The Carnegie Initiative on the Doctorate: The Case of Mathematics

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The Carnegie Foundation commissioned a collection of essays as part of the Carnegie Initiative on the Doctorate (CID). Essays and essayists represent six disciplines that are part of the CID: chemistry, education, English, history, mathematics, and neuroscience. Intended to engender conversation about the conceptual foundation of doctoral education, the essays are a starting point and not the last word in disciplinary discussions. Those faculty members, students, and administrators who work in the discipline are the primary among multiple audiences for each of these essays.

Comments on the essays and on the CID are welcome and may be sent to `cid@carnegiefoundation.org`. Further information may be found at the website `http://www.carnegiefoundation.org/cid` and in the article "The Carnegie Initiative on the Doctorate", by Allyn Jackson, *Notices*, May 2003, pages 566–8.

The other Carnegie essay about mathematics, by Tony Chan, will appear in the September 2003 issue of the *Notices*.

—Allyn Jackson

Introduction[1]

Mathematics is a *discipline*—a domain of knowledge, an intellectual heritage with ancient roots, with language and methods for analysis and understanding of aspects of the worlds that we inhabit and experience. And mathematics is now as well a *profession*—an intellectual community dedicated to knowledge generation, application, conservation, and transmission, interacting with other domains and institutions of learning and with the larger society. My thesis here is that, *historically, the disciplinary perspective of mathematics has dominated and largely shaped the design of doctoral programs.* The professional aspects of mathematics have gradually come into prominence over the past half-century, sometimes haltingly, and with mostly ad hoc adjustments in education and practice. Mathematics departments have mainly responded to immediate environmental pressures—of resource availability and the professional marketplace—without much broad reflection on the proper meaning and purpose of the mathematics doctorate in today's world. I argue here that we should build on the proven strength of the discipline-focused doctoral training and develop scholars who are also professionals with a sense of calling that I shall begin to elaborate below. It is this view rather than transient market and resource pressures that can best guide our rethinking of the doctorate in mathematics.[2] The ideas advanced here are less conclusions than

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[1] *Mathematicians are the primary among multiple audiences for this essay. This is my excuse for occasional use of technical terms of the field in an essay otherwise intended for a broad intellectual audience concerned with doctoral programs.*

[2] *In the COSEPUP (Committee on Science, Engineering, and Public Policy) Report, "Reshaping the Graduate Education of Scientists and Engineers", (National Academy Press, 1995) [2], a related distinction was made between preparation for academic research within the discipline and preparation for applied research in industry, government labs, etc. That report similarly proposed a better-integrated and more versatile preparation for these two kinds of career trajectories. (See [6].) Earlier discussions of the mathematics doctorate can be found, for example, in [8], [3], and [5].*

Selections from the *Oxford English Dictionary*

Doctor:

A teacher, instructor; one who gives instruction in some branch of knowledge, or inculcates opinions or principles.

One who, by reason of his skill in any branch of knowledge, is competent to teach it, or whose attainments entitle him to express an authoritative opinion; an eminently learned man [*sic*].

Philosophy:

(In the original and widest sense) The love, study, or pursuit of wisdom, or of knowledge of things and their causes, whether theoretical or practical.

That more advanced knowledge or study, to which, in the medieval universities, the seven liberal arts were recognized as introductory; it included the three branches of natural, moral, and metaphysical philosophy, commonly called the three philosophies. Hence the degree of Doctor of Philosophy.

That department of knowledge or study which deals with ultimate reality, or with the most general causes and principles of things.

Discipline:

Instruction imparted to disciples or scholars; teaching; learning; education, schooling.

A particular course of instruction to disciples.

A branch of instruction or education; a department of learning or knowledge; a science or art in its educational aspect.

Profession:

The declaration, promise, or vow made by one entering a religious order; hence, the action of entering such an order; the fact of being professed in a religious order.

The action of declaring, acknowledging, or avowing an opinion, belief, intention, practice, etc.; declaration, avowal.

A vocation in which a professed knowledge of some department of learning or science is used in its application to the affairs of others or in the practice of an art founded upon it. Applied spec[ifically] to the three learned professions of divinity, law, and medicine; also to the military profession.

The occupation which one professes to be skilled in and to follow. Now usually applied to an occupation considered to be socially superior to a trade or handicraft; but formerly, and still in vulgar (or humorous) use, including these.

The body of persons engaged in a calling.

prompts for a broad-based professional conversation about the doctorate in mathematics.

