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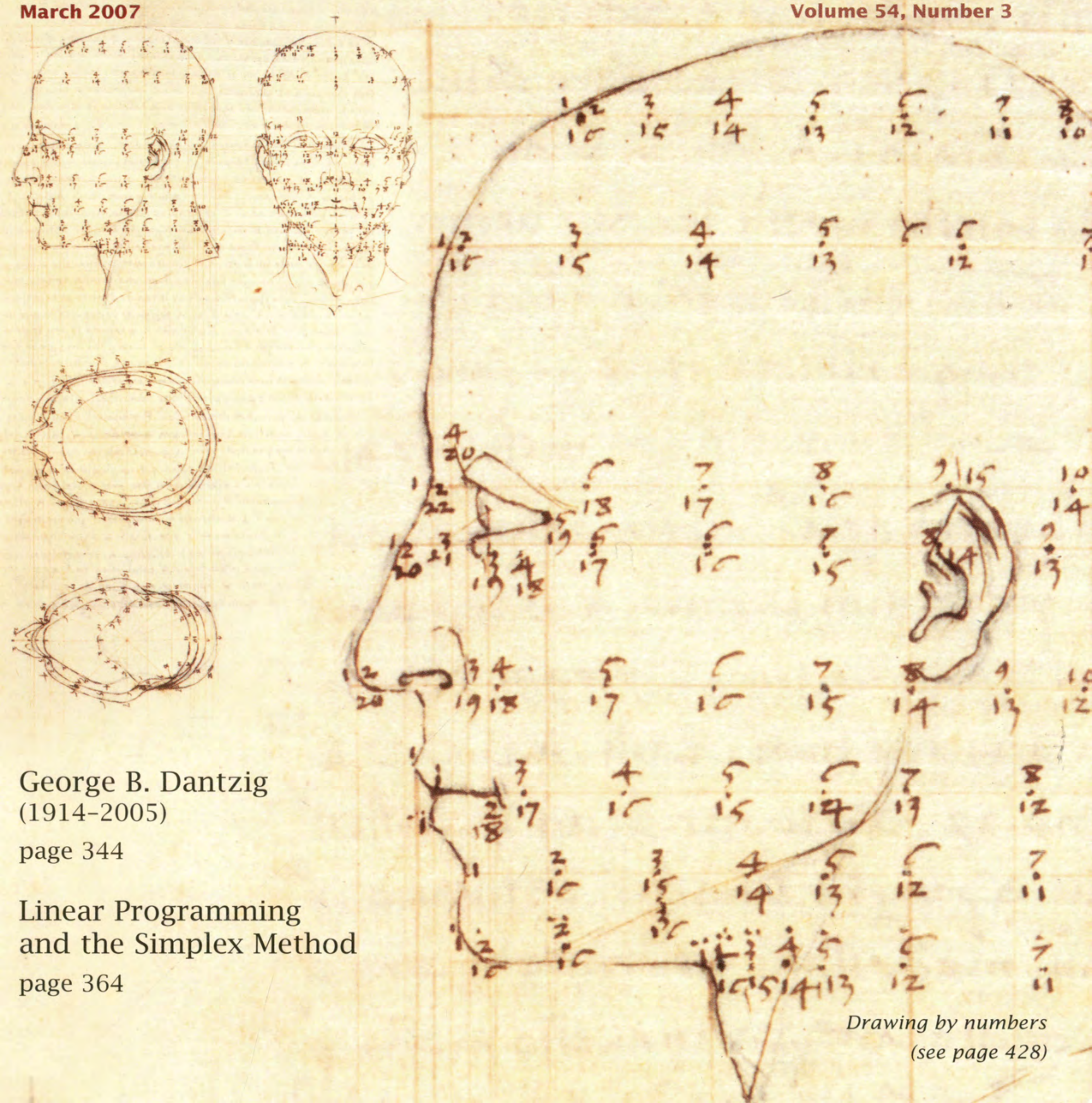

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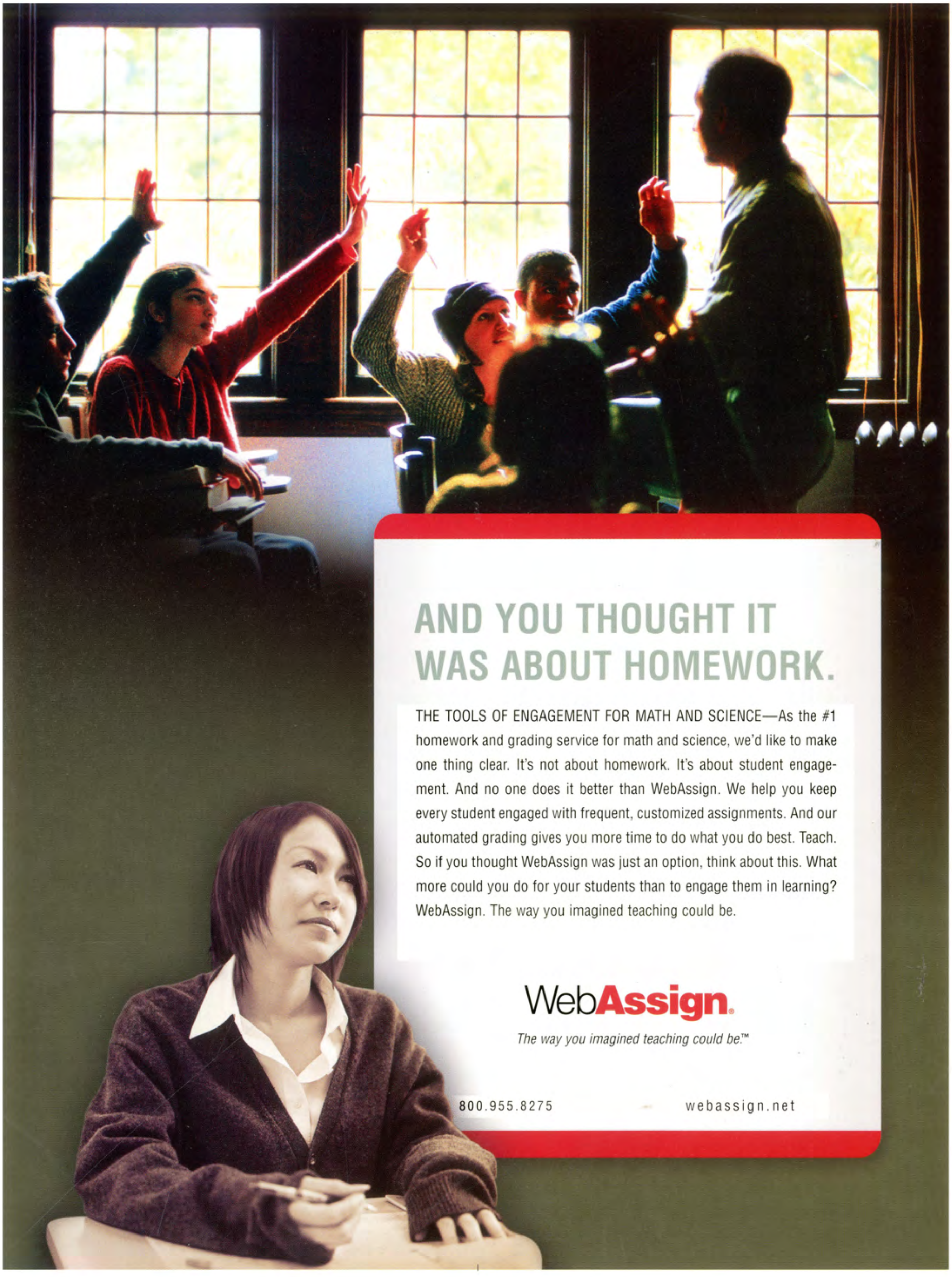
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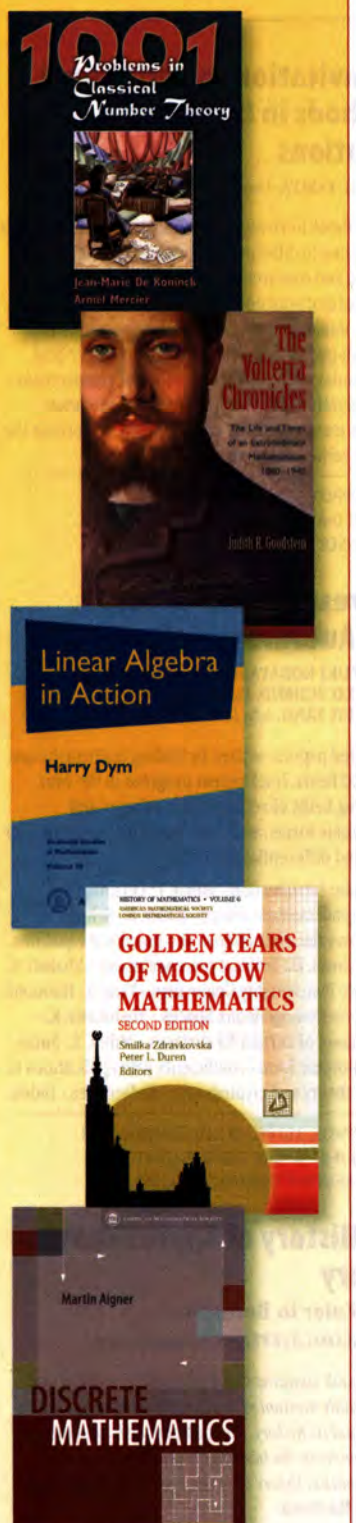

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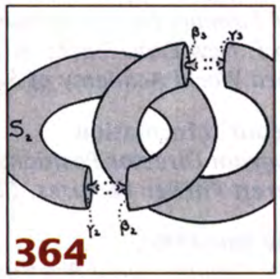
Communications

Commentary



Features

Notices
of the American Mathematical Society

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Departments

From the AMS Secretary

Because Math Matters

The president has recently appointed a National Mathematics Advisory Panel. National newspapers carry lead editorials on math education. Why, and why now? For many years there has been a debate on how to best teach mathematics in our nation's schools. There are a number of reasons that this discussion has gone on so long and become so heated. The first is that there is a great deal at stake. From Sputnik on, we have worried about our ability to compete in science and industry—first with Russia, then with Japan, and now with India and China. Mathematics is at the heart of technological innovation, advances in engineering, physics, medicine, biology, and on and on. Mathematical models can forecast environmental change and monitor energy supply and demand. Without mathematics we wouldn't have MRIs or maps of the human genome.

Second, we are not doing a very good job. U.S. students are falling behind students in most industrial countries as measured on any number of international tests. Again math matters. We know that the careers of the twenty-first century will require more and more quantitative reasoning. We know that in this global economy, companies can and will outsource jobs to countries with more mathematically skilled work forces. To quote CBS news great Fred Friendly, we don't want to become a country "in which we take in each other's laundry".

The third reason the debate is so heated is that it has become political. We hear terms like "back to basics" and "fuzzy math". But what's lost in all of this is the kids. Education debates need at their heart to be about education. We want our children to learn, to understand and be able to use mathematics as they go through school and work. Not all students will go on to be mathematicians, but they will all be called upon to use the mathematics they know.

I can't emphasize this point strongly enough. The half-life of students in mathematics courses remains one year from 10th grade on. In other words, the number of students taking math in 11th grade is half those taking math in 10th and so on for every year up until the Ph.D. What happens to the other half? We simply cannot afford to throw away half of our students each year because they don't have serious prospects of becoming research mathematicians.

We can continue to ask students problems of the form "solve for x in the equation $x^2-3x+1=0$". Or we can ask at what proportion of performance-enhancing drug use in the population is it cheaper to test two athletes by pooling their blood samples—which leads to the same equation. We can teach mathematics through engaging contexts kids will see as real and important or continue to insist on honing skills. Learning to hammer a nail before trying to build a house sounds right. But hammering nails for six years before even knowing that there's such a thing as a house just doesn't make sense. If mathematics is a life skill, then students need to see mathematical skills at work in their lives.

And that brings us back to the National Math Panel. This panel was formed as a part of the president's new American Competitiveness Initiative. The idea was to have a panel with expertise in mathematics education study the issues and give the president their best advice on how to train our students in the subject. This was to be a diverse panel with broad experience representing many points of view. But many panel members have little or no mathematical training. And many subscribe to the philosophy of emphasizing mechanical skills over the ability to meaningfully use the mathematics learned. It is hard not to fear a replay of the National Reading Panel that dealt with the issue of phonics vs. whole language only a few years ago.

But math matters. We cannot afford partisan politics. The National Science Foundation, staffed by independent scientists and mathematicians, has led the effort for innovation in mathematics education since the 1950s. Innovation is desperately needed. We must not go back to methods that have consistently failed us. After all, the reason that the current reform movement began in the first place was that we were unhappy with student performance. What we need are serious people who recognize the importance and difficulties in getting a quantitatively literate citizenry and who are willing to put aside any specific political agenda.

Articles in the *Wall Street Journal* and the *New York Times* have declared that the Math Wars are over and that the Back to Basics movement has won. Well, I have news. The Math Wars are not over because this is still about helping children learn, not about winning a political battle or finding common ground. We will continue to call for the introduction of new and relevant content, the appropriate use of new technologies, showing students important contemporary applications and using innovative pedagogical approaches. And we will do this, because math matters.

—Solomon Garfunkel
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A version of this article appeared previously in the UMAP Journal, *Vol. 27.3.*

Letters to the Editor

Teaching the Romance of Mathematics

It has been refreshing to see *Notices* include increased coverage of the public perception of mathematics, books for lay audiences, and mathematics in the media over the past few years. Sadly, I found the editorial section of the October *Notices* retrograde.

It is not a good sign when the principal math consultant of *Numb3rs* and the past president of the National Council of Teachers of Mathematics both write letters to take issue with *Notices* Opinion pieces regarding this popular show which features mathematics in prominent ways. Rightfully, this show won this year's Carl Sagan award. Do you remember Sagan? All of science and mathematics are still reaping the benefits of Sagan's tremendous achievements in popularizing astronomy, yet in his time he was snubbed by not being elected to the National Academy of Sciences.

Which brings me to my main objection, Daniel Biss' "Communicating the Romance of Mathematics" (*Notices*, October 2006). While agreeing that such communication is "absolutely essential", I found the depth of his essay to be unworthy of a *Notices*' Opinion and his complete omission of the role of teachers to be utterly insulting.

Biss wonders "What can be done" about society's misconceptions of mathematics. What can be done?! Lots of us are doing it—it's called teaching! Many fine mathematicians (AMS members, no less) work on this "daunting problem". We have long embraced teaching core mathematics classes for non-majors, mathematics appreciation courses, and other lower level courses as activities central to our profession. We will never be as well known as Keith Devlin, Ivars Peterson, and the *Numb3rs* producers. But our teaching, scholarship, and the passion we bring to our calling is central to the struggle to change the public's perceptions of mathematics. In this light, Biss' ideas about "communicating the romance of mathematics" are entirely superficial. He needs to get out more. He should come to one of my Mathematics for Liberal Arts classes. Or maybe one of Michael Starbird's. Or one of Annalisa Crannell's. There are thousands of mathematicians across the country successfully inspiring students and challenging their misperceptions of mathematics.

We are accustomed to being marginalized by society, our political leaders, and even our college and university administrations who often fail to see the scholarship involved in teaching. But how dare the *Notices* ignore us? In the future I hope the *Notices* encourages paradigm changes which serve to recognize, nurture and reward the work of these mathematicians. Through our teaching we add tremendous value to the society of mathematics—and not just the American one. Until then, I'll take some small solace in being in the company of Sagan and many fine disrespected colleagues who consider themselves both mathematicians and teachers.

—*Julian F. Fleron, Ph.D.*
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(Received October 4, 2006)

What Did Turing Mean?

On page 1192 of the November 2006 issue is a photo of a piece of scrap paper, containing the following sentence (vii) written by Turing:

"A power series whose coefficients form a computable sequence is computably convergent in the of [sic] its interval of convergence".

The phrase "in the of" being ungrammatical, what did Turing mean to write?

It is true that a power series whose coefficients form a computable sequence is computably convergent in the *interior* of its interval of convergence. And indeed, in sentence (x) of the same page Turing does remember to include the word "interior".

However, a power series whose coefficients form a computable sequence is *not* necessarily computably convergent wherever it is convergent. For example, $n \geq 0$ and let $a_n = 2^{-m}$ where m is such that there are exactly n many positive integers k such that a fixed universal Turing machine, running computations on all possible inputs in parallel ("dovetailed"), halts on input k before halting on input m. Consider the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

This series is convergent, but not computably convergent, at $x = 1$. Indeed, otherwise there would be an algorithm to solve the Halting Problem for Turing machines.

The essence of this example seems perhaps at first to be the fact that the coefficients a_n do not converge computably to zero. However, a minor modification of the example gives a sequence of coefficients that converge monotonically (hence computably) to zero, and nevertheless the sequence of partial sums of the power series does not converge computably.

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(Received October 24, 2006)

Competence of American Math Graduates

Irwin Kra pointed out that the persistent mediocrity of American math education is primarily due to the shortage of knowledgeable and competent math teachers (*Notices* of AMS, December 2006, p. 1301, and *Focus* of MAA, November 2006, p. 18).

Let me pose a less-asked question: why does the American higher education system keep producing unknowledgeable and incompetent math graduates in the first place? It's my personal experience that an average American undergraduate math (or math-education) major near the completion of his degree cannot even write mathematics in a syntactically flawless fashion, let alone the accuracy of semantics. (Do music departments award degrees to pianists who consistently play wrong notes?) Since there is no standardized measure of college math graduates' competence, this widespread problem is not quantified or even publicly acknowledged.

While I don't have an easy answer, let me share some experience of comparative value. I had my secondary education in China; my high school mathematics and physics teachers were smart, knowledgeable, and competent because (I could make this inference in China but not in U.S.) they all held a college degree. One would assume that college students in the wealthiest nation, where they are only required to learn a fraction of what their third-world counter parts have to learn, will have learned that fraction very well; but perplexingly, many don't and we hand them diplomas anyway.

The Math Science Teaching Corps Act introduced in Congress is a very expensive proposition (*Focus* of MAA, Nov. 2006, p. 18). It would have been unnecessary if college diplomas were worth their face value.

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(Received December 4, 2006)

Clay Millenium Prizes

In the January 2007 issue of the *Notices* of the American Mathematical Society, Anatoly Vershik writes in his commentary on the million dollar Clay Millenium Prizes that "this method of promoting mathematics is warped and unacceptable, it does not popularize mathematics as a science, on the contrary, it only bewilders the public and leads to unhealthy interest."

There is no question that the Clay Millenium Prize contributed to the remarkable amount of press coverage received by the resolution of the Poincaré conjecture. In the weeks which represented the apex of the media's interest in the matter, I had at least ten in-depth discussions about topology and the Poincaré conjecture with friends who are not mathematicians but whose interest was piqued by newspaper, magazine, and television reports on the subject. I do not know exactly what Vershik means by "unhealthy interest", but in my view, this represents an unqualified success on the part of the Millenium Prizes.

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Identifications

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Submitting Letters to the Editor

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George B. Dantzig (1914–2005)

Richard Cottle, Ellis Johnson, and Roger Wets

The final test of a theory is its capacity to solve the problems which originated it.

This work is concerned with the theory and solution of linear inequality systems. ...

The viewpoint of this work is constructive. It reflects the beginning of a theory sufficiently powerful to cope with some of the challenging decision problems upon which it was founded.

So says George B. Dantzig in the preface to his book, *Linear Programming and Extensions*, a now classic work published in 1963, some sixteen years after his formulation of the linear programming problem and discovery of the simplex algorithm for its solution. The three passages quoted above represent essential components of Dantzig's outlook on linear programming and, indeed, on mathematics generally. The first expresses his belief in the importance of real world problems as an inspiration for the development of mathematical theory, not for its own sake, but as a means to solving important practical problems. The second statement is based on the theoretical fact that although a linear programming problem, is, prima facie, concerned with constrained optimization, it is really all about solving a linear inequality system. The third statement reveals Dantzig's conviction that constructive methods (in particular, algorithms) are required to obtain the kinds of solutions called for in practical decision problems.

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George Dantzig is best known as the father of linear programming (LP) and the inventor of the simplex method. The practical power of these two contributions is so great that, on these grounds alone, he was arguably one of the most influential mathematicians of the twentieth century. And yet there is much more to the breadth and significance of his work. Our aim in this memorial article is to document the magnitude of George Dantzig's impact on the world by weaving together many strands of his life and professional commitments. In so doing, we bring to light the inadequacy of the "father/inventor" epithet.

The impact we have in mind is of many kinds. The earliest, of course, was on military and industrial planning and production. Dantzig's work greatly impacted economics, mathematics, operations research, computer science, and various fields of applied science and technology. In response to these developments, there emerged the concomitant growth of educational programs. Dantzig himself was a professor for more than half of his professional life and in that capacity had a profound impact on the lives and contributions of his more than fifty doctoral students.

As already suggested, George Dantzig passionately believed in the importance of real world problems as a wellspring of mathematical opportunity. Whether this was a lesson learned or a conviction held since early adulthood is unclear, but it served him very well throughout his long and productive life.

Formation

Our knowledge of George Dantzig's childhood is largely derived from part of an interview [1] conducted in November 1984 by Donald Albers. (Much the same article is available in [2].) George was the son of mathematician Tobias Dantzig (1884–1956) and Anja Ourisson who had met while studying mathematics at the Sorbonne. There, Tobias was greatly impressed by Henri Poincaré (1854–1912) and later wrote a book on him [41], though he is best known for his *Number, The Language of Science* [40].

George Dantzig was born in Portland, Oregon, on November 8, 1914. His parents gave him the middle name "Bernard" hoping that he would become a writer like George Bernard Shaw. George's younger brother Henry (1918–1973), who was given the middle name Poincaré, became an applied mathematician working for the Bendix Corporation.

By his own admission, George Dantzig lacked interest in schoolwork until grade seven. He then became keen about science and mathematics although in ninth grade he made a "poor start" in his first algebra course. "To be precise," he said, "I was flunking." Furious with himself, he buckled down and went on to excel in high school mathematics and science courses. He was particularly absorbed by projective geometry and worked "thousands" of such problems given to him by his father, Tobias, who was then on the mathematics department faculty at the University of Maryland.

As an undergraduate, George concentrated in mathematics and physics at the University of Maryland, taking his A.B. degree in 1936. After this he went to the University of Michigan and obtained the M.A. in mathematics in 1938. In Ann Arbor, he took a statistics course from Harry C. Carver (founding editor of the *Annals of Mathematical Statistics* and a founder of the Institute of Mathematical Statistics). He found the rest of the curriculum excessively abstract and decided to get a job after finishing his master's degree program in 1938.

Dantzig's decision to take a job as a statistical clerk at the Bureau of Labor Statistics (BLS) turned out to be a fateful one. In addition to acquiring a knowledge of many practical applications, he was assigned to review a paper written by the eminent mathematical statistician Jerzy Neyman who was then at University College, London, and soon thereafter at the University of California at Berkeley. Excited by Neyman's paper, Dantzig saw in it a logically based approach to statistics rather than a bag of tricks. He wrote to Neyman (at Berkeley) declaring his desire to complete a doctorate under Neyman's supervision, and this ultimately came to pass.

In 1939 Dantzig enrolled in the Ph.D. program of the Berkeley Mathematics Department where Neyman's professorship was located. Dantzig took only two courses from Neyman, but in one of them he had a remarkable experience that was to become a famous legend. Arriving late to one of Neyman's classes, Dantzig saw two problems written on the blackboard and mistook them for a homework assignment. He found them more challenging than usual, but managed to solve them and submitted them directly to Neyman. As it turned out, these problems were actually two open questions in the theory of mathematical statistics. Dantzig's 57-page Ph.D. thesis [12] was composed of his solutions to these two problems. One of these was immediately submitted for publication and appeared in 1940 as [11]. For reasons that are not altogether clear, the other appeared only in 1951 as a joint paper with Abraham Wald [39].

George B. Dantzig

By June 1941, the content of Dantzig's dissertation had been settled: it was to be on the two problems and their solutions. Although he still had various degree requirements to complete, he was eager to contribute to the war effort and joined the U.S. Air Force Office of Statistical Control. He was put in charge of the Combat Analysis Branch where he developed a system through which combat units reported data on missions. Some of this work also involved planning (and hence modeling) that, in light of the primitive computing machinery of the day, was a definite challenge. The War Department recognized his achievements by awarding him its Exceptional Civilian Service Medal in 1944.

Dantzig returned to Berkeley in the spring of 1946 to complete his Ph.D. requirements, mainly a minor thesis and a dissertation defense. Around that time he was offered a position at Berkeley but turned it down in favor of becoming a mathematical advisor at the U.S. Air Force Comptroller's Office. This second fateful decision set him on a path to the discovery of linear programming and the simplex algorithm for its solution in 1947.

Origins

Dantzig's roles in the discovery of LP and the simplex method are intimately linked with the historical circumstances, notably the Cold War and the early days of the Computer Age. The defense efforts undertaken during World War II and its aftermath included very significant contributions from a broad range of scientists, many of whom had fled the horrors of the Nazi regime. This trend heightened the recognition of the power that mathematical modeling and analysis could bring to real-world problem solving. Beginning in 1946, Dantzig's responsibility at the Pentagon involved the "mechanization" of the Air Force's planning procedures to support time-staged deployment of training and supply activities. Dantzig's approach to the mathematization of this practical problem ushered in a new scientific era and led to his fame in an ever-widening circle of disciplines.

In [19], George Dantzig gives a detached, historical account of the origins of—and influences on—linear programming and the simplex method. Almost twenty years later, as he approached the age of seventy, Dantzig began turning out numerous invited articles [21]–[26] on this subject, most of them having an autobiographical tone. From the mathematical standpoint, there is probably none better than [25] which, unlike the rest, explicitly relates part of his doctoral dissertation to the field of linear programming.

Dantzig was not alone in writing about the discovery of linear programming. One of the most informative articles on this subject is that of Robert Dorfman, a professor of economics at Harvard (now deceased). In telling the story, Dorfman [46] clarifies "the roles of the principal contributors". As he goes on to say, "it is not an especially complicated story, as histories of scientific discoveries go, but neither is it entirely straightforward." These opening remarks are meant to suggest that many elements of linear programming had already been come upon prior to their independent discovery by Dantzig in 1947. For the convenience of readers, the present article will revisit a bit of this lore and will endeavor to establish the point made at the outset that Dantzig's greatness rests not only on the discovery of linear programming and the simplex method, but on the depth of his commitment to their development and extensions.

At the outset of Dantzig's work on linear programming, there already existed studies by two of the principal contributors to its discovery. The first of these was done by the mathematician Leonid V. Kantorovich. The second was by the mathematical statistician/economist Tjalling C. Koopmans. For good reasons, these advances were unknown to Dantzig.

As a professor of mathematics and head of the Department of Mathematics at the Institute of Mathematics and Mechanics of Leningrad State University, Kantorovich was consulted by some engineers from the Laboratory of the Veneer Trust who were concerned with the efficient use of machines. From that practical contact sprung his report on linear programming [57], which, though it appeared in 1939, seems not to have been known in the West (or the East) until the late 1950s and was not generally available in English translation [60] until 1960. (This historically important document is a 68-page booklet in the style of a preprint. Kantorovich [61, p. 31] describes it as a "pamphlet".) Two other publications of Kantorovich deserve mention here. In [58], Kantorovich proposed an approach to solving some classes of extremal problems that would include the linear programming problem. A brief discussion of the second paper [59] is given below. It is worth noting here that these two articles—written in English during World War II—were reviewed by H. H. Goldstine [55] and Max Shiffman [75], respectively. Hence they were not altogether unknown in the West.

In 1940 Koopmans emigrated from the Netherlands to the United States. During World War II he was employed by the Combined Shipping Adjustment Board, an agency based in Washington, D.C., that coordinated the merchant fleets of the Allied governments, chiefly the United States and Britain. Koopmans's first paper of a linear programming nature dates from 1942. For security reasons, the paper was classified. It became available as part of a Festschrift in 1970 [63]. Based on this wartime experience, Koopmans's 1947 paper [64] develops what later came to be called the *transportation problem* [14]. These works are not algorithmic; they emphasize the modeling aspect for the particular shipping application that Koopmans had in mind.

As Koopmans was later to discover, the special class of linear programming problems he had written about (namely, the transportation problem) had already been published in 1941 by an MIT algebraist named Frank L. Hitchcock. The title of Hitchcock's paper, "The distribution of a product from several sources to numerous localities" [56], describes what the transportation problem is about, including its important criterion of least cost. (A review of the paper [56] can be found in [62].) Hitchcock makes only the slightest of suggestions on the application of the model and method he advances in his paper.

Just as Hitchcock's paper was unknown to Koopmans (and to Dantzig), so too the work of Kantorovich on the transportation problem was unknown to Hitchcock. Moreover, it appears that in writing his paper [59] on the "translocation of masses", Kantorovich was not familiar with the very much earlier paper [68] of Gustave Monge (1746–1818) on "cutting and filling", a study carried out in conjunction with the French mathematician's work on the moving of soil for building military fortifications. The formulation in terms of continuous mass distributions has come to be called the "Monge-Kantorovich Problem".

A few others contributed to the "pre-history" of linear programming. In 1826 Jean-Baptiste Joseph Fourier announced a method for the solution of linear inequality systems [52]; it has elements in common with the simplex method of LP. Charles de la Vallée Poussin in 1911 gave a method for finding minimum deviation solutions of systems of equations [81]. John von Neumann's famous game theory paper [70] of 1928, and his book [72] written with Oscar Morgenstern and published in

1944, treated finite two-person zero-sum games, a subject that is intimately connected with linear programming; von Neumann's paper [71] is of particular interest in this regard. In addition to these precursors, there remains Theodor Motzkin's scholarly dissertation *Beiträge zur Theorie der Linearen Ungleichungen* accepted in 1933 at the University of Basel and published in 1936 [69]. (For a loose English translation of this work, see [7].) Apart from studying the general question of the existence (or nonexistence) of solutions to linear inequality systems, Motzkin gave an elimination-type solution method resembling the technique used earlier by Fourier (and Dines [43] who seems to have concentrated on *strict* inequalities).

Initially, Dantzig knew nothing of these precedents, yet he did respond to a powerful influence: Wassily Leontief's work [67] on the structure of the American economy. This was brought to Dantzig's attention by a former BLS colleague and friend, Duane Evans. The two apparently discussed this subject at length. In Dantzig's opinion [19, p. 17] "Leontief's great contribution ... was his construction of a *quantitative model* ... for the purpose of tracing the impact of government policy and consumer trends upon a large number of industries which were imbedded in a highly complex series of interlocking relationships." Leontief's use of "an empirical model" as distinct from a "purely formal model" greatly impressed Dantzig as did Leontief's organizational talent in acquiring the data and his "marketing" of the results. "These things," Dantzig declares, "are necessary steps for successful applications, and Leontief took them all. That is why in my book he is a hero" [1, p. 303].

Carrying out his assignment at the U.S.A.F. Comptroller's Office, George Dantzig applied himself to the mechanization of the Air Force's planning procedures to support the time-staged deployment of training and supply activities. He created a linear mathematical model representing what supplies were available and what outputs were required over a multi-period time horizon. Such conditions normally lead to an under-determined system, even when the variables are required to assume nonnegative values, which they were in this case, reflecting their interpretation as physical quantitities. To single out a "best" solution, Dantzig introduced a linear *objective function*, that is, a linear minimand or maximand. This was an innovation in planning circles, an achievement in which Dantzig took great pride. As he put it in 1957, "linear programming is an anachronism"; here he was alluding to the work of economists François Quesnay, Léon Walras, and Wassily Leontief as well as mathematician John von Neumann all of whom could (and in his mind, should) have introduced objective functions in their work [42, p. 102].

Dantzig's discovery of the linear programming problem and the simplex algorithm was independent of [52], [81], [57], [56], and [63]. Yet, as he has often related, it was not done in isolation. On the formulation side, he was in contact with Air Force colleagues, particularly Murray Geisler and Marshall K. Wood, and with National Bureau of Standards (NBS) personnel. At the suggestion of Albert Kahn at NBS, Dantzig consulted Koopmans who was by then at the Cowles Commission for Research in Economics (which until 1955 was based at the University of Chicago). The visit took place in June 1947 and got off to a slow start; before long, Koopmans perceived the broad economic significance of Dantzig's general (linear programming) model. This might have prompted Koopmans to disclose information about his 1942 work on the transportation problem and Hitchcock's paper of 1941.

Another key visit took place in October 1947 at the Institute for Advanced Study (IAS) where Dantzig met with John von Neumann. Dantzig's vivid account of the exchange is given in [1, p. 309] where he recalls, "I began by explaining the formulation of the linear programming model ... I described it to him as I would to an ordinary mortal. He responded in a way which I believe was uncharacteristic of him. 'Get to the point,' he snapped. ... In less than a minute, I slapped the geometric and algebraic versions of my problem on the blackboard. He stood up and said, 'Oh that.' " Just a few years earlier von Neumann had co-authored and published the landmark monograph [72]. Dantzig goes on, "for the next hour and a half [he] proceeded to give me a lecture on the mathematical theory of linear programs." Dantzig credited von Neumann with edifying him on Farkas's lemma and the duality theorem (of linear programming). On a separate visit to Princeton in June 1948, Dantzig met Albert Tucker who with his students Harold Kuhn and David Gale gave the first rigorous proof of the duality theorem that von Neumann and Dantzig had discussed at their initial meeting.

During the summer of 1947, months before his encounter with von Neumann at the IAS, Dantzig proposed his simplex method (much as described elsewhere in this article); in the process of this discovery he discussed versions of the algorithm with economists Leonid Hurwicz and Tjalling Koopmans. It was recognized that the method amounts to traversing a path of edges on a polyhedron; for that reason he set it aside, expecting it to be too inefficient for practical use. These reservations were eventually overcome when he interpreted the method in what he called the "column geometry", presumably inspired by Part I of his Ph.D. thesis [12], [39], [25, p. 143]. There, the analogue of a so-called convexity constraint is present. With a constraint of the form

$$(1) \qquad x_1 + \cdots + x_n = 1, \quad x_j \geq 0 \quad j = 1, \ldots, n$$

(which is common, but by no means generic) the remaining constraints

$$(2) \qquad A_{\bullet 1}x_1 + \cdots + A_{\bullet n}x_n = b$$

amount to asking for a representation of b as a convex combination of the columns $A_{\bullet 1}, \ldots A_{\bullet n} \in \mathbb{R}^m$. (Of course, the solutions of (1) alone constitute an $n-1$ simplex in $\mathbb{R}^n$, but that is not exactly where the name "simplex method" comes from.) By adjoining the objective function coefficients c_j to the columns $A_{\bullet j}$, thereby forming vectors $(A_{\bullet j}, c_j)$ and adjoining a variable component z to the vector b to obtain (b, z), Dantzig viewed the corresponding linear programming problem as one of finding a positively weighted average of the vectors $(A_{\bullet 1}, c_1), \ldots, (A_{\bullet n}, c_n)$ that equals (b, z) and yields the largest (or smallest) value of z.

It was known that if a linear program (in standard form) has an optimal solution, then it must have an optimal solution that is also an *extreme point* of the *feasible region* (the set of all vectors satisfying the constraints). Furthermore, the extreme points of the feasible region correspond (albeit not necessarily in a bijective way) to *basic feasible solutions* of the constraints expressed as a system of linear equations. Under the reasonable assumption that the system (1), (2) has full rank, one is led to consider nonsingular matrices of the form

$$(3) \qquad B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ A_{\bullet j_1} & A_{\bullet j_2} & \cdots & A_{\bullet j_{m+1}} \end{bmatrix}$$

such that

$$(4) \qquad B^{-1} \begin{bmatrix} 1 \\ b \end{bmatrix} \geq 0.$$

The columns $A_{\bullet j_1}, A_{\bullet j_2}, \ldots, A_{\bullet j_{m+1}}$ viewed as points in $\mathbb{R}^m$ are easily seen to be in *general position.* Accordingly, their convex hull is an m-simplex.

Dantzig conceptualized the n points $A_{\bullet j}$ as lying in the "horizontal" space $\mathbb{R}^m$ and then pictured each $(m+1)$-tuple $(A_{\bullet j}, c_j)$ as a point on the line orthogonal to $\mathbb{R}^m$ and passing through $A_{\bullet j}$ with c_j measuring the vertical distance of the point above or below the horizontal "plane", according to the sign of c_j. The *requirement line* consisting of points (b, z) where $b \in \mathbb{R}^m$ is as above and z is the (variable) value of the objective function is likewise orthogonal to the horizontal plane. For any feasible basis, the requirement line meets σ, the convex hull of the corresponding points $(A_{\bullet j_1}, c_{j_1}), (A_{\bullet j_2}, c_{j_2}), \ldots, (A_{\bullet j_{m+1}}, c_{j_{m+1}})$, in the point (b, z), the ordinate z being the value of the objective function given by the associated basic solution. The $m+1$ vertices of the simplex σ determine a hyperplane in $\mathbb{R}^{m+1}$. The vertical distance from a point $(A_{\bullet j}, c_j)$ to this hyperplane indicates whether this point will improve the objective value of the basic solution that would be obtained if it were to replace one of the basic columns. The convex hull of this new point and σ is an $(m+1)$-simplex τ in $\mathbb{R}^{m+1}$. The requirement line meets the boundary of τ in two points: the previous solution and one other point with a better objective function value. In the (nondegenerate) case where the new point lies in the relative interior of a facet of τ, say ρ, there is a unique (currently basic) column for which the corresponding weight (barycentric coordinate) is zero. This point is opposite the new facet ρ where the requirement line meets the boundary of τ. Note that ρ is an m-simplex and corresponds to a new and improved feasible basis. The process of making the basis change is called *simplex pivoting.* The name can be seen as an apt one when the facets of τ are likened to the (often triangle-shaped) body parts of a hinge.

Beyond the fact that not all linear programming problems include convexity constraints, it was clear that, without significant advances in automatic computing machines, the algorithm—in whatever form it took—was no match for the size of the planning problems for which numerical solutions were needed. One such need is typified by a critical situation that came up less than a year after Dantzig's discoveries: the Berlin Airlift. Lasting 463 days, this program called for the scheduling of aircraft and supply activities, including the training of pilots, on a very large, dynamic scale. During this crisis, Britain, France, and the United States airlifted more than two million tons of food and supplies to the residents of West Berlin whose road, rail, and water contacts with Western Europe had been severed by the USSR.

Initial Impacts

"Mathematical Techniques of Program Planning" is the title of the brief talk in which Dantzig publicly announced his discovery of the simplex method. The presentation took place at a session of the joint annual meeting of the American Statistical Association (ASA) and the Institute of Mathematical Statistics (IMS) on December 29, 1947 [48, p. 134]. As Dorfman reports [46, p. 292], "there is no evidence that Dantzig's paper attracted any particular interest or notice." The paper was not published. The method's next appearance occurred in a session (chaired by von Neumann) at a joint national meeting of the IMS and the Econometric Society on September 9, 1948. Dantzig's talk, titled "Programming in a Linear Structure", aroused more interest than its predecessor. The abstract [13] is remarkable for its visionary scope. In it we find mention of the LP model, the notion of dynamic systems, connections with the theory of games, a reference to computational procedures for "large scale digital computers", and the bold suggestion that the solutions of such problems could be *implemented* and not merely discussed. A lively discussion followed Dantzig's talk; afterwards he participated in a panel discussion beside such distinguished figures as Harold Hotelling, Irving Kaplansky, Samuel Karlin, Lloyd Shapley, and John Tukey. The ball was now rolling.

In an autobiographical piece [66] written around the time he (and Kantorovich) received the 1975 Nobel Prize in Economics, Tjalling Koopmans relates how in 1944 his "work at the Merchant Shipping Mission fizzled" due to a "reshuffling of responsibilities". Renewing his contact with economist Jacob Marschak, Koopmans secured a position at the Cowles Commission in Chicago. He goes on to say "my work on the transportation model broadened out into the study of activity analysis at the Cowles Commission as a result of a brief but important conversation with George Dantzig, probably in early 1947. It was followed by regular contacts and discussions extending over several years thereafter. Some of these discussions included Albert W. Tucker of Princeton who added greatly to my understanding of the mathematical structure of duality."

Oddly, Koopmans's autobiographical note makes no mention of his instrumental role in staging what must be considered one of the most influential events in the development of linear (and nonlinear) programming: the "Conference on Activity Analysis of Production and Allocation" held in Chicago under the auspices of the Cowles Commission for Research in Economics in June 1949. Edited by Koopmans, the proceedings volume [65] of this conference comprises twenty-five papers, four of which were authored—and one co-authored—by Dantzig. The speakers and other participants, approximately fifty in all, constitute an impressive set of individuals representing academia, government agencies, and the military establishment. It is a matter of some interest that the proceedings volume mentions no participants from industry. Exactly the same types of individuals were speakers at the "Symposium on Linear Inequalities and Programming" held in Washington, D.C., June 14–16, 1951. Of the nineteen papers presented, twelve were concerned with mathematical theory and computational methods while the remaining seven dealt with applications. In addition to a paper by George Dantzig and Alex Orden offering "A duality theorem based on the simplex method", the first part of the proceedings contains Merrill Flood's paper "On the Hitchcock distribution problem", that is, the transportation problem. Flood rounds up the pre-existing literature on this subject including the 1942 paper of Kantorovich though not the 1781 paper of Monge. Among the papers in the portion of the proceedings on applications, there appears the abstract (though not the paper) by Abraham Charnes, William Cooper and Bob Mellon (an employee of Gulf Oil Co.) on "Blending aviation gasolines". This—and, more generally, blending in the petrochemical and other industries—was to become an important early application area for linear programming. From this group of papers one can sense the transition of the subject from military applications to its many fruitful applications in the civilian domain. Before long, the writings of Dantzig and others attracted the attention of a wide circle of applied scientists, including many from industry. See [3].

Besides the transportation problem, which continues to have utility in shipping and distribution enterprises, an early direction of applied linear programming is to be found in agriculture, a venerable topic of interest within economics. Our illustration is drawn from a historically important article that provided a natural setting for the LP. In 1945, George J. Stigler [79] published a paper that develops a model for finding a least-cost "diet" that would provide at least a prescribed set of nutritional requirements. [See the companion piece by Gale in this issue for a discussion of the formulation.] Dantzig reports [19, p. 551] how in 1947 the simplex method was tried out on Stigler's nutrition model (or "diet problem" as it is now called). The constraints involved nine equations in seventy-seven unknowns. At the time, this was considered a large problem. In solving the LP by the simplex method, the computations were performed on hand-operated desk calulators. According to Dantzig, this process took "approximately 120 man-days to obtain a solution." Such was the state of computing in those days. In 1953, the problem was solved and printed out by an IBM 701 computer in twelve minutes. Today, the problem would be considered small and solved in less than a second on a personal computer [54]. One may question the palatability of a diet for human consumption based on LP principles. Actually, the main application of this methodology lies in the animal feed industry where it is of great interest.

The Air Force's desire to mechanize planning procedures and George Dantzig's contributions to that effort through his creation of linear programming and the simplex method had a very significant impact on the development of computing machines. The encouraging results obtained by Dantzig and his co-workers convinced the Air Force Comptroller, Lt. Gen. Edwin W. Rawlings, that much could be accomplished with more powerful computers. Accordingly, he transferred the (then large) sum of US$400,000 to the National Bureau of Standards which in turn funded mathematical and electronic computer research (both in-house) as well as the development of several computers such as UNIVAC, IBM, SEAC, and SWAC. Speaking to a military audience in 1956, General Rawlings notes, "I believe it can be fairly said that the Air Force interest in actual financial investments in the development of electronic computers has been one of the important factors in their rapid development in this country" [74]. Dantzig took pride in the part that linear programming played in the development of computers.

