $c = 0$ can be evaluated in terms of the quartic integrals. The fact is

[1] *By now it was* we.

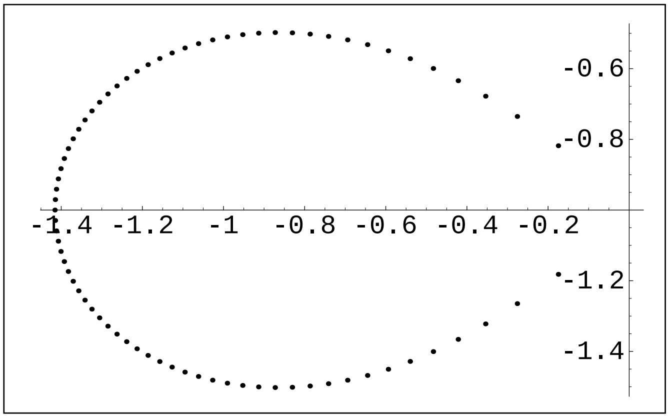

Figure 1: Zeros of the polynomial P_{75}.

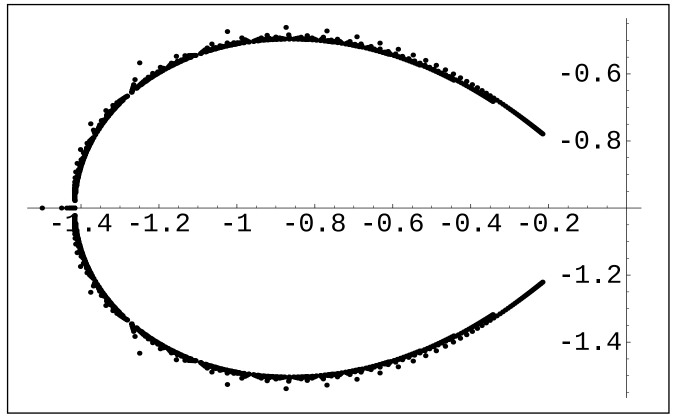

Figure 2: Zeros of the polynomials P_m, $1 \le m \le 50$.

that the integrals $N(a;k)$ are *essentially the coefficients of the Taylor expansion of the double square root* $h(c)$ at $c = 0$.

Theorem. *The Taylor series expansion of* $h(c) = \sqrt{a+\sqrt{1+c}}$, *for* c *in a neighborhood of the origin, is given by*

$$h(c) = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N(a;k-1)c^k.$$

This expansion appears in several classical analysis texts in the particular cases $a = 1$ and $c = a^2$.

The next piece of the puzzle appeared from Ramanujan's work. In particular, Ramanujan's Master Theorem connects the coefficients of a Taylor expansion

$$F(c) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi(n) c^n$$

with the moments

$$M_n = \int_0^{\infty} c^{n-1} F(c)\, dc$$

of the function F via $M_n = (n-1)!\,\varphi(-n)$. Observe that the application of the theorem requires extending the Taylor coefficients $\varphi(n)$ from $n \ge 0$ to $n < 0$. Details of this theorem can be found in Berndt's first volume on Ramanujan's Notebooks [3]. We can apply the theorem to an appropriate derivative of $h(c)$ to establish a relation between the original quartic integral $N(a;m)$ and a new family of integrals

$$B_m(a) := \int_0^{\infty} \frac{x^{m-1}\, dx}{(a+\sqrt{1+x})^{2m+1/2}}.$$

Indeed, Ramanujan's Master Theorem yields

$$B_m(a) = \frac{2^{6m+3/2}}{\pi} \left[m \binom{4m}{2m} \binom{2m}{m} \right]^{-1} N(a;m),$$

so we now need to evaluate $B_m(a)$. A simple change of variables shows that an evaluation of $B_m(a)$ follows from one for the derivatives of the function $u(u^2-1)^{m-1}$ at $u = 1$. To establish the values of these derivatives, we need the following identity for binomial coefficients:

$$\sum_{j \ge 0} (-1)^k \binom{m-1}{j} \binom{2m-2j-1}{k+m-1} = 2^{m-k-1} \frac{k+m}{m} \binom{m}{k}.$$

This identity can be verified by using the powerful WZ-method described in [7]. We conclude that

(2)
$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k,$$

so now the coefficients $d_l(m)$, given by the expression

(3)
$$2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l},$$

are clearly positive.