What is driving the need for change? Partly responsible is the intellectual growth in the discipline, as new ideas, methods, and instruments open up unexplored mathematical landscapes of both theory and application. But at least as important are demographic and economic pressures. Mathematics and science undergird the growth and development of modern technology and industry. Everything from security to commerce to health now rests inextricably on scientific foundations. This involves mathematics both as a direct producer of marketable ideas and applications and as an enabling discipline for all of the other sciences: physical, life, and social. As the whole scientific enterprise thus expands and intertwines with economic and social needs, there is a corresponding growth in the building of human capital and thus of the professional communities webbed in this complex system. The professional mathematics community has fully participated in this growth. Moreover, it has been not only a protagonist but, more intensively than the other sciences, a primary resource and agent for quantitative education at all levels.

The human expression of this growth and evolution is, first of all, a long-term increase in the sheer number of persons who characterize themselves as mathematicians or as involved in mathematically intensive professions.[3] Second, the variety of professional environments in which substantial mathematics is practiced has greatly expanded well beyond the academic settings—themselves now more diverse—that historically employed the vast majority of mathematics doctorates. This demographic change alone already calls for increased professional infrastructure and function (e.g., expanded instructional mission, more publications and journals, more conferences, new institutes, more robust professional organizations, greater

[3] *From 1862 till 1933 there were about 1,300 U.S. mathematics Ph.D.'s earned. Only 16 percent of these published more than five papers, and more than half published none [8]. During the 1950s Ph.D. production increased sevenfold; during the post-Sputnik 1960s it increased from 500 to 1,250 annually. In the 1970s and 1980s there was retrenchment in response to funding reductions and a saturated academic marketplace. About 25 percent of the Ph.D.'s found positions in doctoral-granting departments; the others in liberal arts colleges and nonacademic settings. There was brief but aborted consideration of a nonresearch Doctor of Arts degree. In 1999–2000, 1,127 mathematics Ph.D.'s were earned. Barely half of these were U.S. citizens, and many of these were Asian. The number of women is still too small but has been growing. There are hardly any Blacks or Hispanics. (See [5] and [3].)*

representation of mathematics in public arenas). In turn, there is a concomitant need for mathematicians to take professional responsibility for managing and supporting this more elaborate infrastructure and function. Our doctoral programs in mathematics have yet to prepare students for these expanded professional roles.

In this essay I first offer a brief sketch of mathematics as a *discipline*. This perspective, whose validity endures but whose incompleteness for doctoral preparation is increasingly evident, has historically dominated thinking about the doctorate. Following that I discuss the *profession* of mathematics as it has currently evolved. With this background I then propose one vision of "stewardship" of mathematics. I deliberately choose not to say "of the discipline of mathematics," since I intend this concept (of stewardship) to embrace the professional as well as disciplinary aspect of the field. In a final section I outline some implications of this perspective for the design of doctoral programs in mathematics.

The Discipline of Mathematics

The discipline of mathematics as a deductive science has its roots mainly in Greek antiquity. Geometry for our Greek forebears was considered empirical in content, being a theory of the physical space that we inhabit, but deductive in method. Euclidean geometry took as a logical point of departure a small set of propositions (axioms) deemed to be "self-evident" and thence eschewing all reasoning making appeal to actual physical sense or measurement. The extent to which Euclidean geometry models physical reality is a scientific, not a mathematical, question. But the deductive axiomatic method modeled by the intellectual development of Euclidean geometry remains a cornerstone of the mathematical paradigm. Mathematics today includes deductive explorations of its own internal worlds in their own terms. The extent to which these mathematical worlds reflect, or model, some natural reality may account for the external utility of the mathematics but is not essential to the logical coherence or significance of the mathematics within the discipline. Nonetheless, most mathematical theories have their historical roots in problems arising from empirical science, and even the "purest" of mathematical theories often reconnect in unanticipated ways with the external world, constantly reinforcing Eugene Wigner's evocation of the "unreasonable effectiveness of mathematics" [10]. We are repeatedly reminded that mathematics seems to be the spring of Nature's idiom.

The characteristic that distinguishes mathematics from all other sciences is the nature of mathematical knowledge and its certification by means of mathematical proof. On the one hand, it is the only science that thus pretends to claims of absolute certainty. On the other hand, this certainty, which is self-referential, is gained at the cost of logical disconnection from the empirical world. As Einstein put it, "As far as the properties of mathematics refer to reality, they are not certain: and as far as they are certain, they do not refer to reality" [4].