The list of other industrial applications of linear programming—even in its early history—is impressive. A quick sense of this can be gleaned from Saul Gass's 1958 textbook [53], one of the first to appear on the subject. In addition to a substantial chapter on applications of linear programming, it contains a bibliography of LP applications of all sorts organized under twelve major headings (selected from the broader work [54]). Under the heading "Industrial Applications", Gass lists publications pertaining to the following industries: chemical, coal, commerical airlines, communications, iron and steel, paper, petroleum, and railroad.

Also published in 1958 was the book *Linear Programming and Economic Analysis* by Dorfman, Samuelson, and Solow [47]. "Intended not as a text but as a general exposition of the relationship of linear programming to standard economic analysis" it had "been successfully used for graduate classes in economics." Its preface proclaims that "linear programming has been one of the most important postwar developments in economic theory." The authors highlight LP's interrelations with von Neumann's theory of games, with welfare economics, and with Walrasian equilibrium. Arrow [4] gives a perspective on Dantzig's role in the development of economic analysis.

Early Extensions

From its inception, the practical significance of linear programming was plain to see. Equally visible to Dantzig and others was a family of related problems that came to be known as *extensions of linear programming*. We turn to these now to convey another sense of the impact of LP and its applications.

In this same early period, important advances were made in the realm of *nonlinear* optimization. H. W. Kuhn and A. W. Tucker presented their work "Nonlinear programming" at the Second Berkeley Symposium on Mathematical Statistics and Probability, the proceedings of which appeared in 1951. This paper gives necessary conditions of optimality for the (possibly) nonlinear inequality constrained minimization of a (possibly) nonlinear objective function. The result is a descendant of the so-called "Method of Lagrange multipliers". A similar theorem had been obtained by F. John and published in the "Courant Anniversary Volume" of 1948. For some time now, these optimality conditions have been called the Karush-Kuhn-Tucker conditions, in recognition of the virtually identical result presented in Wm. Karush's (unpublished) master's thesis at the University of Chicago in 1939. Dorfman's doctoral dissertation *Application of Linear Programming to the Theory of the Firm, Including an Analysis of Monopolistic Firms by Non-linear Programming* appeared in book form [45] in 1951. It might well be the first book that ever used "linear programming" in its title. The "non-linear programming" mentioned therein is what he calls "quadratic programming", the optimization of a quadratic function subject to linear inequality constraints. The first book on linear programming *per se* was [9], *An Introduction to Linear Programming* by A. Charnes, W. W. Cooper, and A. Henderson, published in 1953. Organized in two parts (applications and theory), the material of this slender volume is based on seminar lectures given at Carnegie Institute of Technology (now Carnegie-Mellon University).

Organizations and Journals

Linear and nonlinear programming (collectively subsumed under the name "mathematical programming") played a significant role in the formation of professional organizations. As early as April 1948, the Operational Research Club was founded in London; five years later, it became the Operational Research Society (of the UK). The Operations Research Society of America (ORSA) was founded in 1952, followed the next year, by the Institute of Management Sciences (TIMS). These parallel organizations, each with its own journals, merged in 1995 to form the Institute for Operations Research and the Management Sciences (INFORMS) with a membership today of about 12,000. Worldwide, there are now some forty-eight national OR societies with a combined membership in the vicinity of 25,000. Mathematical programming (or "optimization" as it now tends to be called) was only one of many subjects appearing in the pages of these journals. Indeed, the first volumes of these journals devoted a small portion of their space to mathematical programming. But that would change dramatically with time.

Over the years, many scientific journals have been established to keep pace with this active field of research. In addition to the journal *Mathematical Programming*, there are today about a dozen more whose name includes the word "optimization". These, of course, augment the OR journals of OR societies and other such publications.

Educational Programs

Delivering a summarizing talk at "The Second Symposium in Linear Programming" (Washington, D.C., January 27–29, 1955) Dantzig said "The great body of my talk has been devoted to technical aspects of linear programming. I have discussed simple devices that can make possible the efficient solution of a variety of problems encountered in practice. Interest in this subject has been steadily growing in industrial establishments and in government and some of these ideas may make the difference between interest and use." The notion of "making the difference between *interest and use*" was a deeply held conviction of Dantzig's. He knew how much could be accomplished through the combination of modeling, mathematical analysis, and algorithms like the simplex method. And then he added a prediction. "During the next ten years we will see a great body of important applications; indeed, so rich in content and value to industry and government that mathematics of programming will move to a principal position in college curriculums" [15, p. 685].

By the early 1950s, operations research (OR) courses had begun appearing in university curricula, and along with them came linear programming. We have already mentioned the lectures of Charnes and Cooper at Carnegie Tech in 1953. In that year both OR and LP were offered as graduate courses at Case Institute of Technology (the predecessor of Case Western Reserve University). At Stanford OR was first taught in the Industrial Engineering Program in the 1954-55 academic year, and LP was one of the topics covered. At Cornell instruction in these subjects began in 1955, while at Northwestern it began in 1957. In the decades that followed, the worldwide teaching of OR at post-secondary institutions grew dramatically. In some cases, separate departments of operations research and corresponding degree programs were established; in other cases these words (or alternatives like "management science" or "decision science") were added to the name of an existing department, and there were other arrangements as well. The interdisciplinary nature of the field made for a wide range of names and academic homes for the subject, sometimes even within the same institution, such as Stanford, about which George Dantzig was fond of saying "it's wall-to-wall OR." Wherever operations research gained a strong footing, subjects like linear programming outgrew their place within introductory survey courses; they became many individual courses in their own right.

Along with the development of academic courses came a burst of publishing activity. Textbooks for the classroom and treatises for seminars and researchers on a wide range of OR topics came forth. Books on mathematical programming were a major part of this trend. One of the most important of these was George Dantzig's *Linear Programming and Extensions* [19]. Published in 1963, it succeeded by two years the monograph of Charnes and Cooper [8]. Dantzig's book was so rich in ideas that it soon came to be called "The Bible of Linear Programming". Two of the sections to follow treat Dantzig's direct role in academic programs. For now, we return to the chronology of his professional career.

Dantzig at RAND

In 1952 George Dantzig joined the RAND Corporation in Santa Monica, California, as a research mathematician. The RAND Corporation had come into being in 1948 as an independent, private nonprofit organization after having been established three years earlier as the Air Force's Project RAND, established in 1945 through a special contract with the Douglas Aircraft Company. Despite its separation from Douglas—under which Project RAND reported to Air Force Major General Curtis LeMay, the Deputy Chief of Air Staff for Research and Development—the newly formed RAND Corporation maintained a close connection with the Air Force and was often said to be an Air Force "think tank".

The reason for Dantzig's job change (from mathematical advisor at the Pentagon to research mathematician at RAND) is a subject on which the interviews and memoirs are silent. In [1, p. 311] he warmly describes the environment that existed in the Mathematics Division under its head, John D. Williams. The colleagues were outstanding and the organizational chart was flat. These were reasons enough to like it, but there must have been another motivation: the freedom to conduct research that would advance the subject and to write the book [19].

The 1950s were exciting times for research in the Mathematics Division at RAND. There, Dantzig enjoyed the stimulation of excellent mathematicians and the receptiveness of its sponsors. During his eight years (1952-1960) at RAND, Dantzig wrote many seminal papers and internal research memoranda focusing on game theory, LP and variants of the simplex method, large-scale LP, linear programming under uncertainty, network optimization including the traveling salesman problem, integer linear programming, and a variety of applications. Much of this work appeared first in the periodical literature and then later in [19].

The topic of methods for solving large-scale linear programs, which had drawn Dantzig's attention at the Pentagon, persisted at RAND—and for years to come. Although the diet problem described above was considered "large" at the time, it was puny by comparison with the kinds of problems the Air Force had in mind. Some inkling of this can be obtained from the opening of a talk given by M. K. Wood at the 1951 "Symposium on Linear Inequalities and Programming". He says in its proceedings [73, p. 3], "Just to indicate the general size of the programming problem with which we are faced, I would like to give you a few statistics. We are discussing an organization of over a million people who are classified in about a thousand different occupational skills. These people are organized into about ten thousand distince [sic] organizational units, each with its own staff and functions, located at something over three hundred major operating locations. The organizational units use approximately one million different kinds of items of supplies and equipment, at a total annual cost of something over fifteen billion dollars."

The challenge of solving problems on such a grand scale was one that Dantzig took seriously. Speaking at the "First International Conference on Operational Research" in 1957, he acknowledged, "While the initial proposal was to use a linear programming model to develop Air Force programs, it was recognized at an early date that even the most optimistic estimates of the efficiency of future computing procedures and facilities would not be powerful enough to prepare detailed Air Force programs" [17]. Nevertheless, he steadfastly devoted himself to the cause of improving the capabilities of mathematical programming, especially the simplex method of linear programming.

If one had to characterize Dantzig's work on large-scale systems, more specifically, large-scale linear programs, it could be viewed as the design of variants of the simplex method that require only the use of a "compact" basis, i.e., that simplex iterations will be carried out by relying on a basis of significantly reduced size. Early on, it was observed that given a linear program in standard form, i.e., $\min c \cdot x$ subject to $Ax = b$, $x \geq 0$, the efficiency of the simplex method is closely tied to the number of linear constraints ($Ax = b$) that determine the size of the basis rather than to the number of variables. It is part of the linear programming folklore that "in practice" the number of steps required to solve a linear program is of the order of $3m/2$ where m is the number of linear constraints. Moreover, working with a compact basis (i.e., one of small order), which was essentially going be inverted, would make the method significantly more reliable from a numerical viewpoint.

The first challenging test of the simplex method on a "large-scale" problem had come in dealing with Stigler's diet problem; the introduction of upper bounds on the amounts of various foods greatly enlarged the number of constraints, thereby exacerbating an already difficult problem. The need to have a special version of the simplex method to deal with upper bounds came up again at RAND. Researchers were dissatisfied with the turn-around time for the jobs submitted to their computer center, mostly because certain top-priority projects would absorb all available computing resources for weeks. That is when a more flexible priority scheduling method was devised in which the value assigned to a job decreased as its completion date was delayed. The formulation of this scheduling problem turned out to be a linear program having special (assignment-like) structure and the restriction that the number of hours x_{ij} to be assigned to each project i in week j could not exceed a certain upper bound, say α_{ij}. In linear programming terms, the problem was of the following type,

$$\min c \cdot x \text{ subject to } Ax = b,\ lb \leq x \leq ub,$$

with m linear constraints ($Ax = b$) and, rather than just nonnegativity restrictions on the x-variables, the vectors lb and ub imposed lower and upper bounds on x. Here after an appropriate change of variables and inclusion of the slack variables, the problem would pass from one with m linear constraints to one with at least $m + n$ such constraints. Solving the scheduling problem might have been the source of further delays! That is when Dantzig introduced a new pivoting scheme that would be integrated in any further implementation of the simplex method. It would essentially deal with the problem as one with only m linear constraints requiring just a modicum of additional bookkeeping to keep track of variables at their lower bounds, the basic variables, and those at their upper bound, rather than just with basic and nonbasic variables as in the original version of the simplex method.

Dantzig's work on the transportation problem and certain network problems had made him aware that the simplex method becomes extremely efficient when the basis has either a triangular or a nearly triangular structure. Moreover, as the simplex method was being used to solve more sophisticated models, in particular involving dynamical systems, the need for variants of the simplex method to handle such larger-scale problems became more acute. Although the simplex bases of such problems are not precisely triangular, they have a *block triangular structure*, e.g.,

$$\begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ & & \ddots & \\ A_{T1} & A_{T2} & \dots & A_{TT} \end{bmatrix},$$

that can be exploited to "compactify" the numerical operations required by the simplex method. All present-day, commercial-level, efficient implementations of the simplex method make use of the shortcuts proposed by Dantzig in [16], in particular in the choice of heuristics designed to identify a good starting basis known among professionals as a "crash start".

In the late 1950s, Dantzig and Wolfe proposed the *Decomposition Principle* for linear programs on which they lectured at the RAND Symposium on Mathematical Programming in March 1959. Their approach was inspired by that of Ford and Fulkerson on multistage commodity network problems [51]. Actually, the methodology is based on fundamental results dating back to the pioneering work of Minkowski and Weyl on convex polyhedral sets. By the mid-1950s Dantzig had already written a couple of papers on stochastic programming; he realized that a method was needed to handle really large linear programs, with "large" now taking on its present-day meaning.

Because it illustrates so well how theory can be exploited in a computational environment, we present an abbreviated description of the method. Suppose we have split our system of linear constraints in two groups so that the linear program takes on the form,

$$\min c \cdot x \text{ subject to } Ax = b,\ Tx = d,\ x \geq 0$$

where the constraints have been divided up so that $Ax = b$ consists of a relatively small number, say m, of linking constraints, and the system $Tx = d$ is large. As shall be seen later on, stochastic programs will provide good examples of problems that naturally fall into this pattern. The set $K = \{x \in \mathbb{R}^n_+ \mid Tx = d\}$ is a polyhedral set that in view of the Weyl-Minkowski Theorem admits a "dual" representation as a finitely generated set obtained as the sum of a polytope (bounded polyhedron)

$$\{x = \sum_{k=1}^{r} p^k \lambda_k \mid \sum_{k=1}^{r} \lambda_k = 1, \lambda_k \geq 0, k = 1, \dots, r\}$$

and a polyhedral cone

$$\{x = \sum_{l=1}^{s} q^l \mu_l \mid \mu \geq 0, l = 1, \dots, s\}$$

with both r and s finite. Our given problem is thus equivalent to the following linear program, which we shall refer to as the *full master program*,

$$\begin{aligned} \min \quad & \textstyle\sum_{k=1}^{r} \gamma_k \lambda_k + \sum_{l=1}^{s} \delta_l \mu_l \\ \text{subject to} \quad & \textstyle\sum_{k=1}^{r} P^k \lambda_k + \sum_{l=1}^{s} Q^l \mu_l = b, \\ & \textstyle\sum_{k=1}^{r} \lambda_k = 1, \\ & \lambda_k \geq 0, k = 1, \dots, r, \\ & \mu_l \geq 0, l = 1, \dots, s \end{aligned}$$

where $\gamma_k = c \cdot p^k, P^k = Ap^k$, $\delta_l = c \cdot q^l$, and $Q^l = Aq^l$. Because this linear program involves only $m + 1$ linear constraints, the "efficiency and numerical reliability" of the simplex method would be preserved if, rather than dealing with the large-scale system, we could simply work with this latter problem. That is, if we do not take into account the work of converting the constraints $Tx = d, x \geq 0$ to their dual representation, and this could be a horrendous task that, depending on the structure of these constraints, could greatly exceed that of solving the given problem. What makes the Dantzig-Wolfe decomposition method a viable and attractive approach is that, rather than generating the full master program, it generates only a small, carefully selected subset of its columns, namely, those that potentially could be included in an optimal basis. Let us assume that, via a Phase I type-procedure, a few columns of the (full) master program have been generated, say $k = 1, \dots, r_\nu < r$, $l = 1, \dots, s_\nu < s$, such that the resulting (reduced) *master program* is feasible. Clearly, if this master program is unbounded, so is the originally formulated problem. So, let us assume that this problem is bounded, and let (λ^ν, μ^ν) be an optimal solution where $\lambda^\nu = (\lambda_1^\nu, \dots, \lambda_{r_\nu}^\nu)$ and $\mu_\nu = (\mu_1^\nu, \dots, \mu_{s_\nu}^\nu)$ with optimal value ζ_ν. Let π^ν be the vector of multipliers attached to the first m linear constraints and θ_ν be the multiplier attached to the $(m+1)$st constraint $\sum_{k=1}^{r} \lambda_k = 1$. In sync with the (simplex method) criterion to find an improved solution to the full master problem, one could search for a new column of type $(\gamma_k, P^k, 1)$, or of type $(\delta_l, Q^l, 0)$, with the property that $\gamma_k - \pi^\nu \cdot P^k - \theta_\nu < 0$ or $\delta_l - \pi^\nu \cdot Q^l < 0$. Equivalently, if the linear subprogram

$$\min (c - A^\top \pi^\nu) \cdot x \text{ subject to } Tx = d, x \geq 0,$$

is unbounded, in which case the simplex method would provide a direction of unboundedness q^l, that would, in turn, give us a new column of type $(\delta_l, Q^l, 0)$ to be included in our master program. Or, if this linear sub-program is bounded, the optimal solution p^k generated by the simplex method would be a vertex of the polyhedral set $\{x \in \mathbb{R}^n_+ \mid Tx = d\}$. As long as $\theta_\nu < (c - A^\top \pi^\nu) \cdot p^k$, a new column of type $(\gamma_k, P^k, 1)$ would be included in the master problem. On the other hand, if this last inequality was not satisfied, it would indicate that no column can be found that would enable us to improve the optimal value of the problem, i.e.,

$$x^* = \sum_{k=1}^{r_\nu} p^k \lambda_k^\nu + \sum_{l=1}^{s_\nu} q^k \mu_l^\nu$$

is then the optimal solution of our original problem. Clearly, the method will terminate in a finite number of steps; the number of columns generated could never exceed those in the full master program.

An interesting and particular application of this methodology is to linear programs with variable coefficients. The problem is one of the following form:

$$\min \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} P^j x_j = Q,\ x \geq 0,$$

where the vectors (c_j, P^j) may be chosen from a closed convex set $C^j \subset \mathbb{R}^{m+1}$. If the sets C^j are polyhedral, we essentially recover the method described earlier, but in general, it leads to a method for nonlinear convex programs that generates "internal" approximations, i.e., the feasible region, say C, of our convex program is approximated by a succession of polyhedral sets contained in C. The strategy is much the same as that described earlier, viz., one works with a *master (problem)* whose columns are generated from the subproblems to improve the succession of master solutions. For all j, these subproblems are of the following type:

$$\min c_j - \pi^\nu \cdot P^j \text{ subject to } (c_j, P^j) \in C^j.$$

The efficiency of the method depends on how easy or difficult it might be to minimize linear forms on the convex sets C^j; the π^ν correspond to the optimal multipliers associated with the linear constraints of the master problem. Dantzig outlined an elegant application of this method in "Linear control processes and mathematical programming" [20].

In this period, Dantzig co-authored another major contribution: a solution of a large (by the standards of the day) Traveling Salesman Problem (TSP). However, this work was considerably more than just a way to find an optimum solution to a particular problem. It pointed the way to several approaches to combinatorial optimization and integer programming problems.

First, the traveling salesman problem is to find the shortest-distance way to go through a set of cities so as to visit each city once and return to the starting point. This problem was put on the computational mathematical map by the now-famous paper of Dantzig, Fulkerson, and Johnson [28]. It has a footnote on the history of the problem that includes Merrill Flood stimulating interest in the problem, Hassler Whitney apparently lecturing on it, and Harold Kuhn exploring the relation between the TSP and linear programming. The abstract is one sentence: "It is shown that a certain tour of forty-nine cities, one in each of the forty-eight states and Washington, D.C., has the shortest road distance." The distances were taken from a road atlas. The authors reduce the problem to forty-two cities by pointing out that their solution with seven Northeastern cities deleted goes through those seven cities. Thus the solution method is applied to a 42-city problem.

The solution method involves solving the linear program with a 0-1 variable for each link (the terminology they use for undirected arcs) of the complete graph on forty-two nodes. The first set of constraints says the sum of the variables on links incident to a given node must equal two. There are forty-two of these constraints, for the 42-city problem. The second set of constraints is made up of what are called *sub-tour elimination constraints*. These require that the sum of variables over links, with one end in a set S of nodes and the other end not in S, must be greater-than-or-equal to two. There are of order 2^{42} of these constraints. However, they only had to add seven of them to the linear program for the 42-city problem at which point all the rest were satisfied. So despite the fact that there are a huge number of such constraints, only a few were used when they were added on to the linear program as needed.

On several occasions, George Dantzig spoke of a bet he had with Ray Fulkerson. George was convinced that not very many sub-tour elimination constraints would be needed, despite their large number. He was so convinced that he proposed a bet that there would be no more than twelve (the exact number is probably lost forever at this time), but Ray, who was a good poker player, said the bet should be the closest to the actual number and he went lower, saying eleven. In actuality, only seven were needed, plus two more that were presented as being somewhat ad hoc but needed to make the linear programming optimum be integer-valued.

The two additional inequalities are briefly justified and are acknowledged in a footnote, "We are indebted to I. Glicksberg of Rand for pointing out relations of this kind to us." Much of the later research in combinatorial optimization is directed at finding such inequalities that can be quickly identified, when violated. For example, finding a violated sub-tour elimination constraint is equivalent to finding a "minimum cut" in the graph where the weights on the arcs are the primal variables x_{ij}, which may be fractional. The min cut is the one that minimizes that sum of x_{ij} over all edges in the cut; that value is then compared to 2.

The method used is to start with a tour, which can be found by any one of several heuristics. Dual variables on the nodes can then be found, and the paper explains how. Then, non-basic arcs can be priced out and a basis change found. If the solution moves to another tour, then the step is repeated. If it moves to a sub-tour, then a sub-tour elimination constraint is added. If it moves to a fractional solution, then they look for a constraint to add that goes through the current solution vector and cuts off the fractional solution. Thus, it is an integer-primal simplex method, a primal simplex method that stays at integer points by adding cuts as needed.

A device the paper employs is the use of "reduced-cost fixing" of variables. Once a good tour is found and an optimum linear program employing valid cuts, such as sub-tour elimination cuts, the difference between the tour cost and the linear program optimum provides a value that can be used to fix non-basic variables at their current non-basic values. Although the number of variables is not enormous, the 42-city problem has 861 variables, which is a lot if one is solving the linear program by hand, as they did. They were using the revised simplex method so only needed to price out all these columns and enter the best one. Thus, even though the revised simplex method only has to price out over the columns, the process can be onerous when there are 861 variables and computation is being done by hand. For that reason, actually dropping many of these variables from the problem had a real advantage.

The paper also refers to a "combinatorial approach" which depends critically on reducing the number of variables. This approach seems to be related to the reduced-cost fixing referred to above. Although the paper is a bit sketchy about this approach, it seems to be that as the variables are reduced to a small number, some enumeration without re-solving the linear program (but perhaps using the reduced-cost fixing) is required. In this way, the options left open can be evaluated and the optimum solution identified.

According to Bland and Orlin [5], "*Newsweek* magazine ran a story on this 'ingenious application of linear programming' in the 26 July 1954 issue." However, it is highly unlikely that even Dantzig, Fulkerson, and Johnson could have anticipated then the practical impact that this work would ultimately have." This impact has been felt in integer programming generally where strong linear programming formulations are much used. One could say that the general method of finding violated, combinatorially derived cutting planes was introduced in this paper. Dantzig was personally firmly convinced that linear programming was a valuable tool in solving integer programming problems, even hard problems such as the TSP. Practically all of the successful, recent work that has accepted the challenge of solving larger and larger TSPs has been based on the Dantzig, Fulkerson, Johnson approach.

Another contribution of Dantzig in combinatorial optimization is the work based on network flow. Ray Fulkerson and Alan Hoffman were two of his collaborators and co-authors. The crucial observation is that this class of linear programs gives integer basic solutions when the right-hand side and bounds are all integers, so the integer program is solved by the simplex method. In addition, the dual is integer when the costs are integer. In particular the duality theorem provides a proof of several combinatorial optimization results, sometimes referred to as "min-max theorems". An example of a rather complex application of this min-max theorem is the proof of Dilworth's theorem by Dantzig and Hoffman. This theorem says that for a partial order, the minimum number of chains covering all elements is equal to the maximum number of pair-wise unrelated elements.

Of course, George Dantzig was always interested in applications, and developed a rich set of integer programming applications. One of these is what is called today the "fleet assignment model". Current models for this problem are much used in airline planning. As the problems' sizes grow and more details are included in the models, computational challenges abound; nevertheless this model has generally proven to be quite tractable despite being a large, mixed-integer program. In fact, the second of the two papers [50] on this subject was an early example of a stochastic integer programming problem. The problem is to allocate aircraft to routes for a given number of aircraft and routes. The main purpose of the problem is to maximize net revenue by routing the larger aircraft over flights that have more demand or by using more aircraft on such routes. The stochastic version of the problem has simple recourse, i.e., leave empty seats or leave demand unsatisfied depending on the realized demand.

A related problem was posed as finding the minimum number of tankers to cover a schedule. This is used today for example by charter airline companies to cover required flights by the minimum number of aircraft. The schedule of required flights in a charter operation is typically unbalanced requiring ferry flights, or deadhead flights, in order to cover the required flights. Dantzig and Fulkerson [27] modeled this as a network flow problem. With more than the minimum number of planes, the total deadheading distance may be reduced by allowing more than the minimum number of planes.

A notable application paper [34] by G. B. Dantzig and J. H. Ramser introduced what is now called the Vehicle Routing Problem (VRP), an area that has its own extensive literature and solution methods. The problem is a generalization of the TSP having many variations. As put by Toth and Vigo [80, p. xvii], the VRP "calls for the determination of the optimal set of routes to be performed by a fleet of vehicles to serve a given set of customers, and it is one of the most important, and studied, combinatorial optimization problems. More than forty years have elapsed since Dantzig and Ramser introduced the problem in 1959."

In addition to these contributions to network optimization problems, we must also mention Dantzig's article [18] pointing out the rich set of problems that can be modeled as integer programs. This paper motivated computational efforts such as cutting plane methods, enumeration, and branch-and-bound to effectively solve these diverse problems. These efforts have continued up to today.

Dantzig often referred to stochastic programming as the "real problem". He was very well aware that almost all important decision problems are *decision-making problems under uncertainty*, i.e., where at least some of the parameters are at best known in a statistical sense, and sometimes not even reliable statistical information is available. His serious commitment to this class of problems dates from the middle 1950s. As in many other instances, the "spark" probably came from an application, in this case allocating aircraft to routes under uncertain demand. That problem was the concern of Alan Ferguson, one of his colleagues at RAND [49]. After they devised an elegant procedure to solve this particular problem, Dantzig returned to his desk and wrote a fundamental paper that not only introduced the fundamental stochastic programming model, known today as the *stochastic program with recourse*, but also started deriving its basic properties. This certainly reaffirmed the need to deal efficiently with large-scale mathematical programs.

With only a slight reformulation, Dantzig's model was

$$\min c \cdot x + E\{Q(\xi, x)\} \text{ subject to } Ax = b,\ x \geq 0,$$

where $E\{\cdot\}$ denotes the taking of expectation, and

$$Q(\xi, x) = \inf\{q \cdot y \mid Wy = \xi - Tx, y \geq 0\};$$

here $\boldsymbol{\xi}$ is a random vector with values ξ in $\Xi \subset \mathbb{R}^d$; an extension of the model would allow for randomness in the parameters (q, W, T), in addition to just the right-hand sides, defining the function $Q(\cdot, x)$. He proved that in fact this is a well defined convex optimization problem, placing no restrictions on the distribution of the random elements, but requiring that the problem defining Q be solvable for all x and $\xi \in \Xi$, the *complete recourse* condition as it is now called. This means that stochastic programs with recourse, although generally not linear programs, fall into the next "nice" class, viz., convex programs. However, if the random vector $\boldsymbol{\xi}$ has finite support, say $\Xi = \{\xi^l, l = 1, \ldots, L\}$ with $\text{prob}\{\boldsymbol{\xi} = \xi^l\} = p_l$, or the discretization comes about as an approximation to a problem whose random components are continuously distributed, the solution x^* can be found by solving the large-scale linear program:

$$\begin{aligned} \min \quad & c \cdot x + \textstyle\sum_{l=1}^{L} p_l \, q \cdot y^l \\ \text{subject to} \quad & Ax = b \\ & Tx + Wy^l = \xi^l, \; l = 1, \ldots, L \\ & x \geq 0, \; y^l \geq 0, \; l = 1, \ldots, L \end{aligned}$$

"large", of course, depending on the size of L. Just to render this a little bit more tangible, assume that $\boldsymbol{\xi}$ consists of ten independent random variables, each taking on ten possible values; then this linear program comes with at least 10^{11} linear constraints—the 10^{10} possible realizations of the random vector $\boldsymbol{\xi}$ generating 10^{10} systems of 10 linear equations—certainly, a daunting task! To the rescue came (i) the Dantzig-Wolfe decomposition method and (ii) Dantzig's statistical apprenticeship.

It is pretty straightforward that, up to a sign change in the objective, the dual of the preceding problem can be expressed as

$$\begin{aligned} \min \quad & b \cdot \sigma + \textstyle\sum_{l=1}^{L} p_l \, \xi^l \cdot \pi^l \\ \text{subject to} \quad & A^\top \sigma + \textstyle\sum_{l=1}^{L} p_l T^\top \pi^l \leq c, \\ & W^\top \pi^l \leq q, \; l = 1, \ldots; L, \end{aligned}$$

i.e., a problem with a few linking constraints and a subproblem that can, itself, be decomposed into L relatively small (separable) subproblems. The Dantzig-Wolfe decomposition method was perfectly adapted to a structured problem of this type, and this was exploited and explained in [31]. Still, this elegant approach required, at each major iteration, solving (repeatedly) a huge number of "standard" linear programs. And this is how the idea of solving just a sampled number of these subproblems came to the fore. Of course, one could no longer be certain that the absolute optimal solution would be attained, but statisticians know how to deal with this. Dantzig, Glynn, and Infanger [29, 30] relied on the Student-t test to evaluate the reliability of the solutions so obtained. They also proposed a scheme based on importance sampling that reduces sample variance.

It would be misleading to measure the role Dantzig played in this field in terms of just his publications, even taking into account that he was responsible for the seminal work. All along, he encouraged students, associates, colleagues, as well as anybody who would listen, to enter the field, and gave them his full support. He saw the need to deal with the computational challenges, but his major interest seemed to lie in building models that would impact policy-making in areas that would significantly benefit society on a global scale. The "PILOT" project [33] provides an example such an effort. Another is his continued active involvement with the International Institute of Applied Systems Analysis (IIASA) in Laxenburg (Austria) whose mission is to conduct inter-disciplinary scientific studies on environmental, economic, technological, and social issues in the context of human dimensions of global change.

Dantzig at Berkeley

George Dantzig left RAND in 1960 to join the faculty of the University of California in Berkeley; he accepted a post as professor in the Department of Industrial Engineering. Established just four years earlier as a full-fledged department, Industrial Engineering was chaired from 1960 to 1964 by Ronald W. Shephard who, like Dantzig, was born in Portland, Oregon, and had done his doctoral studies under Jerzy Neyman at Berkeley. Asked why he left RAND to return to the academic world, Dantzig told Albers and Reid in their 1984 interview, "My leaving had to do with the way we teamed up to do our research.... Each of us got busy doing his own thing. ... There were no new people being hired to work with us as disciples. ... My stimulus comes from students and working closely with researchers elsewhere" [1, p. 311]. As it turned out, he had disciples aplenty at Berkeley (and later at Stanford).

By 1960 Operations Research was well on its way to becoming a separate (albeit interdisciplinary) field of study having natural ties to mathematics, statistics, computer science, economics, business administration, and some of the more traditional branches of engineering. Dantzig (and Shephard) founded the Operations Research Center (ORC) which coordinated teaching and research in a range of OR topics. For several years, the ORC was inconveniently situated in a small, shabby wood-frame house at the university's Richmond Field Station, some six miles from the main campus. Nevertheless the ORC managed to attract an enthusiastic group of faculty and students who investigated a range of OR themes. Dantzig's research program was certainly the largest at the ORC—and possibly the largest in the IE Department for that matter.

The research Dantzig began at RAND on compact basis techniques for large-scale linear programs led, at Berkeley, to the development of the *Generalized Upper Bound Technique (GUB).* The motivation came from a problem that Dantzig encountered while consulting for the Zellerbach Co., now the Crown Zellerbach Co. Rather than simple lower/upper bound constraints, the m linking constraints $Ax = b$ and nonnegativity restrictions are augmented by a collection of L linear constraints with positive right-hand sides and the property that every variable appears at most in one of these constraints and then, with a positive coefficient. In their paper, Dantzig and Van Slyke [38] show that the optimal solution of a problem of this type has the following properties: (i) from any group of variables associated with one of these L constraints, at least one of these variables must be basic and (ii) among these L equations, the number of those ending up with two or more basic variables is at most $m - 1$. These properties were exploited to show that with an appropriate modification of the pivoting rules of the standard simplex method and some creative bookkeeping, one can essentially proceed to solve the problem as if it involved only m linear constraints. Again, a situation when a compact basis will suffice, and this resulting in much improved efficiency and numerical reliability of the method.

Dantzig offered graduate courses in linear and nonlinear programming in addition to a course in network flows. These were given at a time when his *Linear Programming and Extensions* and the classic work *Flows in Networks* by Ford and Fulkerson were still in one stage of incompleteness or another. At best, some galley proofs were available, but photocopy equipment was then in its infancy (and was a wet process at that). A while later, a corps of Dantzig's students were called upon to proofread and learn from *LP&E.* The richness of the subject augmented by Dantzig's vast store of interesting research problems gave his doctoral students tremendous opportunities for dissertation topics and the resources to carry them out. This included exposure to stacks of technical reports, introductions to distinguished visitors at the ORC, and support for travel to important professional meetings.

Over his six-year period at Berkeley, Dantzig supervised eleven doctoral theses including those written by the authors of the present article. The types of mathematical programming in the dissertations supervised by Dantzig include large-scale linear programming, linear programming under uncertainty, integer programming, and nonlinear programming. A development that evolved from the latter subject (a few years later) came to be called *complementarity theory*, that is, the study of complementarity problems. These are fundamental inequality systems of the form

$$(5) \qquad F(x) \geq 0, \quad x \geq 0, \quad x \cdot F(x) = 0.$$

Besides thesis supervision and book writing, Dantzig busied himself with applied research in the life sciences, large-scale LP, and a bit of work on optimal control problems. With all this, he added "educator" to his large reputation.

Dantzig at Stanford

Dantzig left UC Berkeley in 1966 to join the Computer Science Department and the Program in Operations Research at Stanford University. Always reluctant to discuss his motivation for the switch, Dantzig used to quip that OR program chairman, Gerald J. Lieberman, promised him a parking place adjacent to the office. This arrangement did not survive the relocation of the department to other quarters, but Dantzig remained at Stanford anyway.

Since its creation in 1962, the OR program had been authorized to grant the Ph.D. and was an interschool academic unit reporting to three deans and six departments. Fortunately, this cumbersome arrangement changed in 1967, at which time Operations Research became a regular department in the School of Engineering. Dantzig retained his joint appointment the CS Department, but his principal activity was based in OR. Berkeley followed a slightly different route; in 1966, the IE Department became the Department of Industrial Engineering and Operations Research. But with Dantzig now at Stanford, the West Coast center of gravity for mathematical programming shifted southward.

One of those recruited to the OR faculty was R. W. Cottle who had been Dantzig's student at Berkeley. Together, they produced a few papers on the so-called linear complementarity problem, that being the case where the mapping F in (5) is affine. One of these, "Complementary Pivot Theory of Mathematical Programming" had a considerable readership in the OR field and became something of a classic.

George Dantzig had a knack for planning but harbored little interest in organizational politics or administrative detail. Even so, he began his tenure at Stanford by serving as president of The Institute of Management Sciences (TIMS). In 1968 he and A. F. Veinott Jr. co-directed (and co-edited the proceedings of) a very successful five-week "Summer Seminar on the Mathematics of the Decision Sciences" sponsored by the AMS. He followed this by directing in 1971 the "Conference on Applications of Optimization Methods for Large-Scale Resource Allocation Problems" held in Elsinore, Denmark, under the sponsorship of NATO's Scientific Affairs Division. Still later he was program chairman of the "8^{th} International Symposium on Mathematical Programming", which was held at Stanford in 1973. That year he became the first chairman of the newly established Mathematical Programming Society.

On top of these "distractions", Dantzig concentrated on mathematical programming research, the supervision of doctoral students, and the orchestration of proposal writing so essential to securing its external sponsorship.

Dantzig's leadership and the addition of other faculty and staff contributed greatly to the international stature of the Stanford OR Department. Among Dantzig's many ventures during this period, the Systems Optimization Laboratory (SOL) stands out as one of the most influential and enduring. At the 1971 Elsinore Conference mentioned above, Dantzig discoursed "On the Need for a Systems Optimization Laboratory". A year later, he (and a host of co-authors) put forward another version of this concept at an Advanced Seminar on Mathematical Programming at Madison, Wisconsin. As Dantzig put it, the purpose of an SOL was to develop "computational methods and associated computer routines for numerical analysis and optimization of large-scale systems". By 1973 Stanford had its SOL with Dantzig as director. For many years, the SOL was blessed with an outstanding resident research staff consisting of Philip Gill, Walter Murray, Michael Saunders, John Tomlin, and Margaret Wright. The research and software output of the latter group is world famous. Numerous faculty and students rounded out the laboratory's personnel. Over time, Dantzig's concept of an SOL would be emulated at about twenty other institutions.

**Left to right:
T. C. Koopmans, G. B. Dantzig,
L. V. Kantorovitch, 1975**

While bringing the SOL to fruition, Dantzig collaborated with T. L. Saaty on another altogether different sort of project: a book called *Compact City* [35], which proposes a plan for a liveable urban environment. The multi-story city was to be cylindrical in shape (thereby making use of the vertical dimension) and was intended to operate on a 24-hour basis (thereby making better use of facilities). Although the main idea of this publication does not appear to have been implemented anywhere, the book itself has been translated into Japanese.

During the academic year of 1973–74, Dantzig spent his sabbatical leave at the International Institute for Applied Systems Analysis (IIASA). Located in Laxenburg, Austria, IIASA was then about one year old. IIASA scientists worked on problems of energy, ecology, water resources, and methodology. Dantzig headed the Methodology Group and in so doing established a long association with this institute.

Another memorable feature of 1973 was the well-known mideast oil crisis. This event may have been what triggered Dantzig's interest in energy-economic modeling. By 1975 this interest evolved into something he called the PILOT Model, a passion that was to occupy him and a small group of SOL workers until the late 1980s. ("PILOT" is an acronym for "Planning Investment Levels Over Time".) As Dantzig, McAllister, and Stone explain [32], PILOT aims "to assess the impact of old and proposed new technologies on the growth of the U.S. economy, and how the state of the economy and economic policy may affect the pace at which innovation and modernization proceeds." The PILOT project provided a context combining three streams of research that greatly interested Dantzig: modeling of a highly relevant economic issue, large-scale programming methodology, and the computation of optimal solutions or the solutions of economic equilibrium (complementarity) problems.

The year 1975 brought a mix of tidings to Dantzig and the mathematical programming community. The 1975 Nobel Prize in Economics went to Kantorovich and Koopmans "for their contributions to the theory of the optimum allocation of resources." To the shock and dismay of a worldwide body of well-wishers (including the recipients), George Dantzig was not selected for this distinguished award. In their prize speeches, Kantorovich and Koopmans recognized the independent work of Dantzig; Koopmans felt so strongly about the omission that he donated a third of his prize money to IIASA in Dantzig's honor.

But on a happier note, that same year George Dantzig received the prestigious National Medal of Science from President Gerald Ford, the John von Neumann Theory Prize from ORSA and TIMS, and membership in the American Academy of Arts and Sciences. He had, in 1971, already become a member of the National Academy of Sciences and, indeed, would become, in 1985, member of the National Academy of Engineering. Over his lifetime, many other awards and eight honorary doctorates were conferred on Dantzig. Beginining in 1970, he was listed on the editorial boards of twenty-two different journals.

As a colleague and as a mentor, George Dantzig was a remarkable asset. That he had broad knowledge of the mathematical programming field and a wealth of experience with the uses of its methodology on practical problems goes without saying. What also made him so valuable is that he was a very patient and attentive listener and would *always* have a response to whatever was told him. The response was invariably a cogent observation or a valuable suggestion that would advance the discussion. As a professor at Stanford, Dantzig produced forty-one doctoral students. The subjects of their theses illustrates the range of his interests. The distribution goes about like this: large-scale linear programming (8), stochastic programming (6), combinatorial optimization (4), nonlinear programming (4), continuous linear programming (3), networks and graphs (3), complementarity and computation of economic equilibria (2), dynamic linear and nonlinear programming (2), probability (2), other (7). The skillful supervision Dantzig brought to the dissertation work of these students and the energy they subsequently conveyed to the operations research community shows how effective his earlier plan to have disciples really turned out to be. A nearly accurate version of Dantzig's "academic family tree" can be found at http://www.genealogy.ams.org/.

Retirement

George Dantzig became an emeritus professor in 1985, though not willingly. He was "recalled to duty" and continued his teaching and research for another thirteen years. It was during this period that Dantzig's long-standing interest in stochastic optimization got even stronger. In 1989 a young Austrian scholar named Gerd Infanger came to the OR Department as a visitor under the aegis of George Dantzig. Infanger's doctorate at the Vienna Technical University was concerned with energy, economics, and operations research. He expected to continue in that vein working toward his *Habilitation.* Dantzig lured him into the ongoing research program on stochastic optimization which, as he believed, is "where the real problems are." So began a new collaboration. During the 1990s, Dantzig and Infanger co-authored seven papers reporting on powerful methods for solving stochastic programs. Moreover, Infanger obtained his *Habilitation* in 1993.

Dantzig also established another collaboration around this time. Having decided that much progress had been made since the publication of *LP&E* in 1963, he teamed up with Mukund Thapa to write a new book that would bring his *LP&E* more up to date. They completed two volumes [36] and [37] before Dantzig's health went into steep decline (early 2005), leaving two more projected volumes in an incomplete state.

Dantzig's Mathematical Impact

It will not be possible to give a full account of the mathematical impact of Dantzig's work, some of which is still ongoing. Instead, we focus on a few key points.

Before the 1950s, dealing with systems involving linear—and *a fortiori* nonlinear—inequalities was of limited scope. This is not to make light of the pioneering work of Fourier, Monge, Minkowski, the 1930s-Vienna School, and others, but the fact remains that up to the 1950s, this area was in the domain of an exclusive, albeit highly competent, club of mathematicians and associated economists, with little or no impact in the "practical" world. Dantzig's development of the simplex method changed all that. Perhaps his greatest contribution was to demonstrate that one could deal (effectively) with problems involving inequality constraints. His focus and his determination inspired a breakthrough that created a new mathematical paradigm which by now has flourished in a myriad of directions.

Left to right: George Dantzig, Anne Dantzig, President Gerald Ford (National Medal of Honor ceremony, 1971).

Quickly, practitioners ranging from engineers, managers, manufacturers, agricultural economists, ecologists, schedulers, and so on, saw the potential of using linear programming models now that such models came with an efficient solution procedure. In mathematical circles, it acted as a watershed. It encouraged and inspired mathematicians and a few computer scientists to study a so-far-untouched class of problems, not just from a computational but also from a theoretical viewpoint.

On the computational front, it suffices to visit a site like the NEOS Server for Optimization http://www-neos.mcs.anl.gov/ to get a glimpse at the richness of the methods, now widely available, to solve optimizations problems and related variational problems: equilibrium problems, variational inequalities, cooperative and non-cooperative games, complementarity problems, optimal control problems, etc.