The expression (3) provides an efficient evaluation of $d_l(m)$ if l is close to m. The natural question of formulas that work well when l is small produced an unexpected and interesting problem. An elementary calculation yields the existence of polynomials α_l and β_l of degrees l and $l-1$, respectively, such that $d_l(m) l! m! 2^{m+l}$ can be written in the form

$$\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1).$$

We have conjectured that both families of polynomials have all their zeros on the line where $\operatorname{Re} m = -1/2$.

The graphs of the zeros of $P_m(a)$ suggest some questions about their location. Figure 1 shows the zeros of $P_{75}(a)$ and Figure 2 shows the zeros of all the polynomials $P_m(a)$ from $m = 1$ to $m = 50$.

The Hypergeometric Connection

At this point it was clear to us that we should provide a proof of the formula (2) for $P_m(a)$ based on the theory of special functions. A more careful examination of standard integral tables yielded the formula

$$\int_0^\infty \frac{z^{\nu-1}\,dz}{(z^2+2az+1)^{\mu+1/2}} = \frac{2^\mu \Gamma(1+\mu) B(-\nu+2\mu+1,\nu) P^{-\mu}_{\mu-\nu}(a)}{(a^2-1)^{\mu/2}},$$

[4, 3.252.11], where B is the classical beta integral and $P^\mu_\nu(z)$ is the associated Legendre function. Using the hypergeometric representation of the latter, we can rewrite the right-hand side as

$$\left(\frac{2}{a+1}\right)^\mu B(2\mu+1-\nu,\nu) \times {}_2F_1\left[\nu-\mu, 1+\mu-\nu; 1+\mu; \frac{1-a}{2}\right].$$

The expression (2) now follows directly but with an extra bonus: the polynomials $P_m(a)$ are part of the Jacobi family

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \times \binom{m+k+\alpha+\beta}{k} \left(\frac{a+1}{2}\right)^k$$

corresponding to the parameter values $\alpha = m+1/2$ and $\beta = -(m+1/2)$.

It is safe to say that we would never have found the connection between the quartic integrals and the Taylor expansion of the double square root had we known the most basic results in hypergeometric functions. *Ignorance is bliss.*

Wallis's Formula and Landen Transformations

Wallis's integral formula (1) is completely elementary and is usually proved by showing that

$$J_n := \int_0^{\pi/2} \cos^{2n}\varphi\, d\varphi$$

satisfies the recurrence $J_n = \frac{2n-1}{2n} J_{n-1}$. The recurrence can be used to generate values of J_n for small n from which one can *guess* a general formula. The proof of Wallis's formula is thus reduced to checking the guessed formula in the recurrence.

We stumbled upon a different proof while trying to compute the integral of a rational function. First observe that

$$J_n = \int_0^{\pi/2} \left(\frac{1+\cos 2\varphi}{2}\right)^n d\varphi.$$

Now introduce $\psi = 2\varphi$, expand the power, and simplify the result by observing that the odd powers of cosine integrate to zero. Hence

$$J_n = 2^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} J_k.$$

As before one can use this expression to generate values of J_n and guess the formula

$$J_n = \binom{2n}{n} \pi/2^{2n+1}.$$

The critical point of an inductive proof is the identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} 2^{-2k} \binom{n}{2k}\binom{2k}{k} = 2^{-n}\binom{2n}{n}. \tag{4}$$

Now comes the WZ-method to the rescue, for the identity (4) is precisely the first example used in [7, p. 113] to explain that procedure. Wilf and Zeilberger informed me that they do not recall why they chose this example.

The method of proof described above (double the angle, expand, and use the vanishing of odd powers) yielded an unexpected transformation when applied to integrals of higher degree. This requires a little bit of background. The *Landen transformation* $a \to (a+b)/2$ and $b \to \sqrt{ab}$ leaves the elliptic integral

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

invariant, and iteration of this transformation produces, in the limit, the celebrated *arithmetic-geometric mean* AGM(a,b). It turns out that we can find an analogous transformation to help evaluate integrals of rational functions that are even. Here is an example of such a transformation in the case of degree 6. Let

$$U = \int_0^\infty \frac{b_0x^4 + b_1x^2 + b_2}{x^6 + a_1x^4 + a_2x^2 + 1}\, dx.$$

