WHAT IS...

a Pseudoholomorphic Curve?

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The terminology *pseudoholomorphic curve* (or *J*-holomorphic curve) was introduced by Gromov in 1986. The notion has transformed the field of *symplectic topology* and has a bearing on many other areas such as *algebraic geometry, string theory,* and 4-*manifold theory*; we will return to these later.

We are all familiar with the notion of a "curve"—say a plane curve—at the elementary, and perhaps imprecise, level of ordinary calculus. We can specify a plane curve in two different ways: either as the set of solutions of an equation $f(x, y) = 0$ or via a parametrisation $x = x(t), y = y(t)$. For example, we can specify a circle by the equation $x^2 + y^2 = 1$ or by the parametrisation $x = \cos t, y = \sin t$. Another familiar concept is that of a "family" of curves, for example, the family of lines in the plane.

The theory of curves has, of course, been developed extensively both in differential geometry and algebraic geometry. The relevant branch of the classical theory for us here is that of "complex" or "holomorphic" curves. In the simplest situation, we replace the real variables x, y above by complex variables z, w and consider complex curves in the complex plane. Thus the same equation $z^2 + w^2 = 1$, for example, describes such a complex curve. Or we can consider parametrised complex curves $z = z(\tau), w = w(\tau)$ where $z(\tau), w(\tau)$ are *holomorphic* functions of a complex variable τ. More generally we may consider complex curves in complex manifolds: parametrised by holomorphic maps from Riemann surfaces.

What is a holomorphic map? Think of the simplest case of a map f from $\mathbf{C}$ to $\mathbf{C}$: a holomorphic function. The condition of holomorphicity is characterised by the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This expresses the fact that the derivative of f, in the sense of multivariable calculus, is a complex linear map from $\mathbf{C}$ to $\mathbf{C}$. The concept extends to the case of maps to *almost-complex* manifolds. Let M be a differentiable manifold of dimension $2n$. An almost-complex structure J on M is a family of linear maps $J_x : TM_x \to TM_x$, with $J_x^2 = -1$, on each tangent space TM_x of M, varying smoothly with $x \in M$. Thus the tangent spaces are made into complex vector spaces. Any complex manifold has a natural almost-complex structure, but the converse is not true if $n > 1$: there is an integrability condition which characterises these special almost-complex structures. Many manifolds M which admit almost-complex structures do not have any complex-manifold structure at all.

A pseudoholomorphic curve is just the natural modification of the notion of a holomorphic curve to the case when the ambient manifold is almost-complex. That is, we consider a Riemann surface Σ, an almost-complex manifold (M, J), and a differentiable map $f : \Sigma \to M$ such that for each $\sigma \in \Sigma$ the derivative

$$df_\sigma : (T\Sigma)_\sigma \to TM_{f(\sigma)}$$

is complex-linear with respect to the given complex structures on the tangent spaces. We can spell out more concretely what a pseudoholomorphic map amounts to in the case when we take $\Sigma = \mathbf{C}$ and let M be $\mathbf{C}^n$, with some general almost-complex structure J. It turns out, purely as a matter of linear algebra, that the $\mathbf{R}$-linear maps $J : \mathbf{C}^n \to \mathbf{C}^n$ with $J^2 = -1$ can be neatly parametrised by an open set of $n \times n$ complex matrices $\mu = (\mu_{\alpha\beta})$. Thus our almost-complex structure is represented by a matrix-valued function $\mu(\underline{z})$ of $\underline{z} \in \mathbf{C}^n$. A pseudoholomorphic curve is given by a solution of the system of partial differential equations

$$\frac{\partial z_\alpha}{\partial \overline{\tau}} + \sum_\beta \mu_{\alpha\beta}(\underline{z}) \overline{\frac{\partial z_\beta}{\partial \tau}} = 0,$$

which can be thought of as a deformation of the ordinary Cauchy-Riemann equations, for the vector-valued function $\underline{z}(\tau)$.

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The passage to almost-complex manifolds allows us to move from the classical setting of holomorphic curves in complex manifolds to a much wider, more flexible, world. Crucially, many aspects of the theory do not change greatly when we extend our ideas in this way. We can express this by the slogan *the local theory of pseudoholomorphic curves is closely akin to that of holomorphic curves.* Here, *local* can have two meanings: either that we are studying the situation locally in the manifold M or locally in the space of maps. It is crucial here that we are considering curves, rather than higher-dimensional objects. For any pair of almost-complex manifolds M, N the notion of a (pseudo)holomorphic map $f : N \to M$ makes sense, but if the real dimension of N is greater than 2, this is not a very useful concept. For example, on a generic almost-complex manifold N of dimension greater than 2 there are no nonconstant (pseudo)holomorphic functions, even locally—this is exactly the source of the integrability condition for complex-manifold structures.