This explains a fundamental contrast between mathematics and the scientific disciplines. Mathematics and the physical sciences honor very different epistemological gods. Mathematical knowledge tends much more to be cumulative. New mathematics builds on, but does not discard, what came before.[4] The mathematical literature is extraordinarily stable and reliable. In science, in contrast, new observations or discoveries can invalidate previous models, which then lose their scientific significance. The contrast is sharpest in theoretical physics, which historically has been the science most closely allied with the development of mathematics. The mathematician I. M. Singer once compared the theoretical physics literature to a blackboard that must be periodically erased.

Some theoretical physicists—Richard Feynman, for instance—enjoyed chiding the mathematicians' fastidiousness about rigorous proofs. For the physicist, if a mathematical argument is not rigorously sound but nonetheless leads to predictions that are in excellent conformity with experimental observation, then the physicist considers the claim validated by nature, if not by mathematical logic. For the physicist, nature is the appropriate authority. The physicist P. W. Anderson once remarked, "We are talking here about theoretical physics, and therefore of course mathematical rigor is irrelevant."

On the other hand, some mathematicians have shown a corresponding contempt for this freewheeling approach of the theoretical physicists. The mathematician E. J. McShane once likened the reasoning in a "physical argument" to that of "the

[4] *This monumental growth of usable mathematical knowledge poses a major challenge for the scholar as well as for the research student of mathematics. Relief is found in one of the essential tendencies of mathematics itself—the use of abstraction and generalization—whereby broad swaths of the subject are synthesized and distilled into simple unifying concepts and principles that encompass a variety of complex cases. This process has sometimes been called "compression". Interestingly, this very effective epistemological process presents an obstacle to mathematicians as teachers, in which role they must "decompress" the subject matter in order to connect with their less-initiated students. A more complex discussion of the growth of mathematical and scientific knowledge can be found in [7, Chapter 7].*

woman who could trace her ancestry to William the Conqueror, with only two gaps."[5]

The development of mathematics as a deductive science is more complex than my simple account suggests. Many central mathematical theories were first developed quite far largely on the basis of deep physical or other intuitions prior to being put on rigorous logical foundations. Moreover, even the foundations of mathematics and logic have weathered turbulent shocks and tensions, for example, from the work of Gödel and later the constructivist doctrines. But these foundational crises have not functionally undermined the basic deductive ethos of working mathematicians. The mathematician André Weil once characterized logic and foundations as the "hygiene" of mathematics, not its heart and soul.

A more recent development is the influence of technology and its essential use in the construction of some important mathematical proofs. The notion of mathematical proof is a precise theoretical construct, but it is quite formal, rule bound, and ponderous. Mathematicians typically do not produce such formal proofs, but rather convince expert colleagues essentially that such a proof exists, the presumption being that the conviction carries the belief that under duress and with sufficient time such a proof could be supplied by the proponent. Here we glimpse the boundary of mathematics as a discipline—as a set of theoretical ideas—on the one hand and as a profession—as a human practice—on the other. When a "proof" is reduced to checking a finite but large set of critical cases and this checking is within range of machine computation but beyond reasonable human capacity, what then is the standing of the computer-reliant argument? Notice that this question is not strictly a mathematical one, but one about the intellectual sociology, norms, and methodology of practice.[6]

So much for the deductive methodology of mathematics. What about the *content*, the subject matter of the discipline? How has that changed? Many mathematicians hold deep convictions about the fundamental unity of the discipline. The grand themes—number, space, change, and (more prominently in recent times) chance—are often just different perspectives on or representations of the same phenomena, focusing on somewhat different kinds of questions. These themes are sometimes associated with the respective names—algebra, geometry, analysis, and probability/statistics. What has changed is the appearance of new or rejuvenated areas of investigation within the discipline and also vastly expanded interdisciplinary interaction with other domains of science and technology.

Much of this has been spurred by the availability of powerful computers and sophisticated mathematical software for computation, exploration, modeling, and simulation. The latter, which are founded on mathematically designed software, have become a fundamental paradigm in virtually all of science and industry. For example, aircraft prototypes are now virtually tested on computer screens, not physically built and tested in wind tunnels. It is easy to envisage the cost reductions and design leverage thus gained.

