Mathematics of the Heavens

Robert Osserman

Sponsored each April by the Joint Policy Board for Mathematics, Mathematics Awareness Month provides an opportunity to celebrate mathematics and its uses. The theme for Mathematics Awareness Month 2005 is "Mathematics and the Cosmos". In this article, the *Notices* reproduces three Mathematics Awareness Month "theme essays" written by Robert Osserman. Two other theme essays, plus a variety of resources including a Mathematics Awareness Month poster, are available on the website `http://www.mathaware.org`.

Mathematics and the Cosmos

Introduction and Brief History

The mathematical study of the cosmos has its roots in antiquity with early attempts to describe the motions of the Sun, Moon, stars, and planets in precise mathematical terms, allowing predictions of future positions. In modern times many of the greatest mathematical scientists turned their attention to the subject. Building on Kepler's discovery of the three basic laws of planetary motion, Newton invented the subjects of "celestial mechanics" and dynamics. He studied the "n-body problem" of describing the motion of a number of masses, such as the Sun and the planets and their moons, under the force of mutual gravitational attraction. He was able to derive improved versions of Kepler's laws, one of whose consequences yielded dramatic results just in the past decade when it was used to detect the existence of planets circling other stars.

The two leading mathematicians of the eighteenth century, Euler and Lagrange, both made fundamental contributions to the subject, as did Gauss at the turn of the century, spurred by the discovery of the first of the asteroids, Ceres, on January 1, 1801. The nineteenth century was framed by the publication of Laplace's *Méchanique Céleste* in five volumes from 1799 to 1825 and Poincaré's *Les Méthodes Nouvelles de la Méchanique Céleste* in three volumes from 1892 to 1899. The mid-nineteenth century produced further important contributions from Jacobi and Liouville, among others, as well as two groundbreaking new directions that were to provide the basic tools leading to the two revolutionary breakthroughs of twentieth-century physics; Hamilton's original approach to dynamics became the springboard for quantum mechanics and the general subject of dynamical systems, while Riemann's 1854 "Habilitationsschrift" introduced curved spaces of three and more dimensions as well as the general notion of an n-dimensional manifold, thus ushering in the modern subject of cosmology leading to Einstein and beyond.

The twentieth century saw a true flowering of the subject, as new mathematical methods combined with new physics and rapidly advancing technology. One might single out three main areas, with many overlaps and tendrils reaching out in multiple directions.

First, cosmology became ever more intertwined with astrophysics, as discoveries were made about the varieties of stars and their life histories, as well as supernovae and an assortment of previously unknown celestial objects, such as pulsars, quasars, dark matter, black holes, and even galaxies themselves, whose existence had been suspected but not confirmed until the twentieth century. Most critical was the discovery at the beginning of the century of the expansion of the universe, with its concomitant phenomenon of the Big Bang, and then at the end, the recent discovery of the accelerating universe, with its associated conjectural "dark energy".

Second, the subject of celestial mechanics evolved into that of dynamical systems, with major advances by mathematicians G. D. Birkhoff, Kolmogorov, Arnold, and Moser. Many new discoveries were made about the n-body problem, both general ones, such as theorems on stability and instability, and specific ones, such as new concrete solutions for small values of n. Methods of chaos theory began to play a role, and the theoretical studies were both informed by and applied to the profusion of new discoveries of

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planets, their moons, asteroids, comets, and other objects composing the increasingly complex structure of the solar system.

Third, the advent of actual space exploration, sending artificial satellites and space probes to the furthest reaches of the solar system, as well as the astronaut and cosmonaut programs for nearby study, transformed our understanding of the objects in our solar system and of cosmology as a whole. The Hubble space telescope was just one of many viewing devices, operating at all wave lengths, that provided stunning images of celestial objects, near and far. The 250-year-old theoretical discoveries by Euler and Lagrange of critical points known as "Lagrange points" saw their practical application in the stationing of satellites. The 100-year-old introduction by Poincaré of stable and unstable manifolds formed the basis of the rescue of otherwise abandoned satellites, as well as the planning of remarkably fuel-efficient trajectories.

Finally, the biggest twentieth-century innovation of all, the modern computer, played an ever-increasing and more critical role in all of these advances. Numerical methods were applied to all three of the above areas, while simulations and computer graphics grew into a major tool in deepening our understanding. The ever-increasing speed and power of computers went hand-in-hand with the increasingly sophisticated mathematical methods used to code, compress, and transmit messages and images from satellites and space probes spanning the entire breadth of our solar system.