On the theoretical front, instead of the classical framework for analysis that essentially restricted functions and mappings to those defined on open sets or differentiable manifolds, a new paradigm emerged. It liberated mathematical objects from this classical framework. A comprehensive theory was developed that could deal with functions that are not necessarily differentiable, not even continuous, and whose domains could be closed sets or manifolds that, at best, had some Lipschitzian properties. This has brought about new notions of (sub)derivatives that can be be used effectively to characterize critical points in situations where classical analysis could not contribute any insight. It created a brand new approximation theory where the classical workhorse of pointwise limits is replaced by that of set limits; they enter the picture because of the intrinsic one-sided (unilateral) nature of the mathematical objects of interest. The study of integral functionals was also given some solid mathematical foundations, and much progress was achieved in solving problems that had puzzled us for a very long time.

Although we could go much further in this direction, we conclude with one shining example. The study of (convex) polyhedral sets was immediately revived once Dantzig's simplex method gained some foothold in the mathematical community. At the outset, in the 1950s, it was led by a Princeton team under the leadership of A. W. Tucker. But quickly, it took on a wider scope spurred by the *Hirsch conjecture*: Given a linear program in standard form, i.e., with the description of the feasible polyhedral set as $S = \mathbb{R}^n_+ \cap M$ where the affine set M is determined by m (not redundant) linear equations, the conjecture was that it is possible to pass from any vertex of S to any other one via a feasible path consisting of no more than $m - 1$ edges, or in simplex method parlance, a path requiring no more than m (feasible) pivot steps. This question went to the core of the efficiency of the simplex method. The conjecture turned out to be incorrect, at least as formulated, when $m > 4$, but it led to intensive and extensive research associated with the names of Victor Klee, David Walkup, Branko Grünbaum, David Gale, Micha Perlis, Peter McMullen, Gil Kalai, Daniel Kleitman, among others, that explored the geometry and the combinatorial properties of polyhedral sets. Notwithstanding its actual performance in the field, it was eventually shown that examples could be created, so that the simplex method would visit every vertex of such a polyhedral set S having a maximal number of vertexes. The efficiency question then turned to research on the "expected" number of steps; partial answers to this question being provided, in the early 1980s, by K. H. Borgwardt [6] and S. Smale [77]. An estimate of the importance attached to this subject area, can be gleaned from Smale's 1998 article [78] in which he states eighteen "mathematical problems for the next century". Problem 9 in this group asks the question: *Is there a polynomial-time algorithm over the real numbers which decides the feasibility of the linear system of inequalities $Ax \geq b$?* (This decision version of the problem rests on duality theory for linear programming.) Problem 9 asks for an algorithm given by a real number machine with time being measured by the number of arithmetic operations.

Epilogue

George Dantzig had a fertile mind and was passionately dedicated to his work throughout his adult life. Although his career began with an interest in mathematical statistics, circumstances guided him to becoming a progenitor of mathematical programming (or optimization). In creating the simplex method, he gave the world what was later to be hailed as one of "the top 10 algorithms" of the twentieth century [44]. A selection of Dantzig's research output appears in the anthology [10].

Dantzig built his life around mathematical programming, but not to the exclusion of colleagues around the world. Through his activities he touched the lives of a vast number of mathematicians, computer scientists, statisticians, operations researchers, engineers, and applied scientists of all sorts. Many of these individuals benefited from their bonds of friendship with him. Dantzig's natural warmth engendered a strong sense of loyalty, and he returned it in full.

Many of Dantzig's major birthdays were celebrated by conferences, banquets, Festschrifts, and the like. Even the simplex method had a fiftieth birthday party in 1997 at the 16^{th} International Symposium on Mathematical Programming (ISMP) held in Lausanne, Switzerland. In the year 2000, George Dantzig was honored as a founder of the field at the 17^{th} ISMP in Atlanta, Georgia. Perhaps the most touching of all the festivities and tributes in Dantzig's honor was the conference and banquet held in November 2004. He had turned ninety just a few days earlier. In attendance were colleagues from the past and present, former students of Dantzig, and current students of the department. To everyone's delight, he was in rare form for a man of his age; moreover, he seemed to relish the entire event. Sadly, though, within two months, his health took a serious turn for the worse, and on the following May 13th (a Friday), he passed away.

George B. Dantzig earned an enduring place in the history of mathematics. He will be warmly remembered for years to come by those who were privileged to know him. By some he will be known only through his work and its impact, far into the future. In part, this will be ensured by the creation of the Dantzig Prize jointly by the Mathematical Programming Society and the Society for Industrial and Applied Mathematics (1982), the Dantzig Dissertation Award by INFORMS (1994), and an endowed Dantzig Operations Research fellowship in the Department of Management Science and Engineering at Stanford University (2006).

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Linear Programming and the Simplex Method

David Gale

This exposition of linear programming and the simplex method is intended as a companion piece to the article in this issue on the life and work of George B. Dantzig in which the impact and significance of this particular achievement are described. It is now nearly sixty years since Dantzig's original discovery [3] opened up this whole new area of mathematics. The subject is now widely taught throughout the world at the level of an advanced undergraduate course. The pages to follow are an attempt at a capsule presentation of the material that might be covered in three or four lectures in such a course.

Linear Programming

The subject of linear programming can be defined quite concisely. It is concerned with the problem of maximizing or minimizing a linear function whose variables are required to satisfy a system of linear *constraints*, a constraint being a linear equation or inequality. The subject might more appropriately be called linear optimization. Problems of this sort come up in a natural and quite elementary way in many contexts but especially in problems of economic planning. Here are two popular examples.

The Diet Problem

A list of foods is given and the object is to prescribe amounts of each food so as to provide a meal that has preassigned amounts of various nutrients such as calories, vitamins, proteins, starch, etc. Further, each food has a cost, so among all such adequate meals the problem is to find one that is least costly.

The Transportation Problem

A certain good, say steel, is available in given amounts at a set of m origins and is demanded in specified amounts at a set of n destinations. Corresponding to each origin i and destination j there is the cost of shipping one unit of steel from i to j. Find a shipping schedule that satisfies the given demands from the given supplies at minimum cost.

Almost all linear programming applications, including these examples, can be motivated in the following way. There is a given set of *goods*. For the diet problem these are the various nutrients. For the transportation problem there are $m + n$ goods, these being steel at each of the origins and destinations. There is also a set of processes or *activities* which are specified by amounts of the goods *consumed* as *inputs* or *produced* as *outputs*. In the diet problem, for example, the carrot-consuming activity $\mathbf{c}$ has an output of c_1 units of calories, c_2 units of vitamin A, etc. For the transportation problem the transportation activity $\mathbf{t}^{ij}$ has as input one unit of steel at origin i and as output one unit of steel at destination j. In general, an activity $\mathbf{a}$ is column vector whose positive entries are outputs, negative entries inputs. These vectors will be denoted by boldface letters. It is assumed that activity $\mathbf{a}^j$ may be carried out at any nonnegative *level* x_j. The constraints are given by another activity vector $\mathbf{b}$, *the right-hand side*, which specifies the amounts of the different goods that are to be produced or consumed. For the diet problem it is the list of amounts of the prescribed

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nutrients. For the transportation problem it is the given supplies and demands at the origins and destinations. Finally, associated with each activity $\mathbf{a}^j$ is a cost c_j. Given m goods and n activities $\mathbf{a}^j$ the linear programming problem (LP) is then to find activity levels x_j that satisfy the constraints and minimize the total cost $\sum_j c_j x_j$. Alternatively, c may be thought of as the profit generated by activity $\mathbf{a}$, in which case the problem is to maximize rather than minimize $\sum_j c_j x_j$.

The simplex method is an algorithm that finds solutions of LPs or shows that none exist. In the exposition to follow we will treat only the special case where the constraints are equations and the variables are nonnegative, but the more general cases are easily reduced to this case.

The Simplex Method

In the following paragraphs we describe the simplex algorithm by showing how it can be thought of as a substantial generalization of standard Gauss-Jordan elimination of ordinary linear algebra. To gain intuition as to why the algorithm works, we will refer to the linear activity model of the previous section. Finally we will mention some interesting discoveries in the analysis of the algorithm, including a tantalizing unsolved problem.

Pivots and Tableaus

The basic computational step in the simplex algorithm is the same as that in most of elementary linear algebra, the so-called *pivot* operation. This is the operation on matrices used to solve systems of linear equations, to put matrices in echelon form, to evaluate determinants, etc.

Given a matrix A one chooses a nonzero *pivot entry* a_{ij} and adds multiples of row i to the other rows so as to obtain zeros in the jth column. The ith row is then normalized by dividing it by a_{ij}. For solving linear equations a pivot element can be any nonzero entry. By contrast, the simplex method restricts the choice of pivot entry and is completely described by giving a pair of simple rules, the entrance rule that determines the pivot column j and the exit rule that determines the pivot row i (in theory a third rule may be needed to take care of degenerate cases). By following these rules starting from the initial data the algorithm arrives at the solution of the linear program in a finite number of pivots. Our purpose here is to present these rules and show why they work.

We shall need one other concept.

Definition. The *tableau* X of a set of vectors $A = \{\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^n\}$ with respect to a basis $B = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^m\}$ is the $m \times n$ matrix

	$\mathbf{a}^1$	$\mathbf{a}^2$	$\cdots$	$\mathbf{a}^j$	$\cdots$	$\mathbf{a}^n$
$\mathbf{b}^1$	x_{11}	x_{12}	$\cdots$	x_{1j}	$\cdots$	x_{1n}
$\mathbf{b}^2$	x_{21}	x_{22}	$\cdots$	x_{2j}	$\cdots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}^i$	x_{i1}	x_{i2}	$\cdots$	x_{ij}	$\cdots$	x_{in}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}^m$	x_{m1}	x_{m2}	$\cdots$	x_{mj}	$\cdots$	x_{mn}

where x_{ij} is the coefficient of $\mathbf{b}^i$ in the expression for $\mathbf{a}^j$ as a linear combination of the $\mathbf{b}^i$.

In matrix terms, X is the (unique) matrix satisfying

$$BX = A. \tag{1}$$

(Symbols A, B are used ambiguously to stand either for a matrix or the set of its columns.)

It will often be useful to include the unit vectors $\mathbf{e}^i$ in the tableau in which case it appears as shown below.

	$\mathbf{e}^1$	$\mathbf{e}^2$	$\cdots$	$\mathbf{e}^j$	$\cdots$	$\mathbf{e}^m$
$\mathbf{b}^1$	y_{11}	y_{12}	$\cdots$	y_{1j}	$\cdots$	y_{1m}
$\mathbf{b}^2$	y_{21}	y_{22}	$\cdots$	y_{2j}	$\cdots$	y_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}^i$	y_{i1}	y_{i2}	$\cdots$	y_{ij}	$\cdots$	y_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}^m$	y_{m1}	y_{m2}	$\cdots$	y_{mj}	$\cdots$	y_{mm}

	$\mathbf{a}^1$	$\mathbf{a}^2$	$\cdots$	$\mathbf{a}^j$	$\cdots$	$\mathbf{a}^n$
	x_{11}	x_{12}	$\cdots$	x_{1j}	$\cdots$	x_{1n}
	x_{21}	x_{22}	$\cdots$	x_{2j}	$\cdots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	x_{i1}	x_{i2}	$\cdots$	x_{ij}	$\cdots$	x_{in}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	x_{m1}	x_{m2}	$\cdots$	x_{mj}	$\cdots$	x_{mn}

Note that from (1), $Y = \{y_{ij}\}$ is the solution of $BY = I$ so $Y = B^{-1}$. Multiplying (1) on the left by Y gives the important equation

$$YA = X. \tag{2}$$

It is easy to verify that if one pivots on entry x_{ij}, one obtains a new matrix X' that is the tableau with respect to the new basis in which $\mathbf{a}^j$ has replaced $\mathbf{b}^i$.

	$\mathbf{a}^1$	$\mathbf{a}^2$	$\cdots$	$\mathbf{a}^j$	$\cdots$	$\mathbf{a}^n$
$\mathbf{b}^1$	x'_{11}	x'_{12}	$\cdots$	0	$\cdots$	x'_{1n}
$\mathbf{b}^2$	x'_{21}	x'_{22}	$\cdots$	0	$\cdots$	x'_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{a}^j$	x'_{i1}	x'_{i2}	$\cdots$	1	$\cdots$	x'_{in}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}^m$	x'_{m1}	x'_{m2}	$\cdots$	0	$\cdots$	x'_{mn}

The simplex method solves linear programs by a sequence of pivots in successive tableaus, or, equivalently, by finding a sequence of bases, where each basis differs from its predecessor by a single vector.

Solving Linear Equations

We start by showing how to solve systems of linear equations using the language of pivots and tableaus. Our fundamental result is the following:
Theorem 1. *Exactly one of the following systems has a solution:*

$$A\mathbf{x} = \mathbf{b} \tag{3}$$

or

$$\mathbf{y}^T A = \mathbf{0}^T,\ \mathbf{y}^T\mathbf{b} = 1. \tag{4}$$

(Again, boldface letters are column vectors and are converted to row vectors using the superscript ^{T}.)

Note that rather than giving complicated conditions for (3) not to be solvable this formulation puts the solvable and unsolvable case on an equal footing. This *duality* is an essential element of the theory, as will be seen.

Geometric Interpretation

If (3) has no solution this means that $\mathbf{b}$ does not lie in the linear subspace generated by the $\mathbf{a}^j$. In that case there is a vector $\mathbf{y}$ orthogonal to the $\mathbf{a}^j$ but not to $\mathbf{b}$.

We shall continue to present such geometric pictures as aids to intuition but the mathematical arguments will not depend on these "pictures".

It is immediate that (3) and (4) cannot both hold since multiplying (3) by $\mathbf{y}^T$ and (4) by $\mathbf{x}$ would imply $0 = 1$. The nontrivial part is to show that either (3) or (4) must hold. We do this by giving an algorithm that in at most m pivots finds a solution to either (3) or (4).

The initial tableau consists of the matrix A together with the right-hand side $\mathbf{b}$ all preceded by the identity matrix I. It is represented schematically in the form below.

$$\begin{array}{cccc} & \{\mathbf{e}^i\} & \{\mathbf{a}^j\} & \{\mathbf{b}\} \\ \{\mathbf{b}^i\} & [\ I & A & \mathbf{b}\ ] \end{array}$$

(The symbols like $\{\mathbf{e}^i\}, \{\mathbf{a}^j\}$ in curly brackets are the row and column headings of the tableau.)

We proceed to try replacing the $\mathbf{e}^i$ in order by some $\mathbf{a}^j$. The tableau at any stage has the form

$$\begin{array}{cccc} & \{\mathbf{e}^i\} & \{\mathbf{a}^j\} & \{\mathbf{b}\} \\ \{\mathbf{b}^i\} & [\ Y & X & \mathbf{u}\ ] \end{array}$$

There are now two possibilities.

Case I. All of the unit vectors $\mathbf{e}^i$ can be replaced by some of the $\mathbf{a}^j$. Then a solution of (3) can be read off from the vector $\mathbf{u}$ in the $\mathbf{b}$-column of the tableau above. Namely, each component of $\mathbf{u}$, say u_i, is the value of a variable x_{j_i} whose corresponding column $\mathbf{a}^{j_i} = \mathbf{b}^i$. Every x_j whose corresponding column is not in the current basis has the value zero.

Case II. In trying to replace $\mathbf{e}^k$ there is no available pivot because $x_{kj} = 0$ for all j.

(i) If in addition $u_k = 0$, then leave $\mathbf{e}^k$ in the basis and go on to replace $\mathbf{e}^{k+1}$.

(ii) If $u_k \neq 0$, then the kth row $\mathbf{y}_k$ of Y gives a solution of (4). To see this, note that from (2) $\begin{bmatrix} Y \end{bmatrix}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} X & \mathbf{u} \end{bmatrix}$. Since the kth row of X is zero, we get $\mathbf{y}_k A = 0$, whereas $\mathbf{y}_k\mathbf{b} = u_k \neq 0$, so $\mathbf{y}_k/u_k$ solves (4).

The algebra becomes considerably more complicated if one requires the solutions of (3) to be nonnegative. In this case we have the following existence theorem (known as Farkas's Lemma):
Theorem 2. *Exactly one of the following systems has a solution.*

$$A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \tag{5}$$

or

$$\mathbf{y}^T A \leq \mathbf{0}^T,\ \mathbf{y}^T\mathbf{b} > 0. \tag{6}$$

Again it is not possible for both systems to have solutions since multiplying (5) by $\mathbf{y}^T$ gives $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b} > 0$ whereas multiplying (6) by $\mathbf{x}$ gives $\mathbf{y}^T A\mathbf{x} \leq 0$.

Geometric Interpretation

Equation (5) says that $\mathbf{b}$ lies in the convex cone generated by the columns $\mathbf{a}^j$ of A. If $\mathbf{b}$ does not lie in this cone, then there is a "separating hyperplane" whose normal makes a non-acute angle with the $\mathbf{a}^j$ and an acute angle with the vector $\mathbf{b}$.

The simplex method finds a solution of either (5) or (6) in a finite number of pivots. However, there is no useful upper bound on the number of pivots that may be needed as we shall see shortly.

We may assume to start out that the vector $\mathbf{b}$ is nonnegative (if not, change the signs of some of the equations).

Definition. A basis B will be called *feasible* (*strongly feasible*) if $\mathbf{b}$ is a nonnegative (positive) linear combination of vectors of B. This is equivalent to the condition that the $\mathbf{b}$ column $\mathbf{u}$ of the tableau is nonnegative (positive).

Imitating the algorithm of the previous section we start from the feasible basis $\{\mathbf{e}^i\}$ of unit vectors and try to bring in the $\{\mathbf{a}^j\}$ by a sequence of pivots that maintain feasibility, but to do this the pivot rule of the previous section must be restricted.

Suppose $\mathbf{a}^j$ is to be brought into a new feasible basis. Then in order for the new basis to be feasible the basis vector it replaces must satisfy the following easily derived condition.

Exit Pivot Rule

The pivot element x_{ij} must satisfy

(i) $x_{ij} > 0$,

(ii) $u_i/x_{ij} \leq u_k/x_{kj}$ for all k with $x_{kj} > 0$.

Because of the above restriction it is no longer possible to simply replace the basis vectors e^i one at a time as in the previous section. Some further rule for choosing the pivot column must be given.

We will postpone the proof of Theorem 2 as it will be shown to be a special case of a much more general theorem, which we now describe.

The Standard Minimum Problem

We consider the following important special case of a linear programming problem.

Given an m-vector $\mathbf{b}$, an n-vector $\mathbf{c}^T$ and an $m \times n$ matrix A, find an n-vector $\mathbf{x}$ so as to

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{7}$$

Corresponding to this problem (and using exactly the same data) is a *dual problem,* find an m-vector $\mathbf{y}^T$ so as to

$$\begin{array}{l} \text{maximize } \mathbf{y}^T\mathbf{b} \\ \text{subject to } \mathbf{y}^T A \leq \mathbf{c}^T. \end{array} \tag{8}$$

Terminology

The linear function $\mathbf{c}^T\mathbf{x}$ and $\mathbf{y}^T\mathbf{b}$ are called the *objective functions* of (7) and (8), respectively. The vectors $\mathbf{x}, \mathbf{y}$ that solve these problems are called *optimal solutions*; the numbers $\mathbf{c}^T\mathbf{x}$ and $\mathbf{yb}$ are *optimal values.*

Let $\mathbb{X}$ and $\mathbb{Y}$ be the sets (possibly empty) of all vectors that satisfy the constraints of (7) and (8), respectively. Such vectors are said to be *feasible.*

Key Observation

If $\mathbf{x} \in \mathbb{X}$, $\mathbf{y}^T \in \mathbb{Y}$, then $\mathbf{y}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$.

Proof. Multiply $A\mathbf{x} = \mathbf{b}$ by $\mathbf{y}^T$ and $\mathbf{y}^T A \leq \mathbf{c}^T$ by $\mathbf{x}$.

An important consequence of this observation is

Corollary. *If $\mathbf{x} \in \mathbb{X}$ and $\mathbf{y} \in \mathbb{Y}$ satisfy $\mathbf{y}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions, and the numbers $\mathbf{y}^T\mathbf{b}$=$\mathbf{c}^T\mathbf{x}$ are optimal values for their respective problems.*

The converse of this fact makes up the

Fundamental Duality Theorem. *The dual problems above have optimal solutions $\mathbf{x}, \mathbf{y}^T$ if and only if $\mathbb{X}$ and $\mathbb{Y}$ are nonempty in which case the optimal values $\mathbf{c}^T\mathbf{x}$and $\mathbf{y}^T\mathbf{b}$ are equal.*

Historical Note

The first explicit statement of the duality theorem is due to von Neumann [7] in a manuscript that was privately circulated but never formally published in his lifetime. Moreover it has been hard to verify the validity of the von Neumann proof. The first formally published proof is due to Gale, Kuh, and Tucker [4].

To see how Theorem 2 is a special case of the duality theorem we take $[I \; A \; \mathbf{b}]$ as the given initial tableau of a standard minimum problem and the vector $\mathbf{c}^T = (1, .., 1, 0, ..., 0)$ whose first m entries are 1, the rest 0, as the objective function. The set $\mathbb{X}$ is nonempty since it contains the nonnegative vector $(b_1, .., b_m, 0, .., 0)$, and the set $\mathbb{Y}$ is nonempty since it contains the zero vector. If the optimal value is zero, then a subvector of the optimal solution $\mathbf{x}$ solves (5). If the optimal value is positive, then the dual optimal vector $\mathbf{y}$ solves (6).

We will now see how the simplex method solves problems (7), (8) and gives a constructive proof of the duality theorem.

As a first step we incorporate the objective function $\mathbf{c}^T\mathbf{x}$ as the 0th row of the tableau. That is, we define the *augmented* column vector $\hat{\mathbf{a}}^j$ to be $(-c_j, \mathbf{a}^j)$, and we introduce the 0th unit vector $\mathbf{e}^0$. The *augmented* initial tableau is then,

$$\begin{array}{c} \\ \mathbf{e}^0 \\ \{\mathbf{e}^i\} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{e}^0 & \{\mathbf{e}^i\} & \{\hat{\mathbf{a}}^j\} & \hat{\mathbf{b}} \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -\mathbf{c}^T & 0 \\ 0 & I & A & \mathbf{b} \end{array}\right] \end{array}$$

We may assume that we have found an initial feasible basis, by applying the simplex method to the nonnegative solution problem (5), (6) as described. Then with respect to the general augmented basis $\{\mathbf{e}^0, \hat{\mathbf{b}}^1, \ldots, \hat{\mathbf{b}}^m\}$ the tableau has the form below.

$$\begin{array}{c} \\ \mathbf{e}^0 \\ \{\hat{\mathbf{b}}^i\} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{e}^0 & \{\mathbf{e}^i\} & \{\hat{\mathbf{a}}^j\} & \hat{\mathbf{b}} \end{array} \\ \left[\begin{array}{cccc} 1 & \mathbf{y}^T & \mathbf{z}^T & w \\ 0 & Y & X & \mathbf{u} \end{array}\right] \end{array}$$

Note that by definition of the tableau, $w\mathbf{e}^0 + \sum_i u_i \hat{\mathbf{b}}^i = \hat{\mathbf{b}}$ so for the 0th entry we have $w - \sum_i u_i c_i = 0$, so $w = \sum_i u_i c_i$, which is the value of the objective function for the basic feasible basis $\{\mathbf{b}^i\}$.

Also from the tableau $z_j = \sum_i x_{ij} c_i - c_j$. This number compares the cost associated with activity vector $\mathbf{a}^j$ to that of the corresponding linear combination of the basis vectors $\mathbf{b}^i$. If z_j is positive it means that one could reduce the total cost by bringing the vector $\mathbf{a}^j$ into the next basis. This leads to the following two important observations:

I. If the row $\mathbf{z}^T$ is nonpositive, then $\mathbf{u}$ can be extended to a solution $\mathbf{x}$ of (7) and $\mathbf{y}^T$ solves (8). Namely, using (2) again we have

$$\left[\begin{array}{cc} 1 & \mathbf{y}^T \\ 0 & Y \end{array}\right] \left[\begin{array}{cc} -\mathbf{c}^T & 0 \\ A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc} \mathbf{z}^T & w \\ X & \mathbf{u} \end{array}\right]$$

so $-c_j + \mathbf{y}^T\mathbf{a}^j = z_j \leq 0$. Hence $\mathbf{y}^T A \leq \mathbf{c}^T$ so $\mathbf{y}^T$ satisfies the dual constraints. Similarly we have $\mathbf{y}^T\mathbf{b} = w = \mathbf{c}^T\mathbf{x}$, so from the corollary above, the

Duality Theorem is verified.

II. If for some j we have $z_j > 0$ and $x_{ij} \le 0$ for all i, then problem (7) has no minimum, for from the tableau we have $\mathbf{a}^j - \sum_i x_{ij}\mathbf{b}^i = 0$; and since the $x_{ij} \le 0$, we have

$$\lambda(\hat{\mathbf{a}}^j - \sum_i x_{ij}\hat{\mathbf{b}}^i) + \sum_i x_i\hat{\mathbf{b}}^i = \mathbf{b}$$

giving a new feasible (nonnegative) solution for any $\lambda > 0$. The corresponding change in the objective function is $\lambda(c_j - \sum_i x_{ij}c_i) = -\lambda z_j$; so since $z_j > 0$ the function has no minimum.

In view of I above, one is looking for a feasible solution with z nonpositive, so the following pivot rule suggests itself.

Entrance Pivot Rule

Bring into the basis any column $\hat{\mathbf{a}}^j$ where z_j is positive.

Note that as described the second pivot rule may, in general, require making a choice among many possible positive z_j. A natural choice would be for example to choose the column with the largest value of z_j, a choice originally suggested by Dantzig.

The two pivot rules completely describe the simplex algorithm. If for some j, z_j is positive and some x_{ij} is positive, then bring $\mathbf{a}^j$ into the basis (by a pivot). It remains to show that eventually either I or II will occur. The argument is simple if we make the following

Nondegeneracy Assumption

Every feasible basis is strongly feasible.

Note if this were not the case, then $\mathbf{b}$ would be a positive linear combination of fewer than m of the vectors $\mathbf{a}^j$, a degenerate situation. Although the degenerate case would seem (on mathematical grounds) to be rare, this is not so in practice.

Suppose x_{ij} is the (positive) pivot. Then after pivoting $w' = w - z_j u_i / x_{ij} < w$ since $u_i > 0$ from nondegeneracy, so the value of the objective function decreases with each pivot. This means that no basis can recur (since the basis determines the objective value), and since there are only finitely many bases, the algorithm must terminate in either state I or II.

If the nondegeneracy assumption is not satisfied some further argument is necessary. Indeed, examples have been constructed by Hoffman [5] (for one) using the Dantzig pivot choice rule which can lead to "cycling," meaning the sequence of feasible bases recurs indefinitely. It turns out, however, that the following simple rule due to Bland [1] guarantees that no basis will recur.

Bland's Pivot Selection Rule

Among eligible entering vectors choose the one with lowest index and let it replace the eligible basis vector with lowest index.

While the condition is simple to state, the proof that it avoids cycling is quite subtle. Moreover, it is not as efficient as the customary (Dantzig) pivot choice rule.

Finally, there is a natural economic interpretation of the dual problem. In the model of goods and activities, the vector $\mathbf{y}^T = (y_1, .., y_m)$ can be thought of as a *price vector* where y_i is the price of one unit of the ith good. Then $\mathbf{y}^T\mathbf{a}^j$ is the revenue generated by operating activity $\mathbf{a}^j$ at unit level. The condition of dual feasibility, $\mathbf{y}^T\mathbf{a}^j - c_j \le 0$ states that *profit* (revenue minus cost) cannot be positive. This is an economically natural requirement, for if an activity generated positive profits producers would want to operate it at arbitrarily high levels and this would clearly not be feasible. Duality then says that prices are such that the price vector $\mathbf{y}^T$ maximizes the value $\mathbf{y}^T\mathbf{b}$ of the right-hand side activity $\mathbf{b}$ subject to the condition of no positive profits.

Some Properties of the Simplex Method

Since its creation by Dantzig in 1947, there has been a huge amount of literature on variations and extensions of the simplex method in many directions, several devised by Dantzig himself. We will mention here only one of these, revolving around the question of the number of pivot steps required to solve an LP in the worst case. Based on a vast amount of empirical experience, it seemed that an $m \times n$ program was typically solved in roughly $3m/2$ pivots. However, in 1972, Klee and Minty [6] gave an example of an $m \times 2m$ standard problem that using the Dantzig choice rule required $2^m - 1$ pivots. To appreciate the example, first consider the matrix A consisting of the m unit vectors $\mathbf{e}^i$ and m vectors $\mathbf{a}^j = \mathbf{e}^j$ where the right-hand side is $\mathbf{e}$, the vector all of whose entries are one. By inspection, (see for example the tableau below) the set of feasible solutions $\mathbb{X}$ decomposes into the direct product of m unit intervals, thus, it is a unit m-cube.

$$\begin{array}{ccc} \{\mathbf{e}^i\} & \{\mathbf{a}^j\} & \mathbf{b} \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It is easy to see that every feasible basis, thus every vertex of the cube, must contain either $\mathbf{a}^i$ or $\mathbf{e}^i$, but not both, for all i. This property will be preserved if the $\mathbf{a}^i$ are slightly perturbed. By a clever choice of this perturbation and a suitable objective function, the authors show that the Dantzig pivot rule causes the algorithm to visit all of the

2^m vertices following a well known Hamiltonian path on the m-cube.

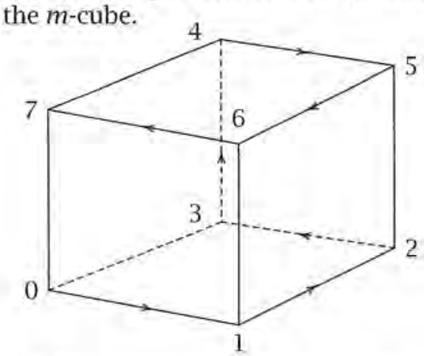

Figure 1. A Hamiltonian path on a 3-cube.

This means, for example, that the vector $\mathbf{a}^1$ enters the basis on the first pivot, stays in the basis for one pivot, then exits and stays out for one pivot, then reenters and stays, then exits and stays out, etc. throughout the computation.

So we know that in the worst case the simplex method may require an exponential number of pivots, although, as mentioned earlier, no naturally occurring problem has ever exhibited such behavior. There are also results on the expected number of pivots of a "random" LP.

Since the Klee-Minty example shows that it is possible for the simplex algorithm to behave badly, it is natural to ask whether there may be some other pivoting algorithms that are guaranteed to find an optimal solution in some number of pivots bounded, say, by a polynomial in n. Let us first consider a lower bound on the number of pivots. If the initial feasible basis is $A = \{\mathbf{a}^1, \dots, \mathbf{a}^m\}$ and the optimal basis is a disjoint set $B = \{\mathbf{b}^1, .., \mathbf{b}^m\}$ then clearly it will require at least m pivots to get from A to B. The very surprising fact is that if the set $\mathbb{X}$ of feasible solutions is bounded, then in all known examples it is possible to go from any feasible basis to any other in at most m pivots. Recall that the set $\mathbb{X}$ is an m-dimensional convex polytope whose vertices are the feasible bases. The above observation is equivalent to the statement that any two vertices of this polytope are connected by an edge path of at most m edges. The conjecture originally posed by Warren Hirsch has been proved by Klee and Walkup through dimension 5 but remains unresolved in higher dimensions.

For Dantzig this "Hirsch conjecture" was especially intriguing since a constructive proof of the conjecture would have meant that there might be an algorithm that solves linear programs in at most m pivots. However, even if such an algorithm existed, it might be that the amount of calculation involved in selecting the sequence of pivots would make it far less computationally efficient than the simplex method. Indeed this remarkable algorithm and its many refinements remains to this day the method of choice for solving most linear programming problems.

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WHAT IS...

a Minimal Model?

János Kollár

Roughly speaking, a compact complex manifold M is a minimal model if the underlying space M is the "best match" to the meromorphic function theory of M. To illustrate what a "best match" is, consider a parallel example from the holomorphic function theory of $\mathbb{C}^2$.

By Hartogs' theorem, every holomorphic function on the spherical shell $r_1^2 < |x|^2 + |y|^2 < r_2^2$ extends to the ball $|x|^2 + |y|^2 < r_2^2$. Thus it does not make much sense to study holomorphic function theory on 2-dimensional spherical shells. By contrast, an open ball turns out to be an optimal domain for function theory.

The precise notion of optimal domain leads to *Stein manifolds*. These are complex manifolds U satisfying the following two properties:

Point separation: For any two points $p \neq q \in U$ there is a holomorphic function f on U such that $f(p) \neq f(q)$.

Maximality of domain: Given any topological space $T \supset U$ containing U as an open subset, for any boundary point $r \in \partial U$ there is a holomorphic function f on U such that $\lim_{p \to r} f(p)$ does not exist.

Minimal models arise when we consider analogous questions for *compact* complex manifolds. The maximum principle implies that on a compact complex manifold every holomorphic function is constant; thus the theory of global holomorphic functions is not interesting. On the other hand, a compact complex manifold may well have many interesting *meromorphic functions*, that is, functions that can locally be written as the quotient of two holomorphic functions. At a point the value of a meromorphic function f can be finite, infinite, or undefined. For instance x/y is undefined at the origin and has value ∞ at the points $(x, 0)$ for $x \neq 0$. The set of points where f is undefined has (complex) codimension 2 (or, very rarely, is empty). This makes it hard to control what happens in codimensions ≥ 2. The guiding principle in dealing with meromorphic functions is: take care of codimension 1 and hope that the higher codimensions do not cause extra problems.

Meromorphic functions on M form a field $\mathbb{C}(M)$, called the *function field* of M. So, following the example of Stein domains, we ask: How tight is the connection between M and $\mathbb{C}(M)$?

In dimension 1, that is, when M is a compact Riemann surface, the correspondence is perfect: M and $\mathbb{C}(M)$ determine each other.

The situation is more complicated in higher dimensions, so let us start with the first condition (with some attention to undefined values).

Point separation: For any two points $p \neq q \in M$ and finite subset $R \subset M$, there is a meromorphic function f on M such that $f(p) \neq f(q)$ and f is defined at all points of R.

By a combination of works of Chevalley, Chow, and Kleiman, such an M is algebraic. That is, there is an embedding of M into some complex projective space $\mathbb{CP}^N$ whose image is defined by polynomial equations and every meromorphic function on M is rational, that is, globally a quotient of two polynomials.

In the algebraic case, the relationship between M and $\mathbb{C}(M)$ is pretty strong. Assume that we have $M_1 \subset \mathbb{CP}^r$ with coordinates $(x_0 : \cdots : x_r)$, $M_2 \subset \mathbb{CP}^s$ with coordinates $(y_0 : \cdots : y_s)$, and an isomorphism $\psi : \mathbb{C}(M_1) \cong \mathbb{C}(M_2)$. Then there are rational functions $\phi_0, \ldots, \phi_r$ on M_2 such that $(y_0 :$

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$\cdots : y_s) \mapsto (\phi_0 : \cdots : \phi_r)$ defines a rational map $\Phi : M_2 \dashrightarrow M_1$ and ψ is induced by pulling back functions by Φ. Similarly, ψ^{-1} leads to the inverse Φ^{-1}. Such an invertible rational map is called *birational*.

A map with an inverse is usually an isomorphism, but this fails here, since Φ and Φ^{-1} are not everywhere defined. To get a typical example, let $\mathbf{Q}^2 \subset \mathbb{CP}^3$ be the quadric surface given by the equation $x^2+y^2+z^2=t^2$. (You can think of it as the complexified sphere.) Let $\pi : (x:y:z:t) \dashrightarrow (x:y:t-z)$ be the projection from the north pole $(0:0:1:1)$ to the equatorial plane $(z=0)$. Its inverse π^{-1} is given by

$$(x:y:t) \dashrightarrow (2xt : 2yt : x^2+y^2-t^2 : x^2+y^2+t^2).$$

These maps show that the meromorphic function theory of $\mathbf{Q}^2$ is the same as that of $\mathbb{CP}^2$. On the other hand, $\mathbf{Q}^2$ and $\mathbb{CP}^2$ are quite different as manifolds. For instance, π contracts the lines $(s : \pm\sqrt{-1}s : 1 : 1)$ to the points $(1 : \pm\sqrt{-1} : 0)$, and π^{-1} contracts the line at infinity $(t=0)$ to the point $(0:0:1:1)$. Neither $\mathbf{Q}^2$ nor $\mathbb{CP}^2$ is simpler than the other.

We say that a birational map $\Phi : M_1 \dashrightarrow M_2$ *contracts* a codimension 1 subset $D \subset M_1$ if $\Phi(D) \subset M_2$ has codimension ≥ 2. Φ is called a *contraction* if it contracts some codimension 1 subset but Φ^{-1} does not contract any.

The simplest examples of contractions are blowdowns. Let $Z \subset M$ be a submanifold of codimension ≥ 2, and let E_Z be the set of all normal directions to Z. For each $p \in Z$ the normal directions at p form a $\mathbb{CP}^{d-1}$ where $d = \dim M - \dim Z$. Thus the projection $\pi_Z : E_Z \to Z$ is a $\mathbb{CP}^{d-1}$-bundle, and so $\dim E_Z = \dim M - 1$. It turns out that $B_Z M := E_Z \cup (M \setminus Z)$ is naturally a compact complex manifold, called the *blow-up* of $Z \subset M$. The projection on E_Z and the identity on $M \setminus Z$ glue together to a birational map $\pi : B_Z M \to M$, called a *blow-down*. It collapses the $\mathbb{CP}^{d-1}$-bundle E_Z to Z. Note that $\pi^{-1}(Z) = E_Z$ has codimension 1, thus π is a contraction. As a first approximation, one can think of any contraction as a composite of blow-downs.

By blowing up repeatedly, starting with any M we can create more and more complicated manifolds with the same function field. Thus here a maximal domain does not exist, but one can look for a minimal one.

Minimality of domain: M is a *minimal model* if every birational map $\Phi : M_1 \dashrightarrow M$ is either a contraction or an isomorphism outside codimension ≥ 2 subsets. We also say that M is a minimal model of any such M_1.

In the first case M is simpler than M_1, at least in codimension 1. In the second case M is about as complicated as M_1.

The map $\pi : \mathbf{Q}^2 \dashrightarrow \mathbb{CP}^2$ shows that neither $\mathbf{Q}^2$ nor $\mathbb{CP}^2$ is a minimal model. In fact, no manifold birational to $\mathbb{CP}^1 \times Y$ has a minimal model. More generally, we exclude all n-folds X that are *uniruled*, that is, for which there is a meromorphic map $\mathbb{CP}^1 \times Y \dashrightarrow X$ with dense image for some $(n-1)$-fold Y. We have other methods to study these; see [2].

Castelnuovo and Enriques proved in 1901 that every smooth, compact, complex algebraic surface S that is not uniruled has a unique minimal model $S^{\min}$. In the past twenty-five years a lot of effort in algebraic geometry has gone into generalizing this result to higher dimensions.

(Minimal model conjecture of Mori-Reid). *Let M be a compact, smooth algebraic n-fold that is not uniruled. Then*

(1) *M has a minimal model $M^{\min}$, and*
(2) *$M^{\min}$ has a Kähler metric whose Ricci curvature is ≤ 0.*

Two caveats are in order. First, we must allow $M^{\min}$ to have certain mild (so called *terminal*) singularities. Algebraic geometers learned to live with these singularities, though their differential geometry is less understood. Second, minimal models are not unique, but birational maps between two minimal models are isomorphisms outside codimension ≥ 2 subsets. In dimension 2 such a map is an isomorphism, but in higher dimensions it can be a *flip* or a *flop* [1].

In dimension 3 the first part is a theorem, due mainly to Mori. For a general introduction, see [3]. In higher dimensions, for manifolds of "general type", the first part is settled by recent work of Hacon, McKernan, and Siu, and the second part by Eyssidieux, Guedj, and Zeriahi, generalizing the work of Aubin and Yau on Monge-Ampère equations.

In applications, part (2) of the conjecture is especially useful. Besides having the simplest codimension 1 geometry, we have very strong global differential geometric properties as well.

Algebraic geometers usually consider a weaker variant, using the *canonical class* K_M, which is defined as the negative of the first Chern class $c_1(M) \in H^2(M, \mathbb{Q})$.

(2′) *The canonical class of $M^{\min}$ is ≥ 0; that is, it has nonnegative cap product with any algebraic curve $C \subset M$. Equivalently, the integral of the Ricci curvature of M on C is nonpositive.*

This is known in dimension 3 and for manifolds of general type in any dimension.

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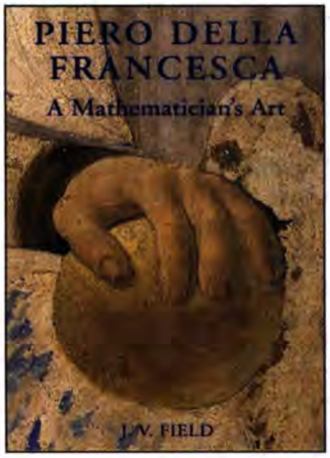
Book Review



Piero della Francesca. A Mathematician's Art

Reviewed by Michele Emmer

Piero della Francesca. A Mathematician's Art
J. V. Field
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256 pages, US$55
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Piero della Francesca (c. 1420–1492) was the painter who "set forth the mathematical principles of perspective in fairly complete form... Piero was the painter-mathematician and the scientific artist *par excellence*, and his contemporaries so regarded him. He was also the best geometer of his time." So writes Morris Kline in his monumental work *Mathematical Thought from Ancient to Modern Times* [11]. Clearly, the title of J. V. Field's book, *Piero della Francesca. A Mathematician's Art*, is amply justified.

The Rediscovery of Piero della Francesca

> Giotto and Paolo Uccello, Signorelli and Piero della Francesca practically did not exist in the first decade of this century. They returned to public attention due to the critical taste characteristic of Italian research, diametrically opposed to solid German philology. Would Lionello Venturi's book (1926) [18] have come to life if the ex-futurist Carlo Carrá had not written of Giotto and Paolo Uccello ten years earlier in *La Voce* (1916)? And could Roberto Longhi [14] have discerned the "problem" of Piero without his curiosity about the avant-garde (I am thinking of his great text on Boccioni's sculptures and on Severini's *pailettes*)? And could the academic historians have ever felt it legitimate to re-engage the 15th century (the pre-Raphaelists, as they were called at the time) without the ongoing debate fostered by the painters' discoveries and the contributions of the *letterati* (Bontempelli, for example)? The reason is that these artists of ours were clearly neither nostalgic nor interested by antiquity: simply, antiquity did not exist and they were perhaps constructing it themselves, searching the territory of a new and unpredictable avant-garde. De Chirico did not look for museographic painting: he himself underlined its value for individuals in the 1920s. No one would have been interested in Luca Pacioli, had it not been for the research of another *pentito* futurist painter, Gino Severini. All in all, Art History itself would have been different without the contribution of these painters and technicians of image.