Then the transformation

$$\begin{aligned} a_1 &\to \frac{9+5a_1+5a_2+a_1a_2}{(a_1+a_2+2)^{4/3}}, \\ a_2 &\to \frac{a_1+a_2+6}{(a_1+a_2+2)^{2/3}}, \\ b_0 &\to \frac{b_0+b_1+b_2}{(a_1+a_2+2)^{2/3}}, \\ b_1 &\to \frac{b_0(a_2+3)+2b_1+b_2(a_1+3)}{a_1+a_2+2}, \\ b_2 &\to \frac{b_0+b_2}{(a_1+a_2+2)^{1/3}} \end{aligned} \tag{5}$$

preserves the integral U. Moreover, the sequence (a_1^n, a_2^n) defined by iteration of (5) converges to $(3,3)$, and there is a value L such that the sequence (b_0^n, b_1^n, b_2^n) converges to $(L, 2L, L)$ precisely when the initial integral converges. The invariance of U shows that $U = L\pi/2$.

One can produce formulas analogous to (5) for rational functions of any even degree, but very soon these expressions become unmanageable. The geometric fact is that these Landen transformations convert an even rational function into its direct image by the Newton map associated to the equation $z^2+1=0$. This interpretation yields a proof of convergence of the process. The fact is that the iterates of the rational Landen transformation converge precisely when the initial data produce a convergent integral.

In the example of degree 6, the determining quantity is the resolvent

$$R(a_1,a_2) = 4a_1^3 + 4a_2^3 - 18a_1a_2 - a_1^2a_2^2 + 27.$$

The locus of $R(a_1,a_2)=0$ consists of two connected components $R_\pm$. The curve R_+ is in the first quadrant and contains the limiting point $(3,3)$. The integral U is finite precisely when $R_-(a_1,a_2)>0$. There is also a dynamical interpretation. The first two equations in (5) form a planar dynamical system that has three fixed points, two of them on the resolvent curve. The point $(3,3)$ is an attractor, explaining in part the convergence of the iterates. The second critical point is a saddle point, and the curve $R_-(a_1,a_2)=0$ is its stable manifold. The dynamics below this curve are quite complicated. Figure 3 shows the first 5,000 iterates starting in this region.

A treatment of the elliptic Landen transformation appears in [6], so my original advice for George's qualifiers paid off.

The Integration of a General Rational Function

The previous section gave some information about how to integrate the even rational functions. Our unsuccessful attempt to extend these methods to the general case produced an interesting map on the space of rational functions.

Consider the splitting of $R(x)$ into its even and odd parts

$$R_e(x) = \frac{R(x)+R(-x)}{2}$$

and

$$R_o(x) = \frac{R(x)-R(-x)}{2}.$$

Ignore the issue of convergence and integrate to produce

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \int_0^\infty R_o(x)\,dx.$$

The integral of the even part can be analyzed, at least partially, by the methods already described. The integral of the odd part can be transformed to

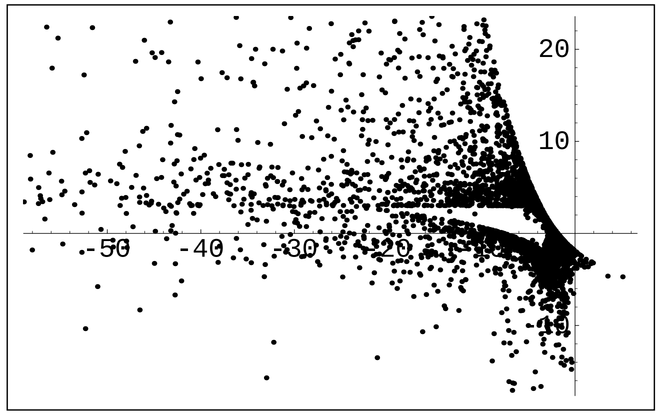

Figure 3: Dynamics associated with the rational Landen transformation.

$$\int_0^\infty R_o(x)\,dx = \frac{1}{2}\int_0^\infty \frac{R_o(\sqrt{x})}{\sqrt{x}}\,dx$$

via $x \to \sqrt{x}$. The new integrand is again rational, and so we have produced a map $\mathfrak{F}$ on the space of rational functions,

$$\mathfrak{F}(R)(x) = \frac{R(\sqrt{x}) - R(-\sqrt{x})}{2\sqrt{x}},$$

with the property

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathfrak{F}(R)(x)\,dx.$$

Observe that even though replacing x by $\sqrt{x}$ decreases the degree of a function, the map $\mathfrak{F}$ itself does not necessarily decrease the degree. The question of explicit integration of a rational function can be separated into two parts:

- explicit integration of even rational functions,
- properties of $\mathfrak{F}$ related to integration.