The new frontiers of mathematics investigation and application are too numerous to list, but we can mention a few noteworthy examples. The theory of dynamics and complex systems—studying the long-term evolution of systems governed by even relatively simple nonlinear laws—stalled in the 1920s for lack of computational capacity. The use of computers here, comparable with the introduction of telescopes into astronomy, has supported an explosive rebirth and expansion of the subject, including the visual discovery of stunning fractal geometries. Coding theory and cryptology, dealing with historic questions of reliability and security in the public transmission of information, now rests on the use of sophisticated tools from number theory and algebraic geometry. Theoretical computer science is founded on discrete mathematics and has given birth to the new mathematical domain of complexity theory which offers precise mathematical measures of the difficulty of certain classes of computations, which in turn is one of the foundations for the design of public key encryption systems. Quantum models of computation (not yet physically realized) are being theoretically developed and shown to support practical algorithms for problems known to be intractable by conventional computers. Methods of geometry and analysis have supported the design of noninvasive medical diagnostics. Signal and image processing have achieved dramatic applications using methods from analysis and statistics. Mathematical biology now incorporates tools from fields like topology and dynamics as well as traditional fields like fluid mechanics. Mathematics of finance has become a thriving field of application, providing widely used mathematical tools to Wall Street. And at the more theoretical end, there has been a virtual merger of fundamental particle physics with some of the most sophisticated branches of geometry and topology, with fascinating shifts in the traditional paradigms of knowledge generation.

[5] *This debate about the norms for mathematical claims based on sophisticated physical heuristics has been recently reawakened by the dramatic and paradigm-challenging commingling of fundamental particle physics with the most advanced levels of geometry. A remarkable record of views on this philosophical issue has been assembled in the* Bulletin of the American Mathematical Society*; see, for example, Vol. 29, No. 1, July 1993; and Vol. 30, No. 2, April 1994.*

[6] *See [9] for an insightful reflection on these issues.*

The overriding message from all of these developments is that mathematics is much more "out in the world" than it was even a quarter of a century ago. There are more directions of exploration within mathematics, with a greater diversity of tools and methods; there are substantial interdisciplinary interventions of mathematics in a variety of fields; the utility of mathematics for many problems of science and society is increasingly evident; and mathematics has a growing presence in administrative and policy environments, both in universities and at the national level. Finally, the mathematics profession has a growing responsibility for helping to improve the quality of quantitative education in the nation's schools, a task that can fruitfully be viewed as another site of interdisciplinary mathematics. Awareness of this outward reach must newly figure in the design of doctoral programs.

The Profession of Mathematics

Mathematics Work Environments

Historically, the doctoral program in mathematics was designed to be an apprenticeship into the research practice of an academic research mathematician. Its general form, if not the fine details of its structure, was remarkably similar across research-intensive universities. Foundational knowledge, typically gained in the first year or two of courses, covered algebra, analysis (real, complex, functional, and differential equations), and topology/geometry. This was certified by passage of qualifying examinations and often earned a master's degree in passing. The next stage was more advanced elective course taking and seminar participation, leading to selection of an area of research and an advisor, perhaps following a second research preliminary examination. The final stage was framing a doctoral research project, carrying out the research, and writing the dissertation under the guidance of one's advisor. In the past there were also requirements for reading knowledge of as many as two of the major languages of the mathematical literature. These have recently been considerably relaxed, if not eliminated. The final passage is the dissertation defense, typically at the conclusion of four to seven years of study.

Imagine that this newly minted doctoral student gains a faculty position in a similar doctorate-granting, research-intensive mathematics department. What are the components of her professional work, life, and responsibility? First and foremost in the culture of her professional formation is the active production and publication of original mathematics research. Second, and at least as demanding in time and effort, is teaching, mostly undergraduate and frequently calculus. Other aspects of her scholarly work might eventually include participating in and running research seminars; mentoring graduate students; writing mathematics papers; interacting with journal editors; peer reviewing of papers and research proposals of others; keeping up with the research literature related to her field; preparing and submitting research proposals for funding support; participating in research conferences, perhaps helping organize them; joining professional organizations; and writing reference letters for students and colleagues. She would be expected to serve on departmental or university committees and to work on committees of state or national organizations. At later stages of her career she might take on major administrative responsibility in the department as chair or director of the graduate or undergraduate program. She might also be enlisted as a journal editor or for work in policy environments or as staff in a federal research agency or as an officer of a professional organization.

It is interesting for us to consider along this spectrum of potential professional activities those for which her doctoral program provided explicit and substantial preparation. Foremost is the preparation for doing original *research*: selecting and framing research questions; assimilating the immediately relevant literature; strategizing the work; using imagination; diligently and productively enduring frustration; and finally, finding new results, organizing and clearly articulating them, and providing for them well-presented documentation and exposition. In the traditional value system of disciplinary mathematics, this performance of creative scholarship far outweighs all others combined. It is considered the most noble of professional achievements.