So wrote Maurizio Fagiolo dell'Arco in the catalogue of the exhibition *Piero della Francesca e il Novecento (Piero della Francesca and the 20th*

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Century) [4]. The 500th anniversary of the artist's death (1991–92) was the occasion of several important conferences and exhibitions. Four volumes worth mentioning were published: *Piero and the 20th Century*, the catalogue of the 1991 exhibition in San Sepolcro (Piero's birthplace) [12]; *Piero and Urbino, Piero and the Renaissance Courts*, the catalogue of the 1992 Urbino exhibition [3]; Milton Glaser's catalogue *Piero della Francesca*, for the exhibition held at Arezzo in 1991 [16]; and finally *Through Piero's Eyes: Clothing and Jewelry in Piero della Francesca's Works* [1]. These constituted an important celebration of a great Italian Renaissance artist who influenced twentieth-century art, as witnessed by De Chirico's metaphysical period (1910–20), with his Ferrara and Modena cathedrals, his trains, those empty piazzas that convey a mysterious and perturbing motionlessness, an enigmatic stillness. Giorgio Morandi and so many others are also clear examples.

Dell'Arco added, in his chapter entitled "Styles of the return to order: perspective and space, light, geometry":

> Right after the [First World] war, research resumes in exactly the opposite way from the past. Painting returns to being precise, mentality to being ancient, structures meditated and color orderly. Giotto and Paolo Uccello were evoked by a master of futurism, Carlo Carrá. Frescoes became the subject of meditation, *cassoni* and *predelle* were reread, and Carrá and De Chirico point a clear finger to Milano's neoclassicism, Mastrini to Etruria (ancient Tuscany) and Oppi to Giovanni Bellini. All of these are clear sources. The most secret, albeit the most fertile, will continue to be the meditation on Piero della Francesca, the name which Roberto Longhi brought to the fore starting in 1914.[1] Many artists welcomed Longhi's book on Piero with memorable texts. In fact, when the topics of proportion, number, color-light, perspective, regular bodies are referred to, Piero della Francesca is always implicit. [4]

I was very happy to have received J. V. Field's book on Piero della Francesca for review, because I too have been influenced by reading Longhi's essays, by de Chirico's metaphysical works, and by the paintings of Morandi and Carrá. Piero has long been one of my favorite masters. I have also been so lucky as to own a house in Senigallia, in the Marche region in the center of Italy, a few hundred meters away from the church where Piero's famous *Madonna di Senigallia* was housed until the beginning of the Second World War, when it was moved to the Palazzo Ducale in Urbino. It has never been returned. But Urbino is quite close to Senigallia, and there, in addition to the Madonna, one can see *The Flagellation of Christ*. I have therefore seen these works by Piero dozens and dozens of times. In my own work, I have cited Longhi on several occasions. As Dell'Arco says, that is where, for the most part, everything started.

Longhi was born in 1890. In 1911 he graduated from the University of Turin with a thesis on Caravaggio. His interest was not only in ancient art but also in the avant-garde, specifically futurism and Boccioni in particular. In 1914 he published a long article on "Piero dei Franceschi [i.e., della Francesca] and the Development of Venetian Painting" in the journal[2] published by Adolfo Venturi, who had been his professor in *Scuola di perfezionamento* (school for advanced studies). In 1927 he revised and republished this article in the journal *Valori Plastici* under the title "Piero della Francesca" [14]. Longhi proposed a look with fresh eyes at the experiments in painting carried out by part of the artistic culture of Tuscany at the beginning of the fifteenth century, "when some artists, too much in love with space to come back to a superficial *spazialitá* (sense of space), too much in love with color to rely on a *chiaroscuro* intended to create an illusion of space, understood that there was only one way to express in a picture shape and color at the same time: perspective." Longhi also wrote: "By constructing, through an effort at synthesis, shapes according to their countable and measurable surfaces, [perspective] succeeded in presenting them all as a projection upon a plane, ready to be clothed in calm, broad areas of color" [14, p. 11]. Giacomo Agosti added, in his essay[3] "From Piero dei Franceschi to Piero della Francesca": "This interpretation of perspective as the art of synthesizing 'shape and color at the same time' allowed him (Longhi) to begin by identifying in Paolo Uccello and Domenico Veneziano the Florentine models closest to Piero della Francesca; and this interpretation brought him to recognize the most evident manifestations of the visual influence of Piero in the works of Antonello da Messina, 'The most *disperato prospettico* (enthusiastic expert in perspective) of the human figure' and those of Giovanni Bellini,

[1] *R. Longhi, Piero dei Franceschi e lo sviluppo della pittura veneziana, L'Arte, XVII (1914), pp. 198–221 and 241–256. Reprinted in [14].*

[2] *L'Arte; see footnote 1.*

[3] *G. Agosti, Da Piero dei Franceschi a Piero della Francesca (qualche avvertenza per la lettura di due saggi longhiani), in [12], p. 199.*

'who, without rejecting the problem of form, considers it as taking to higher levels the problem of color."' Longhi wrote: "Here is what constitutes the unity-distinction of Antonello and Giovanni: their synthesis a priori is Piero" [14, p. 39]. And regarding *The Flagellation*, he added "the perfect union between architecture and painting that emerges should be understood as a mysterious combination of mathematics and painting". Piero is a great painter; Piero is a great mathematician.

Piero as a Painter, Piero as a Mathematician

In the introduction to the book under review, Field writes (p. 1): "Piero della Francesca is now best remembered as a painter. He has a secure position in the history of Italian Renaissance Art. However in his own lifetime, and for some time thereafter, he was also known as a mathematician. In his detailed scholarly monograph on Piero's painting, Eugenio Battisti went so far as to say that there should be a further volume to consider Piero's activity as a mathematician." Field concludes that since the mathematician and the painter were one person, it seems clear that the mathematics and the painting should be taken together, to see what one may have to tell us about the other. "As the title of the present study indicates, I am inclined to think Piero's mathematics does have a recognizable relationship with his painting. Of course there is no avoiding the recognition that Piero is likely to remain a more significant figure in the history of art than in the history of science" (p. 2). It is important to underscore, and Field does so right away, that there is an inevitable shift in style between the historical study of art and that of mathematics. In mathematics the course of history can reasonably be considered in terms of progress. In art this is clearly not the case. Piero's art was forgotten for many centuries because of a change in aesthetic taste.

Field begins by asserting that "since Piero was known to have written on the mathematics of perspective, art historians have generally [following Longhi, I may add—M. E.] been ready to describe his pictures as in some sense 'mathematical', though the quality most often remarked upon is a 'stillness' that is rather hard to define in any precise way. The assertion that Piero's pictures are 'mathematical' is usually so vague that it is understandable that some art historians have preferred largely to ignore it" (pp. 3–4). Even if the greater part of Piero's mathematics is rather elementary, I am not shocked at all as a mathematician by the loose usage of the word "mathematics". We are in the field of art history, where opinions and preferences (such as, for example, the rediscovery of Piero) are important. Field adds that "there has also been a most unfortunate fashion for drawing lines over Piero's pictures with the purpose of exposing their alleged underlying geometrical structure." More frequently than not, these geometric constructions are carried out using reproductions that are evidently quite different from the original, at least in size. Field comments, and one cannot disagree, that "trying to find perspective schemes seems to me like trying to extract the sunbeams from cucumbers." The book is full of imaginative comments, mostly in the notes. I will give a few examples. At the risk of seeming unscholarly, Field forgoes correcting errors in the "apparently relevant literature rather than stir my own and my readers' unhappy memories of maths homework that came back covered with a teacher's comments in red" (p. 5).

In order to have a deeper understanding of Piero's mathematics, the author refers to Luca Pacioli (1445–1517), the mathematician and author of the famous book *De divina proportione* (On the Divine Proportion) [15], with drawings by Leonardo da Vinci, where of course *divina proportione* refers to the so-called Golden Section. The most recent research on his life seems to show that Pacioli was not a pupil of Piero, although both were born in Borgo San Sepolcro. In any case, what is sure is that Pacioli knew the mathematical works of Piero, at least after Piero's death in 1492. They were very likely acquainted: Field argues that in the *Madonna dell'Ovo*, now at the Brera in Milan, there is a portrait of Pacioli done by Piero.

In fact, in Piero's mathematics there is no evidence of an interest in *divina proportione*. So one should first look elsewhere in seeking to identify traces of mathematics in Piero's pictures: "The present study will accordingly look much more carefully at Piero's mathematics than previous studies have done. By doing so it will be possible to offer an explanation of that famous 'stillness'. It is not possible to give mathematical proof, but from my failure to find serious deviations from mathematically correct perspective I strongly suspect that the reason why Piero's pictures look mathematically correct is because they are indeed correct. Thus everything shown is represented as seen from the single ideal viewpoint, though, thanks to the tolerance of the eye, it will of course look convincing even when seen from points at a considerable distance from the ideal viewpoint" (p. 7). The main thesis of Field is that Piero's very considerable mathematical talent does seem to have contributed to his personality as a painter.

Field's Book

The book starts with the background: the training of an artist at the time of Piero, including the study of the abacus. Probably Piero was sent outside of Borgo San Sepolcro to study in an abacus school. "Since Piero never actually taught mathematics, his own interest in the subject has something of the same purely intellectual motivation in it. Piero must have enjoyed mathematics." Piero himself wrote a *Trattato d'abaco* (a treatise on the abacus).

The second chapter is a history of perspective, starting naturally with Filippo Brunelleschi (1377-1446) and Alberti's treatise *De pictura* (1413-1472). Field reconstructs through drawings the perspective constructions of Alberti, later called *costruzione legittima* (correct construction). Investigating the work of Donatello (1386-1466) and Masaccio (1401-1428), Field remarks that "in any picture it is architectural elements that are most likely to provide sets of orthogonals and transversals" and points to Masaccio's fresco of the Trinity in the church of Santa Maria Novella in Florence as providing "a good example in which to search for indications of the method by which the perspective scheme was constructed." Nevertheless, "the process of extending lines is in general to be avoided and its results treated with caution." Finally, as was known to Masaccio, "mathematical correctness was not of the essence in obtaining the required illusion of the third dimension" (p. 50).

The third chapter describes the early life of Piero—his family, his education, and his training as a painter—and refers to the problem of dating Piero's works. The book is full of illustrations, many of them in color. Most of the reproductions are of good quality, with the exception of a few that are shown so small that details are impossible to read. On page 90 Field writes: "Certain elements in his style apparently owe a debt to the occasionally fussy prettiness of detailing found in, say, Pisanello (1395-1455c.) and Gentile da Fabriano (c.1370-1427)." Fabriano is a small town in the region of Umbria in the center of Italy, very famous for its ancient paper industry. Just two weeks before writing this review I visited in Fabriano an exceptional exhibition of the works of Gentile and other artists of the same period, including Masaccio and Donatello. I agree with the opinion [13] of the curator of the exhibition that Gentile, who spent part of his life in Venice, was the binding ring between the painting of Florence, Umbria, and Venice. He was one of the most important and brilliant painters that Italian civilization has ever produced, not just a painter of "occasionally fussy prettiness of detailing".

Chapter 4 bears the title: "The Sense of Space". "In Piero della Francesca's time, space did not exist" (p. 95) is how the chapter begins. "In saying that his paintings convey a sense of space, we mean that each picture conveys a sense of its own pictorial space, that is a particular space. But Piero himself would not have thought in these terms. In his day, space was not considered to be an entity with a separate existence. Space was defined as extension, and measured by body." As an example Field recalls one of Piero's most famous masterpieces, *The Baptism of Christ* (National Gallery, London). She carefully examines the proportions of the painting, measuring from photographs by means of an ordinary ruler, without any claim to precision and taking no notice of the scale of the actual picture. This is not the only occurrence of this practice in the book. But Field reminds readers that the *Baptism* is often and entirely reasonably cited as an example of Piero's beautiful rendering of light. Piero's abacus treatise, quite unusually, treated three-dimensional geometry, as did his famous *Libellus de quinque corporibus regularibus,* incorporated by Pacioli in *De divina proportione* (for which incorporation Vasari accused Pacioli of plagiarism). Piero's mathematical thinking about three-dimensional geometry surely influenced his way of rendering bodies in space. Nevertheless, this need for perspective is clearly not manifest in all of Piero's works. In many of them there is "no perspective": figures stand out against the background landscapes; color and light are used to make bodies seem corporeal. This is the case in the absolute masterpiece I have already spoken about, the *Madonna di Senigallia.* Regarding this work, Field writes (p. 115): "For reasons that may be to do with its delicate color and exquisite degree of finish, reproductions of this picture manage to look what passes for reasonably like it without looking sufficiently like it to be beautiful." I can confirm, as I have seen the painting dozens of times, that the original is always better than any reproductions, given that the picture's texture and its purity are difficult to get across through a book or a postcard. It becomes necessary to immerse oneself in the painting and to feel part of it in order to capture the sense of space that Piero's color makes the viewer perceive. That magical stillness. Little does it matter that "no great subtlety is required to note that the composition presents us with figures against a background from which they are essentially disjoint."

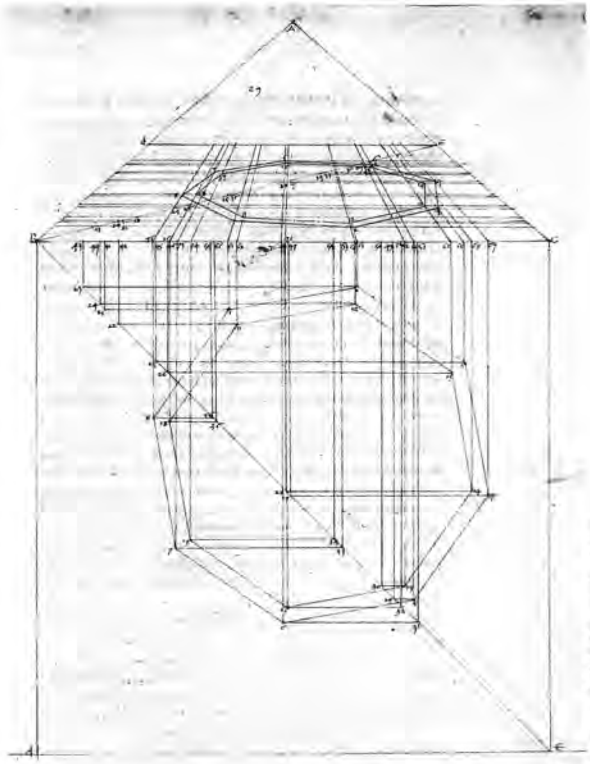
As she examines Piero's geometrical constructions, Field notes his paintings' symmetrical properties. She adds in a note (p. 126, footnote 38) that I am sure will be of interest to mathematicians, "The experts on the mathematics of symmetry in the real world tend to be crystallographers. A crystallographer and a specialist in the relevant branch of mathematics have now written an excellently clear and well-illustrated book on the subject: I. Hargittai and M. Hargittai, *Symmetry: A unifying concept* [8]." I must say, knowing the Hargittais quite well, that István is a chemist and Magdolna is not a mathematician. The book [8] is an elementary one, meant for students. I suggest that the reader look at Hargittai's other books [9] and [10] on symmetry and at what is considered the bible in this area, B. Grünbaum and G. C. Shepard, *Tilings and Patterns* [7]. Being the author of several books and films on M. C. Escher, I do not wish to comment on the last sentence of Field's note: "In addition to its other virtues this book shows a highly commendable tendency to avoid the work of M. C. Escher." In the other books of Hargittai, however, there are plenty of works by Escher.

In Chapter 5 Field goes into the details of *De prospectiva pingendi* [2], Piero's treatise on perspective. "Perspective was a form of study in which it was acknowledged as legitimate to use mathematics in the pursuit of natural philosophy. The use of mathematical methods to give the illusion of depth in pictures clearly belonged to the mathematical end of the subject" (p. 129). The starting point of Piero is the definition of the entities: "A point is that which has no parts, accordingly the geometers say it is imagined; the line they say has length without width" (p. 134). Field carefully examines the results on geometry enunciated and proven by Piero. She reconstructs in detail the diagrams, reproducing whenever possible the artist's original drawings. His drawings become more and more complicated; for example, he shows the construction of the perspective image of an octagonal building. The author states (p. 155) that "the mathematics concerned need not to go beyond the methods we find in Piero's *Trattato d'abaco*." Why not, if it can help us understand?

Piero defends the knowledge of perspective that is necessary for painting: "Therefore it seems to me that I should show how much this science is necessary to painting. I say that perspective literally means things seen at distance, represented as enclosed within given limits [that is, on the picture plane] and in proportion, according to the quantity of the distances, without which nothing can be degraded correctly [Piero calls degradations the deformations performed on the figures]... I say it is necessary [to employ] perspective, which distinguishes all quantities proportionately, as a true science, demonstrating the degradation and magnification of all quantities by means of lines" (p. 163).

The Flagellation of Christ

When talking about the true science of perspective, one cannot overlook *The Flagellation of Christ*. Field writes (p. 174): "It would be perhaps more accurate to describe Piero's *Flagellation* not as an example of correct perspective but as *the* example. Despite longstanding disputes about the overall significance of the picture, and in particular the identification of the three figures in the right foreground, the *Flagellation* is almost invariably chosen as an illustration to elementary lectures on perspective and has had a long career as the subject of reconstructions." An essay exploring new hypotheses on the meaning of this painting,



Courtesy of Yale University Press.

Perspective drawing of an octagon building plan. The horizontal-vertical-diagonal-horizontal line segments from P to P' implement the geometric construction of the perspective transformation. From the Parma MS of *De Prospectiva Pingendi.*

and in particular regarding the identity and the significance of the characters, was recently published in Italy by Ronchey: *L'enigma di Piero* [17] (following the big success of books like *The Da Vinci Code*, the words "code" and "enigma" have become very appreciated by publishers). Pilate, the character on the left with purple boots, would have been John VIII Palaeologos, the next to last emperor of Byzantium. The character with his back turned to the viewer would have been the Turkish sultan. Jesus represents Western Christianity flagellated by the Turks. One of the characters to the right would represent Johannes Bessarion, who later became a cardinal; the gentleman in brocade is Nicolò III d'Este of Ferrara, the site of the 1439 Council, in which John VIII participated; the young man between them is Thomas Palaeologos, the last legitimate heir of the Eastern Empire. Thus the scene, painted in 1459, would have referred to events that took place in 1439. Field's volume also reproduces an image of a three-dimensional reconstruction of Piero's painting carried out by Antonio Criminisi. Field comments (p. 177), "Apart from perspective,

Courtesy of Yale University Press.

***The Montefeltro Altarpiece.* (251 x 172.5 cm) It has been conjectured that the figure second from the right is Luca Pacioli. He was the author of the book *Summa de Arithmetica, Geometrica, Proportioni et Proportionialitate* (1478), best known for its account of double entry book-keeping, and of *De Divina Proportione* (1509), in which are found images of regular polyhedra attributed to Leonardo.**

there is another good reason for using *The Flagellation* as an illustration: its wonderful colour. Even by the standards of the fifteenth century, this is a very pretty picture. It seems that one was meant to go close to the picture in order to admire such details in its finish." This I can confirm. Looking very closely at this picture always stirs strong emotions.

The mathematics used in *De prospectiva pingendi* has been analyzed using mathematical tools that are clearly much more modern than those used by Piero. In a short paper[4] published with the critical edition of *De prospectiva pingendi* edited by Nicço-Fasola, the mathematician Ghione writes "To consider Piero's treatise as a manual, however complete, in which for the first time the rules of drawing with perspective are given through systematically correlated graphic schemes, would be to underestimate not only Piero della Francesca's ideas, which are clearly expressed, but also mathematical ideas that, after following a long and winding road, gave life to the first streams that would end up in the 19th century's great flow of modern Projective Geometry." Field is in agreement with this view. Three examples from Piero's treatises are considered in another essay [6] recently published in Italy, "The mathematics of Piero della Francesca": the volume of a pavilion vault, the surface of a cross vault, and the solution of algebraic equations of degree greater than two.[5] The authors conclude that Piero "is a specialist in both painting and of mathematics. We believe that it is more true to historic reality to consider Piero as one of the very first craftsmen who became scientists, a situation that constituted one of the main engines of the scientific revolution."

In Chapter 6 ("Optics and Illusion") Field examines another important masterpiece of Piero's, *La Madonna dell'Ovo*, also called the *The Montefeltro Alterpiece* (Luca Pacioli probably appears among the characters on the right). Then in Chapter 7 she asks the crucial question, "But is it art?" "The present chapter considers his work as a whole in the context of the intellectual life of his time, and in particular its relation to the learned arts taught in universities." There is a long section on the history of mathematical sciences, considering in particular Nicolaus Cusanus and Johannes Regiomontanus, and the importance of the works of Piero in this context.

Towards Galileo Galilei

In the last chapter, "From Piero della Francesca to Galileo Galilei", Field addresses scientific developments that took place in the period that followed and the role played by Piero's legacy. She concludes (p. 324):

> This description of a revival of mathematics from the fifteenth to the seventeenth centuries is not a complete story in itself, and still less a complete account of the origins of the Scientific revolution. Nor is the sketch of part of a complicated story—a sketch that does no more than trace some strands in a tangled skein—intended to argue for the recognition of a direct debt owned by, say, Galileo to Piero della Francesca. Though Piero's reference to the "force of lines" in his introduction to the

[4] *F. Ghione, Breve introduzione sul contenuto matematico del De prospectiva pingendi di Piero della Francesca, in [2], pp. xxix–xlii.*

[5] *See also M. D. Davis,* Piero della Francesca's Mathematical Treatises, *Longo editore, Ravenna 1977.*

> third book of *De prospectiva pingendi* does indeed seem like a pre-echo of Galileo's later advocacy for mathematics, Piero proposed this "force" as establishing the truth of artificial perspective, that is he was concerned with a matter in which the use of mathematics was entirely accepted. Galileo was extending the use of mathematical methods into areas where they had not been used by his predecessors and were not generally accepted by contemporaries.
>
> My purpose is not to attempt to establish a direct connection between such later work and Piero, but rather to show what developing story Piero's work as a whole, both his paintings and his writings, taken together as a contribution to the cultural history of his time, should be seen to belong to. Piero is a good example of the learned craftsman, and the activities of his intellectual kin were to make notable contributions not only to the development of mathematics but also to the emergence of a mathematical natural philosophy in the two centuries following Piero's death.

I believe that Field's long essay has reached the objective it set for itself, and I believe that it can make mathematicians become interested in Piero's scientific works. But above all, it attests to the strong aesthetic and emotional impact of Piero the artist. I would like to conclude by inviting readers to go and see Piero's original works, to go to Urbino to admire at least the *The Flagellation of Christ* and *La Madonna di Senigallia*. No reproduction (in Piero's case this is absolutely true) can do justice to the physical beauty of the actual works and to the unexplainable "stillness".

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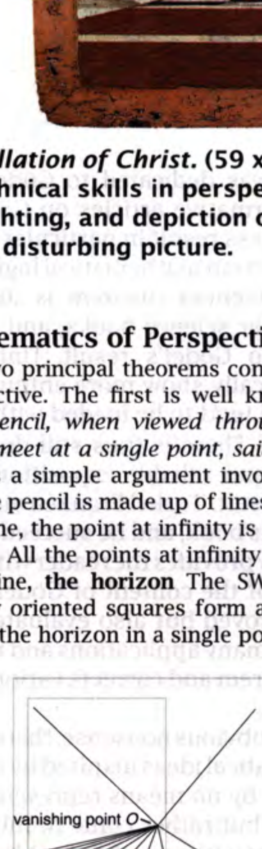
[18] L. VENTURI, *Il gusto dei primitivi* (1926), reprinted Einaudi, Torino, 1981.

Courtesy of Yale University Press.

The Flagellation of Christ. **(59 x 81.5 cm) It was famous in the Renaissance as a demonstration of Piero's technical skills in perspective. Unusual choice of low view point, striking colors, strange interior lighting, and depiction of contemporary figures—all contribute to what is even now a somewhat disturbing picture.**

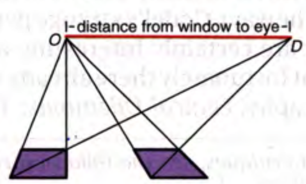
The Mathematics of Perspective

There are two principal theorems concerned with practical perspective. The first is well known: *All lines in a parallel pencil, when viewed through that window, are seen to meet at a single point, said to be at infinity.* The proof is a simple argument involving intersecting planes. If the pencil is made up of lines perpendicular to the view plane, the point at infinity is called the **vanishing point** *O*. All the points at infinity make up a single horizontal line, **the horizon** The SW-NE diagonals of orthogonally oriented squares form a particular pencil intersecting the horizon in a single point *D*.

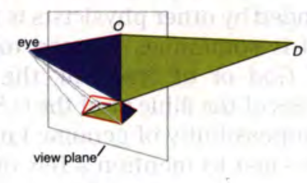


The second is not so well known, and its proof is not so straightforward.

The point D is at the same distance from the vanishing point O as the eye is from the view plane.



This theorem is equivalent to the **distance point construction**, apparently first described—if briefly—by Piero della Francesca. A closely related construction was described by Alberti much earlier in the fifteenth century. This older and clumsier method was presumably that discovered by Brunelleschi, who was as far as we know the first to apply strict perspective in drawing. As for the proof of the Theorem, the following diagram shows that it can be seen easily by picking a particular square, and then rotating part of the diagram out of the view plane.



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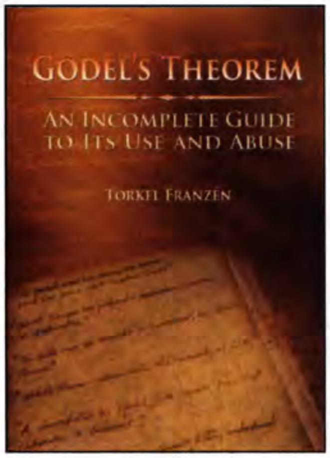
I'd like to thank Tony Phillips for his invaluable and unflagging assistance in preparing this note.

—Bill Casselman



Gödel's Theorem: An Incomplete Guide to Its Use and Abuse

Reviewed by Panu Raatikainen

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse
Torkel Franzén
A K Peters, Wellesley, MA
$24.95, paperback, 2005
182 pages, ISBN 1-56881-238-8

Apparently no mathematical theorem has aroused as much interest outside mathematics as Kurt Gödel's celebrated incompleteness result published in 1931. It is invoked not only by mathematicians, logicians, and philosophers but also by physicists, theologians, literary critics, architects, and others. Some eminent physicists have interpreted it as showing that "the theory of everything" demanded by other physicists is impossible to achieve. It is sometimes claimed to prove the existence of God or of free will, the necessary incompleteness of the Bible or of the U.S. Constitution, or the impossibility of genuine knowledge in mathematics—just to mention a few of the many alleged applications (see also [9]).

Gödel is unquestionably among the greatest mathematicians of our times, and he made many important contributions to mathematical logic and other fields. But it is undoubtedly his incompleteness result that made his reputation. In the year 2006 the one-hundredth anniversary of the birth of Gödel was celebrated all over the world with various conferences. The April 2006 issue of the *Notices* was dedicated to Gödel and contains many informative articles on Gödel and on the incompleteness result in particular by some of the leading experts in mathematical logic.

The incompleteness theorem is discussed in countless popular science books, and several are even devoted to Gödel's result. Unfortunately, such books typically show more enthusiasm than competence and tend to be loaded with inadequacies and errors.[1] There is thus still demand for a knowledgeable and reliable exposition of the incompleteness result. Torkel Franzén aims to fulfill this need with his book, and he succeeds outstandingly. He not only provides the reader with adequate understanding of the content of Gödel's theorem and how it is proved but also evaluates critically and thoroughly many applications and misapplications of the theorem and corrects various common misconceptions.

In addition to obvious nonsense, there are among the nonmathematical ideas inspired by Gödel's theorem many that by no means represent postmodernist excesses, but rather come to mind naturally to many people with very different backgrounds when they think about the theorem. It is especially such naturally occurring misunderstandings that Franzén intends to correct.

The book does not pay much attention to Gödel's life and other scientific achievements—only three pages are devoted to them (bits of history are also given along the way). Gödel's strange person and his eventful life are certainly interesting and deserve attention, but fortunately there already exists an excellent biography, *Logical Dilemmas: The Life and*

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[1]*For some apt critiques, see the following reviews in the* Notices *: [1], [3], [5].*

Work of Kurt Gödel by John W. Dawson ([2]; see [1] for a review).

The Incompleteness Theorems

In order to understand Gödel's theorem, one must first explain the key concepts occurring in it: "formal system", "consistency", and "completeness". Very roughly, a *formal system* is a system of axioms equipped with rules of reasoning which allow one to generate new theorems. The set of axioms must be finite or at least decidable; i.e., there must be an algorithm that enables one to mechanically decide whether a given statement is an axiom or not (otherwise, one might stipulate, e.g., taking all true statements of arithmetic as axioms; such a theory is trivially complete but highly abstract and totally useless in practice).

A formal system is *consistent* if there is no statement for which the statement itself and its negation are both derivable in the system. Only consistent systems are interesting in this context, for it is an elementary fact of logic that in an inconsistent formal system every statement is derivable, and consequently such a system is trivially complete. And a formal system is *complete* if for every statement of the language of the system, either the statement or its negation can be derived (i.e., proved) in the system.

Gödel proved two different though related incompleteness theorems, usually called the first incompleteness theorem and the second incompleteness theorem. "Gödel's theorem" is sometimes used to refer to the conjunction of these two and sometimes to either—usually the first—separately. Accommodating an improvement due to J. Barkley Rosser in 1936, the first theorem can be stated as follows:

First incompleteness theorem. Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; there are statements of the language of F which can neither be proved nor disproved in F.

A common misunderstanding is to interpret Gödel's first theorem as showing that there are truths that cannot be proved. This is, as Franzén points out, incorrect, for the incompleteness theorem does not deal with unprovability in any absolute sense, but only unprovability in some particular axiom system. And for any statement S unprovable in a particular formal system F, there are trivially other formal systems in which S is provable. On the other hand, there is the extremely powerful standard axiom system of set theory (the so-called Zermelo-Fraenkel set theory, which is denoted as ZF, or, with the axiom of choice, ZFC), which is more than sufficient for the derivation of all ordinary mathematics. Now there are, by Gödel's theorem, arithmetical truths that are not provable even in ZFC. Proving them would thus require a formal system that incorporates methods going beyond even ZFC. There is thus a sense in which such truths are not provable using today's "ordinary" mathematical methods and axioms or cannot be proved in a way that mathematicians would today regard as unproblematic and conclusive.

Gödel's second theorem concerns the limits of consistency proofs:

Second incompleteness theorem. For any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself.

It is important to note that this result, like the first incompleteness theorem, is a theorem about formal provability (which is always relative to some formal system). It does not say anything about whether, for a particular theory T, the statement "T is consistent" can be proved in the sense of being shown to be true by a conclusive argument or by an argument acceptable by mathematicians. For many theories, this is perfectly possible.

Franzén describes in some detail but very informally the ideas of the proofs of the incompleteness theorems. Later, he also explains, again quite informally, the basic notions and results of the theory of computability, essential for proper understanding of the incompleteness results. Franzén also clarifies the relation of the incompleteness theorem to another result of Gödel which is often misleadingly called "the completeness theorem" and to the existence of so-called nonstandard models. The book ends with an appendix that gives a slightly more formal yet still easily understandable explanation of the incompleteness theorems. In all these cases, Franzén has done an admirable job. These sections provide an excellent ground for evaluating various alleged consequences of Gödel's theorem, to which we now turn.

Antimechanism, Faith, and Skepticism

There is a popular view according to which Gödel's theorem shows that the human mind cannot be any sort of computing machine but infinitely surpasses any machine. The alleged justification goes like this: For any formal system, which can be viewed as a computing machine generating theorems, Gödel's proof exhibits an unprovable sentence (often called the Gödel's sentence of the system). We humans can know the truth of this sentence, whereas the formal system or its corresponding machine cannot. There is thus—so the argument goes—something noncomputable about human thinking, perhaps even some irreducibly spiritual, nonmaterial component of the human mind. Such antimechanist conclusions have been drawn from Gödel's theorem, for example, by a philosopher, J. R. Lucas [4], and more recently by a distinguished mathematical physicist, Roger Penrose [6], [7], and these conclusions seem to enjoy some popularity. The idea is apparently quite natural and attractive, for it gets reinvented again and again.

Nevertheless, such conclusions are not justified on the basis of the incompleteness theorem. Franzén explains clearly why this is so: in general, we have no idea whether or not the Gödel sentence of an arbitrary system is true. What we can know is only that the Gödel sentence of a system is true if and only if the system is consistent, and this much is provable in the system itself. But in general we have no way of seeing whether a given system is consistent or not. Later in the book Franzén explores in some detail variants and ramifications of the Gödelian antimechanist argument and shows them all wanting.

Franzén then moves on to discuss various attempts to apply Gödel's theorem outside mathematics. It has been claimed that the incompleteness theorem demonstrates the incompleteness of the Bible, the U.S. Constitution, and Ayn Rand's philosophy of objectivism. He points out that such suggestions ignore the essential condition that the system must be capable of formalizing a certain amount of arithmetic. None of the mentioned "systems" have anything to do with arithmetic. Even worse, they are nothing like a formal system: they do not have an exactly specified formal language, a set of axioms, or rules of inference. Therefore, Gödel's theorem simply is not applicable in such contexts.

More reasonable have been attempts to apply the incompleteness theorem to physics. The hypothetical "theory of everything" (TOE) is sometimes taken to be an ideal of theoretical physics. However, such eminent physicists as Freeman Dyson and Stephen Hawking have invoked Gödel's theorem to suggest that there is no such theory of everything to be had. Now it seems more reasonable to assume that a formalization of theoretical physics would be the subject of the incompleteness theorem by incorporating an arithmetical component. Nevertheless, Franzén adds, Gödel's theorem tells us only that there is an incompleteness in the arithmetical component of the theory. Whether a physical theory is complete when considered as a description of the physical world is not something that the incompleteness theorem tells us anything about.

Franzén also discusses various theological conclusions drawn from Gödel's theorem. Abstracts from the *Bibliography of Christianity and Mathematics* declare, for example, that Gödel's theorem demonstrates that physicists will never be able to formulate a theory of physical reality that is final or that the human mind is more than just a logical machine. Such theological appeals to Gödel's theorem only recycle the above-discussed and deficient Gödelian arguments against the mechanist theory of mind and TOE. But there are some more specifically theological appeals to the incompleteness theorem. Some of these are simply preposterous, and others at best are based on analogies. Sometimes it is suggested that Gödel's theorem shows that the only possible way of avowing an unprovable truth is faith. But, first, Gödel did not exhibit any absolutely unprovable truths, only relative ones; and, second, if we have, on the basis of mathematical reasoning, absolutely no idea whether a given highly complex formal system is consistent or not, it is quite unclear how Christian faith (or anything else) could help.

Gödel's theorem is often thought to support some form of skepticism with regard to mathematics: it is contended that we cannot, strictly speaking, prove anything or that the consistency of our fundamental theories (such as ZFC) is shown to be doubtful. Franzén argues against such claims that nothing in Gödel's theorem in any way contradicts the view that we have absolutely certain knowledge about the truth of the axioms of the system and, consequently, of their consistency. We don't need Gödel's theorem to tell us that we must adopt some basic principles without proof. If we have no doubts about the consistency of, say, ZFC, there is nothing in the second incompleteness theorem to give rise to any such doubts. And if we do have doubts about the consistency of ZFC, we have no reason to believe that a consistency proof of ZFC given in ZFC itself would do anything to remove those doubts.

Franzén also devotes a brief chapter to the variants of incompleteness results arising from the so-called Algorithmic Information Theory, or the theory of Kolmogorov complexity, and especially the various philosophical interpretations of these results by Gregory Chaitin (one of the founders of this theory). For example, Chaitin claims that his results not only explain Gödel's incompleteness theorem but also are the ultimate, or the strongest possible, incompleteness results. Franzén first explains these results and then shows that such claims are in no way justified by mathematical facts (see also [8]).

Concluding Remarks

This is a marvelous book. It is both highly competent and yet enjoyably readable. At some points there are even glimpses of humor, as when Franzén declares in the preface: "For any remaining instances of incompleteness or inconsistency in the book, I consider myself entirely blameless, since after all, Gödel proved that any book on the incompleteness theorem must be incomplete or inconsistent. Well, maybe not" (p. ix). At last there is available a book that one can wholeheartedly recommend for anyone interested in Gödel's incompleteness theorem—one of the most exciting and wide-ranging achievements of scientific thought ever.

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Two Poems by Bill Parry (1934–2006)

Editor's Note: It is common to find poetry and mathematics cited as mutual metaphors, for example in Ducasse's famous maxim "la poésie est la géométrie par excellence." But real poetry by real mathematicians about real mathematics is rare. The *Notices* is pleased to offer two poems by the late Bill Parry for their intrinsic interest and as a memorial.

—Andy Magid

Bill Parry, emeritus professor at the University of Warwick in England, died at age 72 on August 20, 2006. He was a student of Yael Dowker who introduced ergodic theory to the United Kingdom, and Bill played a major role in establishing the subject in this country. His many former students occupy key positions in universities in the UK and elsewhere. He gave an invited address at the International Congress of Mathematicians in 1970 and was elected a Fellow of the Royal Society in 1984. Although he was the first in his family to attend university, he was a well- (though self-) educated man, with a deep interest in politics, literature, science, cinema, theatre, and other topics. He was a firm socialist all his life. Towards the end he became increasingly interested in poetry, and several of his poems have been published. Two previously unpublished poems, with mathematical themes, are published here with the permission of his estate.

The picture above was taken at Oberwolfach and dates from 1968. It is published with the permission of the Mathematisches Forschungsinstitut Oberwolfach.

—David Epstein,
University of Warwick

David Epstein is emeritus professor at the University of Warwick. His email address is David.Epstein@warwick.ac.uk.

Argument

As he cleaned the board,
chalk-dust rose like parched mist.
A dry profession, he mused as morosely
they shuffled settling tier upon tier.

Now, almost half-way through the course,
(coughs, yawns and automatic writing)
the theorem is ready.

Moving to the crucial point,
the sly unconventional twist,
a quiver springs his voice and breast;

soon the gambit will appear
opposed to what's expected.
The ploy will snip one strand
the entire skein sloughing to the ground.

His head turns sympathetically
from board to class.
They copy copiously.
But two, perhaps three pause and frown,

wonder will this go through,
questioning this entanglement
— yet they nod encouragement.
Then the final crux; the ropes relax and fall.

His reward: two smile, maybe three,
and one is visibly moved.
Q.E.D., the theorem is proved.

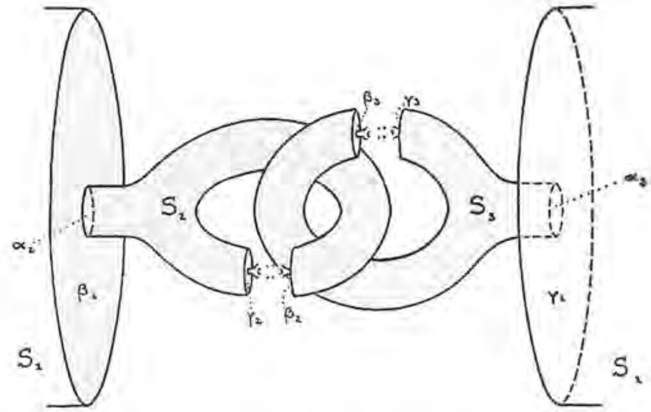
This was his sole intent.
Leaving the symbols on the board
he departs with a swagger of achievement.

Alexander's Horned Sphere

The sea was the first to propose it,
anticipating Alexander's thought.
The whited branch which clawed the ocean floor,
the echoing shell, the time-worn stone,
anemone and jell of sea-flower;
had he seen all this before?
Did he find his inspiration
along this sepulchered beach
or in some anatomy lesson?

Strands of bladderwrack,
distend to dentritic kelp. He saw
this could go on forever,
each increment a crab's claw.

So friend, hand in hand, we must keep close.
For should you stray,
perhaps along that promontory,
there's no going back.
Our lonely paths shall intersect
only by leap of synapse.



Alexander's horned sphere. This picture is taken from J. W. Alexander's original paper ("An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected," *Proceedings of the National Academy of Sciences*, 10 (1924), 8–10). Copyright J. W. Alexander.

The Mathematical Work of the 2006 Fields Medalists

The *Notices* solicited the following articles about the works of the four individuals to whom Fields Medals were awarded at the International Congress of Mathematicians in Madrid, Spain, in August 2006. (Grigory Perelman was awarded the medal but declined to accept it.) The International Mathematical Union also issued news releases about the medalists' work, and these appeared in the October 2006 *Notices*.

—*Allyn Jackson*

The Work of Andrei Okounkov

*Nicolai Reshetikhin**

Perhaps two basic words that can characterize the style of Andrei Okounkov are clarity and vision.

His research is focused on problems that are at the junction of several areas of mathematics and mathematical physics. If one chooses randomly one of his papers, it will almost certainly involve more than one subject and very likely will have a solution to a problem from one area of mathematics by techniques from another area. Many of his papers opened up new perspectives on how geometry, representation theory, combinatorics, and probability interact with each other and with other fields.

Typically, one also finds among the results of each of his papers a beautiful explicit formula.

The variety of tools Okounkov uses is very impressive. He has the rather unique quality of moving freely from analysis and combinatorics to algebraic geometry, numerical computations, and representation theory.

Let me focus on some representative examples.

One of his remarkable results is the Gromov-Witten and Donaldson-Thomas correspondence, which is an identification of two geometric enumerative theories. These results are intrinsically related to his works on Gromov-Witten invariants for curves, on random matrices, and on dimer models.

Works on Dimers

Dimer configurations are well known in combinatorics as perfect matchings on vertices of a graph where matched vertices are connected by edges. In the 1960s, Kasteleyn and Fisher computed the partition function of a dimer model on a planar graph and the local correlation functions in terms of Pfaffians of the so-called Kasteleyn matrix [3]. Since that time, dimer models have played a prominent role in statistical mechanics.

A pair of dimer configurations on a bipartite planar graph defines a stepped surface that projects bijectively to the graph. Thus, random dimer configurations on bipartite planar graphs can be regarded as random stepped surfaces. A snapshot of such a random surface for dimer configurations on a domain in a hexagonal lattice is shown in Figure 1.

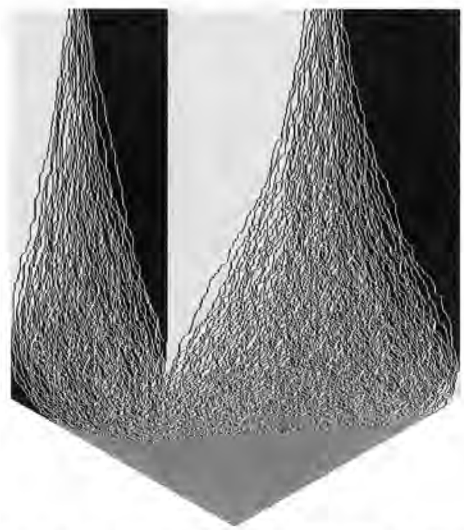


Figure 1. A random surface representing a random dimer configuration on a region of hexagonal lattice.

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Looking at this picture, the following is clear: on the scale comparable to the size of the system, such random stepped surfaces are deterministic. This phenomenon is by its nature close to deterministic limits in statistical mechanics (also known as hydrodynamical limits), to the semiclassical limit in quantum mechanics, and to the large deviation phenomenon in probability theory. It is also clear from this snapshot that fluctuations take place on the smaller scale.