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Space Exploration

Starting in the twentieth century, the mathematical exploration of the cosmos became inextricably entwined with the physical exploration of space. On one hand, virtually all the methods of celestial mechanics that had been developed over the centuries were transformed into tools for the navigation of rockets, artificial satellites, and space probes. On the other hand, almost all of those space vehicles were equipped with scientific instruments for gathering data about the Earth and other objects in our solar system, as well as distant stars and galaxies going back to the cosmic microwave background radiation. Furthermore, the deviations in the paths of satellites and probes provide direct feedback on the gravitational field around the Earth and throughout the solar system.

Beyond these direct effects, there are many other areas of interaction between the space program and mathematics. We list just a few:

- GPS: the global positioning system,
- data compression techniques for transmitting messages,
- digitizing and coding of images,
- error-correcting codes for accurate transmissions,
- "slingshot" or "gravitational boosting" for optimal trajectories,
- exploitation of Lagrange points for strategic placement of satellites,
- dynamical systems methods for energy-efficient orbit placing,
- finite element modeling for structures such as spacecraft and antennas.

Some of the satellites and space probes that have contributed to cosmology and astrophysics are

- the Hubble space telescope,
- the Hipparcos mission to catalog the positions of a million stars to new levels of accuracy,
- the COBE and WMAP satellites for studying the cosmic microwave background radiation,
- the Genesis mission and SOHO satellite for studying the Sun and solar radiation,
- the ISEE3/ICE space probe to study solar flares and cosmic gamma rays before going on to visit the Giacobini-Zimmer comet and Halley's comet,
- the LAGEOS satellites to test Einstein's prediction of "frame dragging" around a rotating body.

Rather than trying to cover all or even most of the mathematical links, we focus on two that are absolutely essential and central to the whole endeavor: first, navigation and the planning of trajectories; and second, communication and the transmission of images.

Navigation, Trajectories, and Orbits

When the U.S. space program was set up in earnest—a process described in detail in the recent History Channel documentary *Race to the Moon*—a notable feature was the introduction of the Mission Control Center. The first row of seats in mission control was known as "the trench", and it is from there that the mathematicians whose specialty is orbital mechanics kept track of trajectories and fed in the information needed for navigation. Their role is particularly important for operations involving rendezvous between two vehicles, in delicate operations such as landings on the Moon and in emergencies that call on all their skills, the most notable of which was bringing back alive the crew of Apollo 13 after they had to abandon the command module and were forced to use the lunar landing module—never designed for that purpose—to navigate back to Earth.

The first thing that an astronaut or former astronaut will tell you about navigating in a spaceship is that no amount of experience piloting a plane will be of any help. On the contrary, previous experience may be a hindrance, since it reinforces one's natural intuition that if you want to catch up with an object ahead, you go faster, and conversely. But if you are orbiting at a certain speed and have to rendezvous with something ahead, then "stepping on the accelerator" (translate as "applying a forward thrust") will lift you into a higher orbit where first of all, the vertical distance between you and the object orbiting ahead will increase, and second, your average angular velocity will decrease, by Kepler's third law, and you will find yourself getting further and further behind. In fact, the only way in practice to effect a rendezvous and docking maneuver is to feed the data on both vehicles into a computer and apply the methods of orbital mechanics to plan a trajectory that brings both vehicles to the same place at the same time at essentially the same velocities. Mathematically, such maneuvers are best described by working in phase space, where each point has six coordinates: three describing its position and three describing components of its velocity vector. One must find paths for the two vehicles that come together in phase space.

The ability to navigate started with Isaac Newton. Not only did he formulate his laws of motion and of gravity, but he also developed the calculus which allowed him to put those laws into the language of mathematical equations. Today, our knowledge of the physics involved has been improved with the addition of relativity and other factors that may play a role, and calculus has been further developed into many different branches of mathematics.

What kind of mathematics? Newton's equations involve the gravitational forces acting upon one of the participating bodies, arising from all of the other bodies. Since force is mass times acceleration and since acceleration is simply the second derivative of position with respect to time, it is the differential calculus that describes the accelerations. Then, once the accelerations are given, it is necessary to use integral calculus in order to get from the second derivatives to the positions.