Teaching, which typically occupies about half of her working time, is belatedly gaining an improved status in the professional value system. But traditionally it was considered a professional duty (one spoke of "teaching loads" but not of "research burdens") whose most dignified aspect was the instruction of mathematically talented and motivated students whom one tried to nurture and induct toward advanced mathematics study. Currently, the quality of mathematics instruction for *all* students is taken much more seriously, not least because of external pressures. Mathematics faculty members are now expected to provide high-quality mathematics instruction across the board, and they are held accountable for this in the prevailing hiring, promotion, and reward system. At the same time, it is often tacitly assumed that rigorous and deep understanding of disciplinary mathematics, coupled with injunctions to communicate it clearly and coherently to students, suffices to produce quality instruction. The "transfer model" of learning implicit in this Platonic way of thinking treats knowledge as a commodity that the professor carefully delivers to the student, considered as a vessel expectantly waiting to be filled. Until recently it was hardly acknowledged that teaching entails knowledge and skills that are *more than* academic subject matter knowledge combined with formally lucid exposition and a sympathetic disposition toward students. In fact, it involves a kind of knowledge of mathematics itself that is distinct from what research mathematicians require for their research or typically know. Moreover, it is only recently recognized that this knowledge and skill can be taught and learned. Apart from a minimally mentored apprenticeship through teaching assistantships or graduate instructorships, scant professional development for the work of teaching has been provided to doctoral students in most mathematics departments. Similarly, the skills of mentoring graduate students are, like those of teaching, typically (and imperfectly) gained by imitating the observed models of one's own mentors.

Beyond these two domains—research and teaching—what preparation was provided for the other aspects of our new Ph.D.'s professional life and work? A good initiation into tracking the research literature and participation in research seminars will have been provided. These activities, integral to the dissertation research, are among the vital practices of the ongoing intellectual life of a research mathematician. But there remain numerous other basic professional functions for which little or no mentoring may have been provided. These include the more refined skills of scientific writing, interaction with editors, preparing peer-review evaluations and letters of reference, and the preparation and submission of research proposals. Some mentoring for these activities may be picked up as part of a postdoctoral appointment, but unevenly so. Virtually no preparation, nor even consciousness-raising, is made for possible administrative or other leadership or public roles. Nor is there much cultivation of an expected participation in the larger mathematics community, for example, in the professional organizations or as staff in federal research agencies.[7]

The preceding discussion was predicated on our new Ph.D. having joined a doctoral-granting, research-intensive mathematics department. Two other major kinds of career launches have to be considered as well: academic appointment in a less research-intensive mathematics department—for example, in a liberal arts college—or nonacademic appointment as a mathematics specialist in some industrial or other private sector setting. Moreover, her lifetime career trajectory may well include passages in a mix of such environments. They each place many of the same demands on our new Ph.D. that were discussed above, but perhaps with different emphases and priorities. In the first kind of example, research and intellectual vitality remain important to varying degrees, but much greater emphasis tends to be placed on teaching, interaction with students, and service to the department and to the university or college. In the second, nonacademic, types of settings, the mathematical activity tends to be interdisciplinary, and then of course there is need to gain some functional knowledge of one or more outside fields of mathematical application. This kind of work is often part of a collaborative team effort, so that relational skills come into play. Moreover, the demands of effective communication of technical knowledge among others with a very different professional culture and language present a major challenge. It is worth noting that these demands are not unlike those of effective teaching. Again, the doctoral program likely provided very little professional development for these kinds of skills, dispositions, and sensibilities.

Mathematicians in the World at Large

Our portrait of the professional life of a mathematics Ph.D. has consisted so far of a survey of the diverse demands and responsibilities of the kinds of *work environments* in which she would likely find herself. Missing from this is a sense of what is, or might be, the sense of personal agency and professional identity that our Ph.D. carries into the *outside world at large*. Neither our profession nor our doctoral departments have devoted much conscious reflection to these issues, and hence these have not been cultivated in our doctoral programs.