The nature of fluctuations changes depending on the point in the limit shape. For example it is clear that the fluctuations in the bulk of the limit shape are quite different from those near a generic point at the boundary of the limit shape, those near singular points at the boundary of the limit shape, or those near the points where the limit shape touches the boundary. These empirical observations are confirmed now and quantified by precise mathematical statements. Okounkov made an essential contribution to these results.

In joint work with Kenyon and Sheffield [4] Okounkov proved that the limit shape for periodically weighted dimers is the graph of the Ronkin function of the spectral curve of the model. A similar description was obtained for local correlation functions in the bulk. This work was based on Kasteleyn's results.

In subsequent papers with Kenyon, Okounkov gave a complete description of real algebraic curves that describe the boundary of the limit shape in a dimer model. It turns out that natural equivalence classes of such curves form the moduli space of Harnack curves [5]. It also turned out that such curves describe a special class of solutions to the complex Burgers equation [6].

Fluctuations near the generic and special points of the boundary are described in [12] and [13]. In this work the dimer model was reformulated in terms of the Schur process and then in terms of vertex operators for gl_∞. Some key ideas used in these papers go back to previous works of Okounkov on representation theory.

All these results are characteristic of the style of Okounkov: the problem in statistical mechanics is solved using tools from algebraic geometry and representation theory. The moduli space of Harnack curves is described in terms of Gibbs measures on dimer configurations. The same space is identified as the space of certain algebraic solutions to the complex Burgers equation.

Gromov-Witten Theory

Gromov-Witten theory studies enumerative geometry of moduli spaces of mappings of algebraic curves (Riemann surfaces) into some fixed algebraic variety.

Let us recall the idea of the Gromov-Witten invariant. Let X be a nonsingular complex projective variety, and let $\overline{M}_{g,n}(X,\beta)$ be the space of isomorphism classes of triples $\{C, p_1; \dots, p_n; f\}$ where C is a complex projective connected nodal curve of genus g with n marked smooth points $p_1, \dots, p_n$ and $f : C \to X$ is a stable mapping such that $[f(C)] = \beta$. Here stability means that components of C that are pre-images of points with respect to the mapping f have finite automorphism groups. Let $\mathrm{ev}_j : (C, p_1; \dots, p_n; f) \to f(p_j)$ be the evaluation mapping. Denote by $ev_j^* \alpha \in H^*(\overline{M}_{g,n}(X,\beta))$ the pull-back of a class $\alpha \in H^*(X)$. Let L_j be a line bundle on $\overline{M}_{g,n}(X,\beta)$ whose fiber over the point $\{C, p_1; \dots, p_n; f\}$ is $T^*_{p_j}C$. The *Gromov-Witten invariants* of X are intersection numbers

$$(1) \quad <\tau_{k_1}(\alpha_1)\dots\tau_{k_n}(\alpha_n)>^X_{\beta,g} = \int_{[\overline{M}_{g,n}(X,\beta)]} \wedge_{j=1}^n c_1(L_j)^{k_j}\mathrm{ev}_j^*(\alpha_j).$$

Here the (virtual) fundamental class $[\overline{M}_{g,n}(X,\beta)]$ was constructed in the works of Behrend-Fantechi and Li-Tian.

When X is a point, Witten [15] conjectured that the generating function

$$(2) \quad f(t_0, t_1, \dots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1+\dots+k_n=3g-3+n} <\tau_{k_1}\dots\tau_{k_n}>_g \prod_{j=1}^n t_{k_j}$$

is a tau-function for the KdV integrable hierarchy satisfying the additional "string equation". The first proof of this conjecture was given by Kontsevitch in [7]. An alternative derivation of the key formula from [7] was later given by Okounkov and Pandharipande.

When X is a curve, the Gromov-Witten invariants were described completely by Okounkov and Pandharipande [10][11]. They showed that when $X = \mathbb{P}^1$ the generating function for the Gromov-Witten invariants is a tau-function for the Toda hierarchy, again with a special constraint similar to the "string equations". They also showed that the case when X is a point can be obtained by taking a limit of the $X = \mathbb{P}^1$ case.

Another remarkable result by Okounkov and Pandharipande is the following explicit formula for GW-invariants when X is a curve. Let $\beta = d[X]$, and suppose all α_i are equal to ω (the Poincaré dual of a point). Then

$$(3) \quad <\tau_{k_1}(\omega)\dots\tau_{k_n}(\omega)>^X_{d[X],g} = \sum_{|\lambda|=d} \left(\frac{\dim(\lambda)}{d!}\right)^{2-2g} \prod_{j=1} \frac{p_{k_j+1}(\lambda)}{(k_i+1)!}.$$

Here the sum is taken over all partitions of d and

$$(4) \quad p_k(\lambda) = \sum_{j\geq 1}\left((\lambda_j - j + 1/2)^k - (-j+1/2)^k\right) + (1-2^{-k})\zeta(-k).$$

This formula for GW-invariants of curves is rooted in the relation between the GW-invariants and Hurwitz numbers. Recall that the latter are the numbers of branched coverings of X with given ramification type at given points. The branched coverings of X were studied in [1], [2], where the problem was resolved essentially by using representation theory of $S(\infty)$.

Donaldson-Thomas Invariants

Let X be a three-dimensional algebraic variety. Algebraic curves $C \subset X$ of arithmetic genus g with the fundamental class $\beta \in H_2(X)$ are parameterized by the Hilbert scheme $\mathrm{Hilb}(X;\beta,1-g)$. Let $c_2(\gamma)$ be the coefficient of $\gamma \in H^*(X)$ in the Künneth decomposition of the second Chern class of the universal ideal sheaf $\mathcal{J} : \mathrm{Hilb}(X;\beta,\chi)\times X \to X$. The Donaldson-Thomas invariants are:

$$(5) \quad <\gamma_1,\ldots,\gamma_n>_{\beta,\chi} = \int_{[\mathrm{Hilb}(X;\beta,\chi)]}\prod_{j=1}^{n} c_2(\gamma_j).$$

Here $[\mathrm{Hilb}(X;\beta,\chi)]$ is the (virtual) fundamental class of $\mathrm{Hilb}(X;\beta,\chi)$ constructed by R. Thomas. It is of dimension $-\beta K_X$ where K_X is the fundamental class of X. This is the same dimension as the dimension of $\overline{M}_{g,0}(X,\beta)$.

The conjecture of Maulik, Nekrasov, Okounkov, and Pandharipande [8] relates suitably normalized generating functions of Gromov-Witten invariants for $\overline{M}_{g,0}(X,\beta)$ and Donaldson-Thomas invariants of $\mathrm{Hilb}(X;\beta,\chi)$. This conjecture was proved for local curves and when X is the total space of the canonical bundle of a toric surface.

One can argue that these results were inspired by observations from [14] and from the computation of the Seiberg-Witten potential in [9], and that they can be regarded as a generalization of the formula (3) for Gromov-Witten invariants of a curve. It is also remarkable that combinatorics of partitions came up as a computational tool in this topological subject.

This brief and incomplete description of the achievements of Andrei Okounkov might underestimate one important feature of his work, namely, its unifying quality: Seemingly unrelated subjects have become parts of one field. In this sense his contributions also have an important organizing value.

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The Work of Andrei Okounkov

*A. M. Vershik**

In spite of his youth, Andrei Okounkov has gone through several periods in his fruitful creative work: he started in asymptotic representation theory, and then extremely quickly went one by one through several areas, subsequently enlarging his knowledge of a number of subjects in algebraic geometry, symmetric functions, combinatorics, topology, integrable systems, statistical physics, quantum field theory, etc.—modern topics of great interest to all mathematicians. Each change in topic had a prehistory in the previous one and brought useful erudition. The influence of the first period of Okounkov's activity—symmetric functions, asymptotic representation theory, Young diagrams, partitions, and so on—one can clearly feel. I want to say a few words only about his first achievements (and not even about all of them), which are perhaps not so well known and which are right now a little bit in the shadow of his great subsequent successes of the last several years. These subjects also are closer to me because in some sense I started with them.

I will mention several of the first of Okounkov's results: a new proof of Thoma's theorem giving the solution of the problem about admissible representations of $\mathfrak{S}_\infty$; some distinguished formulas; contributions to the asymptotic Plancherel measure on Young tableaux, in particular, the analysis of fluctuations of random Young diagrams and the link with random matrices and matrix problems; and several results on symmetric functions and characters.

Asymptotic Representation Theory: Finite and Infinite Symmetric Groups

The modern theory of the representations of infinite symmetric groups started with the celebrated and fundamental theorem of E. Thoma (1964) about normalized characters of $\mathfrak{S}_\infty$ (the group of all finite permutations of a countable set). It asserts that an indecomposable normalized character (= a positive-definite central function $\chi(.), \chi(\mathbf{1}) = 1$) has the form

$$\chi_{\alpha,\beta}(g) = \prod_{n=2}^{\infty} s_n(\alpha,\beta)^{r_n(g)},$$

where $r_n(g), n > 1$, is the (finite) number of cycles of length n in the permutation $g \in \mathfrak{S}_\infty$; the parameters of the character ("Thoma's simplex") are $\alpha = (\alpha_1, \dots)$ and $\beta = (\beta_1, \dots)$, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq 0$. The sums $\sum_{k=1}^{\infty} \alpha_k + \beta_k \leq 1$ and $s_n(\alpha,\beta) = \sum_{k=1}^{\infty}[\alpha_k^n + (-1)^{n-1}\beta_k^n], n > 1$, are supernewtonian sums. The characters are multiplicative with respect to decomposition of the permutation on the cycles, and the value on a cycle of length n of the previous character is $s_n(\alpha,\beta)$.

Thoma's proof was based on hard analysis and the theory of positive-definite functions. It had nothing in common with proper representation theory. In the 1970s the author with S. Kerov gave a "representation-theoretic" proof of the theorem in the framework of the general approach to the approximations of those characters with the irreducible characters of finite symmetric groups, some ergodic ideas, and an important interpretation of those parameters α, β above as frequencies of the rows and columns in growing random Young tableaux. Then, in 1995, Okounkov suggested in his thesis a completely new idea and gave a third proof, based on a nice analysis of the elementary operators in the space of the representation. This proof immediately gives a new interpretation of the parameters as eigenvalues of some operators. Moreover, his goal was the solution of a more general problem that had been posed by his advisor G. Olshansky, namely, the description of all so-called admissible representations of the *bisymmetric* group (which includes the problem about the characters above). This is a problem about the complete list of the irreducible representations of the group $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$, whose restriction to the diagonal subgroup is a direct sum of tensor representations of $\mathfrak{S}_\infty$; or perhaps it is better to say that those restrictions could be extended to the group of all (infinite) permutations $\mathfrak{S}^\infty$. A typical example of such a representation is the irreducible right-left (regular) representation of $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$ in $l^2(\mathfrak{S}_\infty)$, whose restrictions to each multiplier are type II_1 factor representations. Olshansky gave a method of description, but the whole list of admissible representations and a model for it was obtained by Okounkov. The answer is very natural and related to actions of the bisymmetric group in a groupoid. This was one of the first examples of a non-locally compact ("big") group of type 1 with a very rich set (of infinite dimension) of irreducible representations. By the way, the identification of the left and right parts of some of those representations with von Neumann factors is still an open question.

The description of the characters in Thoma's theorem mentioned above is equivalent to the description of so-called central measures of the space of Young tableaux (= space of paths of Young graphs)—or to the description of homomorphisms of the ring of symmetric functions Λ to the scalars that are positive on the Schur functions $s_\mu(.) \in \Lambda$. Each deformation of a Young graph (or one can substitute other symmetric functions for the Schur polynomials) gives a new problem

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of this type. With S. Kerov and G. Olshansky, Okounkov gave the solution of such a problem for Jack polynomials $P_\mu(x,\theta)\theta \geq 0$; in this case the formula above is changed as follows. For the supernewtonian sums $s_n(\alpha,\beta)$ one must substitute the sums $s_{n,\theta} = \sum_{k=1}^{\infty}[\alpha_k^n + (-\theta)^{n-1}\beta_k^n]$ where α and β have the same meaning and where the parameter $\theta \in [0,\infty)$. Similar problems about central measures on the space of paths of the graded graphs (Bratelli diagram)—in another words, the question of how to describe the traces on locally semisimple algebras—are difficult. Such problems were extensively studied by S. Kerov, G. Olshansky, A. Okounkov, the author, and others, but are not solved for many natural examples. They are very important for the asymptotic representation theory of classical groups and for the theory of symmetric functions. It is worth mentioning that the asymptotic point of view on the infinite symmetric group (also on the other groups of that type), produced a new approach to the theory of representations of the finite symmetric groups—in some sense this is a look from infinity to finiteness. This approach was started in the paper of Okounkov and the author (1996) and gave later many natural explanations of the classical results.

Several Formulas

Okounkov is the author (or coauthor) of papers containing remarkable concrete formulas that concern very classical subjects. Among other results by him about symmetric functions and their application to various problems of analysis and representations, I want to mention two examples of important formulas. In 1996-7 with G. Olshansky (following Olshansky-Kerov and other authors), Okounkov systematically studied the theory of shifted Schur functions and its role in representation theory and obtained an elegant new formula for the number of standard tableaux of a skew Young diagram $\lambda \vdash n, \mu \vdash k, \lambda \supset \mu$:

$$\dim \lambda/\mu = \frac{s_\mu^*(\lambda)}{n(n-1)\dots(n-k+1)}.$$

This formula is mainly based on the surprising vanishing theorem of Okounkov:

$$s_\mu^*(\lambda) = 0 \quad \text{unless} \quad \mu \subset \lambda \quad s_\mu^*(\mu) = \prod_{\alpha\in\mu} h(\alpha).$$

Here $h(\alpha)$ is the hook length of the cell α of the diagram μ. The reformulation of that theory of shifted Schur functions in terms of Frobenius coordinates was given later (by Olshansky-Regev-Vershik) and led to the so-called inhomogeneous Frobenius-Schur symmetric functions. These are nothing but reformulations of the shifted Schur functions in Frobenius coordinates.

Another example is a formula that attracted the attention of many mathematicians because of its importance. This formula came out of Okounkov's and A. Borodin's beautiful answer to the question posed by Deift-Its about the existence of a formula for the determinant of a Toeplitz matrix as a Fredholm determinant. Borodin and Okounkov gave such a formula with an elegant proof. It happened that it had been proved before (with a different proof) by Jeronimo-Case, but this does not reduce the importance of their result.

Asymptotic Statistics of Young Diagrams with Respect to Plancherel Measure

This is one of the most impressive developments during the last few years in analysis and representation theory.

The study of the statistics of Young diagrams with Plancherel measure and the limit shape of typical diagrams was started in the papers of the author and S. Kerov in the 1970s. At the end of the 1990s, the remarkable link between those questions and the behavior of the eigenvalues of Gaussian random matrices was discovered. Great progress has been made in the last ten years, and the area has simultaneously been connected to random matrices, matrix problems, orthogonal polynomials, Young diagrams, integrable systems, determinant point processes, etc.

Many mathematicians took part in this progress: J. Baik, P. Deift, K. Johansson, C. Tracy, H. Widom, and others. Using serious analytical tools, those authors obtained striking and deep results about distributions of the fluctuations of the maximal eigenvalue of Gaussian matrices (Tracy-Widom) and the first and second row of a Young diagram with respect to Plancherel measure (Baik-Deift-Johansson). The contribution of Okounkov to subsequent progress was very important: *he suggested a strategy for considering all such problems.* This strategy, together with previous ideas of his coauthors G. Olshansky and A. Borodin, led to completely new proofs and simplifications of previous results on the one hand, and on the other hand provided a jumping-off point for him to move from those problems to random surfaces, to algebraic curves, to the Gromov-Witten/Hurwitz correspondence, etc. He was able to join together problems that had previously been very far from each other. One can say that this was a continuation of the "representation-theoretic" and the "partition-combinatorial" approach to the problems, together with ideas from modern mathematical physics.

I will mention only the main result. The concrete problem was to refine old results about the limit distribution of the spectrum of random Gaussian matrices (Wigner's semicircle law) and the limit shape of random Young diagrams with Plancherel measure (Vershik-Kerov, Logan-Shepp), and to find the distribution of the fluctuations of the maximal eigenvalue of random matrices (or the first several rows of Young diagrams). In a few preliminary papers Okounkov prepared the exclusively natural approach to the precise calculation of the correlation functions.

The correlation functions for random point process had been studied earlier for so-called z-measures on Young diagrams, in several papers of Borodin-Olshansky. The idea was to extend their calculation to Plancherel measure. Consider "poissonization" (=passage to a grand canonical ensemble) of the Plancherel measure on the diagrams. The so-called "Russian way" to look at a Young diagram is to turn the diagram 135°, and then to project it onto the lattice $\mathbb{Z}$, or, more exactly, onto the lattice $\mathbb{Z} + (1/2)$. When diagrams are equipped with poissonization of the Plancherel measure, the "Russian way" gives a remarkable random point process on the lattice, which is a determinant process (even before the limit procedure!). This is the main point, and it gives the possibility of calculating correlation functions of the processes and of obtaining a remarkable solution of the problem about the fluctuation of the Plancherel Young diagrams in the middle of the diagrams and near the edges. It gives correspondingly the Airy and Bessel ensemble as a determinant process on the lattice $\mathbb{Z} + (1/2)$ and produces as a partial result the proof of Deift's conjecture about joint distributions of the fluctuations of finitely many of the rows of a Young diagram with respect to Plancherel measure. This technique avoids the complicated tools of the Riemann-Hilbert problem as well as other tools that were used before, and brings to light the essence of the effects. Okounkov's idea about a direct link with random matrices was realized via matrix problems and ramified covering of surfaces. We can say now that the asymptotic theory of the Plancherel measure on Young diagrams is more or less complete.

In order to give a flavor of the ideas in those papers of Okounkov I mention some links and preliminary and hidden ideas: the action of $SL(2)$ on the partitions (started by S. Kerov), the infinite wedge model, boson-fermion correspondence; the very important new idea of Schur measures on the diagrams and, later, its generalization to Schur processes, which allowed one to consider three-dimensional Young diagrams and their limit shapes, asymptotics of enumeration of the ramified coverings of surfaces, etc. Of course this gigantic body of work still is not in complete order, and many ideas must be made clearer. But the number of results and future prospects are impressive.

The Work of Grigory Perelman

*John W. Morgan**

Introduction

I will report on the work of Grigory Perelman for which he was awarded the Fields Medal at the International Congress in the summer of 2006. Perelman posted three preprints on the arXiv between November 2002 and July 2003, [14, 16, 15]. In these preprints he gave a complete, albeit highly condensed, proof of the Poincaré Conjecture. Furthermore, at the end of the second preprint he stated a theorem about three-manifolds with curvature bounded below which are sufficiently collapsed. He showed how, from what he had established in the first two of his preprints, this collapsing result would imply the vast generalization of the Poincaré Conjecture, known as Thurston's Geometrization Conjecture. He stated that he would provide another manuscript proving the collapsing result, and in private conversations, he indicated that the proof used ideas contained in an earlier unpublished manuscript of his from 1992 and a then recently circulated manuscript of Shioya-Yamaguchi (which has now appeared [20]). To date, Perelman has not posted the follow-up paper establishing the collapsing result he stated at the end of his second preprint.

In this article, I will explain a little of the history and significance of the Poincaré Conjecture and Thurston's Geometrization Conjecture. Then I will describe briefly the methods Perelman used to establish these results, and I will discuss my view of the current state of the Geometrization Conjecture. Lastly, I will speculate on future directions that may arise out of Perelman's work. For a survey on the Poincaré Conjecture see [12]. For more details of the ideas and results we sketch here, the reader can consult [10], [2], and [13] and [23].

The Poincaré Conjecture and Thurston's Geometrization Conjecture

In 1904 Poincaré asked whether every closed (i.e., compact without boundary) simply connected three-manifold is homeomorphic to the three-sphere, see [18]. (It has long been known that this is equivalent to asking whether a simply connected smooth three-manifold is diffeomorphic to the three-sphere.) What has come to be known as the Poincaré Conjecture is the conjecture that the answer to this question is "yes". Since its posing, the Poincaré Conjecture has been a central problem in topology, and most of the advances in study of the topology of manifolds, both in dimensions three and in higher dimensions, over the last one hundred

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years have been related to the Poincaré Conjecture or its various generalizations. Prior to Perelman's work, analogues of the conjecture had been formulated in all dimensions and the topological versions of these analogues had been established (see [21] and [6]) in all dimensions except dimension three. Counterexamples to the analogue of the Poincaré Conjecture for smooth manifolds of higher dimension were first given by Milnor, in [11], where he constructed exotic smooth structures on the 7-dimensional sphere. The smooth four-dimensional Poincaré Conjecture remains open. Perelman's work resolves one remaining case of the topological version of the question. The Poincaré Conjecture is also the first of the seven Clay Millennium problems to be solved.

Poincaré was motivated to ask his question by an attempt to characterize the simplest of all three-manifolds, the three-sphere. But there is no reason to restrict only to this three-manifold. In 1982 Thurston formulated a general conjecture, known as Thurston's Geometrization Conjecture, which, if true, would (essentially) classify all closed three-manifolds, see [24] and [19]. Unlike the Poincaré Conjecture, which has a purely topological conclusion, Thurston's Conjecture has a geometric conclusion: In brief, it says that every closed three-manifold has an essentially unique two-step decomposition into simpler pieces. The first step is to cut the manifold open along a certain family of embedded two-spheres and cap off the resulting boundaries with three-balls. The second step is to cut the manifold open along a certain family of two-tori. The conjecture posits that at the end of this process each of the resulting pieces will admit a complete, finite-volume, Riemannian metric locally modeled on one of the eight homogeneous three-dimensional geometries. All manifolds modeled on seven of the eight geometries are completely understood and easily listed. The eighth homogeneous three-dimensional geometry is hyperbolic geometry. It is the richest class of examples and is the most interesting case. Manifolds modeled on this are given as quotients of hyperbolic three-space by torsion-free lattices in $SL(2,\mathbf{C})$ of finite co-volume. The classification of these manifolds is equivalent to the classification of such lattices, a problem that has not yet been completely solved. Thurston's Conjecture includes the Poincaré Conjecture as a special case, since the only geometry that can model a simply connected manifold is the spherical geometry (constant positive curvature), and simply connected manifolds of constant positive curvature are easily shown to be isometric to the three-sphere.

Over the last one hundred years there have been many attempts to prove the Poincaré Conjecture. Most have been direct topological attacks which involve simplifying surfaces and/or loops in a simply connected three-manifold in order to show that the manifold is the union of two three-balls, which then implies that it is homeomorphic to the three-sphere. While these types of topological arguments have proved many beautiful results about three-dimensional manifolds, including for example that a knot in the three-sphere is trivial (i.e., unknotted) if and only if the fundamental group of its complement is isomorphic to $\mathbb{Z}$, they have said nothing about the Poincaré Conjecture. For a description of some of these approaches and why they failed see [22]. More recently, there have been approaches to the Geometrization Conjecture along the following lines. Given any three-manifold, the complement of a sufficiently general knot in the manifold will have a complete hyperbolic structure of finite volume. Work from here back to the original manifold by navigating in the space of hyperbolic structures with cone-like singularities. This approach, pioneered by Thurston, proved a beautiful result, called the orbifold theorem [5], that proves a special case of the Geometrization Conjecture, but has not led to a proof of the full conjecture.

Perelman's Method: Ricci Flow

How did Perelman do it? He used, and generalized, the work of Richard Hamilton on Ricci flow. For an introduction to Ricci flow see [3] and for a collection of Hamilton's papers on Ricci flow see [1]. Hamilton showed, at least in good cases, that the Ricci flow gives an evolution of a Riemannian metric on a 3-manifold that converges to a locally homogeneous metric (e.g., a metric of constant positive curvature), see [7] and [9]. But Hamilton also showed (see [8]) that there are other possibilities for the effect of Ricci flow. It can happen, and does happen, that the Ricci flow develops finite-time singularities. These finite-time singularities impede the search for a good metric; for, in order to find the good metric one needs to continue the flow for all positive time—the Riemannian metrics one is looking for in general only appear as limits as time goes to infinity. In fact, the finite-time singularities are needed in order to do the first layer of cutting (along two-spheres and filling in balls) as required in Thurston's Conjecture. These cuttings along two-spheres and filling in balls have been long known to be necessary in order to produce a manifold that admits a Riemannian metric modeled on one of the homogeneous geometries. As we shall see presently, the second step in the decomposition proposed by Thurston, along two-tori, happens as t limits to infinity.

The Ricci flow equation, as introduced and first studied by Hamilton in [7], is a weakly parabolic partial differential equation for the evolution of a Riemannian metric g on a smooth manifold. It is given by:

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)),$$

where $\text{Ric}(g(t))$ is the Ricci curvature of $g(t)$. One should view this equation as a (weakly) nonlinear version of the heat equation for symmetric two-tensors on a manifold. Hamilton laid down the basics of the theory of this equation—short-time existence and uniqueness of solutions. Furthermore, using a version of the maximum principle he established various important differential inequalities, including a Harnack-type inequality, that are crucial in Perelman's work. Hamilton also understood that, in dimension three, surgery would be a crucial ingredient for two reasons: (i) it is the way to deal with the finite-time singularities and (ii) because of the necessity in general to do cutting on a given three-manifold in order to find the good metric. With this motivation, Hamilton introduced the geometric surgeries that Perelman employs.

From the point of view of the previous paragraph, an n-dimensional Ricci flow is a one-parameter family of metrics on a smooth manifold. It is also fruitful to take the perspective of space-time and define an n-dimensional Ricci flow as a horizontal metric on $M \times [a, b]$ where the metric on $M \times \{t\}$ is the metric $g(t)$ on M. This point of view allows for a natural generalization: a generalized n-dimensional Ricci flow is an $(n+1)$-dimensional space-time equipped with a time function which is a submersion onto an interval, a vector field which will allow us to differentiate in the "time direction", and a metric on the "horizontal distribution" (which is the kernel of the differential of the time function). We require that these data be locally isomorphic to a Ricci flow on a product $M \times (a, b)$, with the vector field becoming $\partial/\partial t$ in the local product structure.

Perelman's First Two Preprints

Here we give more details on Perelman's first two preprints, [14] and [16]. As we indicated above, the work in these preprints is built on the foundation of Ricci flow as developed by Hamilton. A central concept is the notion of an n-dimensional Ricci flow, or generalized Ricci flow, being κ-non-collapsed on scales $\le r_0$ for some $\kappa > 0$. Let p be a point of space-time and let t be the time of p. Denote by $B(p, t, r)$ the metric ball of radius r in the t time-slice of space-time, and denote by $P(x, t, r, -r^2)$ the backwards parabolic neighborhood consisting of all flow lines backwards in time from time t to time $t - r^2$ starting at points of $B(x, t, r)$. This means that for any point p and any $r \le r_0$ the following holds. If the norm of the Riemannian curvature tensor on $P(p, t, r, -r^2)$ is at most r^{-2} then the volume of $B(x, t, r)$ is at least κr^n. Here are the main new contributions of his first two preprints:

(1) He introduced an integral functional, called the reduced $\mathcal{L}$-length for paths in the space-time of a Ricci flow.

(2) Using (1) he proved that for every finite time interval I, a Ricci flow of compact manifolds parameterized by I are non-collapsed where both the measure of non-collapsing and the scale depend only on the interval and the geometry of the initial manifold.

(3) Using (1) and results of Hamilton's on singularity development, he classified, at least qualitatively, all models for singularity development for Ricci flows of compact three-manifolds at finite times. These models are the three-dimensional, ancient solutions (i.e., defined for $-\infty < t \le 0$) of non-negative, bounded curvature that are non-collapsed on all scales ($r_0 = \infty$).

(4) He showed that any sequence of points (x_n, t_n) in the space-time of a three-dimensional Ricci flow whose times t_n are uniformly bounded above and such that the norms of the Riemannian curvature tensors $Rm(x_n, t_n)$ go to infinity (a so-called "finite-time blow-up sequence") has a subsequence converging geometrically to one of the models from (3). This result gives neighborhoods, called "canonical neighborhoods", for points of sufficiently high scalar curvature. These neighborhoods are geometrically close to corresponding neighborhoods in non-collapsed ancient solutions as in (3). Their existence is a crucial ingredient necessary to establish the analytic and geometric results required to carry out and to repeat surgery.

(5) From the classification in (3) and the blow-up result in (4), he showed that surgery, as envisioned by Hamilton, was always possible for Ricci flows starting with a compact three-manifold.

(6) He extended all the previous results from the category of Ricci flows to a category of certain well-controlled Ricci flows with surgery.

(7) He showed that, starting with any compact three-manifold, repeatedly doing surgery and restarting the Ricci flow leads to a well-controlled Ricci flow with surgery defined for all positive time.

(8) He studied the geometric properties of the limits as t goes to infinity of the Ricci flows with surgery.

Let us examine these in more detail. Perelman's length function is a striking new idea, one that he has shown to be extremely powerful, for example, allowing him to prove non-collapsing results. Let $(M, g(t))$, $a \le t \le T$, be a Ricci flow. The reduced $\mathcal{L}$-length functional is defined on paths $\gamma\colon [0, \bar{\tau}] \to M \times [a, T]$ that are parameterized by backwards time, namely satisfying

$\gamma(\tau) \in M \times \{T - \tau\}$ for all $\tau \in [0, \bar\tau]$. One considers

$$l(\gamma) = \frac{1}{2\sqrt{\bar\tau}} \int_0^{\bar\tau} \sqrt{\tau} \left(R(\gamma(\tau)) + |X_\gamma(\tau)|^2 \right) d\tau.$$

Here, $R(\gamma(\tau))$ is the scalar curvature of the metric $g(T - \tau)$ at the point in M that is the image under the projection into M of $\gamma(\tau)$, and $X_\gamma(\tau)$ is the projection into TM of the tangent vector to the path γ at $\gamma(\tau)$. The norm of $X_\gamma(\tau)$ is measured using $g(T - \tau)$. It is fruitful to view this functional as the analogue of the energy functional for paths in a Riemannian manifold. Indeed, there are $\mathcal{L}$-geodesics, $\mathcal{L}$-Jacobi fields, and a function $l_{(x,T)}(q,t)$ that measures the minimal reduced $\mathcal{L}$-length of any $\mathcal{L}$-geodesic from (x,T) to (q,t). Fixing (x,T), the function $l_{(x,T)}(q,t)$ is a Lipschitz function of (q,t) and is smooth almost everywhere.

Perelman proves an extremely important monotonicity result along $\mathcal{L}$-geodesics for a function related to this reduced length functional. Namely, suppose that W is an open subset in the time-slice $T - \bar\tau$, and each point of W is the endpoint of a unique minimizing $\mathcal{L}$-geodesic. Then he defined the reduced volume of W to be

$$\int_W (\bar\tau)^{-n/2} e^{-l_{(x,T)}(w, T-\bar\tau)} dvol.$$

For each $\tau \in (0, \bar\tau]$ let $W(\tau)$ be the result of flowing W along the unique minimal geodesics to time $T - \tau$. Then the reduced volume of $W(\tau)$ is a monotone non-increasing function of τ. This is the basis of his proof of the non-collapsing result: Fix a point (x,T) satisfying the hypothesis of non-collapsing for some $r \le r_0$. The monotonicity allows him to transfer lower bounds on reduced volume (from (x,T)) at times near the initial time (which is automatic from compactness) to lower bounds on the reduced volume times near T. From this, one proves the non-collapsing result at (x,T).

Perelman then turned to the classification of ancient three-dimensional solutions (ancient in the sense of being defined for all time $-\infty < t \le 0$) of bounded, non-negative curvature that are non-collapsed on all scales. By geometric arguments he established that the space of based solutions of this type is compact, up to rescaling. Here the non-collapsed condition is crucial: After we rescale to make the scalar curvature one at the base point, this condition implies that the injectivity radius at the base point is bounded away from zero. There are several types of these ancient solutions. A fixed time section is of one of the following types—(i) a compact round three-sphere or a Riemannian manifold finitely covered by a round three-sphere, (ii) a cylinder which is a product of a round two-sphere with the line or a Riemannian manifold finitely covered by this product, (iii) a compact manifold diffeomorphic to S^3 or $\mathbb{R}P^3$ that contains a long neck which is approximately cylindrical, (iv) a compact manifold of bounded diameter and volume (when they are rescaled to have scalar curvature 1 at some point), and (v) a non-compact manifold of positive curvature which is a union of a cap of bounded geometry (modulo rescaling) and a cylindrical neck (approximately S^2 times a line) at infinity.

Using the non-collapsing result and delicate geometric limit arguments, Perelman showed that in a Ricci flow on compact three-manifolds any finite-time blow-up sequence has a subsequence which, after rescaling to make the scalar curvature equal to one at the base point, converges geometrically to a non-collapsed, ancient solution of bounded, non-negative curvature. (Geometric convergence means the following: Given a finite-time blow-up sequence (x_n, t_n), after replacing the sequence by a subsequence, there is a model solution with base point and scalar curvature at the base point equal to one such that for any $R < \infty$ the balls of radius R centered at the x_n in the n^{th} rescaled flow converge smoothly to a ball of radius R centered at the base point in the given model.) For example, we could have a component of positive Ricci curvature. According to Hamilton's result [7] this manifold contracts to a point at the singular time and as it does so it approaches a round (i.e., constant positive curvature) metric. Thus, the model in this case is a round manifold with the same fundamental group. Perelman's result implies that there is a threshold so that all points of scalar curvature above the threshold have canonical neighborhoods modeled on corresponding neighborhoods in non-collapsed ancient solutions. This leads to geometric and analytic control in these neighborhoods, which in turn is crucial for the later arguments showing that surgery is always possible.

Now let us turn to surgery. An n-dimensional Ricci flow with surgery is a more general object. It consists of a space-time which has a time function. The level sets of the time function are called the time-slices and they are compact n-manifolds. This space-time contains an open dense subset that is a smooth manifold of dimension $n + 1$ equipped with a vector field and a horizontal metric that make this open dense set a generalized n-dimensional Ricci flow. At the singular times, the time-slices have points not included in the open subset which is a generalized Ricci flow. This allows the topology of the time-slices to change as the time evolves past a singular time. For example, in the case just described above, when a component is shrinking to a point, the effect of surgery is to remove entirely that component. A more delicate case is when the singularity is modeled on a manifold with a long, almost cylindrical tube in it. In this case, at the singular time, the singularities are developing inside the tube. Following Hamilton, at the singular time Perelman cuts off the tube near its ends and sews in a predetermined metric on the three-ball, using a partition of unity. The effect of surgery is to produce a new, usually topologically distinct manifold at the singular, or surgery, time. After having produced this new closed, smooth manifold, one restarts the Ricci flow using that manifold as the initial conditions. This then is the surgery process: remove some components of positive curvature and those fibered over circles by manifolds of positive curvature (all of which are topologically standard) and surger others along two-spheres, and then restart Ricci flow.

Perelman then showed the entire Ricci flow analysis described above extends to Ricci flows with surgery, provided that the surgery is done in a sufficiently controlled manner. Thus, one is able to repeat the argument ad infinitum and construct a Ricci flow with surgery defined for all time. Furthermore, one has fairly good control on both the change in the topology and the change in the geometry as one passes the surgery times. Here then is the main result of the first two preprints taken together:

Theorem. *Let (M, g_0) be a compact, orientable Riemannian three-manifold. Then there is a Ricci flow with surgery $(\mathcal{M}, G)$ defined for all positive time whose zero-time slice is (M, g_0). This Ricci flow with surgery has only finitely many surgery times in any compact interval. As one passes a surgery time, the topology of the time-slices changes in the following manner. One does a finite number of surgeries along disjointly embedded two-spheres (removing an open collar neighborhood of the two-spheres and gluing three-balls along each of the resulting boundary two-spheres) and removes a finite number of topologically standard components (i.e., components admitting round metrics and components finitely covered by $S^2 \times S^1$).*

A couple of remarks are in order:
(i) It is clear from the construction that if Thurston's Geometrization Conjecture holds for the manifold at time t then it holds for the manifolds at all previous times.
(ii) In this theorem one does not need orientability, only the weaker condition that every projective plane has non-trivial normal bundle. To extend the results to cover manifolds admitting projective planes with trivial normal bundle one takes a double covering and works equivariantly. This fits perfectly with the formulation of the Geometrization Conjecture for such manifolds.

In addition, Perelman established strong geometric control over the nature of the metrics on the time-slices as time tends to infinity. In particular, at the end of the second preprint he showed that for t sufficiently large, the t time-slice contains a finite number of incompressible tori (incompressible means π_1 injective) that divide the manifold into pieces. Each component that results from the cutting process has a metric of one of two types. Either the metric is, after rescaling, converging to a complete constant negatively curved metric, or the metric is arbitrarily collapsed on the scale of its curvature. Components of the first type clearly support hyperbolic metrics of finite volume. It is to deal with the components of the second type that Perelman states the proposed result on collapsed manifolds with curvature bounded below.

Completion of the Proof of the Poincaré Conjecture

As we have already remarked, the Poincaré Conjecture follows as a special case of Thurston's Geometrization Conjecture. Thus, the results of the first two of Perelman's preprints together with the collapsing result stated at the end of the second preprint, give a proof of the Poincaré Conjecture. In a third preprint [15] Perelman gave a different argument, avoiding the collapsing result, proving the Poincaré Conjecture but not the entire Geometrization Conjecture. For a detailed proof along these lines, see Chapter 18 of [13]. One shows that if one begins with a homotopy three-sphere, or indeed any manifold whose fundamental group is a free product of finite groups and infinite cyclic groups, then the Ricci flow with surgery, which by the results of Perelman's first two preprints is defined for all positive time, becomes extinct after a finite time. That is to say, for all t sufficiently large, the manifold at time t is empty. We conclude that the original manifold is a connected sum of spherical space-forms (quotients of the round S^3 by finite groups of isometries acting freely) and S^2-sphere bundles over S^1. It follows that if the original manifold is simply connected, then it is diffeomorphic to a connected sum of three-spheres, and hence is itself diffeomorphic to the three-sphere. This shows that the Poincaré Conjecture follows from this finite-time extinction result together with the existence for all time of a Ricci flow with surgery.

Let us sketch how the finite-time extinction result is established in [15]. (There is a parallel approach in [4] using areas of harmonic two-spheres of non-minimal type instead of area minimizing 2-disks.) We consider the case of a homotopy three-sphere M (though the same ideas easily generalize to cover all cases stated above). The fact that the manifold is a homotopy three-sphere implies that $\pi_3(M)$ is non-trivial. Perelman's argument is to consider a non-trivial element in $\xi \in \pi_3(M)$. Represent this element by a two-sphere family of homotopically trivial loops. Then for each loop take the infimum of the areas of spanning disks for the loop, maximize over the loops in family and then minimize over all representative families for ξ. The result is an invariant $W(\xi)$. One asks what happens to this invariant under Ricci flow. The result (following similar arguments that go back to Hamilton [9]) is that

$$\frac{dW(\xi)}{dt} \leq -2\pi - \frac{1}{2}R_{\min}(t)W(\xi).$$

Here $R_{\min}(t)$ is the minimum of the scalar curvature at time t. Since one of Hamilton's results using the maximum principle is that $R_{\min}(t) \geq -6/(4t+\alpha)$ for some positive constant α, it is easy to see that any function satisfying Equation (6) goes negative in finite time. Perelman shows that the same equation holds in Ricci flows with surgery as long as the manifold continues to exist in the Ricci flow with surgery. On the other hand, $W(\xi)$ is always non-negative. It follows that the homotopy three-sphere must disappear in finite time.

This then completes Perelman's proof of the Poincaré Conjecture. Start with a homotopy three-sphere. Run the Ricci flow with surgery until the manifold disappears. Hence, it is a connected sum of manifolds admitting constant positive curvature metrics. That is to say it is a connected sum of three-spheres, and hence itself diffeomorphic to the three-sphere.

Status of the Geometrization Conjecture

What about the general geometrization theorem? Perelman's preprints show that proving this result has been reduced to proving a statement about manifolds with curvature locally bounded below that are collapsed in the sense that they have short loops through every point. There are results along these lines by Shioya-Yamaguchi [20]. The full result that Perelman needs can be found in an appendix to [20], except for the issue of allowing a boundary. The theorem that Perelman states in his second preprint allows for boundary tori, whereas the statement in [20] is for closed manifolds. Perelman has indicated privately that the arguments easily extend to cover this more general case. But in any event, invoking deep results from three-manifold topology, it suffices to consider only the closed case in order to derive the full Geometrization Conjecture. The paper [20] relies on an earlier, unpublished work by Perelman.

All in all, Perelman's collapsing result, in the full generality that he stated it, seems eminently plausible. Still, to my mind the entire collapsing space theory has not yet received the same careful scrutiny that Perelman's preprints have received. So, while I see no serious issues looming, I personally am not ready to say that geometrization has been completely checked in detail. I am confident that it is only a matter of time before these issues are satisfactorily explored.

The Effect of Perelman's Work

Let me say a few words about the larger significance of what Perelman has accomplished and what the future may hold. First of all, an affirmative resolution of the one-hundred-year-old Poincaré Conjecture is an accomplishment rarely equaled or surpassed in mathematics. An affirmative resolution of the Geometrization Conjecture will surely lead to a complete and reasonably effective classification of all closed three-dimensional manifolds. It is hard to overestimate the progress that this represents. The Geometrization Conjecture is a goal in its own right, leading as it (essentially) does to a classification of three-manifolds. Paradoxically, the effect of Perelman's work on three-manifold topology will be minimal. Almost all workers in the field were already assuming that the Geometrization Conjecture is true and were working modulo that assumption, or else they were working directly on hyperbolic three-manifolds, which obviously satisfy the Geometrization Conjecture.

The largest effects of Perelman's work will lie in other applications of his results and methods. I think there are possibilities for applying Ricci flow to four-manifolds. Four-manifolds are terra incognita compared to three-manifolds. In dimension four there is not even a guess as to what the possibilities are. Much less is known, and what is known suggests a far more complicated landscape in dimension four than in dimension three. In order to apply Ricci flow and Perelman's techniques to four-manifolds there are many hurdles to overcome. Nevertheless, this is an area that, in my view, shows promise.

There are also applications of Ricci flow to Kähler manifolds, where already Perelman's results are having an effect—see for example [17] and the references therein.

As the above description should make clear, Perelman's great advance has been in finding a way to control and qualitatively classify the singularities that develop in the Ricci flow evolution equation. There are many evolution equations in mathematics and in the study of physical phenomena that are of the same general nature as the Ricci flow equation. Some, such as the mean curvature flow, are related quite closely to the Ricci flow. Singularity development is an important aspect of the study of almost all of these equations both in mathematics and in physical applications. It is not yet understood if the type of analysis that Perelman carried out for the Ricci flow has analogues in some of these other contexts, but if there are analogues, the effect of these on the study of those equations could be quite remarkable.

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The Work of Terence Tao

*Jean Bourgain**

The work of Terence Tao includes major contributions to analysis, number theory, and aspects of representation theory. Perhaps his most spectacular achievement to date is the proof that the set of prime numbers contains arbitrarily long arithmetic progressions (jointly with Ben Green). A reasonably detailed account on this result would already easily make up the full article. But it certainly would not give a picture of the unique scope and diversity in Tao's opus. The number of areas that he has marked either by solving the main questions or making them progress in a decisive way is utterly astonishing. This unique problem-solving ability relies not only on supreme technical strength but also on a deep global understanding of large parts of mathematics. To illustrate this, I plan to report below also on some of his contributions to harmonic analysis, partial differential equations, and representation theory. Because of lack of space, the results will be formulated with little or no background discussion or how they fit in the larger picture of Tao's research.