In a more general context, where the mass may be changing with time, such as happens with an extended application of thrust to a vehicle, with the gradual reduction of weight as fuel is used up, or in cases of relativistic speeds, the force is given by the first derivative of momentum, but the principle is the same.

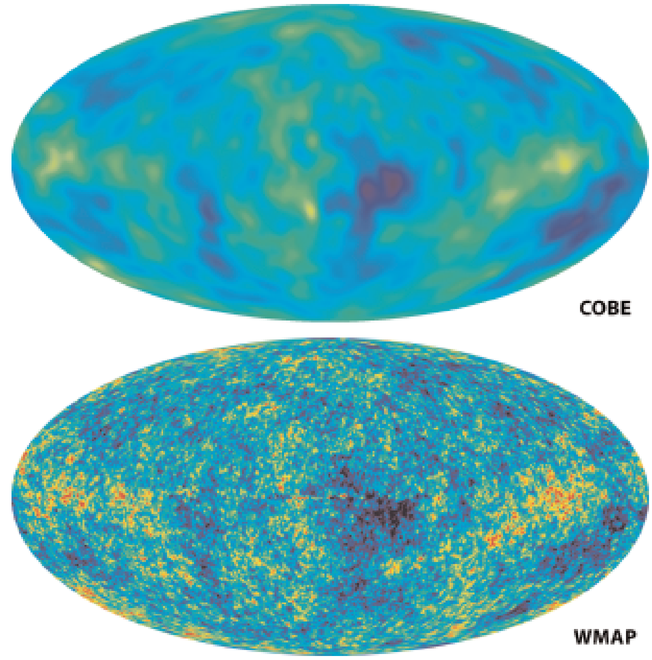

Image by NASA/WMAP Science Team.

An all-sky image of the infant universe, 380,000 years after the Big Bang. In 1992, NASA's COBE mission first detected tiny temperature fluctuations (shown as color variations) in the infant universe, a landmark discovery. The WMAP image brings the COBE picture into sharp focus. The new, detailed image provides firm answers to age-old questions.

In the case of the 2-body problem, where the only force involved is the gravitational attraction between the two bodies, it is frequently said that Newton was able to give a complete solution. That is not, strictly speaking, the case, if one means by "a solution" of a differential equation, an expression for the unknown function whose derivatives appear in the equation. In this case, it would mean finding an expression for the position as a function of time. However, what Newton showed was that the orbit of each of the bodies lies on a conic section (in a fixed inertial frame of reference), and in the case considered by Kepler, where the orbit is an ellipse, there is an explicit expression for the *time* as a function of the *position*. Commonly known as "Kepler's equation", it is of the form $t = x - e \sin x$, in suitable units of time t, where x is the polar angle from the center of the ellipse and e is the eccentricity. What one wants, of course, is x as a function of t, and much effort and ingenuity have gone into finding effective means of solving Kepler's equation for x in terms of t. Lagrange did extensive work on the problem, in the course of which he developed both Fourier series and

Bessel functions, named after later mathematicians who investigated these concepts in greater detail. Both Laplace and Gauss made major contributions, and succeeding generations continued to work on the subject.

When there are more than two bodies involved, the problem cannot be solved analytically; instead, the integration (positions from accelerations) must be done numerically: now, with high-speed computers. So, numerical integral calculus is a major factor of spacecraft navigation.

One may picture navigation as being the modeling of mother nature on a computer. At some time, with the planets in their orbits, a spacecraft is given a push outward into the solar system. Its subsequent orbit is then determined by the gravitational forces upon it due to the Sun and planets. We compute these, step-by-step in time, seeing how the (changing) forces determine the motion of the spacecraft. This is very similar to what one may picture being done in nature.

How does one get an accurate orbit in the computer? The spacecraft's orbit is measured as it progresses on its journey, and the computer model is adjusted in order to best fit the actual measurements. Here one uses another type of calculus: estimation theory. It involves changing the initial "input parameters" (starting positions and velocities) in the computer in order to make the "output parameters" (positions and velocities at subsequent times) match what is being measured: adjusting the computer model to better fit reality.

Also in navigation, one must "reduce" the measurements. Usually, the measurements do not correspond exactly with the positions in the computer; one must apply a few formulae before a comparison can be made. For instance, the positions in the computer represent the centers of mass of the different planets; a radar echo, however, measures the path from the radio antenna to the spot on a planet's surfaces from which the signal bounces back to Earth. This processing involves the use of trigonometry, geometry, and physics.

