WHAT IS...

A Worm?

Harold P. Boas

My topic is not renegade computer programs, nor Write-Once-Read-Many optical storage devices, nor convoluted plane curves.[1] The "worm" in my title is an important example in multidimensional complex analysis.

A first course on holomorphic (that is, complex analytic) functions of *one* variable introduces two flavors of function theory: namely, functions holomorphic in the entire complex plane $\mathbb{C}$ and functions holomorphic in the unit disc. If one restricts attention to topologically trivial planar domains, then there are no other one-dimensional theories of holomorphic functions. Indeed, the Riemann mapping theorem says that every simply connected planar domain other than the whole plane can be mapped onto the unit disc by a one-to-one holomorphic mapping with a holomorphic inverse, and hence function theory on that domain is equivalent to function theory on the unit disc.

In contrast, the theory of holomorphic functions of two (or more) variables comes in an infinite variety of flavors. Two domains in multidimensional complex space typically are holomorphically inequivalent: each domain supports its own individual theory of holomorphic functions. For example, a bidisc (the product of two one-dimensional discs) cannot be mapped onto a ball in $\mathbb{C}^2$ by a one-to-one, invertible holomorphic mapping; indeed, Walter Rudin has written both a book titled *Function Theory in Polydiscs* and a book titled *Function Theory in the Unit Ball of* $\mathbb{C}^n$.

The author is professor of mathematics at Texas A&M University and the editor of the Notices. *His email address is* `boas@math.tamu.edu`.

[1] *Leo Moser's worm problem asks for a planar convex set of minimal area that contains a congruent copy of every rectifiable arc of length* 1. *In the November 1973 issue of* Scientific American, *Martin Gardner discussed Paterson's worms, which are certain paths on a planar grid.*

For general domains that lack symmetry, there is little hope to understand the function theory in such an explicit way as one can on polydiscs and on balls. One may hope, however, to get some qualitative information. For example, *is function theory stable under perturbations of the domain?*

One of several remarkable discoveries of Fritz Hartogs in his seminal 1905 dissertation was a propagation phenomenon in two-dimensional function theory. An example is furnished by perturbations of the Hartogs triangle T, the set of points (z, w) in $\mathbb{C}^2$ such that $|z| < |w| < 1$. One may visualize this domain as being fibered over a punctured unit disc in the w-plane, the fiber over a point w being a disc in the z-variable of radius $|w|$. For small positive ϵ, let the perturbation T_ϵ be the union of T and the set of points (z, w) such that $\max(|z|, |w|) < \epsilon$ (the bidisc of radius ϵ). A function that is holomorphic in T_ϵ has a two-variable Maclaurin series that converges in a neighborhood of the origin, and a little trickery with the one-dimensional Cauchy integral formula shows that this series must actually converge in the whole bidisc of radius 1. Thus, *all* holomorphic functions on T_ϵ extend to be holomorphic functions on the unit bidisc, a much larger domain! No such extension phenomenon occurs for holomorphic functions of a single variable.

A domain for which it is *not* the case that all holomorphic functions on the domain extend to a larger domain is called a *domain of holomorphy*. Domains of holomorphy are the natural domains on which to study function theory; they are of interest also in mathematical physics. Some examples of domains of holomorphy are polydiscs, balls, and (more generally) convex domains.

The Hartogs triangle T turns out to be a domain of holomorphy too. Yet every function holomorphic in a neighborhood of the *closure* of T is holomorphic on some T_ϵ and hence extends to be holomorphic on the unit bidisc. Consequently, every domain of holomorphy that contains the closure of T also contains the unit bidisc. Thus the domain of holomorphy T cannot be approximated from outside by domains of holomorphy.

In 1933, when H. Behnke and P. Thullen raised the general question of when a domain of holomorphy can be approximated from outside by domains of holomorphy, the only negative examples they could offer were variations of the Hartogs triangle T. The boundary of T has a bad singularity at the origin, and one might hope that exterior approximation would be possible for a domain of holomorphy whose boundary is, say, a C^∞-smooth manifold. In 1976 Klas Diederich and John Erik Fornæss constructed a counterexample in their paper "A strange bounded smooth domain of holomorphy". Their example came to be known as "the worm domain" (actually they constructed a family of domains), because it winds in a way reminiscent of a spiral staircase.

For simplicity I specialize the real parameter in their example to be equal to 25 and discuss one specific worm. A preliminary nonsmooth version is the set of points (z, w) in $\mathbb{C}^2$ such that

$$\left| z + e^{i \log |w|^2} \right| < 1 \quad \text{and} \quad 1 < |w| < 25.$$

This domain is fibered over an annulus in the w-plane, each fiber in the z-variable being a disc of radius 1 with center at a point of modulus 1. As $|w|$ varies, the center of the fiber moves along a unit circle, and so these disc fibers wind around a central axis.

Since $\log 25^2 > 2\pi$, the fibers wind all the way around a circle, and the projection of this domain into the z-plane is a punctured disc of radius 2. The projection of a neighborhood of the *closure* of the domain, however, covers a full disc of radius 2 in the z-plane. This filling in of the puncture gives a hint why, as in the case of the Hartogs triangle, one cannot approximate the domain from outside by domains of holomorphy.

The domain just described, in which $|w|$ is chopped off flat at the ends, does not have smooth boundary, but Diederich and Fornæss showed how to add suitable caps to the ends to create counterexample domains of holomorphy with smooth boundary. The smooth domains constructed in this way are the worm domains.

Since these domains depend on the variable w only through the modulus $|w|$, one can represent the worms in ordinary three-dimensional space by collapsing together all the points with equal values of $|w|$. The illustration, with the central axis fattened for clarity, was created by James T. Hoffman for the 1995–96 program in several complex variables at the Mathematical Sciences Research Institute in Berkeley.

The worm is a counterexample to another stability property: Diederich and Fornæss showed that there exist holomorphic functions on the worm that are continuous on the closure of the domain yet cannot be approximated by functions holomorphic in a neighborhood of the closure. Two decades later, the worm was discovered to be a counterexample also to a regularity property in partial differential equations.

Holomorphic functions are the solutions of the homogeneous Cauchy-Riemann equations. When solving the inhomogeneous Cauchy-Riemann equations, one often is interested in the "canonical" solution, the solution whose modulus squared has minimal integral. If the inhomogeneous Cauchy-Riemann equations on a domain of holomorphy with smooth boundary have data whose derivatives of all orders extend continuously to the boundary, must the canonical solution have the same regularity? In 1980 Steve Bell and Ewa Ligocka showed that an affirmative answer has important consequences for determining whether two domains are holomorphically equivalent.

The class of domains for which the answer to the question is yes includes, for example, the convex domains (as Emil J. Straube and I proved). Work of Christer O. Kiselman for the chopped-off worms and of David E. Barrett for the smooth worms suggested that the answer might be negative in general. Finally, a 1996 paper of Michael Christ proved that the smooth worms, originally constructed for a different purpose, are indeed counterexamples to this regularity property in partial differential equations.

For Further Reading

[1] JOHN ERIK FORNÆSS and BERIT STENSØNES, *Lectures on Counterexamples in Several Complex Variables*, Princeton University Press, 1987.

[2] SO-CHIN CHEN and MEI-CHI SHAW, *Partial Differential Equations in Several Complex Variables*, American Mathematical Society and International Press, 2001.