1) Tao has the strongest results to date on several central conjectures in higher-dimensional Fourier analysis, first formulated by E. Stein. These conjectures express mapping properties of the Fourier transform when restricted to hypersurfaces (in particular, the Fourier restriction conjecture to spheres and the Bochner-Riesz conjectures belong to this class of problems). Those issues were basically understood in dimension 2 already in the 1970s, but they turn out to be much more resistant starting from dimension 3 (see [1] for the current state of affairs). They are intricately connected to combinatorial questions such as the dimension of higher-dimensional "Kakeya sets". Those are compact subsets in R^n containing a line segment in every direction. Such sets may have zero Lebesgue measure (the well-known Besicovitch construction in the plane) but it is believed that the Hausdorff dimension is always maximal (i.e., equals n). This

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problem, which underlies Stein's conjectures, has been the object of intensive research over the last few decades. Although still unsolved even for $n = 3$, also here Tao's work has led to some of the most significant developments, often of interest in their own right. Among these is his joint paper with N. Katz and I. Laba [2] and the discovery of the "sum-product phenomenon" in finite fields [3].

2) Tao's work on nonlinear dispersive equations is too extensive to describe here in any detail. But the two papers [4], [5] particularly stand out. Both prove a final result, contain new ideas, and are technically speaking a real tour de force. The paper [4] (jointly with J. Colliander, M. Keel, G. Staffilani and H. Takaoka) solves the longstanding conjecture of global wellposedness and scattering for the defocusing energy-critical Schrödinger equation in R^3, thus the equation $iu_t + \Delta u - u|u|^4 = 0$. This is the counterpart for the nonlinear Schrödinger equation of M. Grillakis' theorem [6] on nonlinear wave equations. In both cases, the local theory was understood for at least a decade, and the main difficulty was to rule out "energy concentrations". Tao and his collaborators resolve this issue by proving a new type of "Morawetz-inequality" where the solution u is studied simultaneously in physical and frequency space. In the paper [5] the wave map equation with spherical target is studied in dimension 2, in which case the energy norm is the critical Sobolev norm for the local wellposedness of the Cauchy problem. This result is the main theorem of the paper. Its proof (which builds upon several earlier works) involves a careful study of the "null-structures" in the nonlinearity and a novel renormalization argument. One should mention that large data may create blowup behavior although this is not expected to happen if the sphere is replaced by a hyperbolic target (S. Klainerman's conjecture).

3) Tao's proof (in joint work with B. Green) that the prime numbers contain arbitrarily long arithmetic progressions (a problem considered out of reach even for length-four progressions) has come as a real surprise to the experts, because the initial breakthrough here is one in ergodic theory rather than in our analytical understanding of prime numbers. Of great relevance is Szemerédi's theorem, which roughly states that arbitrary sets of integers of positive density contain arithmetic progressions of any length (as conjectured by Erdős and Turán). Szemerédi's original proof was combinatorial. Subsequent developments came with H. Furstenberg's new proof through ergodic theory and the concept of multiple recurrence and later T. Gowers' approach using harmonic analysis, closer in spirit to K. Roth's work. All of them are crucial for the discussion of the Green-Tao achievement. In their first paper [7], the existence of progressions in the primes is derived from a remarkable extension of Szemerédi's theorem, claiming that the same conclusion holds if now we consider subsets of positive density in a "pseudo-random" set. Here one should be more precise about what "pseudo-random" means, but let us just say that the prime numbers fit the model well, and the required pseudo-random "container" is simply provided by looking at an appropriate set of pseudo-primes (these are integers without small divisors). Following Furstenberg's structural approach (in a finite setting), Green and Tao construct in this pseudo-random setting a "characteristic factor" (in the sense of Furstenberg's theory) that is of positive density in the integers and hence allows application of Szemerédi's theorem. Their analysis uses strongly the so-called "Gowers norms" introduced in [8] to manipulate higher-order correlations. The importance of [8] is even more prominent in Green and Tao's subsequent papers (see [9] in particular) where [7] is combined with Gowers' analysis involving generalized spectra and classical techniques from analytic number theory. The authors establish in particular the validity of the expected asymptotic formula for the number of length 4 progressions in the primes. Such a formula had been obtained by Van der Corput in 1939 for prime triplets, using the circle method and Vinogradov's work. Besides solving questions of historical magnitude, the work of Green and Tao had a strong unifying effect on various fields.

4) I conclude this report with one more line of research in a completely different direction: Tao's solution [10] (jointly with A. Knutsen) of the "saturation conjecture" for the Littlewood-Richardson coefficients in representation theory, formulated by A. A. Klyachko in 1998, and Horn's conjecture for the eigenvalues of hermitian matrices under addition. This was one of Tao's earlier achievements (with a later follow-up). Again it has greatly impressed experts in the field.

So much in such short time. To paraphrase C. Fefferman, "What's next?"

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The Work of Wendelin Werner

*Harry Kesten**

Wendelin Werner received the Fields Medal for work which is on the borderline of probability and statistical physics, and which confirmed and made rigorous quite a number of conjectures made by mathematical physicists studying critical phenomena. In particular he proved so-called power laws and gave the precise value of corresponding critical exponents.

Many of Werner's papers are joint papers with one or more of G. Lawler, O. Schramm, and S. Smirnov. Together they have developed entirely new tools for determining limits of lattice models as the lattice spacing goes to 0, and for establishing conformal invariance of such limits. Even though there are still many open problems, the work of Werner and his co-workers opened up an area in which no rigorous arguments were known.

Werner received a Fields Medal for a theory and a general method, rather than for a specific theorem. We can only discuss some of this work; basically we restricted ourselves to Werner's work from the year 1999 on. More details, and especially references to other contributors than Werner and his co-workers can be found in [11], [12], and [3].

Background: Stochastic Loewner Evolutions or Schramm-Loewner Evolutions

Schramm [8] invented a new kind of stochastic process to model certain two-dimensional random processes, in which the state at time t is the initial piece of a curve $\gamma : [0,t] \to \mathbb{C}$, or more generally, a set $S(t) \subset \mathbb{C}$ which increases with t. A principal motivating example was the so-called scaling limit of loop-erased random walk on $\delta\mathbb{Z}^2$ as $\delta \downarrow 0$. Loop-erased random walk on $\delta\mathbb{Z}^2$ starting from $a \in \delta\mathbb{Z}^2$ and stopped at a set K is obtained as follows: First we choose a simple random walk path from a to K. Let $S_0 = a$. If $S_0, S_1, \dots, S_k \in \delta\mathbb{Z}^2 \setminus K$ have already been chosen, pick S_{k+1} uniformly from the four neighbors of S_k in $\delta\mathbb{Z}^2$. Stop this process at the random time τ when $S_.$ first reaches a point in K. Then $S_0, \dots, S_\tau$ is a realization of simple random walk from a to K. A realization of loop-erased random walk is obtained by erasing the loops in $S_0, \dots, S_\tau$ in the order in which they arise. A formal description of this process can be found in [11].

The resulting path will be a self-avoiding path $LE_0, LE_1, \dots, LE_\sigma$ from a to K with $LE_j = S_{t(j)}$ for some $t(0) = 0 < t(1) < \cdots < t(\sigma) = \tau$. The process of loop-erased walks obtained in this way is completely described by its distribution function which is some probability measure on self-avoiding paths from a to K, which we denote by $LERW(a,K)$. Usually one wants to study the path as long as it is inside some domain $D \subset \mathbb{C}$, so one often takes K to be $\mathbb{C} \setminus D$ or an approximation to ∂D. How does a "typical" loop-erased path look? Does there exist a limit in a suitable topology of $LERW(a,K)$ as $\delta \downarrow 0$ and is the limit simple to describe? Such a limit is usually called a "scaling limit". Note that $\delta \downarrow 0$ automatically means that one studies walks of more and more steps. It seems reasonable to expect that the lattice structure of $\delta\mathbb{Z}^2$ is no longer visible in the limit and that the limit is rotation invariant. In fact, it was believed, in the first place by physicists, that the scaling limit exists and is even "conformally invariant". As Schramm puts it "conformal invariance conjectures [were] 'floating in the air'." [8] gives the following precise formulation to the conjecture (we skip any discussion of the topology in which the scaling limit should be taken):

> *Let $D \subsetneq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$, and let $a \in D$. Then the scaling limit of $LERW(a, \partial D)$ exists. Moreover, suppose that $f : D \to D'$ is a conformal homeomorphism onto a domain $D' \subset \mathbb{C}$. Then $f_*\mu_{a,D} = \mu_{f(a),D'}$, where $\mu_{a,D}$ is the scaling limit measure of $LERW(a,\partial D)$ and $\mu_{f(a),D'}$ is the scaling limit measure of $LERW$ $(f(a), \partial D')$.*

Schramm in [8] could not prove this conjecture, but assuming the conjecture to be true he did determine what $\mu_{a,D}$ had to be. (The conjecture has been proven now in [7].) Let $D \subsetneq \mathbb{C}$ be a simply connected domain containing 0 and let $\mathbb{U}$ be the open unit disc. By the Riemann mapping theorem there exists a unique conformal homeomorphism $\psi = \psi_D$ which takes D onto $\mathbb{U}$ and is such that $\psi(0) = 0$ and $\psi'(0)$ is real and strictly positive (the prime denotes differentiation with respect to the complex variable z). Now let $\eta : [0,\infty] \to \overline{\mathbb{U}}$ and set $U_t = \mathbb{U} \setminus \eta[0,t]$ and $g_t = \psi_{U_t}$. Then one can parametrize the path η in such a way that $g_t'(0) = e^t$. One can show that $W(t) := \lim_{z \to \eta(t)} g_t(z)$ exists and lies in

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$\partial\mathbb{U}$, where in this limit z approaches $\eta(t)$ from inside U_t. Loewner's slit mapping theorem implies that in the above parametrization of η, $g_t(z)$ satisfies the differential equation

$$\frac{\partial}{\partial t}g_t(z) = -g_t(z)\frac{W(t)+g_t(z)}{W(t)-g_t(z)},$$

$$g_0(z) = z \text{ for all } z \in \mathbb{U}.$$

Here $W : [0,\infty) \to \partial\mathbb{U}$ is some continuous function which is called the *driving function* of the curve η. Properties of η are tied to properties of W. If η is a random curve, then W is also random. To get back to loop-erased random walk, Schramm first showed that $\mu_{0,\mathbb{U}}$ has to be concentrated on simple curves. Using a kind of Markov property of loop-erased random walks and assuming the conjecture in italics above, Schramm then shows that if η (or rather its reversal) is chosen according to $\mu_{0,\mathbb{U}}$, then $W(t)$ must be $\exp[iB(\kappa t)]$ for a Brownian motion B whose initial point is uniformly distributed on $\partial\mathbb{U}$ and $\kappa \ge 0$ some constant. In a way one can get back from the driving force to a curve η. An SLE_κ process is then the process of the $\eta[0,t]$ (or of the sets $\mathbb{U}\setminus\eta[0,t]$) when the driving function $W(t)$ equals $B(\kappa t)$. What we described here is so-called *radial SLE*. There is also a variant called *chordal SLE* which we shall not define here.

What value of κ should one take for the scaling limit of loop-erased random walk? Schramm compares the winding number of SLE with that of $LERW$ to conclude that one should take $\kappa = 2$.

The Work of Werner on Brownian Intersection Exponents

It has turned out that the representation of certain models by SLE_κ is very helpful. Various questions reduce to questions about hitting probabilities of sets by SLE or behavior of functionals of SLE, and in many instances these can be reformulated into questions about stochastic differential equations and stochastic integrals. One now clearly needs techniques to show in concrete cases that a model can be analyzed by relating it to SLE. A case where Werner and his co-workers have been quite successful is the calculation of Brownian intersection exponents.

Without using SLE Lawler and Werner had already made good progress on determining the asymptotic behavior of the "non-intersection probability"

(2)

$$f(p_1,p_2) := P\Big\{\bigcup_{i=1}^{p_1} B_i^{(1)}[0,t] \cap \bigcup_{j=1}^{p_2} B_j^{(2)}[0,t] = \emptyset\Big\},$$

where $B_i^{(\ell)}, 1 \le i \le p_\ell, \ell = 1,2$, are independent planar Brownian motions with $B_i^{(\ell)}(0) = a_\ell \in \mathbb{H} := \mathbb{R}\times(0,\infty)$ with $a_1 \ne a_2$. They also studied the analogous non-intersection probabilities $f(p_1,\dots,p_q)$ for q "packets" of Brownian motions. It was known that there is some exponent $\xi(p_1,\dots,p_q) > 0$ such that $f(p_1,\dots,p_q) \approx t^{-\xi(p_1,\dots,p_q)/2}$. ($a \approx b$ means that $[\log b]^{-1}\log a$ tends to 1 as an appropriate limit is taken; here it is the limit as $t \to \infty$.) Lawler and Werner also considered $\tilde{f}(p_1,\dots,p_q)$ which is defined by adding the condition that all $B_{i_\ell}^{(\ell)}[0,t] \subset \mathbb{H}$ in (2). This has similar exponents $\tilde{\xi}$ such that $\tilde{f}(p_1,\dots,p_q) \approx t^{-\tilde{\xi}(p_1,\dots,p_q)/2}$. Duplantier and Kwon predicted (on the basis of non-rigorous arguments from quantum field theory and quantum gravity) explicit values for many of the exponents. This model has conformal invariance built in, because if $B(\cdot)$ is a planar Brownian motion in a domain D and Φ is a conformal homeomorphism from D onto D', then $\Phi \circ B(\cdot)$ is a time change of a Brownian motion in D'. In [4], [5] the authors manage to compute the exponents $\tilde{\xi}$ and ξ by relating Brownian excursions to SLE_6. An important ingredient of their proof is the singling out of SLE_6 as the only SLE process with the "locality property". Roughly speaking, this property says that SLE_6 does not "feel the boundary" of its domain as long as it does not hit it.

Earlier Lawler had shown that the Hausdorff dimension of various sets defined in terms of planar Brownian motions could be expressed by means of the exponents $\xi, \tilde{\xi}$ or similar exponents. For instance, the Hausdorff dimension of the outer boundary of a planar Brownian motion equals $2-\eta_2$. Here η_2 is the "disconnection" exponent defined by

$$P\{B^{(1)}[0,\sigma_R^{(1)}] \cup B^{(2)}[0,\sigma_R^{(2)}] \text{ does not disconnect 0 from } \infty\} \approx R^{-\eta_2},$$

with $B^{(1)}(0), B^{(2)}(0)$ two independent two-dimensional Brownian motions starting at 1, and $\sigma_R^{(i)}$ is the first hitting time by $B^{(i)}$ of the circle of radius R and center at the origin.

In [4], [5] it is shown that η_2 equals $2/3$, thereby proving a conjecture by Mandelbrot of more than twenty years' standing.

Background: Conformal Invariance in Percolation

Percolation is one of the simplest models which exhibits a phase transition and because of its simplicity is a favorite model among probabilists and statistical physicists for studying critical phenomena. One considers an infinite connected graph $\mathcal{G}$ and takes each edge of $\mathcal{G}$ open or closed with probability p and $1-p$, respectively. All edges are independent of each other. In the simple case this model has just the one parameter p. We write P_p for the corresponding probability measure on configurations of open and closed edges. Let v_0 be some fixed vertex of G and consider the so-called percolation probability $\theta(p) = \theta(p,v_0) := P_p\{$there is an infinite self avoiding open path starting at $v_0\}$, where a path is called open if all its edges are open. If $\mathcal{G} = \mathbb{Z}^d$ or $\mathcal{G}$ is the triangular lattice in the plane, then it was shown by Broadbent and Hammersley that there is some $p_c \in (0, 1)$ such that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. One can also attach the randomness to the vertices instead of the edges of $\mathcal{G}$. Then the vertices are open or closed with probability p or $1 - p$, respectively. The critical probability is then defined as before. The different critical probabilities are sometimes denoted as $p_c(\mathcal{G}, edge)$ and $p_c(\mathcal{G}, site)$. The study of "critical phenomena" usually refers to the study of singularities of various functions near or at p_c. For instance, let $\mathcal{G}$ be the cluster of v_0, that is the set of all points which can be reached by an open path starting at v_0. Let $|C|$ denote the number of vertices in C, and $E_p|C|$ its expectation. Then it is believed that $E_p|C| \approx (p_c - p)^{-\gamma}$ as $p \uparrow p_c$. Also, at criticality, $P_{p_c}\{|C| \geq n\}$ is expected to behave like $n^{-\rho}$ for some ρ. Constants like γ and ρ are called critical exponents. Physicists not only believe that these exponents exist, but also that they are universal, that is, that they depend only on some simple characteristic of the graph $\mathcal{G}$, such as the dimension. For instance the exponent γ is believed to be the same for the edge and site problems on $\mathbb{Z}^2$ and the edge and site problem on the triangular lattice. This is not the case for the critical probabilities which usually vary with $\mathcal{G}$. In order to explain this universality physicists proposed the "renormalization group", but this has not been made rigorous in the case of percolation. However, this led people to consider whether there existed a scaling limit for critical percolation. More specifically, if one considers percolation at $p_c(\mathcal{G})$ on $\delta\mathcal{G}$ for a periodically imbedded graph $\mathcal{G}$ in $\mathbb{Z}^2$, can one find a limit of the pattern of open paths as $\delta \downarrow 0$? A concrete question is the following: Let D be a "nice" simply connected domain in $\mathbb{C}$ and let A and B be two arcs in ∂D. Let $P_\delta(D, A, B)$ be the probability that in critical percolation on $\delta\mathcal{G}$ there is an open connection in D between A and B. Does $P_0(D, A, B) := \lim_{\delta\downarrow 0} P_\delta(D, A, B)$ exist and what is its value? Here too, it was conjectured that there would be conformal invariance in the limit. For the question at hand that would mean that if $f : D \to D'$ is a conformal homeomorphism from D onto a domain D', then $P_0(D, A, B) = P_0(f(D), f(A), f(B))$. Using conformal invariance and some arguments which have not been made rigorous, John Cardy ([2]) even found explicit values for $P_0(D, A, B)$. Cardy typically took the upper half plane for D and his answers contained various hypergeometric functions. Lennart Carleson realized that the answers would look much simpler when D is an equilateral triangle. This inspired S. Smirnov ([9]) to take the triangular lattice for $\mathcal{G}$ and he succeeded in proving the conformal invariance for this choice.

A connection between critical percolation and SLE is suggested by Schramm in [8]. Schramm writes that assuming a conformal invariance hypothesis, the exploration process should be distributed as SLE_6. The simplest version of the exploration process for critical percolation is as follows: Consider percolation on the hexagonal lattice (the dual of the triangular lattice) and let each hexagonal face be independently open or closed with probability 1/2. Assume that the origin is the center of some hexagon and all hexagons which intersect the open negative real axis (the closed positive real axis) are open (respectively, closed). Then the exploration curve is a curve γ on the boundaries of the hexagonal faces and starting on the boundary of the hexagon containing the origin, such that γ always has an open hexagon on its left and a closed hexagon on its right.

Werner's Work: Power Laws and Critical Exponents for Percolation

Many power laws had been conjectured in the physics literature for the asymptotic behavior of functions at or near the critical point for percolation. In fact, for $\mathcal{G}$ a two-dimensional lattice even the exact value of the corresponding exponents had been predicted. In the two articles [6], [10] Werner and co-authors established most of these power laws and rigorously confirmed the predicted values for the critical exponents in the case of site percolation on the triangular lattice. Before this, no approach which mathematicians deemed rigorous existed for these problems. Even now, the results are restricted to percolation on the triangular lattice because this is the only lattice for which conformal invariance of the scaling limit has been proven.

Previous work, partly by the present author, reduces the problem to the following: Consider critical site percolation on δ times the triangular lattice; this is percolation at $p = 1/2$ for this lattice. Let $b_j(\delta, r, R)$ be the probability that there exist j disjoint crossings of the annulus $\{r \leq |z| < R\}$. Here a crossing may be either an open crossing or a closed crossing, but for $j \geq 2$ we require in b_j that among the j crossings there are some open ones and some closed ones. Then show that $b_j(r, R) = \lim_{\delta\downarrow 0} b_j(\delta, r, R)$ exists and that $\lim_{R\to\infty}[\log R]^{-1} \log b_j(r, R) = -(j^2 - 1)/12$ if $j \geq 2$, and equals 5/48 if $j = 1$. An analogous half-plane problem is to set $a_j(\delta, r, R) =$ the probability that there exist j disjoint open crossings of the semi-annulus $\{z \in \mathbb{H} : r \leq |z| < R\}$. Then show the existence of $a_j(r, R) = \lim_{\delta\downarrow 0} a_j(\delta, r, R)$ and show that $\lim_{R\to\infty}[\log R]^{-1} \log a_j(r, R) = -j(j + 1)/6$. It is much easier to deal with the problem in the half-plane than in the full plane, because it is possible to define the open crossing of the semi-annulus which is closest to the positive real axis. The fact that one cannot single out such a first crossing of the full annulus is also the reason why in b_j with $j \geq 2$ one has to require the existence of both open and closed crossings, but in a_j they can be all of the same type (or, in fact, any mixture of types). The proofs of these limit relations in [6], [10] are based on the result of [9] that the scaling limit for the exploration process in critical percolation on the triangular lattice is an SLE_6 curve. Actually, one needs convergence in a stronger sense than given in [9], but this convergence has now been provided in [1]. The limit relation for percolation is then reduced to finding certain crossing probabilities for SLE_6 which already had been tackled in [4], [5].

Over the years the proofs have been simplified quite a bit. [7] gives a general outline how to prove that a given process has SLE_κ as a scaling limit. One model for which this is still an open problem is self avoiding walk. It is believed that two-dimensional self avoiding walk has $SLE_{8/3}$ as scaling limit. An indication for this is the fact that $SLE_{8/3}$ is the only SLE_κ process which has the so-called restriction property. This property relates SLE conditioned to stay in D with the same SLE process conditioned to stay in D', for a domain D and a subdomain D' (see [12]).

Now that the power of SLE to rigorously analyze some models motivated by physics has been demonstrated, the properties of SLE are also being developed for their own sake (see [3]).

There is a list of open problems in Section 11.2 of Werner's St. Flour Lecture Notes [11]. We may add to this that many of the problems mentioned here can also be formulated in dimension > 2 and the two-dimensional tools of conformal invariance and SLE are probably of little help there.

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Interviews with Three Fields Medalists

Interviewed by Vicente Muñoz and Ulf Persson

Andrei Okounkov, Terence Tao, and Wendelin Werner received Fields Medals at the 2006 International Congress of Mathematicians in Madrid, Spain (Grigory Perelman was also awarded a medal but declined to accept it). These interviews with Okounkov, Tao, and Werner were conducted via email by Vicente Muñoz and Ulf Persson in the fall of 2006 and originally appeared in the December 2006 issue of the *European Mathematical Society Newsletter*.

Andrei Okounkov

Muñoz & Persson: *How did you get interested in mathematics?*

Okounkov: The most important part of becoming a mathematician is learning from one's teachers. Here I was very fortunate. Growing up in Kirillov's seminar, I had in its participants, especially in Grisha Olshanski, wonderful teachers who generously invested their time and talent into explaining mathematics and who patiently followed my first professional steps. I can't imagine becoming a mathematician without them. So it must be that in this respect my professional formation resembles everybody else's.

What was perhaps less usual is the path that led me to mathematics. I didn't go through special schools and olympiads. I came via studying economics and army service. I had a family before papers. As a result, my mind is probably not as quick as it could have been with an early drilling in math. But perhaps I also had some advantages over my younger classmates. I had a broader view of the universe and a better idea about the place of mathematics in it. This helped me form my own opinion about what is important, beautiful, promising, etc.

It also made mathematics less competitive for me. Competition is one of those motors of human society that will always be running. For example, we are having this interview because of the outcome of a certain competition. But I believe it distracts us from achieving the true goal of science, which is to understand our world.

M & P: *So, you wouldn't say that competition is the best way to do mathematics?*

Okounkov: I think it is a mistake that competition is actively promoted on every level of math. While kids take solving puzzles perhaps a bit too seriously, grown-ups place the ultimate value on being the first to prove something. A first complete proof, while obviously very important, is only a certain stage in the development of our knowledge. Often, pioneering insights precede it and a lot of creative work follows it before a particular phenomenon may be considered understood. It is thrilling to be the first, but a clear proof is for all and forever.

M & P: *How do you prefer to work on mathematical problems? Alone or in collaboration?*

Okounkov: Perhaps you can guess from what I said before that I like to work alone, I equally like to freely share my thoughts, and I also like to perfect my papers and talks.

There may well be alternate routes, but I personally don't know how one can understand something without both thinking about it quietly over and over and discussing it with friends. When I feel puzzled, I like long walks or bike rides. I like to be alone with my computer playing with formulas or experimenting with code. But when I finally have an idea, I can't wait to share it with others. I am so fortunate to be able to share my work and my excitement about it with many brilliant people who are at the same time wonderful friends.

And when it comes to writing or presenting, shouldn't everyone make an effort to explain? Wouldn't it be a shame if something you understood were to exist only as a feeble neuron connection in your brain?

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M & P: Do you prefer to solve problems or to develop theories?

Okounkov: I like both theory and problems, but best of all I like examples. For me, examples populate the world of mathematics. Glorious empty buildings are not my taste. I recall my teacher Kirillov saying that it is easier to generalize an example than to specialize a theory. Perhaps he did not mean this 100 percent seriously, but there is a certain important truth in those words. Understanding examples links with ability to compute. Great mathematicians of the past could perform spectacular computations. I worry that, in spite of enormous advances in computational methods and power, this is a skill that is not adequately emphasized and developed. Any new computation, exact or numeric, can be very valuable. The ability to do a challenging computation and to get it right is an important measure of understanding, just like the ability to prove is.

M & P: Much of your work has deep connections to physics. Does that mean that you find it essential that mathematics is related to the natural world, or that you would even think of it as subservient to the other natural sciences?

Okounkov: When I said "our world" earlier I didn't mean just the tangible objects of our everyday experience. Primes are as real as planets. Or, in the present context, should I say that celestial bodies are as real as primes? Throughout their history, natural sciences were a constant source of deep and challenging mathematical problems. Let's not dwell now on the obvious practical importance of these problems and talk about something else, namely the rich intuition that comes with them. This complex knowledge was derived from a multitude of sources by generations of deep thinkers. It is often very mathematical. Anyone looking to make a mathematical discovery needs a problem and a clue. Why not look for both in natural sciences?

This doesn't make mathematics a subordinate of other sciences. We bring, among other things, the power of abstraction and the freedom to apply any tools we can think of, no matter how apparently unrelated to the problem at hand. Plus what we know we really do know. So we can build on firmer foundations, hence higher. And look—mathematics is the tallest building on campus both in Princeton and in Moscow.

M & P: There is a common view of the public that computers will make mathematicians superfluous. Do you see a danger in that? And in particular what is your stand on computer-assisted proofs? Something to be welcomed or condemned?

Okounkov: Computers are no more a threat to mathematicians than food processors are a threat to cooks. As mathematics gets more and more complex while the pace of our lives accelerates, we must delegate as much as we can to machines. And I mean both numeric and symbolic work. Some people can manage without dishwashers, but I think proofs come out a lot cleaner when routine work is automated.

This brings up many issues. I am not an expert, but I think we need a symbolic standard to make computer manipulations easier to document and verify. And with all due respect to the free market, perhaps we should not be dependent on commercial software here. An open-source project could, perhaps, find better answers to the obvious problems such as availability, bugs, backward compatibility, platform independence, standard libraries, etc. One can learn from the success of TeX and more specialized software like Macaulay2. I do hope that funding agencies are looking into this.

M & P: The age of the universalists is gone. Nowadays mathematics is very diverse and people tend to get mired in subspecialities. Do you see any remedy to this?

Okounkov: Mathematics is complex. Specialization, while inevitable, doesn't resolve the problem. Mathematics is a living organism; one cannot simply chop it up. So how do we both embrace and resist specialization?

We can be better neighbors. We shouldn't build high fences out of sophisticated words and a "you wouldn't understand" attitude. We should explain what we know in the simplest possible terms and minimal generality. Then it will be possible to see what grows in the next field and use the fruits of your neighbor's labor.

Good social contact makes good neighbors. Effective networks are hard to synthesize but they may be our best hope in the fight against fragmentation of mathematics. I, personally, wouldn't get anywhere without my friends/collaborators. I think there is a definite tendency in mathematics to work in larger groups, and I am certain this trend will continue.

Terence Tao

M & P: When did you become interested in mathematics?

Tao: As far as I can remember I have always enjoyed mathematics, though for different reasons at different times. My parents tell me that at age two I was already trying to teach other kids to count using number and letter blocks.

M & P: Who influenced you to take the path of mathematics?

Tao: I of course read about great names in mathematics and science while growing up, and perhaps had an overly romanticized view of how progress is made; for instance, E.T. Bell's *Men of Mathematics* had an impact on me, even though nowadays I realize that many of the stories in that book were overly dramatized. But it was my own advisors and mentors, in particular my undergraduate advisor Garth Gaudry and my graduate advisor Eli Stein, who were the greatest influence on my career choices.

M & P: What was your feeling when you were told about being a medalist?

Tao: I had heard rumors of my getting the medal a few months before I was officially notified—which meant that I could truthfully deny these rumors before they got out of hand. It was still of course a great surprise, and then the ceremony in Madrid was an overwhelming experience in many ways.

M & P: Do you think that the Fields Medal will put too high expectations on you, thus coming to have an inhibiting influence?

Tao: Yes and no. On the one hand, the medal frees one up to work on longer-term or more speculative projects, since one now has a proven track record of being able to actually produce results. On the other hand, as the work and opinions of a medalist carry some weight among other mathematicians, one has to choose what to work on more carefully, as there is a risk of sending many younger mathematicians to work in a direction that ends up being less fruitful than first anticipated. I have always taken the philosophy to work on the problems at hand and let the recognition and other consequences take care of themselves. Mathematics is a process of discovery and is hence unpredictable; one cannot reasonably try to plan out one's career, say by naming some big open problems to spend the next few years working on. (Though there are notable exceptions to this, such as the years-long successful attacks by Wiles and Perelman on Fermat's last theorem and Poincaré's conjecture respectively.) So I have instead pursued my research organically, seeking out problems just at the edge of known technology whose answer is likely to be interesting, lead to new tools, or lead to new questions.

M & P: Do you feel the pressure of having to obtain results quickly?

Tao: I have been fortunate to work in fields where there are many more problems than there are people, so there is little need to competitively rush to grab any particular problem (though this has happened occasionally, and has usually been sorted out amicably, for instance via a joint paper). On the other hand, most of my work is joint with other collaborators, and so there is an obligation to finish the research projects that one starts with them. (Some projects are over six years old and still unfinished!) Actually, I find the "pressure" of having to finish up joint work to be a great motivator, more so than the more abstract motivation of improving one's publication list, as it places a human face on the work one is doing.

M & P: What are your preferences when attacking a problem?

Tao: It depends on the problem. Sometimes I just want to demonstrate a proof of concept, that a certain idea can be made to work in at least one simplified setting; in that case, I would write a paper as short and simple as possible and leave extensions and generalizations to others. In other cases I would want to thoroughly solve a major problem, and then I would want the paper to become very systematic, thorough, and polished, and spend a lot of time focusing on getting the paper just right. I usually write joint papers, but the collaboration style varies from co-author to co-author; sometimes we rotate the paper several times between us until it is polished, or else we designate one author to write the majority of the article and the rest just contribute corrections and suggestions.

M & P: Do you spend a lot of time on a particular problem?

Tao: If there is a problem that I ought to be able to solve, but somehow am blocked from doing so, that really bugs me and I will keep returning to the problem to try to resolve it. (Then there are countless problems that I have no clue how to solve; these don't bother me at all).

M & P: Do you prefer to solve problems or to develop theories?

Tao: I would say that I primarily solve problems, but in the service of developing theory; firstly, one needs to develop some theory in order to find the right framework to attack the problem, and secondly, once the problem is solved it often hints at the start of a larger theory (which in turn suggests some other model problems to look at in order to flesh out that theory). So problem-solving and theory-building go hand in hand, though I tend to work on the problems first and then figure out the theory later.

Both theory and problems are trying to encapsulate mathematical phenomena. For instance, in analysis, one key question is the extent to which control on inputs to an operation determines control on outputs; for instance, given a linear operator T, whether a norm bound on an input function f implies a norm bound on the output function Tf. One can attack this question either by posing specific problems (specifying the operator and the norms) or by trying to set up a theory, say of bounded linear operators on normed vector spaces. Both approaches have their strengths and weaknesses, but one needs to combine them in order to make the most progress.

M & P: How important is physical intuition in your work?

Tao: I find physical intuition very useful, particularly with regard to PDEs [partial differential equations]—I need to see a wave and have some idea of its frequency, momentum, energy, and so forth, before I can guess what it is going to do—and then, of course, I would try to use rigorous mathematical analysis to prove it. One has to keep alternating between intuition and rigor to make progress on a problem, otherwise it is like tying one hand behind your back.

M & P: What point of view is helpful for attacking a problem?

Tao: I also find it helpful to anthropomorphize various mathematical components of a problem or argument, such as calling certain terms "good" or "bad", or saying that a certain object "wants" to exhibit some behavior, and so forth. This allows one to bring more of one's mental resources (beyond the usual abstract intellectual component of one's brain) to address the problem.

M & P: Many mathematicians are Platonists, although many may not be aware of it, and others would be reluctant to admit it. A more "sophisticated" approach is to claim that it is just a formal game. Where do you stand on this issue?

Tao: I suppose I am both a formalist and a Platonist. On the one hand, mathematics is one of the best ways we know to try to formalize thinking and understanding of concepts and phenomena. Ideally we want to deal with these concepts and phenomena directly, but this takes a lot of insight and mental training. The purpose of formalism in mathematics, I think, is to discipline one's mind (and filter out bad or unreliable intuition) to the point where one can approach this ideal. On the other hand, I feel the formalist approach is a good way to reach the Platonic ideal. Of course, other ways of discovering mathematics, such as heuristic or intuitive reasoning, are also important, though without the rigor of formalism they are too unreliable to be useful by themselves.

M & P: Is there nowadays too much a separation between pure and applied mathematics?

Tao: Pure mathematics and applied mathematics are both about applications, but with a very different time frame. A piece of applied mathematics will employ mature ideas from pure mathematics in order to solve an applied problem today; a piece of pure mathematics will create a new idea or insight that, if the insight is a good one, is quite likely to lead to an application perhaps ten or twenty years in the future. For instance, a theoretical result on the stability of a PDE may lend insight as to what components of the dynamics are the most important and how to control them, which may eventually inspire an efficient numerical scheme for solving that PDE computationally.

M & P: Mathematics is often described as a game of combinatorial reasoning. If so, how would it differ from a game, say like chess?

Tao: I view mathematics as a very natural type of game, or conversely games are a very artificial type of mathematics. Certainly one can profitably attack certain mathematical problems by viewing them as a game against some adversary who is trying to disprove the result you are trying to prove, by selecting parameters in the most obstructive way possible, and so forth. But other than the fact that games are artificially constructed, whereas the challenge of proving a mathematical problem tends to arise naturally, I don't see any fundamental distinctions between the two activities. For instance, there are both frivolous and serious games, and there is also frivolous and serious mathematics.

M & P: Do you use computers for establishing results?

Tao: Most of the areas of mathematics I work in have not yet been amenable to systematic computer assistance, because the algebra they use is still too complicated to be easily formalized, and the numeric work they would need (e.g., for simulating PDEs) is still too computationally expensive. But this may change in the future; there are already some isolated occurrences of rigorous computer verification of things such as spectral gaps, which are needed for some arguments in analysis.

M & P: Is a computer-assisted proof acceptable from your point of view?

Tao: It is of course important that a proof can be verified in a transparent way by anyone else equipped with similar computational power. Assuming that is the case, I think such proofs can be satisfactory if the computational component of the proof is merely confirming some expected or unsurprising phenomenon (e.g., the absence of sporadic solutions to some equation, or the existence of some parameters that obey a set of mild conditions), as opposed to demonstrating some truly unusual and inexplicable event that cries out for a more human-comprehensible analysis. In particular, if the computer-assisted claims in the proof already have a firm heuristic grounding then I think there is no problem with using computers to establish the claims rigorously. Of course, it is still worthwhile to look for human-readable proofs as well.

M & P: Is mathematics becoming a very dispersive area of knowledge?

Tao: Certainly mathematics has expanded at such a rate that it is no longer possible to be a universalist such as Poincaré and Hilbert. On the other hand, there has also been a significant advance in simplification and insight, so that mathematics that was mysterious in, say, the early twentieth century now appears routine (or more commonly, several difficult pieces of mathematics have been unified into a single difficult piece of mathematics, reducing the total complexity of mathematics significantly). Also, some universal heuristics and themes have emerged that can describe large parts of mathematics quite succinctly; for instance, the theme of passing from local control to global control, or the idea of viewing a space in terms of its functions and sections rather than by its points, lend a clarity to the subject that was not available in the days of Poincaré or Hilbert. So I remain confident that mathematics can remain a unified subject in the future, though our way of understanding it may change dramatically.

M & P: What fields of mathematics do you foresee will grow in importance, and maybe less positively, fade away?

Tao: I don't think that any good piece of mathematics is truly wasted; it may get absorbed into a more general or efficient framework, but it is still there. I think the next few decades of mathematics will be characterized by interdisciplinary synthesis of disparate fields of mathematics; the emphasis will be less on developing each field as deeply as possible (though this is of course still very important), but rather on uniting their tools and insights to attack problems previously considered beyond reach. I also see a restoration of balance between formalism and intuition, as the former has been developed far more heavily than the latter in the last century or so, though intuition has seen a resurgence in more recent decades.

M & P: There are lots of definitions of randomness. Do you think there is a satisfying way of thinking of randomness?

Tao: I do see the dichotomy between structure and randomness exhibiting itself in many fields of mathematics, but the precise way to define and distinguish these concepts varies dramatically across fields. In some cases, it is computational structure and randomness that is decisive; in other cases, it is a statistical (correlation-based) or ergodic concept of structure and randomness, and in other cases still it is a Fourier-based division. We don't yet have a proper axiomatic framework for what a notion of structure or randomness looks like (in contrast to, say, the axioms for measurability or convergence or multiplication, which are well understood). My feeling is that such a framework will eventually exist, but it is premature to go look for it now.

M & P: If you were not to have been a mathematician, what career would you have considered?

Tao: I think if I had not become a mathematician, I would like to be involved in some other creative, problem-solving, autonomous occupation, though I find it hard to think of one that matches the job satisfaction I get from mathematics.

Wendelin Werner

M & P: Were you always interested in mathematics?

Werner: Well, as far as I can remember, math was always my preferred topic at school, and I was a rather keen board-games player in my childhood (maybe this is why I now work on 2-dimensional problems?). As a child, when I was asked if I knew what I wanted to be later, I responded "astronomer". In high school, just because of coincidences, I ended up playing in a movie and having the possibility to try to continue in this domain, but I remember vividly that I never seriously considered it, because I preferred the idea of becoming a scientist, even if at the time, I did not know what it really meant. When it was time to really choose a subject, I guess I realized that mathematics was probably closer to what I liked about astronomy (infiniteness, etc.).

M & P: Have you known about the Fields Medal since an early age, and did it in any way motivate you? In particular what was your feeling when you were told about being a medalist?

Werner: I learned about the existence of the Fields Medal quite late (when I graduated roughly). In fact, I remember some friends telling me half-jokingly, half-seriously, that "you will never get the Fields Medal if you do this" when I told them that I was planning to specialize in probability theory (it is true that this field had never been recognized before this year). It is of course a nice feeling to get this medal today, but it is also very strange: I really do not feel any different or "better" than other mathematicians, and to be singled out like this, while there exist so many great mathematicians who do not get enough recognition is almost embarrassing. It gives a rather big responsibility, and I will now have to be careful each time I say something (even now). But again, it is nice to get recognition for one's work, and I am very happy. Also, I take it as a recognition for my collaborators (Greg Lawler and Oded Schramm) and for the fact that probability theory is a nice and important field in contemporary mathematics.

I guess that all these feelings and thoughts were present in my head when I hung up the phone after learning from John Ball in late May that I was awarded the medal. I knew that it was a possibility, but nevertheless it took me by surprise.

M & P: Are there some mathematicians who you admire particularly?

Werner: I am not a specialist of history of mathematics, but I find it amazing what the great nineteenth century mathematicians (Gauss, Riemann, ...) managed to work out—I certainly feel like a dwarf compared to giants. I have also the greatest respect for those who shaped probability theory into what it is now (Kyoshi Itô, Paul Lévy, Ed Harris, Harry Kesten, to mention just a few). Also, I owe a lot to the generation of probabilists who are just a little older than I am (just look at the list of Loève prize winners for instance. I really felt very honored to be on that list!) and opened so many doors.

M & P: Do you fear that the Fields Medal will inhibit you by putting up too high expectations for future work?

Werner: It is true that in a way, the medal puts some pressure to deliver nice work in the future and that it will probably be more scrutinized than before. On the other hand, it gives a great liberty to think about hard problems, to be generous with ideas and time with Ph.D. students for instance. We shall see how it goes.

***M & P:** As you pointed out, your chosen subject has never been awarded a medal before. Is it because it has been considered as "applied mathematics"? Would you call yourself an "applied mathematician"?*

Werner: Probability theory has long been considered as part of applied mathematics, maybe also because of some administrative reasons (in the U.S., probabilists often work in statistics departments that are disjoint from the mathematics departments). This has maybe led to a separate development of this field, slightly isolated. Now people realize how fruitful interactions between probabilistic ideas and other fields in mathematics can be, and this automatic "applied" notion is fading away (even if probability can be indeed fruitfully applied in many ways). In a way, the field that I am working on has been really boosted by the combination of complex analytic ideas with probability (for instance Schramm's idea to use Loewner's equation in a probabilistic context to understand conformally invariant systems). I personally never felt that I was doing "applied" mathematics. It is true that we are using ideas, intuitions, and analogies from physics to help us to get some intuition about the concepts that we working on. Brownian motion is a mathematical concept with something that resonates in us, gives us some intuition about it, and stimulates us.

***M & P:** Is there any risk that computers will make mathematicians obsolete, say by providing computerized proofs? Or do you believe this will stimulate mathematics instead?*

Werner: Well, one of my brothers is working precisely on computer-generated or computer-checked proofs. I have to be careful about what I say, especially since my own knowledge on this is very thin. I personally do not really use the computer in my work, besides TEX and the (too long) time spent with emailing. It can very well be that some day soon, computers may be even more efficient than now in helping understanding and proving things. The past years have shown how things that looked quite out of reach ended up being possible with computers.

***M & P:** Do you have any other interests besides mathematics?*

Werner: I often go to concerts (classical music) and play (at a nonprofessional level, though) the violin. Often, I hear people saying "yes, math and music are so similar, that is why so many mathematicians are also musicians". I think that this is only partially true. I cannot forget that many of those I was playing music with as a child simply had to stop playing as adults because their profession did not leave any time or energy to continue to practice their instruments: doctors usually have many more working hours than we do. Also, music is nicely compatible with mathematics because—at least for me—it is hard to concentrate on a math problem more than 4-5 hours a day, and music is a good complementary activity: it does not fill the brain with other concerns and problems that distract from math. It is hard to do math after having had an argument with somebody about non-mathematical things, but after one hour of violin scales, one is in a good state of mind.