Finally, there is error analysis, or "covariance" calculus. In the initial planning stages of a mission, one is more interested in how accurately we will know the positions of the spacecraft and its target, not in the exact positions themselves. With low accuracy, greater amounts of fuel are required, and it could be that some precise navigation would not even be possible. Covariance analysis takes into account (1) what measurements we will have of the spacecraft: how many and how good, (2) how accurately we will be able to compute the forces, and (3) how accurately we will know the position of the target. These criteria are then used in order to determine how closely we can deliver the spacecraft to the target. Again, poor accuracy will require more fuel to correct the trajectory once the spacecraft starts approaching its final target.

One of the mathematical tools used to optimize some feature of a flight trajectory, such as fuel consumption or flight time, is a maximum principle introduced by Pontryagin in 1962. Pontryagin's theorem characterizes the optimum values of certain parameters, called the *controllers*, that determine a trajectory.

In recent decades, ingenious new methods have been developed to extract the maximum effect from the least amount of fuel. One such method is known as the "slingshot" or "gravity-assisted trajectory". By aiming a space vehicle in a way that crosses the orbit of another planet or moon just behind that body, the path of the vehicle will be deflected, sending it on its way to the next target with minimum expenditure of fuel. A number of space probes, such as Cassini-Huygens, have benefited from carefully calculated trajectories that make multiple use of the slingshot effect. Gravity-assist methods are equally important for sending a probe toward the inner planets: Venus and Mercury. In that case, one sends the probe to a point on the orbit just in advance of the body it is passing. In both cases, the effect of the fly-by is to alter the velocity, changing the direction of flight and leaving the end speed *relative to the body it is passing* unchanged. However, that body will be moving with considerable momentum relative to the Sun, and there will be an exchange of momentum in which the body will be slowed down or speeded up by an infinitesimal amount, while the probe will be speeded up or slowed down by a considerable amount *relative to the Sun.*

More modern, twentieth-century mathematical methods of dynamical systems have proved invaluable in designing complicated fuel-efficient orbits. These methods include the theory of stable and unstable manifolds, pioneered by Poincaré, leading to the subject now known as chaotic dynamics, and the KAM theory, due to Kolmogorov, Arnold, and Moser, of invariant tori and stability. One of the first achievements of the new methods was the 1991 rescue of a Japanese spacecraft *Hiten* that was stranded without enough fuel to complete a planned mission when a second satellite, intended to work in tandem with the first, failed to operate. The mathematician Edward Belbruno had designed highly fuel-efficient orbits using methods derived from chaos theory, and that turned out to be just what was needed for this rescue operation. Belbruno's methods were incorporated into the design used by Giuseppe Racca and the European Space Agency in sending "SMART-1", their first satellite to the Moon, in 2004. Newspaper headlines trumpeted "spacecraft reaches moon on 5 million miles a gallon" as a dramatic way of underlining the astonishing fuel efficiency of the method.

In the past few years, other applications of the theory of stable and unstable manifolds have been invoked in trajectory planning. Martin Lo and his colleagues at the Jet Propulsion Laboratory developed ways to apply the theory in order to place satellites such as Genesis in an orbit around the Lagrange point between Earth and the Sun and then return it to Earth. More recently, the same group, together with Jerry Marsden at Caltech, have expanded the method for use in interplanetary travel, along what they call the "interplanetary superhighway", a route derived from the ever-changing configurations of stable and unstable manifolds in the phase space of our solar system or selected parts of it. A beautifully illustrated article by Douglas Smith describing this work can be found in the journal *Engineering and Science*.

Communication and Image Transmission

For space exploration and interplanetary probes, navigational techniques and orbital mechanics may get you where you want to go, but it is not worth much if the data collected cannot be successfully transmitted back to Earth. In the case of the Cassini spacecraft at Saturn, signals have to travel distances on the order of a billion miles or more. Data transmitted from across the planetary system with very limited power are received on Earth as a very faint signal (as low as a billionth of a billionth of a watt) embedded in noise. Only through miracles of modern technology operating in tandem with ever-improving mathematical methods is one able to receive the striking and detailed images that are now on display.

Two critical processes come into play for transmitting messages of all sorts. The first is compression, to be able to transmit the maximum amount of information with the least number of bits, and the second is the use of error-correcting codes, to overcome problems of noise and distortion. Basically, one wants to eliminate redundancy to obtain compression of the data, and then one has to introduce redundancy in order to catch and correct errors of transmission. The two operations may at first seem to cancel each other out, but in fact the types of redundancies involved in the two cases are quite different.