What professional identity does a traditionally trained mathematics Ph.D. carry, at least ideally? Being a mathematician incorporates a deep and expert knowledge of some significant domain of mathematics, including its epistemology and research methods. This is situated within a broader knowledge of the history and grand intellectual currents of the discipline, including their historic connections with the allied natural sciences, particularly physics. Today, more and more, the mathematician should have some functional knowledge of active interfaces with other disciplines and of important areas of application of mathematical methods. This ensemble of resources provides the mathematician with a rich cultural awareness of the discipline, with the tools and skills for the generation of new mathematical knowledge, and with the

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[7] *This portrait of traditional professional development in doctoral programs has been substantially improved in selected departments that have benefited from the NSF VIGRE (Vertical Integration of Research and Education in the Mathematical Sciences) program. In fact the VIGRE program incorporates a vision of a departmental culture that resonates remarkably well with the vision of the CID. For an informative account and assessment of current implementations of the VIGRE program, see "The Report of the AMS, ASA, MAA, and SIAM Workshop on Vertical Integration of Research and Education in the Mathematical Sciences" (American Mathematical Society, 2002 [1]).*

skills and expert knowledge for selected interdisciplinary work environments or institutional settings. Finally, the mathematician should be a competent teacher in academic settings and communicator in interdisciplinary settings. This collectively describes a professional identity founded on a deep enculturation into the intellectual traditions of the discipline and, further expanded by the demands and vicissitudes of the job market, the mathematician serving as supplier of expert skill and communicator of technical knowledge to needy consumers.

This portrait includes some, but not all, of the roles which "stewardship of mathematics" might encompass. What significance does being a mathematician carry in the larger society and culture? In what ways does a mathematician function as a representative of the discipline in public arenas? What are its dimensions of social responsibility and of cultural and aesthetic expression? To prompt our thinking about these questions, we might consider other professions in which this sense of professional belonging, presence, and purpose is more easily recognized and appreciated. Medicine is a profession of health and healing guided by the Hippocratic Oath. While each physician pursues a specialized practice, this practice is situated in a larger sense of belonging to a professional community with a collective social mission of human betterment, of which each practitioner is a contributor, advocate, and representative. Law is similarly a profession of diverse expert practices but which carries with it a sense of stewardship of the political institutions on which our system of social organization and justice is founded. Architects provide expert technical and aesthetic skills, but these are expressions of an ancient legacy of the design and function of public and private physical environments to support and harmonize with human need and social purpose. And of course creative artists—writers, composers, performers—have a deep sense of how their well-honed craft serves large goals and needs of human social and cultural enrichment, of awakening emotions, and of elevating an awareness of the basic human condition.

What is the larger social and cultural significance of mathematics that the public should know and appreciate and that professional mathematicians could represent? Many current conditions beg for some compelling answers to this question: the existence and growing scale of our professional community, supported by public resources; the pervasive, though not highly visible, enabling roles of mathematics in every domain of science and technology; the many years of mathematics instruction as a basic literacy required of all school children. These conditions all speak to the implied importance of mathematics. Yet few mathematicians can furnish intrinsic explanations of this that would be compelling to most well-educated adults who, in the U.S., often boast of their mathematical weakness. Indeed, mathematicians may well be challenged even to provide convincing arguments to themselves, arguments that go much beyond the comfortable celebration of the beauty and depth of the core ideas and intellectual architecture of mathematics.

This relatively undeveloped sensibility and skill among professional mathematicians is not merely a matter of benign neglect of their social development and responsibility. It has a direct bearing on the overall long-term well-being of the field and of the quality of mathematics education at all levels. It is at the root of the pressing problems of sustaining public resources and of enlisting more, and more diverse, domestic U.S. talent into mathematics and into mathematically intensive professions. It is certainly germane to any conception of "stewardship" of mathematics.

By what means could these larger senses of professional identity find expression beyond the routines of professional practice? In many ways. Op-ed pieces or other public writing and exposition. Public presentations or performances. Participation in civic enterprises such as school boards or cultural organizations. Contributing well-informed advice or service in policy environments and government agencies. And, perhaps above all, through teaching and communicating with the same professional attention and skill that we dedicate to our scholarly research. In each of these instances, one crucial resource is having deep and well-articulated expert knowledge and communicating clearly how this knowledge bears on the issues at hand. But much more skill and sensibility of a more subtle kind come into play. This includes a sense of audiences, of their knowledge and beliefs, and of the kind of language, contexts, and representations they can find comprehensible and persuasive. It includes a sense of norms for disciplined and respectful interaction with people very different from oneself. It includes an appreciation that in civil or cross-disciplinary (as opposed to some scholarly mathematical) discourse, adversarial postures are not a primary virtue but are a last recourse when constructive and collaborative approaches have failed.