The map $\mathfrak{F}$ itself is an object worthy of study. In particular, the orbit $\{\mathfrak{F}^j(R) : j = 0, 1, 2, \dots\}$, starting at an arbitrary rational function R, is interesting. In order to keep the coefficients of the orbit under some control, we were led to study the orbit of a rational function with all its poles of modulus 1. The simplest case is the function $x^j/(x^{a_1}-1)$, where a_1 is an odd integer. A simple calculation shows that

$$\mathfrak{F}\left(\frac{x^j}{x^{a_1}-1}\right) = \frac{x^{\alpha_1(j)}}{x^{a_1}-1},$$

where

$$\alpha_1(j) = \begin{cases} (a_1-1+j)/2 & \text{if } j \text{ is even,} \\ (j-1)/2 & \text{if } j \text{ is odd.} \end{cases}$$

In this case the study of the map $\mathfrak{F}$ reduces to that of $\alpha_1 : \mathbb{Z} \to \mathbb{Z}$. The dynamics of α_1 are quite interesting: for any initial $j \in \mathbb{Z}$, the iterates $\alpha_1^n(j)$,

$n = 0, 1, \ldots$ reach either the *invariant set* $\{0, 1, 2, \ldots, a_1 - 2\}$ or the fixed-point set $\{-1, a_1 - 1\}$ in a finite number of steps. On the invariant set, the *inverse* of α_1 is given by

$$\begin{cases} 2k+1 & \text{if } 0 \le k \le (a_1-3)/2, \\ 2k+1-a_1 & \text{if } (a_1-1)/2 \le k \le a_1 - 2. \end{cases}$$

This has amusing number theoretical consequences: if a_1 is prime, then all orbits of $\mathfrak{F}$ starting in the invariant set have the same length. Moreover, there is a single orbit if and only if 2 is a primitive root modulo a_1, that is, 2 is a generator of the group $\{1, 2, \ldots, a_1 - 1\}$ under multiplication modulo a_1.

More generally, suppose $a_1, \ldots, a_m$ are *odd integers*, and define

$$T_m(x) := \prod_{k=1}^{m} (x^{a_k} - 1)$$

and

$$S_{m,j}(x) := \frac{x^j}{T_m(x)}.$$

An elementary calculation shows that there are polynomials $V_{p,j}(x)$ such that the iterates of $\mathfrak{F}$ applied to $S_{m,j}$ have the form

$$\mathfrak{F}^{(p)}(S_{m,j})(x) = \frac{V_{p,j}(x)}{T_m(x)}.$$

In the summer of 2001, Roopa Nalam, an undergraduate at Tulane on her way to medical school, proposed the following result, which remains open.

Conjecture. *Assume $gcd(a_1, a_2, \ldots, a_m) = 1$, and let $LC(P)$ denote the leading coefficient of a polynomial P. Then*

$$\lim_{p\to\infty} \frac{V_{p,j}(x)}{LC(V_{p,j}(x))} \times \frac{(x-1)^m}{T_m(x)} = \frac{A_{m+1}(x)}{x},$$

where $A_m(x)$ is the Eulerian polynomial defined by the generating function

$$\frac{1-x}{1 - x\exp[\lambda(1-x)]} := \sum_{m=0}^{\infty} A_m(x) \frac{\lambda^m}{m!}.$$

The situation in which $gcd(a_1, \cdots, a_m) \ne 1$ seems more complicated.

Many other aspects of $\mathfrak{F}$ are quite interesting. For instance, every fixed point of $\mathfrak{F}$ is a linear combination of functions of the form $x^{q_j - 1}/(1 - x^{q_j})$, where the q_j are odd integers. On the other hand, for each positive integer n, the rational functions of the form

$$\sum_{k=1}^{n} x^{2^{k-1}-1} R_k(x^{2^k}),$$

where $R_1, \ldots, R_n$ are arbitrary rational functions, vanish after precisely n iterations of $\mathfrak{F}$.