A variety of mathematical techniques are used to compress the spacecraft data into fewer bits prior to transmission to the ground. A simple one avoids transmitting all sixteen bits of every data element of a data stream. The value of the first data element is sent, but for the rest of the elements, only the difference from the first is sent. The value of the first element might require sixteen bits, but the differences are so small they might only need two or three bits. Once on the ground the first value can be added to all the others to restore the original content. For a large data stream, techniques such as this can save hours of transmission time and much storage capacity.

"Entropy coding" is a technique that takes into account the probability distribution of different sets of data in order to encode more probable data with shorter sequences, just as in Morse code where the letter "E" is represented by a single dot. Image data compression techniques rely on mathematical image probability models that exploit the similarities between neighboring small picture elements to minimize the number of bytes needed to describe the image.

The possibility of detecting and even correcting errors in transmission was first pointed out in a groundbreaking paper of Richard Hamming in 1950. Since then, an entire field has grown up in which, on one hand, ever more refined methods have been devised for practical applications, and on the other hand, the theory of error-correcting codes has turned out to have fascinating links with sphere packing and simple groups, beautifully described in the book of Thomas Thompson.

More recent mathematical innovations that have proved to be of both theoretical interest and great practical use in this connection are the subjects of fractals and wavelets. An excellent survey of all aspects of image analysis, transmission, and reconstruction is the theme essay on Mathematics and Imaging on the 1998 Mathematics Awareness Week website.

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Trajectories, Orbits, and Space Navigation

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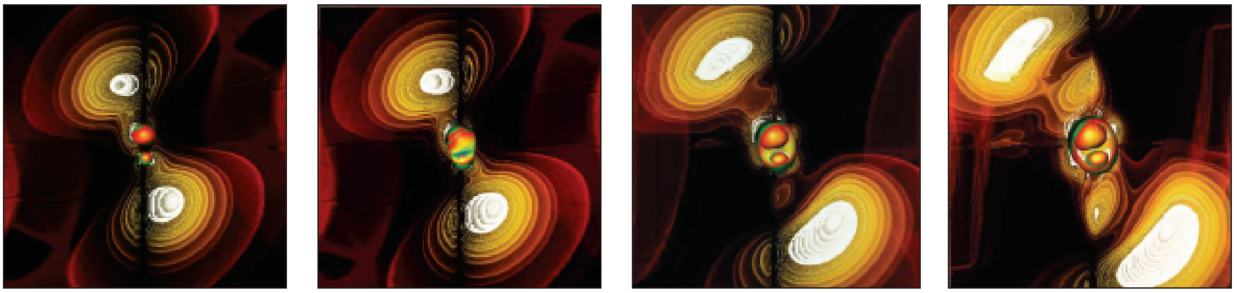

This sequence of images shows one of the first 3D simulations, carried out by members of the European Union Training Network "Sources of Gravitational Waves". The sequence shows two spinning black holes merging in the final stages of a rapidly decaying orbit. The leftmost image shows the two individual black holes about to merge. The individual horizons are shown in the center of the image (the larger black hole is just above a smaller one). The developing burst of radiation is shown shooting out towards the upper left direction and to the lower right. The final image on the right shows the final black hole, with the two original black hole horizon surfaces still seen inside, and it also shows the developing and intensifying burst of gravitational waves. Images by Ed Seidel and Werner Benger of the Albert-Einstein-Institut, the Zuse-Institut-Berlin, and the Center for Computation & Technology at Louisiana State University. Reproduced with permission of the European Union Training Network "Sources of Gravitational Waves".

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Martin Lo main page: `http://www.gg.caltech.edu/~mwl/`.

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From Black Holes to Dark Energy: Cosmology in the Twenty-first Century

Cosmology in the twentieth century was almost in its entirety the outgrowth of Einstein's foundational paper in 1915 on general relativity. Two years later he presented his first model of the universe based on general relativity together with Riemann's notion of the three-sphere.