Stewardship of Mathematics

As I emphasized at the outset, mathematics is both a discipline and a profession. The discipline of mathematics is a domain of knowledge, with finely developed methods of generation and validation of new knowledge, a noble intellectual heritage with ancient roots, and an unending source of language and concepts for quantitative description and understanding of the world. The profession of mathematics, on the other hand, is a community of human practice, one that generates, validates, synthesizes, conserves, and disseminates mathematical knowledge and practices. "Stewardship" of the field of mathematics must attend to both the strength and integrity of the disciplinary culture and to the health and integrity of the professional community. This may seem self-evident, since each essentially depends on and reinforces the other. However, a main purpose here is to give the professional face of this a visibility that has been largely lacking.

A steward of mathematics must have a deeply developed sense of intellectual and professional mission and community. This is operative in an expanding progression of spheres of professional life and activity.

At the most immediate and familiar level, a mathematician belongs to the community of *research* scholars in her area of *specialization*, colleagues with whom she intellectually identifies and communicates via correspondence, shared manuscripts, conferences, etc. This is situated within the *larger mathematics research enterprise*. Though any single mathematician is active in no more than a small number of sites of this work, mathematicians recognize the global cohesion and interdependence of the ensemble of this work, and this manifests itself in the lack of parochialism in the advocacy (for example, to federal agencies) of support for the field. One expression of this community participation is active membership in and support of the work of the professional organizations in the field.

In a similar vein, as a member of a *university mathematics department*, mathematicians face a different terrain of collective mission, this time embracing not only diverse mathematical specialties but also interdisciplinary connections with other units and programs, the immense teaching enterprise for which every mathematics department is responsible; and the larger intellectual, instructional, and administrative needs of the university environment. The individual mathematician is actively engaged with only a few components of this vast portfolio of responsibilities. A mathematics department is not a single purposeful agent with a focused agenda. What makes it a cohesive intellectual and professional community is that each member feels the collective responsibility for and commitment to the whole departmental mission, in the sense that each member respects and supports each aspect, whether directly involved in it or not.

Finally, the mathematician as scholar and teacher, may function within the *larger society* in diverse areas of policy and outreach—federal funding of research; education at all levels; technical legislation and regulatory policies; public communication about the nature, significance, and evolution of mathematics and its applications.

If doctoral programs are to produce "stewards of mathematics", then what are the capacities that comprise this? In the view that I have espoused here, the following should be prominent in that list:

- Mastery of the core foundational knowledge of the discipline, including a broad sense of its historical evolution.
- Command of the methods of mathematical inquiry and of certification of new knowledge.
- A deep and expert knowledge of at least one specialized area of mathematics at a level supporting the capacity for original research, including a knowledge of how this area is situated in the larger mathematical landscape in relation to other fields, and a thorough knowledge of the immediately relevant literature.
- A sense of discrimination and judgment of the significance and depth of new mathematical problems and results.
- Skills of scientific documentation and written exposition.
- Facility in the use of the mathematical literature, including an informed awareness of its scope, organization, and editorial and reviewing practices.
- Knowledge, and mastery in some cases, of some basic uses of technology in mathematics, including uses of the Web, electronic manuscript preparation, computationally supported research, and instructional uses of technology.
- The ability to frame and draft proposed programs of research for outside funding.
- Finely developed and adaptable skills for teaching mathematics at diverse levels, from introductory undergraduate courses to advanced graduate research courses and seminars.
- The ability and disposition to mentor research students and young faculty.
- A general cultural knowledge of the range of mathematically intensive fields and of the ways that mathematics is used in various human endeavors, and with what applications, perhaps including some in-depth knowledge in one or two cognate areas.
- Skills of communication of and about mathematics to diverse audiences.

The Design of Mathematics Doctoral Programs

What does all of this say or imply about the design of doctoral programs in mathematics? Let us summarize some of the main features proposed here.

The strength and soundness of the traditional research training in the core areas of the discipline should be preserved. This is training that emphasizes both a broad and unified global view of the discipline and the need for deep knowledge and original scholarship in some specialized area. In addition, opportunities for interdisciplinary learning and research should be available and sanctioned, if not required. Because of the increasing role of data analysis in applications of mathematics, some exposure to probability and statistics should now be a part of every mathematician's preparation.

Doctoral programs should further recognize the critical role of teaching in a mathematician's career. For this it does not suffice to provide graduate teaching experience. Much more serious attention needs to be paid to professional development for this work, and this is an area where involvement of expertise from mathematics education would be useful and appropriate.