Also, but this is a more personal feeling, with the years, I guess that what I am looking for in music becomes less and less abstract and analytical and more and more about emotions—which makes it less mathematical...

But I should also mention that, as far as I can see it, mathematics is simultaneously an abstract theory and also very human: When we work on mathematical ideas, we do it because in a way, we like them, because we find something in them that resonates in us (for different reasons, we are all different). It is not a dry subject that is separate from the emotional world. This is not so easy to explain to nonmathematicians, for whom this field is just about computing numbers and solving equations.

Presidential Views: Interview with James Glimm

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with James G. Glimm, who began a two-year term as president on February 1, 2007. The interview was conducted in fall 2006 by *Notices* senior writer and deputy editor Allyn Jackson. Glimm is a Distinguished Professor of Applied Mathematics and Statistics at Stony Brook University.

An interview with past-president James G. Arthur appeared in the February 2007 issue of the *Notices*.

***Notices:** How do you see your role as AMS president?*

Glimm: The AMS has a very rich committee structure, and the committees properly want to have authority to make their own decisions and recommendations. The most direct influence that the president has is through appointments to committees. The single most important thing the president does is to make quality appointments to all of the committees that the AMS runs. But beyond that, we all have opinions, and the president, in a leadership role, can articulate points of view that, if they turn out to be shared by other people, will help shape events.

I am concerned about preserving the breadth of the AMS to serve all of its constituencies. Mathematics is a growing enterprise, and some of the activities of the AMS should grow to keep up with that. We have good relations with other parts of the mathematical sciences, in particular SIAM [Society for Industrial and Applied Mathematics] and the statistics societies. But different subjects gravitate to one place or another, and we want to make sure we keep our doors open, and perhaps our activities have to expand to do that. So I don't think we want to keep our activities fixed in size, but rather they should be able to expand if there is activity to support our expanding and if the quality remains high.

***Notices:** You have been very active in SIAM. How do you view cooperation between the AMS and other organizations like SIAM?*

Glimm: Science in general across all subjects is changing quite dramatically in its organization, and much more cooperation is needed among societies, among research groups, among institutions, among research teams, and so on. The AMS has to be part of that and cooperate with its natural colleagues.

James Glimm

Federal policymakers do not perceive big differences between different parts of mathematics—and in fact if they did, it would be only to our detriment, because it would be seen as rivalry, which is not healthy to the conduct of research. We want to continue to present a unified picture to Congress and to the federal policymakers, so arriving at that with SIAM and with the statistics associations is quite essential for having an effective policy.

At another level, I think that there are many areas where we can cooperate with other societies by participating in one another's meetings. Each suggestion like that has to be evaluated on its merits and succeed on its merits, but I think there are probably cases where that would be helpful for everybody.

***Notices:** What about cooperation with the MAA [Mathematical Association of America]?*

Glimm: I am very interested in the education issue, although I don't have any prior involvement with it. Mathematics education is a huge issue, vastly important to the country. It breaks up into three ranges: graduate, undergraduate, and kindergarten through twelfth grade. I am not sure that there are particular problems in the graduate area. In any case, if there are, addressing them would not involve cooperation with other societies. At the K-12 level the problems go way beyond the math community, because they involve high school teachers, boards of education, parents, and others. The solutions probably will involve aspects that deal with all of those different communities. Mathematicians will have a contributory but essential role. But within that limitation, I think there are important things mathematicians can do, so we have to cooperate quite broadly with other people who are trying to solve the same problems and certainly cooperate with the MAA in the process.

The mission of the MAA is in undergraduate education. However, this is an area where the AMS also has a responsibility. At many schools the AMS members are the leading faculty, and they are teaching undergraduates, so they have direct responsibility for their own students. I have been talking to many people who are involved in this issue, and it turns out that everyone thinks that *they* don't have a problem but maybe someone else does. But deans and students, when surveyed, do think there is a problem in this area. So I think you could say that, at the very least, there is a disconnect between different people with a direct interest in undergraduate mathematics education. It would be good to look into that and see where those differences in perception arise.

***Notices:** How do you see the role of the AMS internationally?*

Glimm: The whole world has certainly become globalized, and mathematics has been in the forefront of that. It's been international for decades. I think most mathematicians primarily feel loyalty to mathematics as an international force, and I think we should welcome that point of view. The International Congress is a very strong activity with a lot of vitality. But I don't have any particular agenda. I think that things are going well as far as international cooperation is concerned. We have special relations with our two neighbors, Canada and Mexico, and that's a good thing.

***Notices:** One major challenge facing the AMS is attracting young mathematicians to join. Why aren't young people joining, and what can be done to attract them?*

Glimm: I'm not sure about the answer to that. In general, people will join when they see an advantage in the AMS being part of their professional life. I have to become better informed about what is being tried and what could be tried.

***Notices:** Meetings and publications are central activities of the AMS. What can be done to ensure that these activities remain vital and relevant to members?*

Glimm: That's absolutely the most central issue, I think, because if the center doesn't work, the rest is not going to work either. I think that the meetings should grow in size. I understand that everyone wants a small meeting, and I am reminded that everyone wants a small house with a large number of large rooms. The small meeting model is predicated on the idea that one's friends would be there. But the Society has gotten very big, and to serve the breadth of mathematics and to have everyone's friends there probably will require some increase in the meeting size. The failure to do that will simply be to restrict inwards the scope of mathematics, which I don't think would be very healthy. Parts of the Joint Meetings have grown; they have become much more of a professional activity in the sense that nonresearch issues, such as education and policy, are on the agenda. I think that's good, and we should be pleased about that, but as those activities expand, I think the research program should expand also. By doing that we can increase the opportunities for young people to participate and attend the meetings. I hope they would then decide to join the Society also.

***Notices:** And what about publications?*

Glimm: The AMS is presently considering an expansion in the number of journal pages. Because the private-sector journals have increased publication, to have a reasonable influence on the market it is argued that we should try to keep our share somewhat constant. I think it is a good idea. We want to maintain quality, but I think that that is possible because there are simply more people doing mathematics, and they are doing it in an increased number of areas. To keep a static publication policy is probably not in the interest of anybody.

These discussions are responding to the same forces, which is that mathematics is simply a bigger activity. If you try to fit a bigger activity into a box that doesn't expand, then something gets crushed, and that's undesirable. So in order to represent the breadth of opportunities that are really natural to mathematics, meetings and publications should grow at a moderate rate. It's always necessary to assure that you preserve quality while you do that, but I think a modest growth in a range of Society activities would be good.

***Notices:** Do you think the AMS should start new journals?*

Glimm: New journals come up from time to time. I think the most successful journals are not started by a decision at the top. They are started by a dedicated group of people who see a need for something, and they fight for their case. There are many ways for a journal to flounder, and you don't want to get started on one that is going to flounder. But if a proposal for a new journal is strong and cogent and well argued, then the Publications Committee can consider it. I wouldn't rule that out, but as the president I wouldn't want to start a search for areas where new journals could be started. It is a fact that starting a successful new journal is very difficult at the present time, because libraries have trouble paying for their current subscription base.

Notices: *In your election statement you wrote: "Fundamental advances in mathematical reasoning have seldom been as pervasively important to society as they are today, and at the same time they are more at risk of being compromised." Can you expand on this? What exactly is at risk?*

Glimm: A lot of this is tied to computation and the way that computation is replacing experiments in many cases. Not only are the physicists using mathematical reasoning, which they have done since the days of Isaac Newton, but with the computer they are actually solving equations and using them to build things. They are relying on mathematical models, as opposed to trial and error. The Edisonian style of invention is not the same driving force as it was one hundred years ago. Analytic models are the workaday method of engineering, and you see this across many industries. So the role of mathematics is certainly increasing. Even outside the engineering and physical sciences, entire new areas are becoming mathematized. Economics and finance is an example, and biology is another example, where twenty years ago the mathematicians wouldn't have gotten in the front door. Now they often have a leading intellectual role in making decisions in those areas and guiding the direction. The human genome project is wildly successful, and strong mathematical modeling and analysis were part of both of the teams that competed to complete the project. The importance of mathematics to society is unquestioned. This has been well documented: for instance, in the promotional materials created by the AMS, the "Mathematical Moments", which are on the bulletin boards of math departments around the world and have been translated into many languages.

The second part of the question was, What is at risk? Actually, mathematics is in danger of drowning in its own success, because as it succeeds, people are trying to capture, control, and direct it. This is probably more true for the applied than for the pure mathematician, but there are certainly forces that would take away the freedom to control our research agenda. These forces are very pervasive, and they are subtle; they sort of lap up like rising water, so you don't notice it from year to year, and people even welcome it because it comes with good news attached as well as bad. But it is definitely a danger. I think it is worth some thought.

Notices: *Where do you see these forces operating?*

Glimm: They operate across the board. I think to a large extent mathematicians have been on the periphery, so they probably do not see them. But where they do operate, there is a cooperation between the federal funding agencies and university administrations across the entire structure of science. These changes for the most part have not reached the mathematical community at the present time, but they certainly are coming rather close, so it's not too early to start thinking about these issues.

Notices: *You said that mathematicians could lose the freedom to set their own research agenda. How would that happen?*

Glimm: It hasn't happened in pure mathematics, but in the more applied sciences you can see this happening. As science gets more expensive, it can be conducted only with the permission of the federal agencies that fund the science, and they have their own ways of deciding what they want to support and what they don't. In general, other people in the system tend to swim along with the tide.

New York City Programs Provide a Model for National Teaching Corps

Allyn Jackson

When James Simons received his Ph.D. in mathematics in 1962 from Berkeley, he was the first person to finish a doctorate under the National Defense Education Act. The NDEA had been launched just four years earlier and was one of several initiatives taken by the U.S. government after the Soviet Union's launching of the Sputnik satellite. "I was a beneficiary of a program that worked," Simons remarked. Today he is known not only for the Chern-Simons invariants, which he developed in the 1970s with Shiing-Shen Chern and for which he received the 1976 AMS Veblen Prize, Simons has also become well known as an enormously successful financial entrepreneur and the founder of the investment firm Renaissance Technologies. *Forbes* magazine estimates Simons's net worth to be US$2.6 billion, making him the 278th richest person in the world.

Simons is trying to spur the creation of another government program that would work. He wants to see a government-funded nationwide effort to put teachers with solid mathematics knowledge into the nation's classrooms. As the first step, he created Math for America (MfA), a New York City-based project that could serve as a model for a national program. The main part of MfA is the Newton Fellowship program, which provides teacher education to people with strong mathematics and science backgrounds and gives them a generous salary supplement in return for a four-year commitment to teach mathematics in the New York City public schools. As the second crop of Newton Fellows gets through its first year of teaching in the 2006–2007 academic year, a bill to establish a similar program at the national level is wending its way through Congress.

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Two Curves that Don't Match

About ten years ago Simons began to sense that the nation was not producing enough highly trained mathematicians and scientists. He can observe this phenomenon firsthand at Renaissance Technologies, which hires primarily people with science and mathematics backgrounds, many of them foreign-born. As opportunities for those with good knowledge of mathematics and science have increased, the incentive to become teachers in those areas has continued to diminish. "So you have two curves that don't make any sense: Fewer and fewer people are teaching for a field that wants more and more people," Simons noted. "The economy in the long run is really going to suffer."

So what did he do? Naturally, he played poker. A few years back, as a way to raise money for the Mathematical Sciences Research Institute in Berkeley, Simons organized a charity poker tournament with some of his colleagues from the finance world. Charity poker tournaments have since become commonplace, "but we were the first," Simons said. And it worked: the tournament raised US$1 million for MSRI. Poker tournaments have also been a main source of funding for the MfA; the most recent such tournament, held in spring 2006, raked in almost US$2 million. The generosity of Simons's poker buddies shows that they share his deep concern about the poor state of mathematics education in the United States.

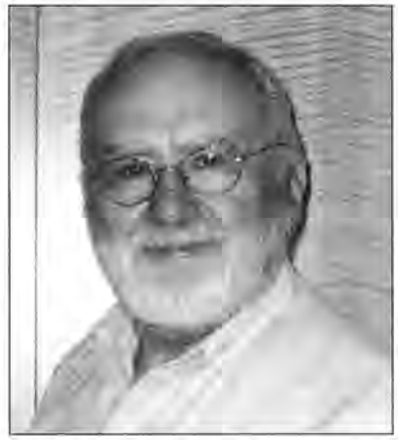
The money generated by the poker tournaments goes directly to support the MfA's programs, such as the Newton Fellowships. Funds are also contributed by the Simons Foundation, a philanthropic organization established in 1994 by Simons and his wife, Marilyn Hawrys Simons, to support basic research in mathematics and science (including research into causes of and possible cures for autism). Through these sources, US$25 million has been committed to the MfA. The operating expenses of the MfA are paid separately by the Simons Foundation. The small MfA staff of eight people is headed by executive director Irwin Kra, a mathematician now retired from Stony Brook University. Simons was on the faculty at Stony Brook from 1968 until 1978, when he moved into finance. Kra and Simons have known each other for almost forty years. When asked how he got involved with the MfA, Kra states simply, "Jim called me and asked me to get involved."

An Elegant Solution

Kra came on board in November 2003, before the Newton Fellowship program was in place, and he developed the idea together with some other members of the MfA board. Kra called the program "an extremely elegant solution" to the problem of getting people with good mathematics backgrounds into the teaching profession. Step one is ensuring that fellowship candidates really know mathematics through an exhaustive interview and testing procedure that Kra characterized as being "perhaps as rigorous as the admission procedure to any Ph.D. program in the United States." Step two is to pay a stipend and tuition for a masters of teaching program, where the fellows learn about pedagogy. Step three is to supplement the fellows' salaries while they teach for four years in the New York City public schools. Typical starting salaries for the city's public school teachers are in the range of US$40,000 to US$50,000. The Newton Fellowship program provides salary supplements totaling US$62,000 over the four years.

Of course, this simple outline masks many aspects of the program that are critical to its success. One is the choice of the teacher training programs. The MfA has selected a small number of partner institutions where the fellows enter the masters of teaching program; right now the partners are New York University, Teachers College at Columbia University, and Bard College. The partner institutions make a financial contribution by covering part of the fellows' tuition (between 20 and 55 percent), and the MfA pays the rest. By keeping the number of partner institutions small, the MfA can work closely with them to create a course of study that will be challenging and meaningful for the fellows. For example, one requirement the MfA has insisted on is that the mathematics courses the fellows take must be in mathematics departments, not in education departments. Because the MfA pays a sizable chunk of the fellows' tuition, it has some leverage to make demands.

Because the Newton Fellows are clustered together in the teacher training programs, they form a peer network, and the MfA reinforces the group's closeness with monthly meetings and social events. The meetings continue after the fellows start teaching, with discussions shifting to professional development topics such as classroom management, lesson planning, educational technology, and so forth. According to MfA associate director Dawn Techow, "These meetings are an essential element of the design of the program, because we feel that building a community of math educators who can learn from and support each other is key to retaining them."



Irwin Kra

The MfA pays close attention to where the fellows get their first positions. On the one hand, as Kra explained, "We try to have a program for all schools, so our fellows may not teach in the elite schools," meaning those with entrance examinations, such as the Bronx High School of Science or Stuyvesant High School. On the other hand, as first-time teachers, the fellows should not plunge immediately into the most difficult classrooms. "We do not want our fellows to work in schools where 90 percent of their time will be devoted to discipline problems rather than teaching," Kra said. Even with these constraints, the fellows have all found suitable jobs, and most have had two or three offers.

James Simons

What do these special job requirements for Newton Fellows mean for the MfA's relations with the New York City Board of Education? "It's a complicated situation," Kra acknowledged. "Clearly we are making more work for the Board of Education. They have to think of us as a new idea.... However, they are getting a product that they are very happy with." Teachers' unions, which ordinarily might balk at extra pay for a special coterie of teachers, have reacted positively to the MfA's programs, because the source of the extra pay is private money, not money from the coffers of the Board of Education.

In 2005 the MfA received about one hundred fifty applications to the Newton Fellowship program and chose forty fellows. In the current academic year, there are eighty Newton Fellows teaching in New York, and forty are in masters of teaching programs in the partner institutions. About half of the 2005

Newton Fellowship applicants were from New York State and the rest from across the nation. Those from elsewhere might very well leave New York City at the end of their teaching commitment, and Kra said, "We have no problem with that." The MfA sees the Newton Fellowship program as addressing a national concern. "We want to help New York City tremendously, we are all committed New Yorkers," he said. "But our program is national, and if someone leaves and teaches in San Francisco or Idaho, fantastic."

In addition to the Newton Fellowships, which are aimed at recent mathematics and science graduates and career-changers, the MfA has the Newton Master Teachers program. This program is intended to support people who are already teaching and who have shown themselves to be outstanding in what they do and to have the potential to become educational leaders. The MfA has designated twenty Newton Master Teachers in New York City, providing them with a US$50,000 award spread over four years as well as support for professional development activities. One of the roles of the master teachers is to help the Newton Fellows who are in their first year of teaching. "They are helping to change the atmosphere in the schools," Kra said of the master teachers. "We have extremely high expectations for them, and we want to have as few demands as possible. ... So far the master teachers have exceeded our expectations. They have organized seminars, they have organized presentations for each other and for other teachers, they are working on curricular material—they are really involved."

A Classy Enterprise for Brainy People

Talking with some of the Newton Fellows, one gets the impression that they see the program as an elite, classy enterprise for brainy, high-powered people. David Griswold was an undergraduate in computer science at Caltech when he gradually came to the realization that he did not want a career in research and academia but wanted to teach high school. Up to that time he had always thought he would teach school only after retiring. "I realized that the only reason I had decided that I wanted to teach when I retired was because... high school teaching is a looked-down-upon profession," he said. "It's a low-status job." When he finally did commit to the idea of becoming a teacher, he applied to masters in teaching programs in New York City and found out about the Newton Fellowship program when the MfA contacted him. He finished his teacher training in the summer of 2006 and is now in his first teaching job. "It has been a fantastic program," he said. "I love the fact that I got here and immediately had a peer group of people like me... who are very interested in math and also very interested in the teaching." He plans to keep a foot in the door of computer science by doing programming during the summer.

Jesse Johnson studied mathematics and filmmaking at Hampshire College and after graduating worked for three years in film schools. She had considered teaching but had not committed to it as a career choice. After a year in the Newton Fellowship program she has no doubts that she made the right choice. "The prestige of the fellowship and the ease of being able to get so much support made it easy to decide to do it," she said. And the financial benefit "makes it easy to stay committed." One impression she had after completing her first year in the program was a lack of interaction with mathematicians. "That is a call that I want to put out," she said, "that I want to know more mathematicians. ... I want them to be involved in curriculum change and policy, but also to be informed by having put some of their time and energy into finding out what it is like to be in [school] classrooms." She said that the MfA could help to stimulate such interactions, because the Newton Fellows come primarily from backgrounds in academic mathematics, giving them natural points of contact with college and university mathematicians.

Going National

In February 2006 bills were introduced into the U.S. Senate and House of Representatives to establish the Math Science Teaching Corps, known by its acronym MSTC, pronounced "mystic". The basic aim is to create a cadre of outstanding mathematics and science teachers by giving them extra pay and support, much as is done in the MfA's Newton programs. In fact, the main goal behind the establishment of the MfA was to provide pilot projects that could serve as a model for a national program. The bill was sponsored in the Senate by Charles Schumer (D-NY), a longtime friend of James Simons, and in the House by Congressman Jim Saxton (R-NJ). The bill is now in committee: the Health, Education, Labor, and Pensions Committee in the Senate, and the Education and the Workforce Committee in the House. Exactly when Congress might act on the MSTC bill is difficult to predict, but in any case it is unlikely the money would be appropriated in the current fiscal year.

And serious money is involved. The aim is to have about 20 percent of all secondary mathematics and science teachers be members of the corps, so that MSTC would at any given time have about 80,000 members. The program would ramp up to that number, and in the steady state the estimated yearly cost would be US$1.75 billion. The bill spells out a proposed structure for the program, including requiring MSTC candidates to pass a standardized test that would be approved and reviewed periodically by the National Academy of Sciences. "It's a fairly complicated program," Kra said, "but we think it needs to happen."

Kra compared a teacher becoming a member of MSTC to a mathematician receiving a grant from the National Science Foundation (NSF). Mathematicians apply to the NSF for grants for a fixed term, and the most outstanding get funded. They can also reapply. In the same way, teachers could apply to become MSTC members and receive a salary supplement for four years. At the end of that period, they can reapply to MSTC. Thus the MSTC stipends are not a permanent addition to salary, just as summer salary on an NSF grant is not permanent. MSTC teachers must perform at a high level to be readmitted. As Kra explained, "We expect the corps members not only to be superb teachers but also to be leaders in the field, mentor other teachers, work on curricula if necessary, influence boards of education, and so forth."

Support of the major teachers' unions, such as the National Education Association and the American Federation of Teachers, is sure to be a major factor in getting the MSTC bill passed. Kra said that the MfA has been talking with the unions about the bill, and while the unions have suggested some modifications, these have not affected the main principles of the MSTC program. Another important factor will be whether the MSTC bill fits in with the educational agenda of the Bush administration, noted Samuel M. Rankin III, director of the AMS Washington office. "Whether the president would sign such a bill depends on many issues," he said, "for example, the budget deficit, whether or not he sees this as a complement or an intrusion to NCLB [Bush's "No Child Left Behind" agenda], whether or not other bills are introduced regarding K-12 education that would compete for new funds, and whether or not this would help him or the Republicans holistically."

The United States has a strong tradition of local control of education and a concomitant suspicion of big national educational programs, so there could be some opposition at the local level to MSTC. "Our counterargument to that is, we are not mandating anything," Kra said. "We are not insisting that anyone become a member of the Math Science Teaching Corps. Teachers would be foolish not to try to become members, because it gives them lots of benefits, but it is a completely voluntary thing. We are not insisting that school districts hire these members of the corps. They would be foolish not to hire them if the corps really produces better teachers."

Good Teachers Rise above the Curriculum

Regardless of the fate of the MSTC bill, its introduction into Congress has highlighted a growing recognition that teacher content knowledge is the key to improving mathematics and science education. This recognition has come in the wake of a wave of educational reform efforts that tended to emphasize curricular change over teacher qualifications. Several recent reports, such as *Rising Above the Gathering Storm: Energizing and Employing America for a Brighter Economic Future,* issued in 2006 by the National Academy of Sciences, have strongly emphasized the need for science and mathematics teachers who really know their subjects. Indeed, many of the recommendations in the NAS report express the same principles embodied in the MfA's programs and the MSTC bill.

Simons put it this way. Suppose you wanted to learn to fly an airplane and your instructor tells you that he is not really a pilot, but he knows how to take off and land. You would say no thanks, even if he told you he would be using the most up-to-date flight school curriculum. "If you have a person who does not know mathematics, I don't care what the curriculum is, the person is not going to do a very good job of teaching," Simons remarked. "But if you have a person who does know mathematics, then he or she has a reasonable chance of imparting some of that knowledge to the students, no matter what the curriculum is." The MfA is banking on the idea that better pay will attract such people. It's not a bad bet.

Mathematics People

Mathematics Student Wins Siemens Competition

For the second straight year, a high school mathematician has won the grand prize in the Siemens Competition in Math, Science, and Technology. The top honors for the 2006–2007 competition went to DMITRY VAINTROB, a senior at South Eugene High School in Eugene, Oregon. He won a US$100,000 scholarship in the individual category.

Vaintrob's project, "The String Topology BV Algebra, Hochschild Cohomology, and the Goldman Bracket on Surfaces", involves the new mathematical field of string topology. Focusing on mathematical shapes, his work offers insights that are universal and applicable in any field. His research could provide knowledge that mathematicians and physicists might apply to help understand the fundamental forces of nature: electricity, magnetism, and gravity.

Left to right: Bettina von Siemens; Dmitry Vaintrob; Secretary of Education Margaret Spellings; and George Nolen, president and CEO of Siemens Corporation. *Photo courtesy of Siemens Corporation.*

"Mr. Vaintrob found a very beautiful formula for describing the way shapes combine in string theory," said competition judge Michael Hopkins of Harvard University. "His work is at the Ph.D. level, publishable and already attracting the attention of researchers."

Vaintrob was introduced to this topic of research by his mentor, Pavel Etingof of the Massachusetts Institute of Technology, who proposed a problem that came out of his own recent work. "It was an insanely difficult problem, which he solved within weeks and then came up with an important additional development," said Hopkins. "This brilliant young mathematician showed amazing maturity and perspective which would be surprising in a graduate student, let alone a high school senior."

Vaintrob is a volunteer in his high school library and in the mathematics library at the University of Oregon. He also organized the math club in his high school. He is a pianist and enjoys reading classical literature and memorizing poetry, a Russian tradition. He is fluent in Russian, French, and English. He plans a career teaching mathematics on the college level.

The Siemens Competition, a program of the Siemens Foundation, is administered by the College Board. The awards were presented by U.S. Secretary of Education Margaret Spellings.

—From a Siemens Competition announcement

Lebowitz Receives Max Planck Medal

JOEL LEBOWITZ of the Center for Mathematical Sciences Research at Rutgers University has been awarded the Max Planck Medal of the German Physical Society (DPG).

Lebowitz is being honored for his life's work in statistical physics.

Lebowitz's work focuses on heat transport, magnetism, and hydrodynamics, among many other areas. He has also studied the "time's arrow" phenomenon, the question of why time always continues unstoppably and why physical processes always take place in a single direction.

According to the award citation, Lebowitz was honored for "promoting new streams of physics and for his great enthusiasm in bringing several generations of scientists to the field," as well as for his important contributions to the statistical physics of equilibrium and nonequilibrium systems, especially his contributions to the theory of phase transitions, the dynamics of infinite systems, and stationary states in nonequilibrium.

The Max Planck Medal is awarded for outstanding achievements in the field of theoretical physics. Previous award winners include Max Planck, Albert Einstein, and Carl Friedrich von Weizsäcker.

—From a German Physical Society announcement

Prizes of the Mathematical Society of Japan

The Mathematical Society of Japan (MSJ) awarded a number of prizes in autumn 2006.

HIROSHI ISOZAKI of Tsukuba University was awarded the Autumn Prize for his contributions to the study of scattering theory and inverse problems, especially for his proof of the eigenfunction expansion theorem for 3-particle Schrödinger equations and for his approach to the inverse boundary value problems in Euclidean space based on hyperbolic geometry. The Autumn Prize is awarded to a mathematician who has made outstanding contributions within the preceding five years to mathematics in the highest and broadest sense.

TOSHIKI MABUCHI of Osaka University and TAKASHI SHIOYA of Tohoku University were awarded the Geometry Prizes. Mabuchi was recognized for his fundamental research work on the existence problem for Kähler-Einstein metrics, in particular his study of the so-called Mabuchi functional defined on the space of admissible Kähler metrics. Shioya was honored for his research work on the geometry of Alexandrov spaces, in particular his study of the singular set of these spaces when the curvature is bounded below, and for initiating a new direction in the theory of the Laplacian on these singular spaces.

The Analysis Prizes were awarded to NARUTAKA OZAWA and NAKAHIRO YOSHIDA, both of the University of Tokyo, and JUN KIGAMI of Kyoto University in recognition of their outstanding contributions in analysis. Ozawa was honored for a series of results on operator algebras arising from discrete groups and related topics, particularly his work on hyperbolic group von Neumann algebras and its applications. Yoshida was honored for his work on a theory of asymptotic expansion and statistical inference for stochastic processes and its applications. Kigami was honored for his fundamental theory of analysis on fractals and for recent developments in resistance forms and local Nash inequalities.

The Takebe Senior Prize was awarded to MAKOTO NAKAMURA of Tohoku University for his study of initial boundary value problems for nonlinear hyperbolic equations. Takebe Junior Prizes were awarded to the following five mathematicians. TAKESHI KATSURA of Hokkaido University was honored for his study of topological dynamical systems and C^*-algebras. KENTARO SAJI of Hokkaido University received an award for his studies on geometry of fronts and their singularities. EIGE FUJIKAWA of Sophia University was honored for her study of dynamics of quasiconformal mapping class groups acting on infinite-dimensional Teichmüller spaces. TARO YOSHINO of Kyoto University was selected for his solution of the Lipsman conjecture and his study of discontinuous transformation groups on non-Riemannian homogeneous spaces. TERUYUKI YORIOKA of Shizuoka University was honored for his combinatorial investigation of the continuum. The Takebe Prizes were established to encourage young mathematicians in their research and are awarded to recipients chosen from nominations by members of the Mathematical Society of Japan.

—From a Mathematical Society of Japan announcement

Sofya Kovalevskaya Award Recipients Announced

Two mathematical scientists are among the twelve recipients of the Sofya Kovalevskaya Award of the Alexander von Humboldt Foundation of Germany for 2006.

OLGA HOLTZ of the University of California, Berkeley, works in numerical analysis, using both pure and applied mathematics to develop a method of matrix multiplication that, in the words of the prize citation, "should provide the solution to a multitude of computational calculations in science and engineering." She received her doctorate from the University of Wisconsin, Madison. Her primary research areas are numerical analysis and scientific computing, matrix and operator theory, approximation theory, orthogonal polynomials, wavelets, and splines. Her host institution for the Kovalevskaya Prize is Berlin Technical University.

BENJAMIN SCHLEIN of the University of California, Davis, received his doctoral degree from the Swiss Federal Institute of Technology (ETH). His work involves developing mathematical methods that will make it possible to derive simpler equations to describe the dynamics of macroscopic systems and creating a mathematical basis for assessing and developing further applications in quantum mechanics. His host institution for the Kovalevskaya Prize is the University of Munich.

The Kovalevskaya Prize is funded by the Federal Ministry of Education and Research of Germany. Recipients receive awards of up to €1.2 million (approximately US$1,500,000), which allow them to concentrate on high-level, innovative research work in Germany, virtually without administrative constraints, and which give them the ability to finance their own work groups at German universities and research institutes of their choice for up to four years.

—From a Humboldt Foundation announcement

Rhodes Scholarships Awarded

Two students in the mathematical sciences are among the thirty-two American men and women who have been selected as Rhodes Scholars by the Rhodes Scholarship Trust. The Rhodes Scholars were chosen from among 896 applicants who were endorsed by 340 different colleges and universities in a nationwide competition. The names and brief biographies of the mathematics scholars follow.

MICHELLE M. SIKES of Lakewood, Ohio, is a senior at Wake Forest University, where she majors in mathematical economics. She is captain of the Wake Forest track and cross-country teams and is an NCAA All-American. Michelle has done comparative research in organ donation systems. She plans to study for an M.Sc. in global health science at Oxford University.

AVI FELLER of Scottsdale, Arizona, is a senior at Yale University majoring in political science and applied mathematics. He has interned at the State Department in international environmental policy and has done research on comparative welfare and health care policies. Avi has also sung leading roles in four operas, is the president of the Yale Alley Cats, an a capella group, and is a soloist in the Yale Collegium Musicum. At Oxford he plans to study for an M.Sc. in applied statistics.

Rhodes Scholarships provide two or three years of study at the University of Oxford in England. The value of the Rhodes Scholarship varies depending on the academic field, the degree (bachelor's, master's, doctoral), and the Oxford college chosen. The Rhodes Trust pays all college and university fees and provides a stipend to cover students' necessary expenses while they are in residence in Oxford, as well as during vacations, and transportation to and from England. The total value averages approximately US$40,000 per year.

—From a Rhodes Scholarship Trust announcement

AAAS Fellows Chosen

Four mathematicians have been elected as new fellows to the Section on Mathematics of the American Association for the Advancement of Science (AAAS). The new fellows are: AMY COHEN, Rutgers University; EVANS M. HARRELL II, Georgia Institute of Technology; WARREN PAGE, Yeshiva University; and ALAN S. PERELSON, Los Alamos National Laboratory.

—From an AAAS announcement

Mathematics Opportunities

NSF Support for Interactions with Computer Sciences

The National Science Foundation (NSF) offers opportunities for support through its Mathematical Sciences Priority Area (MSPA) program for research involving interactions between mathematical sciences and computer science (MSPA-MCS). The goal of the MSPA-MCS program is to deepen support of collaborative research involving fundamental mathematics and statistics, together with computer science. The primary focus is on the areas of mathematical and statistical challenges posed by large data sets, managing and modeling uncertainty, and modeling complex nonlinear systems.

Grants are available through the Division of Mathematical Sciences (DMS) of the Directorate for Mathematical and Physical Sciences (MPS) and the Division of Computing and Communication Foundations (CCF) of the Directorate for Computer and Information Science for research and development teams that focus on mathematical and computational innovations relevant to areas of specific interest. These areas include statistical learning in nonlinear spaces, transductive learning, models inspired by statistical physics, and game theoretic ideas as applied to learning theory; also algorithms, techniques, and theories for the modeling, reduction, and visualization of large, real-time data sets.

The deadline for full proposals is **March 12, 2007**. See the website `http://www.nsf.gov/publications/pub_summ.jsp?ods_key=nsf07534`.

—From an NSF announcement

Project NExT: New Experiences in Teaching

Project NExT (New Experiences in Teaching) is a professional development program for new and recent Ph.D.'s in the mathematical sciences (including pure and applied mathematics, statistics, operations research, and mathematics education). It addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, and participating in professional activities. It also provides the participants with a network of peers and mentors as they assume these responsibilities. Each year about sixty faculty members from colleges and universities throughout the country are selected to participate in a workshop preceding the Mathematical Association of America (MAA) summer meeting, in activities during the summer MAA meetings and the Joint Mathematics Meetings in January, and in an electronic discussion network.

Faculty for whom the 2007–2008 academic year will be the first or second year of full-time teaching (post-Ph.D.) at the college or university level are invited to apply to become Project NExT Fellows.

The application deadline is **April 16, 2007**. For more information, see the Project NExT website, `http://archives.math.utk.edu/projnext/`.

Project NExT is a program of the MAA. It receives major funding from the ExxonMobil Foundation, with additional funding from the Dolciani-Halloran Foundation, the Educational Advancement Foundation, the American Mathematical Society, the American Institute of Mathematics, the American Statistical Association, the National Council of Teachers of Mathematics, Texas Instruments, the Association of Mathematics Teacher Educators, the Association for Symbolic Logic, the W. H. Freeman Publishing Company, Maplesoft, MAA Sections, and the Greater MAA Fund.

—Christine Stevens, Project NExT

Call for Nominations for Third World Academy of Sciences Prizes

The Third World Academy of Sciences (TWAS) Prizes will be awarded to individual scientists in developing countries in recognition of outstanding contributions to knowledge in eight fields of science.

Eight awards are given each year in the fields of mathematics, basic medical sciences, biology, chemistry, physics, agricultural sciences, earth sciences, and engineering sciences. Each award consists of a prize of US$10,000 and a plaque. Candidates for the awards must be nationals of developing countries and, as a rule, must be living and working in those countries.

The deadline for nominations for the 2007 prizes is **March 31, 2007**. Nomination forms should be sent to: TWAS Prizes, c/o The Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, 1-34014 Trieste, Italy; fax: 39-040-224559. Further information is available on the World Wide Web at `http://www.twas.org/`.

—From a TWAS announcement

For Your Information

Program Director Positions at NSF

The Division of Mathematical Sciences (DMS) announces a nationwide search for program director positions at the National Science Foundation (NSF).

NSF Program Directors bear the primary responsibility for carrying out the NSF's overall mission: to support innovative and merit-reviewed activities in basic research and education that contribute to the nation's technical strength, security, and welfare. To discharge this responsibility requires not only knowledge in the appropriate disciplines but also a commitment to high standards, a considerable breadth of interest and receptivity to new ideas, a strong sense of fairness, good judgment, and a high degree of personal integrity.

Applicants should have a Ph.D. or equivalent training in a field of the mathematical sciences, a broad knowledge of one of the relevant disciplinary areas of the DMS, some administrative experience, a knowledge of the general scientific community, skill in written communication and preparation of technical reports, an ability to communicate orally, and several years of successful independent research normally expected of the academic rank of associate professor or higher. Skills in multidisciplinary research are highly desirable.

Qualified individuals who are women, ethnic/racial minorities, and/or persons with disabilities are strongly urged to apply. No person shall be discriminated against on the basis of race, color, religion, sex, national origin, age, or disability in hiring by the NSF.

Program director positions recruited under this announcement may be filled under one of the following appointment options:

Visiting Scientist Appointment: Appointment to this position will be made under the Excepted Authority of the NSF Act. Visiting scientists are on unpaid leave status from their home institutions and appointed to NSF's payroll as federal employees. NSF withholds Social Security taxes and pays the home institution's contributions to maintain retirement and fringe benefits (i.e., health benefits and life insurance) either directly to the home institution or to the carrier. Appointments are usually made for up to one year and may be extended for an additional year by mutual agreement.

Intergovernmental Personnel Act (IPA) Assignment: Individuals eligible for an IPA assignment with a federal agency include employees of state and local government agencies or institutions of higher education, Indian tribal governments, and other eligible organizations in instances in which such assignments would be of mutual benefit to the organizations involved. Initial assignments under IPA provisions may be made for a period of up to two years, with a possible extension for up to an additional two-year period. The individual remains an employee of the home institution, and NSF provides funding toward the assignee's salary and benefits. Initial IPA assignments are made for a one-year period and may be extended by mutual agreement.

Temporary Excepted Service Appointment: Appointment to this position will be made under the Excepted Authority of the NSF Act. Candidates who do not have civil service status or reinstatement eligibility will not obtain civil service status if selected. Candidates currently in the competitive service will be required to waive competitive civil service rights if selected. Usual civil service benefits (retirement, health benefits, life insurance) are applicable for appointments of more than one year. Temporary appointments may not exceed three years.

For additional information on NSF's rotational programs, see "Programs for Scientists, Engineers and Educators" on the NSF website at `http://www.nsf.gov/about/career_opps`.

Applicants should send a letter of interest and vita to Deborah F. Lockhart, Executive Officer, Division of Mathematical Sciences, National Science Foundation, 4201 Wilson Boulevard, Suite 1025, Arlington, VA 22230; phone:

703-292-8870; fax: 703-292-9032; email: dlockhart@nsf.gov.

NSF is an Equal Opportunity Employer committed to employing a highly qualified staff that reflects the diversity of our nation. This announcement can also be found at http://www.nsf.gov/publications/vacancy.jsp?org=DMS&nsf_org=DMS.

—DMS announcement

New Math Institute in Wales

In December 2006 the new Wales Institute of Mathematical and Computational Sciences (WICMS) was launched in Swansea. The institute is supported by a government grant of £5 million (approximately US$10 million), phased over four years starting in December 2006. The institute, which involves departments in the University of Wales Swansea, Cardiff University, the University of Wales Aberystwyth, and the University of Wales Bangor, is set to boost the research base in mathematical and computational sciences and to increase income for research in these areas in Wales. WIMCS will also play an important role in fostering collaboration with industry and commerce and enhancing the interaction between universities and schools. The patron professor of WIMCS is Sir Michael Atiyah.

The work of WIMCS will be organized in clusters linking different subject areas and/or universities. Initially the following clusters will be introduced: Analysis, Stochastic Processes and Stochastic Analysis, Mathematical Physics, Computational Modeling, and Statistics and Operational Research.

The funding is for appointing seven research professors and twenty fixed-term research fellows. The research professors will be permanent posts located in one of the departments with the appointees working at least for the first four years as research professors in WIMCS. A search for these positions will soon be advertised.

For more information see the website http://WIMCS.swan.ac.uk.

—Niels Jacob, University of Wales Swansea

Everett Pitcher Lectures

The next series of Everett Pitcher Lectures will be held April 16–18, 2007, on the campus of Lehigh University in Bethlehem, Pennsylvania. The speaker will be George E. Andrews of Pennsylvania State University. The titles of his lectures are: "The Indian Genius, Ramanujan: His Life and the Excitement of His Mathematics" (lecture for a general audience), "Rogers and Ramanujan", and "The Lost Notebook of Ramanujan". The lectures, which are open to the public, are held in honor of Everett Pitcher, who was secretary of the AMS from 1967 until 1988. Pitcher served in the mathematics department at Lehigh from 1938 until 1978, when he retired as Distinguished Professor of Mathematics. He passed away in December 2006 at the age of 94. For further information, contact the Everett Pitcher Lecture Series, Department of Mathematics, Lehigh University, Bethlehem, PA, 18015; telephone 610-758-3788; or see the website http://www.lehigh.edu/~math/pitcher.html.

—From a Lehigh University announcement

Correction

The January 2007 issue of the *Notices*, pages 20–29, carried an interview with Joan S. Birman. On page 22 the picture of Birman with her husband Joe bears a caption erroneously stating that the photo is from 1954. The photo is from around 1974.

—Allyn Jackson

Inside the AMS

AMS Congressional Briefing

AMY LANGVILLE, professor of mathematics at the College of Charleston, spoke to congressional representatives on Capitol Hill on November 16, 2006, about the role mathematics plays in some of today's technologies. Her presentation included illustrations of the mathematics behind Google, Sudoku, counterterrorism, email surveillance, and military flight plans.

Mathematics is the enabling discipline for many of the technologies that we enjoy today, as well as being critical to national security. Mathematics is needed for deciphering the complicated and tangled communication networks of terrorists; for managing the world's largest document collection, the World Wide Web; and for formulating something as routine as the daily Sudoku puzzle found in newspapers around the globe.

—Anita L. Benjamin, AMS Washington office

Photos: Top, speaker Amy Langville; below, audience at congressional briefing.

AMS Sponsors NExT Fellows

Each year the AMS sponsors six Project NExT (New Experiences in Teaching) Fellows who are affiliated with Ph.D.-granting institutions and who show promise in mathematics research. The names, affiliations, and areas of research of the 2006–2007 NExT Fellows are: GERARD AWANOU, Northern Illinois University, numerical analysis of partial differential equations; ELENA DIMITROVA, Virginia Polytechnic Institute and State University, mathematical biology; GIZEM KARAALI, Pomona College, representation theory and quantum groups; PETER MCNAMARA, Bucknell University, combinatorics; HOLLY SWISHER, Oregon State University, number theory and combinatorics; and ANDREW WHITTLE, Kennesaw State University, mathematical biology and optimal control.

Project NExT (New Experiences in Teaching) is a professional development program for new or recent Ph.D.'s in the mathematical sciences (including pure and applied mathematics, statistics, operations research, and mathematics education). It addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, and participating in professional activities. It also provides the participants with a network of peers and mentors as they assume these responsibilities. Each year sixty to seventy new Ph.D.'s receive Project NExT Fellowships, which allow them to attend special events at the summer MathFest of the Mathematical Association of America and at the Joint Mathematics Meetings. The AMS also holds activities for the AMS NExT Fellows at the Joint Mathematics Meetings.

For further information about Project NExT, visit the website `http://archives.math.utk.edu/projnext/`.

—Elaine Kehoe

Reference and Book List

*The **Reference** section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.ou.edu` in the case of the editor and `notices@ams.org` in the case of the managing editor. The fax numbers are 405-325-7484 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Upcoming Deadlines

February 16, 2007: Nominations for CMI Liftoff fellowships. See `http://claymath.org/fas/liftoff_fellows/`; telephone: 617-995-2600; email: `nominations@claymath.org`.