Side-by-side with the theoretical advances, observational astronomy led to great leaps in astrophysics, as the life cycles of stars were discovered and elaborated, the existence of other galaxies outside our own was confirmed, and the expansion of the universe was demonstrated to the satisfaction of nearly all. Einstein was able to use general relativity on the one hand to explain earlier observations, such as the amount of precession of the planet Mercury, and on the other hand to make new predictions for the observers to confirm or refute. The first and most widely heralded of those was the prediction of the bending of light as it passed close to a large mass such as the Sun. Others, such as gravitational red-shift, gravitational lensing, and "frame-dragging" around a rotating body, were confirmed one by one over the course of the century. Still others, like the existence of gravity waves, remain a high priority for twenty-first-century experimentalists.

The Nobel Prize-winning work of Russell Hulse and Joseph Taylor stemmed from their discovery in 1974 of a pulsar whose "pulses" varied in a regular fashion, leading them to conclude that it had an invisible companion, the pair forming a familiar binary system each one circling the other (or actually, their common center of gravity) in an approximately elliptical orbit. In this case, the pair consisted of two bodies each as massive as the Sun, but compressed into a tight ball whose diameter was the size of a small town, and each completed its orbit around the other in about eight hours. Under such extreme conditions, the relativistic effects would be considerable. One of those effects would be the production of gravity waves, and Einstein's equations predicted that those waves would radiate energy in a way that causes the two bodies to gradually get closer, which would in turn speed up the rate that they completed each orbit by a very precise amount. After observing the variations in the pulse rate over a period of four years, Hulse and Taylor were able to show that the speed-up was indeed taking place at the rate predicted, to within less than 1 percent deviation.

That provided the first experimental evidence for the existence of gravity waves.

That evidence, however, was indirect. In fact, the strength of the predicted waves in the case of the binary pulsars was far too weak for any hope of direct detection on Earth. However, the same general principles would apply to a binary pair of black holes, and there the calculations indicated that the strength of the waves could be just within the limits of possible detectability with suitably crafted apparatus. Attempts at detection had already begun in the late 1950s with Joseph Weber. At that time, not only was the reality of gravity waves in doubt, but the existence of black holes was generally greeted with skepticism.

The idea of black holes (although not the name) arose very soon after Einstein formulated general relativity. Karl Schwarzschild, despite the fact that he was in the German army stationed in Russia and that it was in the midst of World War I, read Einstein's paper and almost immediately was able to solve Einstein's equations for the case of the gravitational field surrounding a (nonrotating) spherically symmetric body. A few weeks later he was able to solve the equations and describe the space-time curvature in the interior of the body. One of the consequences of the Schwarzschild solution seemed to be that a sufficiently massive body compressed within a sufficiently small radius (where "sufficiently" was made precise by the Schwarzschild equations) would have the property that no radiation or matter could ever escape. Oddly, a very similar conclusion was reached by purely Newtonian methods in 1783 by John Michell in England and became widely known through Laplace's famous five-volume *Le Système du Monde*. In both cases, however, the question remained whether it was possible for a real-world physical body to exist within those parameters. The first theoretical evidence was adduced in a 1939 paper by Robert Oppenheimer and Hartland Snyder, who calculated the space-time geometry around an imploding massive star, under certain simplifying assumptions, and concluded that the star would eventually become invisible.

As for the reality of black holes, it was hard for the experts, much less the general public, to decide whether they represented science or science fiction. Many leaders in the field, from Einstein to John Wheeler, had serious doubts. It was not until the advent of X-ray astronomy that the balance was tilted in favor of science. Since X-rays from outer space do not penetrate our protective atmosphere, this research developed hand-in-hand with rocket science. The big discovery was the existence of a powerful X-ray source in the constellation Cygnus, designated Cyg X-1. This discovery was made in a rocket flight in 1964. The first X-ray satellite, *Uhuru*, was launched in 1970, while its successor, *Einstein*, launched in 1978, was an X-ray telescope that was able to make X-ray images as sharp as an optical telescope. Gradually, the scientific community became convinced that Cyg X-1 was indeed a real-life black hole whose physical characteristics corresponded closely to those predicted by the theory. Evidence has accumulated for other X-ray sources arising from the vicinity of black holes, as well as black holes in the center of quasars and large galaxies, such as our own.

One tool that has become increasingly more important in the study of black holes as in the rest of astronomy and cosmology has been computer simulations. By 2001, such simulations were able to predict the nature of the gravitational waves that we might be able to detect from the collision and merging of two black holes and to display the results in dramatic images. The reality of black holes and, in particular, their role in the production of gravity waves are now widely enough accepted that large amounts of money are being invested in experimental devices, such as the LIGO project, to detect associated gravity waves.