Development of competencies with the diverse uses of technology for document preparation, for research, and for instruction should be provided. Attention should be given to the development of skills of scientific documentation and written exposition. Students should learn to navigate and use the mathematical literature and the protocols of scientific publication and reviewing. Mentoring should be provided for the process of framing a research program and of preparing and submitting a proposal to a funding agency for its support.

Students should be given more explicit awareness of the infrastructure of their diverse professional environments, and of the resources and services that sustain them. This includes the mathematics department within the university environment and, on the outside, the disciplinary community, its organized activities, and the organizations and institutions that sponsor them. Students should be able to anticipate and appreciate the professional roles that they will eventually play in these spheres.

Finally, doctoral programs should more self-consciously and creatively confer a strong sense of cultural awareness in students of the significance of their discipline in the larger worlds of science and society and of the expectation that they will serve as emissaries of their discipline in the outside world. One concrete way that this might be done is in the form of a professional development seminar. The themes of the seminar could include questions in education—for example, a serious inquiry into the nature of teaching, learning, and assessment—or critical evaluation of some curriculum materials. Or the seminar might examine some current area of public policy of concern to mathematicians, in which assignments might include composition of an op-ed piece or a letter to a congressman.

This ambitious list poses a challenging task of program design. Some, but not all, of the items can be treated in a curricular framework, through appropriate course or seminar development. Other aspects might more appropriately be addressed through other professional development kinds of formats. Possibilities include special supervised projects or brief internships, perhaps in the context of a one-credit professional development seminar. These might take such forms as immersion in an interdisciplinary project; time spent in an industrial setting or in a school mathematics program; or a small project of analysis and writing about some policy area, for example, in education, in research funding, or about the infrastructure of the profession.

At this point we must confront an obvious and fundamental dilemma. Reform agendas, of which this essay represents one, typically know how to add but not subtract. To an already demanding model of the mathematics doctorate, we have proposed added conditions of performance. Yet the traditional model has already been criticized for the excessive time required. This is a major design challenge. No simple solution exists. This difficulty cannot, however, be an excuse for inaction. The challenge merits broad discussion in the professional community, with perhaps the development of diverse models emphasizing different kinds of orientation. We can likely profit from study of how other fields manage within a fixed time frame to prepare doctoral candidates to enter the profession as well as earn admission to the practice of the discipline.[8]

Conclusion

This essay argues that the traditional doctorate in mathematics has been fashioned almost exclusively on a *disciplinary* view of the field and that the strength of that model needs to be expanded to encompass the modern evolution of mathematics as a *profession*. Stewards of mathematics must attend not only to the traditional disciplinary missions of knowledge generation, validation, representation, and dissemination but also to the needs and infrastructure of the professional community of mathematicians, and to the responsibilities of that community to the discipline, to the

[8] *Phillip Griffiths has suggested that we have something to learn here from the design of programs in the professional schools.*

institutional environments in which it functions, and to the needs of the larger society.

Works Cited

[1] American Mathematical Society, The Report of the AMS, ASA, MAA and SIAM Workshop on Vertical Integration of Research and Education in the Mathematical Sciences (Reston, VA), American Mathematical Society, Providence, RI, 2002; http://www.ams.org/amsmtgs/VIGRE-report.pdf. See also footnote 9.

[2] Committee on Science Engineering and Public Policy, Reshaping the Graduate Education of Scientists and Engineers, National Academy Press, Washington, DC, 1995.

[3] William Duren Jr., Are there too many Ph.D.'s in mathematics?, Amer. Math. Monthly **77** (6) (1970), 641–6.

[4] Albert Einstein, Geometry and experience (an address to the Prussian Academy of Sciences, January 27, 1921), in *Sidelights on Relativity*, New York, Dover, 1983.

[5] John Ewing, *Graduate and Postdoctoral Mathematics Education*, American Mathematical Society, Providence, RI, 2002; http://www.ams.org/ewing/Grad-Education.pdf.

[6] Allyn Jackson, Should doctoral education change?, *Notices* Amer. Math. Soc. **43** (1) (1996), 19–23.

[7] Philip Kitcher, *The Nature of Mathematical Knowledge*, Oxford University Press, New York, 1984.

[8] R. G. D. Richardson, The Ph.D. degree and mathematical research, Amer. Math. Monthly **43** (14) (1936), 199–215.

[9] William P. Thurston, On proof and progress in mathematics, Bull. Amer. Math. Soc. (N.S.) **30** (1994), 161–177.

[10] Eugene Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Commun. Pure Appl. Math. **13** (1960), 1–14.