Our more precise form of our slogan is the statement that if Σ is a compact Riemann surface, there is a *nonlinear Fredholm theory* which describes the deformations of a given pseudoholomorphic curve $f : \Sigma \to (M, J)$. This means, roughly, that the deformations are parametrised by a finite-dimensional manifold or moduli space $\mathcal{M}$, whose dimension can be computed from standard topological data. Moreover, again roughly, the moduli space will deform smoothly with variations in the almost-complex structure J or the Riemann surface structure on Σ. For example, suppose we take M to be the complex projective plane with its standard complex-manifold structure and Σ to be the Riemann sphere. Then any "line" (in the sense of projective geometry) in M, together with a choice of parametrisation, gives a pseudoholomorphic curve. Thus the moduli space $\mathcal{M}$ is a bundle over the dual plane with fibre $PGL(2, \mathbf{C})$—the group of Möbius maps. The nonlinear Fredholm theory tells us that if we deform the almost-complex structure slightly, while we probably cannot describe the pseudoholomorphic maps explicitly, we get a moduli space of the same general character.

Gromov's insight was that the local understanding of the pseudoholomorphic maps furnished by the Fredholm theory extends to good global theory in the situation where the almost-complex structure on M is compatible with a *symplectic structure*. Recall that a symplectic structure is given by an exterior 2-form ω satisfying two conditions. One is pointwise and algebraic: at each point ω is a nondegenerate skew-symmetric form on the tangent space of M. The other is more global and differential geometric: the form ω is closed. We say that J is compatible with ω if the bilinear form on tangent vectors

$$g(v, w) = \omega(v, Jw)$$

is symmetric and positive definite. Then g is a Riemannian metric on M. Let $f : \Sigma \to M$ be a pseudoholomorphic map. Then we can think of the integral

$$I = \int_\Sigma f^*(\omega)$$

in two ways. On the one hand, the pointwise compatibility between the structures means that I is essentially the *area* of the image of f, measured in the Riemannian metric g. On the other hand, the condition that ω is closed means that I is a topological (homotopy) invariant of the map f. So the areas of pseudoholomorphic curves, in this situation, are controlled by straightforward topological data. This allowed Gromov to prove a partial compactness theorem for the moduli spaces. For example, consider as before the maps from the Riemann sphere to the complex projective plane. If we allow large and arbitrary deformations of the standard almost-complex structure, then we cannot say much, because the pseudoholomorphic curves may degenerate in some very complicated way as we deform the structure and perhaps "disappear". But if we restrict to almost-complex structures compatible with a symplectic form, the curves cannot degenerate, because their area is controlled, and in fact Gromov showed that in this case the curves must persist, however large the deformation.

These two properties—the Fredholm theory and compactness—lay the foundations for Gromov's theory, in which the pseudoholomorphic curves are used as a tool in *symplectic topology*. The curves have been used in two main ways. The first way is as geometric probes to explore symplectic manifolds: for example in Gromov's result (later extended by Taubes) on the uniqueness of the symplectic structure on the complex projective plane, proved by sweeping out the manifold by "lines" (i.e., the pseudoholomorphic curves of the same topological type as lines in the standard case). The second way is as the source of numerical invariants: *Gromov-Witten invariants*. In the simplest case, where our moduli space has dimension zero and consists of a finite set of points, we might get an integer invariant by counting these points. This second direction has been developed most extensively in the years following Gromov's paper. The theory of *Floer homology* is based on pseudoholomorphic curves with boundary lying on a Lagrangian submanifold. This leads on to the notion of the *Fukaya category*. In *four dimensions*, Taubes discovered that the Gromov-Witten invariants coincide with the Seiberg-Witten invariants, defined in a completely different way. In the case when the manifold M is in fact a complex manifold, say an algebraic variety, the invariants are related to classical enumerative problems in *algebraic geometry*. The same invariants also appear in topological *string theory*, arising from Feynman integrals. This has provided completely new insights and uncovered wonderful and intricate algebraic structures in the invariants such as *quantum cohomology*. The Fukaya category is related to the phenomenon of *mirror symmetry*, as formulated by Kontsevich.

Further Reading

Dusa McDuff and Dietmar Salamon, *J-holomorphic Curves and Symplectic Topology*, Amer. Math. Soc. Colloq. Publ., Vol. 52, 2004.