Given the extent to which theoretical predictions about black holes appear to be confirmed by observations, why the continued hesitancy about their wholehearted acceptance? One answer is that each of the predictions is based on certain simplifying assumptions and continuing unknowns. For example, the early models were for a spherically symmetric nonrotating body, whereas physical reality almost certainly corresponds to rotating bodies with concomitant bulges at the equator. But most importantly, what was missing in the early studies and what is conspicuously absent in the above discussion is the central role of quantum effects. That, however, would lead us far afield and can be found in many of the references given below. Instead, we indicate briefly two further subjects of particular mathematical interest.

In 1953, a young differential geometer, Eugenio Calabi, made a study of complex manifolds and was led to conjecture that under very general conditions there should be a metric on each manifold of a particularly symmetric nature. This Calabi conjecture was a subject of great interest and was finally proved in 1977 by Shing-Tung Yau. Although of considerable mathematical interest, the Calabi-Yau manifolds, as they came to be known, had no obvious connection to cosmology until the advent of string theory introduced a whole new dimension—or, more precisely, set of dimensions—into play. What the theory required was that, in addition to our familiar four-dimensional space-time, there would be six additional "curled-up" space dimensions. Furthermore, the equations of string theory imply that this six-dimensional component must have a very particular structure, and in 1984 it was proved that the Calabi-Yau manifolds have precisely the structure needed.

The gift of mathematics to physics provided by Calabi-Yau manifolds was amply repaid when physicists discovered what was termed "mirror symmetry" between pairs of geometrically distinct but physically linked pairs of Calabi-Yau manifolds. Using this link, Philip Candelas and his collaborators were able to suggest precise numerical answers to a problem in algebraic geometry that had seemed far beyond the capabilities of any known method to provide: the number of rational curves of given degree on a fifth-degree algebraic hypersurface in projective four-space. Those numbers that algebraic geometers were able to calculate directly confirmed the predictions arising from physics.

The other circle of ideas involves what is known as "curvature flow". The simplest example consists of starting with a smooth closed curve in the plane and defining a "flow" by moving the curve in a direction orthogonal to itself at each point and at a speed proportional to the curvature at the point. Intuition suggests that the curve should become progressively more circular. In 1986, Michael Gage and Richard Hamilton were able to prove the result, starting from an arbitrary convex curve and normalizing the flow to fix the area enclosed. In a rather different situation, Hamilton was led to define and study a "Ricci flow" on an arbitrary Riemannian manifold, in which the rate of change of the metric tensor is proportional to the Ricci tensor. After some rather spectacular successes in which Hamilton was able to use his method to prove that under certain assumptions such a flow tended toward a constant curvature metric, Grigori Perelman announced in 2003 that he used extensions of the method to give complete proofs of Poincaré's conjecture and the Thurston "Geometrization Conjecture". Perelman's proof is still under review by the mathematical community before being fully endorsed by the experts. Possible cosmological implications relate to characterizing shapes of three-dimensional manifolds that may constitute the universe as it evolves in time.

In another direction, the curvature flow for curves was generalized to higher-dimensional hypersurfaces, leading to a proof of the "Riemannian Penrose inequality", first by Huisken and Ilmanen in 1997 for a single black hole and then for multiple black holes in 1999 by Hubert Bray. Roger Penrose was led to the inequality in 1973 by a physical argument about the nature of black holes.

It need hardly be said that the number of topics touched upon here represents a minuscule portion of the activity in recent decades in astrophysics, cosmology, and related parts of mathematics. In some cases, theory has led the way, suggesting observations that might be made and what to look for in those observations. In others, the results of the observations have forced theorists to rethink some of their fundamental assumptions. One of the most striking examples along those lines was the discovery in 1998, in the course of examining a certain class of supernovae, that the expansion of the universe, rather than slowing down under the restraining force of gravity, appeared to be speeding up as a result of some mysterious, hitherto undreamed of force, dubbed "dark energy". One immediate thought was that this was due to Einstein's notorious "cosmological constant". But even if it worked mathematically, that would be no more of a physical explanation than when Einstein originally inserted it into his equation for what turned out to be the wrong reason: his equations seemed to imply that the universe was expanding or contracting, rather than static in time, and this was just before the realization that the universe actually *was* expanding. The search for a satisfactory explanation of dark energy is sure to occupy a central place in the mathematics of cosmology for some time to come.

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