Unimodality and Logconcavity

The symbolic study of the coefficients $d_l(m)$, and in particular of their graph, suggested that these coefficients are *unimodal*. A finite sequence of real numbers $\{d_0, d_1, \ldots, d_m\}$ is said to be unimodal if there is an index m^*, called the *mode* of the sequence, such that d_j increases up to $j = m^*$ and decreases from then on. Our first proof of the unimodality of $d_l(m)$ was elementary but long. Soon after, we were able to give a very simple criterion for unimodality: *If $P(x)$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal with mode* $\lfloor \frac{m-1}{2} \rfloor$. With the speed of electronic publishing, our simpler proof appeared before the original one.

A property stronger than unimodality is that of *logarithmic concavity* (or logconcavity for short), meaning that $d_{j+1}d_{j-1} \le d_j^2$. We have conjectured that $\{d_l(m) : 0 \le l \le m\}$ is logconcave, but much more seems to be true. Define the operator

$$\mathfrak{L}\left(\{d_j\}\right) := \{d_j^2 - d_{j+1}d_{j-1}\},$$

so that logconcave sequences are those positive sequences $\{d_j\}$ for which $\mathfrak{L}(\{d_j\})$ is also positive. We say that $\{a_j\}$ is *infinitely logconcave* if $\mathfrak{L}^p(\{a_j\})$ is a positive sequence for every natural number p. The conjecture is that $\{d_l(m)\}$ is infinitely logconcave. The prototype sequence in issues of unimodality and logconcavity is the sequence of binomial coefficients. A reasonable first step would be to prove that $\{\binom{n}{k} : 0 \le k \le n\}$ is infinitely logconcave.

SACNAS, SIMU, Puerto Rico, and Convergence of Landen

The 1999 annual meeting of the Society for the Advancement of Chicanos and Native Americans (SACNAS) took place in Portland, Oregon. My colleague Ricardo Cortez has been involved with this association since his days as a graduate student. That year he had organized a special session for which he asked me to give a presentation.

At the end of my talk, two participants at the conference, Ivelisse Rubio and Herbert Medina, wanted to know if I would be interested in their REU (Research Experiences for Undergraduates) Program SIMU (Summer Institute in Mathematics for Undergraduates). They told me that the program has as a mission "to increase the number of Chicanos/Latinos and native Americans earning Master and Ph.D. degrees and pursuing research careers in the mathematical sciences." They invited me to direct a group of twelve students during the summer of 2000 at the University of Puerto Rico at Humacao. The idea sounded very interesting, so I agreed to do it. They warned me that "it is a lot of work." They were right. The program is structured so that there are lectures during the first three weeks, and students work on research projects for three and one-half weeks. The students were fantastic, and the following generalization of the unimodality criterion [2] came out of one of their projects:

If $P(x)$ is a polynomial with positive nondecreasing coefficients, and n is a natural number, then $P(x+n)$ is unimodal with mode $\lfloor \frac{m}{n+1} \rfloor$. In the discussion of this problem, we proved that *unimodality of a sequence* plus *negative second derivative*, that is, $d_{j+1} - 2d_j + d_{j-1} \leq 0$, *implies logconcavity.*

Every Friday SIMU has an invited speaker, and the next day there is a field trip. That summer, one of the speakers was John Hubbard from Cornell University. During a trip to Arecibo's observatory, John asked me about the projects for the students. I remember saying, "I won't tell you; I would like the students to solve them." He then asked me about my area of work. My standard response used to be "classical analysis," but I simply told him: "I compute integrals for a living." The geometric interpretation of the rational Landen transformations came out of my argument to convince him that not everything was done in the subject.

Acknowledgments

The author thanks G. Boros for many suggestions and the editor for considerably improving an original version of this work. The author acknowledges the partial support of NSF-DMS 0070567, Project number 540623.

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About the Cover

This month's cover is a variation of Figure 3 of Victor Moll's article—it zooms in on a smaller region than displayed in Moll's article, and plots many more iterations of the 2D Landen transformation in formula (5) of his article, starting with a more or less random point in the chaotic domain of the transformation. The color changes from red to violet as the iterations proceed. The fine structure one sees is striking, and also striking is that there does not seem to be any simple or even satisfactory way to account for all that is seen. Some Java applets allowing one to explore these and related phenomena can be found at http://www.math.ubc.ca/people/faculty/cass/covers/2002/march/landen.html.

—*Bill Casselman* (covers@ams.org)