March 1, 2007: Applications for National Academies Christine Mirzayan Graduate Fellowships for the summer program. See `http://www7.nationalacademies.org/policyfellows` or contact the National Academies Christine Mirzayan Science and Technology Policy Graduate Fellowship Program, 500 Fifth Street, NW, Room 508, Washington, DC 20001; telephone: 202-334-2455; fax: 202-334-1667; email: `policyfellows@nas.edu`.

March 12, 2007: Proposals for NSF program for research involving interactions between mathematical sciences and computer science. See "Mathematics Opportunities" in this issue.

March 31, 2007: Nominations for Third World Academy of Sciences Prizes. See "Mathematics Opportunities" in this issue.

April 15, 2007: Applications for AMS "Math in Moscow" Scholarships for fall 2007. See `http://www.mccme.ru/mathinmoscow` or contact Math in Moscow, P.O. Box 524, Wynnewood, PA 19096; fax +7095-291-65-01; email: `mim@mccme.ru`. For information and application forms for the AMS scholarships see `http://www.ams.org/outreach/mimoscow.html` or contact Math in Moscow Program, Membership and Programs Department, American Mathematical Soci-

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

ety, 201 Charles Street, Providence, RI 02904-2294; email: student-serv@ams.org.

April 16, 2007: Applications for Project NExT: New Experiences in Teaching. See "Mathematics Opportunities" in this issue.

May 1, 2007: Applications for AWM Travel Grants. See http://www.awm-math.org/travelgrants.html; telephone 703-934-0163; email: awm@math.umd.edu; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

June 1, 2007: Applications for National Academies Christine Mirzayan Graduate Fellowships for the fall program. See http://www7.nationalacademies.org/policyfellows or contact the National Academies Christine Mirzayan Science and Technology Policy Graduate Fellowship Program, 500 Fifth Street, NW, Room 508, Washington, DC 20001; telephone: 202-334-2455; fax: 202-334-1667; email: policy-fellows@nas.edu.

June 5, 2007: Proposals for Enhancing the Mathematical Sciences Workforce in the Twenty-First Century. See http://www.nsf.gov/publications/pub_summ.jsp?ods_key=nsf05595.

June 30, 2007: Nominations for 2007 Fermat Prize. See http://www.math.ups-tlse.fr/Fermat/.

October 1, 2007: Applications for AWM Travel Grants. See http://www.awm-math.org/travelgrants.html; telephone 703-934-0163; email: awm@math.umd.edu; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

Board on Mathematical Sciences and Their Applications, National Research Council

The Board on Mathematical Sciences and Their Applications (BMSA) was established in November 1984 to lead activities in the mathematical sciences at the National Research Council (NRC). The mission of BMSA is to support and promote the quality and health of the mathematical sciences and their benefits to the nation. Following are the current BMSA members.

Massoud Amin, University of Minnesota

Marsha Berger, New York University

Philip Bernstein, Microsoft Corporation

Patricia Brennan, University of Wisconsin

Patrick L. Brockett, University of Texas, Austin

Debra Elkins, General Motors

Lawrence Craig Evans, University of California, Berkeley

John Geweke, University of Iowa

Darryll Hendricks, UBS Investment Bank

John E. Hopcroft, Cornell University

C. David Levermore (Chair), University of Maryland

Charles M. Lucas, American International Companies

Charles Manski, Northwestern University

Joyce R. McLaughlin, Rensselaer Polytechnic Institute

Jill P. Mesirov, Broad Institute

Andrew Odlyzko, Digital Technology Center

John Rice, University of California, Berkeley

Stephen M. Robinson, University of Wisconsin, Madison

George Sugihara, University of California, San Diego

Edward J. Wegman, George Mason University

Lai-Sang Young, Courant Institute of Mathematical Sciences

The postal address for BMSA is: Board on Mathematical Sciences and Their Applications, National Academy of Sciences, Room K974, 500 Fifth Street, NW, Washington, DC 20001; telephone: 202-334-2421; fax: 202-334-2422/2101; email: bms@nas.edu; World Wide Web: http://www7.nationalacademies.org/bms/BMSA_Members.html.

Book List

The Book List highlights books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. When a book has been reviewed in the Notices, *a reference is given to the review. Generally the list will contain only books published within the last two years, though exceptions may be made in cases where current events (e.g., the death of a prominent mathematician, coverage of a certain piece of mathematics in the news) warrant drawing readers' attention to older books. Suggestions for books to include on the list may be sent to* notices-booklist@ams.org.

*Added to "Book List" since the list's last appearance.

An Abundance of Katherines, by John Green. Dutton Juvenile Books, September 2006. ISBN 0-525-47688-1.

Alan Turing's Automatic Computing Engine: The Master Codebreaker's Struggle to Build the Modern Computer, edited by B. Jack Copeland. Oxford University Press, June 2005. ISBN 0-198-56593-3.

Analysis and Probability: Wavelets, Signals, Fractals by Palle E. T. Jorgensen. Springer, September 2006. ISBN 0-387-29519-4.

Arthur Cayley: Mathematician Laureate of the Victorian Age, by Tony Crilly. Johns Hopkins University Press, December 2005. ISBN 0-801-88011-4.

The Artist and the Mathematician: The Story of Nicolas Bourbaki, the Genius Mathematician Who Never Existed, by Amir D. Aczel. Thunder's Mouth Press, August 2006. ISBN 1-560-25931-0.

A Beautiful Math: John Nash, Game Theory, and the Modern Quest for a Code of Nature, by Tom Siegfried. Joseph Henry Press, October 2006. ISBN 0-309-10192-1.

**Bourbaki, A Secret Society of Mathematicians*, Maurice Mashaal. AMS, June 2006. ISBN 0-8218-3967-5.

The Coxeter Legacy: Reflections and Projections, edited by Chandler Davis and Erich W. Ellers. AMS, March 2006. ISBN 0-8218-3722-2.

Dark Hero of the Information Age: In Search of Norbert Wiener, by Flo Conway and Jim Siegelman. Basic Books, December 2004. ISBN 0-738-20368-8. (Reviewed May 2006.)

Decoding the Universe: How the New Science of Information Is Explaining Everything in the Cosmos, from Our Brains to Black Holes, by Charles Seife. Viking Adult, February 2006. ISBN 0-670-03441-X.

Descartes: A Biography, by Desmond Clarke. Cambridge University Press, March 2006. ISBN 0-521-82301-3.

Descartes: The Life and Times of a Genius, by A. C. Grayling. Walker & Company, November 2006. ISBN 0-8027-1501-X.

The Equations: Icons of Knowledge, by Sander Bais. Harvard University Press, November 2005. ISBN 0-674-01967-9.

The Essential Turing, edited by B. Jack Copeland. Oxford University Press, September 2004. ISBN 0-198-25080-0. (Reviewed November 2006.)

Euclid in the Rainforest: Discovering Universal Truths in Logic and Math, by Joseph Mazur. Pi Press, October 2004. ISBN 0-131-47994-6.

Euler through Time: A New Look at Old Themes, by V. S. Varadarajan. AMS, June 2006. ISBN 0-8218-3580-7.

**Evolutionary Dynamics: Exploring the Equations of Life*, by Martin Nowak. Belknap Press, September 2006. ISBN 0-674-02338-2.

**The Fabulous Fibonacci Numbers*, by Alfred S. Posamentier and Ingmar Lehmann. Prometheus Books, February 2007. ISBN 1-591-02475-7.

Fearless Symmetry: Exposing the Hidden Patterns of Numbers, by Avner Ash and Robert Gross. Princeton University Press, May 2006. ISBN 0-691-12492-2. (Reviewed January 2007.)

**From Cosmos to Chaos: The Science of Unpredictability*, by Peter Coles. Oxford University Press, August 2006. ISBN 0-198-56762-6.

From Zero to Infinity: What Makes Numbers Interesting, by Constance Reid. Fiftieth anniversary edition, A K Peters, February 2006. ISBN 1-568-81273-6. (Reviewed February 2007.)

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse, by Torkel Franzen. A K Peters, May 2005. ISBN 1-568-81238-8. (Reviewed in this issue.)

**Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever*, by Hal Hellman. Wiley, September 2006. ISBN 0-471-64877-9.

Hiding in the Mirror: The Mysterious Allure of Extra Dimensions, from Plato to String Theory and Beyond, by Lawrence M. Krauss. Viking Adult, October 2005. ISBN 0-670-03395-2.

**How Mathematics Happened*, by Peter S. Rudman. Prometheus Books, October 2006. ISBN 1-591-02477-3.

How to Cut a Cake: And Other Mathematical Conundrums, by Ian Stewart. Oxford University Press, November 2006. ISBN 0-199-20590-6.

Incompleteness: The Proof and Paradox of Kurt Gödel, by Rebecca Goldstein. W. W. Norton, February 2005. ISBN 0-393-05169-2. (Reviewed April 2006.)

Infinite Ascent: A Short History of Mathematics, by David Berlinski. Modern Library, September 2005. ISBN 0-679-64234-X.

It's about Time: Understanding Einstein's Relativity, by N. David Mermin. Princeton University Press, September 2005. ISBN 0-691-12201-6.

John von Neumann: Selected Letters, edited by Miklós Rédei. AMS, November 2005. ISBN 0-8218-3776-1.

King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry, by Siobhan Roberts. Walker & Company, September 2006. ISBN 0-802-71499-4.

The Lifebox, the Seashell, and the Soul: What Gnarly Computation Taught Me about Ultimate Reality, the Meaning of Life, and How to Be Happy, by Rudy Rucker. Thunder's Mouth Press, October 2005. ISBN 1-560-25722-9.

A Madman Dreams of Turing Machines, by Janna Levin. Knopf, August 2006. ISBN 1-400-04030-2.

The Man Who Knew Too Much: Alan Turing and the Invention of the Computer, by David Leavitt. Great Discoveries series, W. W. Norton, December 2005. ISBN 0-393-05236-2. (Reviewed November 2006.)

The Math Instinct: Why You're a Mathematical Genius (along with Lobsters, Birds, Cats, and Dogs), by Keith Devlin. Thunder's Mouth Press, March 2005. ISBN 1-56025-672-9.

Mathematical Illustrations: A Manual of Geometry and PostScript, by Bill Casselman. Cambridge University Press, December 2004. ISBN 0-521-54788-1. (Reviewed January 2007.)

**Mathematics and Common Sense: A Case of Creative Tension*, by Philip J. Davis. A K Peters, October 2006. ISBN 1-568-81270-1.

**Measuring the World*, by Daniel Kehlmann. Pantheon, November 2006. ISBN 0-375-42446-6.

More Mathematical Astronomy Morsels, by Jean Meeus. Willmann-Bell, 2002. ISBN 0-943396-743.

Musimathics: The Mathematical Foundations of Music, Volume 1, by Gareth Loy. MIT Press, September 2006. ISBN 0-262-12282-0.

Mystic, Geometer, and Intuitionist: The Life of L. E. J. Brouwer. Volume 2: Hope and Disillusion, by Dirk van Dalen. Oxford University Press, October 2005. ISBN 0-198-51620-7.

The Newtonian Moment: Isaac Newton and the Making of Modern Culture, by Mordechai Feingold. New York Library and Oxford University Press, December 2004. ISBN 0-195-17735-5.

Not Even Wrong: The Failure of String Theory and the Continuing Challenge to Unify the Laws of Physics, by Peter Woit. Jonathan Cape, April 2006. ISBN 0-224-07605-1.

Once upon Einstein, by Thibault D'Amour. A K Peters, March 2006. ISBN 1-568-81289-2.

The Pea and the Sun: A Mathematical Paradox, by Leonard M. Wapner. A K Peters, April 2005. ISBN 1-568-81213-2. (Reviewed October 2006.)

Piano Hinged Dissections: Time to Fold!, by Greg Frederickson. A K Peters, October 2006. ISBN 1-568-81299-X.

Piero della Francesca: A Mathematician's Art, by J. V. Field. Yale University Press, August 2005. ISBN 0-300-10342-5. (Reviewed in this issue.)

**Prince of Mathematics: Carl Friedrich Gauss*, by M. B. W. Tent. A K Peters, January 2006. ISBN 1-568-81261-2.

Pursuit of Genius: Flexner, Einstein, and the Early Faculty at the Institute for Advanced Study, by Steve Batterson. A K Peters, June 2006. ISBN 1-568-81259-0.

Reality Conditions: Short Mathematical Fiction, by Alex Kasman. Mathematical Association of America, May 2005. ISBN 0-88385-552-6. (Reviewed August 2006.)

Reflections: V. I. Arnold's Reminiscences, by V. I. Arnold. Springer, April 2006. ISBN 3-540-28734-5.

The Road to Reality: A Complete Guide to the Laws of the Universe, by Roger Penrose. Knopf, February 2005. ISBN 0-679-45443-8. (Reviewed June/July 2006.)

The Secret Life of Numbers: 50 Easy Pieces on How Mathematicians Work and Think, by George G. Szpiro. Joseph Henry Press, March 2006. ISBN 0-309-09658-8.

Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought, by Tony Robbin. Yale University Press, March 2006. ISBN 0-300-11039-1.

The Shoelace Book: A Mathematical Guide to the Best (and Worst) Ways to Lace Your Shoes, by Burkard Polster. AMS, June 2006. ISBN 0-8218-3933-0. (Reviewed December 2006.)

Stalking the Riemann Hypothesis: The Quest to Find the Hidden Law of Prime Numbers, by Dan Rockmore. Pantheon, April 2005. ISBN 0-375-42136-X. (Reviewed September 2006.)

Symmetry and the Monster: The Story of One of the Greatest Quests of Mathematics, by Mark Ronan. Oxford University Press, May 2006. ISBN 0-192-80722-6. (Reviewed February 2007.)

The Three Body Problem, by Catherine Shaw. Allison and Busby, March 2005. ISBN 0-749-08347-6. (Reviewed October 2006.)

The Trouble with Physics: The Rise of String Theory, the Fall of a Science, and What Comes Next, by Lee Smolin. Joseph Henry Press, October 2006. ISBN 0-309-10192-1.

Unknown Quantity: A Real and Imaginary History of Algebra, by John Derbyshire. Joseph Henry Press, May 2006. ISBN 0-309-09657-X.

Yearning for the Impossible: The Surprising Truths of Mathematics, by John Stillwell. A K Peters, May 2006. ISBN 1-568-81254-X.

About the Cover

The cover on this issue is from the manuscript of Piero della Francesca's *De Prospectiva Pingendi* now in the Palatina Library of Parma. It is one of a series of illustrations showing how to draw the human head in perspective, an extraordinarily complicated task if done as meticulously as done here. In the series, Piero maps the head, using numbers as rather arbitary labels. This is discussed by J. V. Field in the book reviewed by Michele Emmer in this issue, but in more detail in her earlier book *Invention of Infinity,* reviewed by Tony Phillips in the January, 2001, issue of the *Notices.*

Images on the cover and on this page are from ArtResource and the Palatina Library.

—*Bill Casselman, Graphics Editor*
(notices-covers@ams.org)

Leroy P. Steele Prizes

Call for Nominations

The selection committee for these prizes requests nominations for consideration for the 2008 awards. Further information about the prizes can be found in the November 2005 *Notices*, pp. 1251-1255 (also available at http://www.ams.org/prizes-awards).

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through Ph.D. students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2008 the prize for Seminal Contribution to Research will be awarded for a paper in discrete mathematics.

Nominations with supporting information should be submitted to the Secretary, Robert J. Daverman, American Mathematical Society, 312D Ayres Hall, University of Tennessee, Knoxville, TN 37996-1330. Include a short description on the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included. The nominations will be forwarded by the Secretary to the prize selection committee, which will, as in the past, make final decisions on the awarding of prizes.

Deadline for nominations is March 31, 2007.

AMS
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American Mathematical Society

AMS EXEMPLARY PROGRAM PRIZE

At its meeting in January 2004, the AMS Council approved the establishment of a new award called the AMS Award for an Exemplary Program or Achievement in a Mathematics Department. It is to be presented annually to a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

The prize amount is $1,200. All departments in North America that offer at least a bachelor's degree in the mathematical sciences are eligible.

The Prize Selection Committee requests nominations for this award, which will be announced in Spring 2008. Letters of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements that make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

All nominations should be submitted to the AMS Secretary, Robert J. Daverman, American Mathematical Society, 312D Ayres Hall, University of Tennessee, Knoxville TN 37996-1330. Include a short description of the work that is the basis of the nomination, with complete bibliographic citations when appropriate. The nominations will be forwarded by the Secretary to the Prize Selection Committee, which will make the final decision on the award.

Deadline for nominations is April 1, 2007.

Mathematics Calendar

The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at `http://www.ams.org/mathcal/`.

March 2007

1-4 **Workshop on University Mathematics Courses for School Teachers**, University of Arizona, Tucson, Arizona. (Dec. 2006, p. 1376)

3-4 **AMS Southeastern Section Meeting**, Davidson College, Davidson, North Carolina. (Jun/Jul. 2006, p. 713)

4-7 **3rd International Conference on 21st Century Mathematics 2007**, School of Mathematical Sciences, Lahore, Pakistan. (Aug. 2006, p. 823)

4-8 **Twelfth International Conference on Approximation Theory**, Menger Hotel, San Antonio, Texas. (Jun/Jul. 2006, p. 713)

5-9 **School: Combinatorics on Words & Workshop on Recent Progress in Combinatorics on Words**, Centre de recherches mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 303)

10-13 **2007 ASL Annual Meeting**, University of Florida, Gainesville, Florida. (Jan. 2007, p. 62)

10-13 **Complex Cobordism in Homotopy Theory: its Impact and Prospects**, Johns Hopkins University, Baltimore, Maryland. (Oct. 2006, p. 1090)

12-15 **2007 MBI Workshop for Young Researchers in Mathematical Biology**, The Ohio State University, Columbus, Ohio. (Sept. 2006, p. 958)

12-16 **Geometric Evolution Equations**, Mathematical Sciences Research Institute, Berkeley, California. (Aug. 2006, p. 824)

12-16 **Workshop on Recent Progress in Combinatorics on Words**, Centre de recherches mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 303)

16-17 **AMS Central Section Meeting**, Miami University, Oxford, Ohio. (Jun/Jul. 2006, p. 713)

19-23 **Motives and Algebraic Cycles, A Conference Dedicated to the Mathematical Heritage of Spencer J. Bloch**, Fields Institute, Toronto, Canada. (Feb. 2007, p. 303)

19-23 **Representations of Surface Groups**, AIM Research Conference Center, Palo Alto, California. (Jun/Jul. 2006, p. 713)

19-23 **Stochastic Dynamical Systems and Control**, Mathematical Sciences Research Institute, Berkeley, California. (Jun/Jul. 2006, p. 713)

21-23 **The 2007 IAENG International Conference on Scientific Computing**, Regal Kowloon Hotel, Kowloon, Hong Kong. (Oct. 2006, p. 1090)

22-23 **DIMACS Workshop on Auctions with Transaction Costs**, DIMACS Center, CoRE Bldg, Rutgers University, Piscataway, New Jersey. (Feb. 2007, p. 303)

22-25 **Analysis on Homogeneous Spaces**, Tucson, Arizona. (Jan. 2007, p. 62)

*24-25 **59th midwest partial differential equations seminar**, University of Kentucky, Lexington, Kentucky.
Speakers: Pascal Auscher, University of Paris, XI; Tanya Christiansen, University of Missouri, Columbia; Marianne Korten, Kansas State University; Yanyan Li, Rutgers University; Arshak Petrosyan, Purdue University; Antônio Sá Barreto, Purdue University; Monica Visan, The Institute for Advanced Study; Yuxi Zheng, Pennsylvania State University.
Organizers: Russell Brown (`Russell.Brown@uky.edu`), Peter Hislop, John Lewis and Changyou Wang.

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.

An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.

In general, announcements of meetings and conferences held in North America carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. Meetings held outside the North American area may carry more detailed information. In any case, if there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to `notices@ams.org` or `mathcal@ams.org`.

In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.

The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.

The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: `http://www.ams.org/`.

Information: http://www.ms.uky.edu/~rbrown/midwpde07/.

25–29 **The 4th International Conference (SETIT 2007): Sciences of Electronic, Technology of Information and Telecommunications**, Tunisia, North Africa. (Sept. 2006, p. 959)

25–31 **Talbot Workshop 2007: Topological modular forms**, North Conway, New Hampshire. (Dec. 2006, p. 1376)

26 **World Congress on Computational Finance: The First Decade**, London, England. (Aug. 2006, p. 824)

26–28 **Perspectives on Mathematical Practices 2007**, Vrije Universiteit Brussel, Brussels, Belgium. (Feb. 2007, p. 303)

26–29 **International Colloquium On Stochastic And Potential Analysis**, Abounawas hotel, Hammamet, Tunisia. (Jan. 2007, p. 62)

26–30 **Buildings and Combinatorial Representation Theory**, AIM Research Conference Center, Palo Alto, California. (May 2006, p. 612)

26–30 **Homotopy theory of schemes (March-April)**, Fields Institute, Toronto, Canada. (Feb. 2007, p. 303)

* 28–30 **3rd International Symposium on Computational Intelligence and Intelligent Informatics**, Agadir, Morocco.
Description: While continuing to to encourage high quality innovative papers, the SOC remains committed to facilitate encounters and exchanges among the top researchers in the field and rising stars. Indeed, the venue was chosen to enhance interaction between participants as well as give the events an appropriate setting.
Information: email: hbouzahir@yahoo.fr; http://www.bmf.hu/conferences/isciii2007/.

29–31 **Spring Topology and Dynamical Systems Conference 2007**, University of Missouri-Rolla, Rolla, Missouri. (Dec. 2006, p. 1377)

* 29–April 3 **Curves, abelian varieties and their interactions**, University of Georgia, Athens, Georgia.
Aim: To review recent developments on curves, abelian varieties and their interactions, and to encourage new ones.
Speakers: Arnaud Beauville (Nice, France), Michel Brion (Grenoble, France), Lucia Caporaso (University of Rome-3, Italy), Sebastian Casalaina-Martin (Harvard University), Herb Clemens (Ohio State), Robert Friedman (Columbia University), Phillip A. Griffiths (Institute for Advanced Study, Princeton), Matt Kerr (University of Chicago), Igor Krichever (Columbia University), Herbert Lange (Erlangen, Germany), David Lehavi (University of Michigan), Alina Marian (Yale University), Martin Olsson (University of California, Berkeley), Giuseppe Pareschi (University of Rome-2, Italy), Ravi Vakil (Stanford University), Alessandro Verra (University of Rome-3, Italy), Claire Voisin (University of Paris-7, France), Ilia Zharkov (Harvard University).
Organizers: Valery Alexeev and Elham Izadi.
Support: Support is available for graduate students and recent Ph.D.s to attend.
Information: http://www.math.uga.edu/~valery/conf07/conf07.html.

30–31 **33rd Annual New York State Regional Graduate Mathematics Conference**, Syracuse University, Syracuse, New York. (Dec. 2006, p. 1377)

30–April 1 **Texas Algebraic Geometry Symposium**, University of Texas, Austin, Texas. (Feb. 2007, p. 303)

* 31–April 2 **Learning Technologies and Mathematics Middle East Conference**, Sultan Qaboos University, Muscat, Sultanate of Oman.
Program: Invited lectures related to reforming the mathematics undergraduate curriculum, introduction of learning technologies into the classroom and their effects on undergraduate and graduate education, teacher training, e-learning, innovative methods of teaching, use of the web in the mathematics classroom, etc.
Deadlines: Abstracts and Early Registration: 15 February 2007
Financial Support: Depending on the availability of funds, the organizing committee may award some attendees financial support in terms of one or more of the following: Waiving the registration fees, and/or Providing local accommodations, and/or Providing limited support for travel expenses.
Information: Please visit the website http://math.arizona.edu/~atp-mena/conference/.

31–April 3 **International Conference on Industrial & Applied Mathematics**, Jammu University, Jammu, India. (Dec. 2006, p. 1377)

April 2007

2–5 **Quantum Graphs and their Applications**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Dec. 2006, p. 1377)

9–13 **Geometry of Integrable Systems**, Hanoi University of Education, Hanoi, Vietnam. (Jan. 2007, p. 62)

* 9–14 **International conference: Modern Analysis and Applications (MAA 2007): Dedicated to the centenary of Mark Krein**, Odessa National I.I. Mechnikov University, Odessa, Ukraine.
Information: For information about the MAA 2007 consult the web site URL: http://maa2007.onu.edu.ua or contact the MAA 2007 Working Committee at maa2007@onu.edu.ua.

10–12 **The 3rd International Conference on Research and Education in Mathematics 2007 (ICREM07)**, Dynasty Hotel, Kuala Lumpur, Malaysia. (Dec. 2006, p. 1377)

10–13 **Graph Models of Mesoscopic Systems, Wave-Guides and Nano-Structures**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Dec. 2006, p. 1377)

10–13 **PADGE 2007, International Congress on Pure and Applied Differential Geometry**, Brussels, Belgium. (Jan. 2007, p. 62)

* 10–13 **Random Matrices and Higher Dimensional Inference**, AIM Research Conference Center, Palo Alto, California.
Description: This workshop, sponsored by AIM and the NSF, concerns issues arising in high-dimensional inference that can be served by recent developments in random matrix theory. This follows on from an intensive program held at SAMSI (http://www.samsi.info) during this past academic year. This workshop at AIM will serve to sum up the program and set an agenda for future research in this area. We intend to have presentations by members of each of working groups from the SAMSI program. Each of these presentations will be followed by a discussion session in which open problems and future directions will be discussed. This will enable us to achieve our goal of, on the one hand, understanding clearly what has been achieved in each sub-area and seeing what interdisciplinary endeavors have been initiated and, on the other, setting an agenda for future work in this area.
Organizers: Peter Bickel, Christopher Jones, Helene Massam, and Don Richards.
Deadline: February 1, 2007.
Information: http://aimath.org/ARCC/workshops/rmtinference.html.

10–14 **The Geometric Langlands Correspondence**, Mathematical Institute, Oxford, England. (Feb. 2007, p. 303)

10–14 **Workshop on Control Theory & Finance**, Instituto Superior de Economia e Gestão (ISEG), Technical University of Lisbon, Lisbon, Portugal. (Oct. 2006, p. 1090)

14–15 **AMS Eastern Section Meeting**, Stevens Institute of Technology, Hoboken, New Jersey. (Jun/Jul. 2006, p. 713)

14–18 **Maghrebian Conference of Applied Mathematics**, USTHB, Algiers, Algeria. (Oct. 2006, p. 1090)

16–17 **4th Montreal Scientific Computing Days**, Centre de Recherches Mathématiques, Montréal, Quebec, Canada. (Feb. 2007,

p. 303)

18–20 **DIMACS Workshop on Discrete Mathematical Problems in Computational Biomedicine**, DIMACS Center, CoRE Bldg, Rutgers University, Piscataway, New Jersey. (Jan. 2007, p. 62)

18–20 **SMAI Conference of Optimization and Decision Making(CODE 2007)**, Institut Henri Poincaré, Paris, France. (Dec. 2006, p. 1377)

18–21 **2007 ASL Spring Meeting (with APA)**, Chicago, Illinois. (Jan. 2007, p. 63)

20–22 **Riviere-Fabes Symposium on Analysis and PDE**, University of Minnesota, Minneapolis, Minnesota. (Nov. 2006, p. 1253)

21–22 **AMS Western Section Meeting**, University of Arizona, Tucson, Arizona. (Jun/Jul. 2006, p. 713)

21–22 **Graduate Student Topology Conference**, University of Chicago, Chicago, Illinois. (Jan. 2007, p. 63)

* 21–22 **Random Combinatorial Structures**, University of Nebraska-Lincoln, Lincoln, Nebraska.
Program: Nine plenary talks on various areas of probabilisitic combinatorics.
Organizers: Jamie Radcliffe, Jonathan Cutler (UN-L).
Speakers: Béla Bollobás, Prasad Tetali, Alan Frieze, Boris Pittel, Zoltán Füredi, Paul Balister, Catherine Yan, József Balogh and Joshua Cooper.
Information: email: rcs@math.unl.edu; http://www.math.unl.edu/pi/events/rcs.

21–26 **The First IPM Conference on Algebraic Graph Theory (AGT 2007)**, School of Mathematics, Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran. (Feb. 2007, p. 303)

22–27 **Industry workshop & short course: The mathematics of electricity supply and pricing**, Holiday Inn, Surfers Paradise, Queensland, Australia. (Feb. 2007, p. 303)

23–27 **Dynamics in Perturbations**, Hasselt University, Campus Diepenbeek, and Brussels, Belgium. (Oct. 2006, p. 1090)

23–27 **Problems in Geometric Group Theory**, AIM Research Conference Center, Palo Alto, California. (May 2006, p. 612)

25–27 **International Conference on Nonlinear Analysis and Optimization**, Isfahan, Iran. (Feb. 2007, p. 304)

25–28 **Conference on Ordered Rings, Baton Rouge 2007 ("Ord007")**, Louisiana State University, Baton Rouge, Louisiana. (Nov. 2006, p. 1253)

28–30 **Multi-scale Modeling and Simulation in Materials Science**, University of Tennessee, Knoxville, Tennessee. (Dec. 2006, p. 1377)

28–May 5 **Advances in Algebra and Geometry**, Mathematical Sciences Research Institute, Berkeley, California. (Aug. 2006, p. 824)

30–May 4 **School: Macdonald Polynomials & Workshop: Combinatorial Hopf Algebras and Macdonald Polynomials**, Centre de recherches mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 304)

May 2007

2–5 **Foundations of the Formal Sciences VI (FotFS VI): Reasoning about probabilities and probabilistic reasoning**, Universiteit van Amsterdam, Amsterdam, The Netherlands. (Feb. 2007, p. 304)

6–12 **Symposium in Algebraic Geometry and its applications (cryptography, coding theory)**, Faa'a, French Polynesia. (Dec. 2006, p. 1377)

7–11 **Instructional conference solution methods for Diophantine equations**, Lorentz Center, Leiden, The Netherlands. (Feb. 2007, p. 304)

7–11 **Rational Curves on Algebraic Varieties**, AIM Research Conference Center, Palo Alto, California. (May 2006, p. 612)

7–11 **Workshop: Combinatorial Hopf Algebras and Macdonald Polynomials**, Centre de recherches mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 304)

* 7–12 **New Trends in Complex and Harmonic Analysis: an International Conference on Analysis and Mathematical Physics**, University of Bergen (UiB), Norwegian University of Science and Technology (NTNU), Voss, Norway.
Description: The purpose of the conference is to bring together specialists in analysis with experts in mathematical physics, mechanics and adjacent areas of applied sciences and numerical analysis. The participants will present their results and discuss further developments of the frontier research exploring the bridge between complex, real analysis, potential theory, PDE and modern topics of fluid mechanics and mathematical physics. The conference will feature invited expository 1-hour lectures, 45-min. talks and short 25-min. communications.
Invited speakers: J. M. Anderson (London), D. Ch. Chang (Washington, DC), A. Eremenko (West Lafayette), H. Hedenmalm (Stockholm), D. Khavinson (Tampa), P. Malliavin (Paris), L. Pastur (Kharkov), S. P. Novikov (College Park), A. Olevskii (Tel-Aviv), Ch. Pommerenke (Berlin), D. Prokhorov (Saratov), M. Sodin (Tel-Aviv), L. Takhtajan (Stony Brook), X. Tolsa (Barselona), P. Wiegmann (Chicago).
Information: http://analysis2007.uib.no.

14–16 **Workshop solution methods for Diophantine equations**, Lorentz Center, Leiden, The Netherlands. (Feb. 2007, p. 304)

14–18 **Conference on Cryptography and Digital Content Security**, Centre de Recerca Matemàtica, Barcelona, Spain. (Jan. 2007, p. 63)

14–18 **Stacks in geometry and topology (May-June)**, Fields Institute, Toronto, Canada. (Feb. 2007, p. 304)

14–July 13 **Braids**, Institute for Mathematical Sciences, National University of Singapore, Singapore. (Dec. 2005, p. 1383)

15–20 **The 27th Annual Great Plains Operator Theory Symposium (GPOTS-2007)**, University of Nebraska, Lincoln, Nebraska. (Nov. 2006, p. 1253)

16–20 **Illinois Number Theory Fest**, University of Illinois at Urbana-Champaign, Illinois. (Feb. 2007, p. 304)

16–20 **Linear algebraic groups and cohomology**, Emory University, Atlanta, Georgia. (Dec. 2006, p. 1378)

18–20 **The 2007 Midwest Geometry Conference (MGC 07)**, University of Iowa, Iowa City, Iowa. (Jun/Jul. 2006, p. 713)

18–20 **Workshop on p-adic Methods and Rational Points**, Alfréd Rényi Institute of Mathematics, Budapest, Hungary. (Jan. 2007, p. 63)

* 20–21 **8th Colloquium of Algebra and Number Theory**, Faculty of Maths USTHB, Babezzouar, Algiers/Algeria.
Organizers: Department of Algebra and Number Theory, Faculty of Mathematics, University of Sciences and Technology Houari Boumedienne
Topics: Umbral calculus and applications, number theory and applications, , elliptic functions, modular forms and applications, multi-valued maps and applications, algebraic manifolds, elliptic curves and applications, combinatorics, cryptography, coding theory and algorithmics.
Important dates: Registration: January 10th to March 10th, 2007. Abstract submission deadline: March 10th, 2007. Paper submission deadline: April 10th, 2007.
Information: Contact: M.O. Hernane, mhernane@usthb.dz; http://www.usthb.dz.

* 20–23 **IMST2007-FIMXV: Fifteenth International Conference of Forum for Interdisciplinary Mathematics on Interdisciplinary Mathematical and Statistical Techniques**, Shanghai Institute of

Advanced Studies, University of Science and Technology of China, Shanghai, China.

Theme: The major concentration of the conference's academic activities will be in mathematical and statistical sciences including, but not limited to, Actuarial and Financial Mathematics, Statistics and Applications, Biostatistics, Combinatorics, Computer and Information Sciences, Distribution Theory, Econometrics and Mathematical Economics, Environmental Statistics, Experimental Designs, Extreme Values, Graph Theory, Linear Statistical Inference, Mathematics, Multivariate Statistics, Nonparametric Statistical Inference, Operations Research, Probability/Stochastic Processes, Public Health, Quality Control, Reliability and Life Testing, Sampling, partner areas. Those wishing to contribute outside these areas are welcome to submit their abstracts.

Information: `http://imst2007.southalabama.edu/`.

20–26 **Braids and their ramifications**, Cortona, Italy. (Feb. 2007, p. 304)

21–23 **Applications of Analysis to Mathematical Biology**, Duke University, Durham, North Carolina. (Oct. 2006, p. 1090)

21–June 1 **Algebraic Geometry and Algebraic Combinatorics**, Centre de Recherches Mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 304)

21–25 **Mathematical Issues in Stochastic Approaches for Multiscale Modeling**, Mathematical Sciences Research Institute, Berkeley, California. (Aug. 2006, p. 824)

22–26 **Extremal problems in complex and real analysis**, Peoples Friendship University of Russia, Moscow, Russia. (Aug. 2006, p. 824)

22–26 **Inverse Problems in Stochastic Differential Equations**, University of Southern California, Los Angeles, California. (Dec. 2006, p. 1378)

23–26 **Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems**, Northern Arizona University, Flag Staff, Arizona. (Dec. 2006, p. 1378)

* 24–26 **Geometry Operator Theory and Applications**, Faculty of Sciences, Oujda, Morocco.

Speakers: P. Aiena (Palermo, Italy), B. Aqzzouz (Kenitra, Morocco), C. Badea (Lille, France), A. Baklouti (Monastir, Tunisia), M. Berraa (Marrakech, Morocco), N. Boudi (Meknes, Morocco), M. Boumazgour (Agadir, Morocco), Raul E. Curto(Iowa, USA), M. Elhouari (Lille, France), Y. Hantout (Lille, France), A. Kaidi (Almeria, Spain), M. Mbekhta (Lille, France), B. Ndombol (Dschang, Cameroon), Peralta (Granada, Spain), H. Queffelec (Lille, France), Y. Rami (Meknes, Morocco), A. M. Rodrigues (Seville, spain), F. Leon Saavedra (Cadiz, Spain), J.-C. Thomas (Angers, France), A. M. F. H. Vasilescu (Lille, France), A. Zeghib (E.N.S. Lyon, France), E. H. Zerouali (Rabat, Morocco).

Deadline: Of inscription: February 28, 2007.

Information: `http://sciences1.ump.ma/`.

* 27–30 **Third Symposium on Analysis and PDEs**, Purdue University, West Lafayette, Indiana.

Description: The Symposium will focus on recent developments in Partial Differential Equations and their applications. It is intended to introduce prospective and young researchers to a larger mathematical community and help them to establish professional connections with key figures in their areas of interest. It will also provide an opportunity to summarize some of the most recent progress in the fields, exchange ideas towards the solution of open questions, and formulate new problems and avenues of research.

Support: Some funds are available to help support the participation of graduate students, postdoctoral faculty, and active senior researchers who do not have grant support. We especially encourage people who belong to currently underrepresented groups (women and minorities) to apply.

Information: `http://www.math.purdue.edu/~danielli/symposium.html`.

27–June 1 **Analysis on Graphs and Fractals (A satellite meeting for the June 2007 Isaac Newton Institute program "Analysis on Graphs and its Applications")**, University of Wales, Cardiff, UK. (Feb. 2007, p. 304)

27–June 2 **Spring School on Analysis: Function Spaces, Inequalities and Interpolation**, Paseky nad Jizerou, Czech Republic. (Jan. 2007, p. 63)

28–June 1 **SIAM Conference on Applications of Dynamical Systems (DS07)**, Snowbird Ski and Summer Resort, Snowbird, Utah. (Oct. 2006, p. 1090)

* 28–June 1 **Polya-Schur-Lax problems: hyperbolicity and stability preservers**, AIM Research Conference Center, Palo Alto, California.

Description: This workshop, sponsored by AIM and the NSF, will be devoted to bringing together researchers working on the following topics and their interplay: Polya-Schur problems: classification of linear preservers of polynomials and entire functions in one or several variables with prescribed zero sets. Lax-type problems: determinantal representations of multivariate Garding-hyperbolic polynomials and related objects. Properties and applications of stable polynomials and polynomials with the half-plane property.

Organizers: Julius Borcea, Petter Branden, George Csordas, and Victor Vinnikov.

Deadline: March 1, 2007.

Information: `http://aimath.org/ARCC/workshops/polyaschurlax.html`.

28–June 2 **Advanced Course on Group-Based Cryptography**, Centre de Recerca Matemàtica, Barcelona, Spain. (Jan. 2007, p. 63)

28–June 2 **Differential Equations, Theory of Functions, and Applications**, Novosibirsk State University, Novosibirsk, Russia. (Feb. 2007, p. 305)

28–June 2 **Workshop on Finsler Geomertry and its Applications**, Hotel Jogar, Balatonfoldvar, Hungary. (Aug. 2006, p. 824)

29–30 **DIMACS Workshop on Computational Methods for Predicting Outcome in Cancer**, DIMACS Center, CoRE Bldg, Rutgers University, Piscataway, New Jersey. (Jan. 2007, p. 63)

29–June 1 (REVISED DATE) **Days on Diffraction–2007**, St. Petersburg, Russia. (Jan. 2007, p. 63)

29–June 1 **The Fourth International Conference on Mathematical Biology**, Wuyishan city, Fujian, P.R. China. (Oct. 2006, p. 1091)

29–June 1 **XIIth Applied Stochastic Models and Data Analysis International Conference (ASMDA2007)**, Chania, Crete, Greece. (Oct. 2006, p. 1091)

* 30–June 2 **The Fifth International Conference on Dynamic Systems and Applications**, Morehouse College, Atlanta, Georgia.

Topics: Dynamical Systems, Computational Mathematics, Simulation, Stochastic/Deterministic: Differential Equations, Partial Differential Equations, Integral Equations, Integro-Differential Equations, Difference Equations, and related topics.

Call for Papers: Authors are requested to submit an abstract of their research presentation in `http://atlas-conferences.com/cgi-bin/abstract/submit/catb-01` on or before March 31, 2007. Or please e-mail the abstract to `icdsa5@yahoo.com`.

Deadlines: Submission of abstract (on or before): March 31, 2007. Acceptance of abstract: April 15, 2007. Motel cut off date: April 16, 2007.

Information: Please visit `http://www.dynamicpublishers.com/icdsa5.htm` for conference information. Conference Coordinator: M. Sambandham, ICDSA 05, Department of Mathematics, 830 Westview Dr, S.W., Morehouse College, Atlanta, GA 30314; Ph: (404) 215-2614; email: `icdsa5@yahoo.com`.

June 2007

1-5 **Sampling Theory and Applications 07 (SAMPTA07),** Aristotle Univ. of Thessaloniki, Thessaloniki, Greece. (Dec. 2006, p. 1378)

* 1-7 **International Conference—Leonhard Euler and Modern Combinatorics: Applications to Logic, Representation Theory, Mathematical Physics**, Euler IMI, St. Petersburg, Russia.
Organizers and Sponsors: St. Petersburg Department of Steklov Institute of Mathematics of RAS (POMI RAN), Euler International Mathematical Institute (EIMI), Russian Foundation for Basic Research (RFBR), National Science Foundation, USA (NSF) St. Petersburg Mathematical Society.
Information: email: lemc@imi.ras.ru; http://www.pdmi.ras.ru/EIMI/2007/LEMC/.

2-9 **Symmetry and Perturbation Theory**, Otranto (Lecce), Italy. (Feb. 2007, p. 305)

4-8 **Arithmetic Harmonic Analysis on Character and Quiver Varieties**, AIM Research Conference Center, Palo Alto, California. (Jun/Jul. 2006, p. 714)

4-15 **Moduli spaces of Riemann surfaces and related topics**, Centre de Recherches Mathematiques, Montreal, Canada. (Jan. 2007, p. 63)

* 7-9 **International Conference—Mathematical Hydrodynamics: Euler Equations and Related Topics**, Euler IMI, St. Petersburg, Russia.
Organizing Committee: Claude Bardos, Peter Constantin, Susan Friedlander, Alex Mahalov (Co-Chairman), Basil Nicolaenko, Sergey Repin, Gregory Seregin (Co-Chairman), Vsevolod Solonnikov (Honorary Chairman).
Information: Contact Tim Shilkin and Alexander Mikhaylov (EEC300@pdmi.ras.ru); http://www.pdmi.ras.ru/~eec300/.

7-9 **Ordinal and Symbolic Data Analysis (OSDA) 2007**, Ghent University, Ghent, Belgium. (Jan. 2007, p. 64)

8-13 **The Ninth International Conference on Geometry, Integrability and Quantization**, Sts. Constantine and Elena resort, Varna, Bulgaria. (Feb. 2007, p. 305)

* 8-14 **33rd International Conference: Applications of Mathematics in Engineering and Economics**, Sozopol, Bulgaria.
Goal: The main goal of the conference is to bring together experts and young talented scientists from Bulgaria and abroad, to discuss the modern trends, and to ensure exchange of views in various applications of mathematics in engineering, physics, economics, biology, etc.
Organizer: The Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Bulgaria.
Topics: Potential Theory and Partial Differential Equations; Mathematical; Analysis and Applications; Differential Equations and Differential Geometry; Numerical Methods in Mathematical Modeling, Algebraic Methods in Informatics; Software Innovations in Scientific Computing.
Preliminary List of Keynote Speakers: P. Caithamer (USA), C. I. Christov (USA), A. Haghighi (USA), H. Kojouharov (USA), M. Konstantinov (Bulgaria), R. Lazarov (USA), A. Loskutov (Russia), P. Minev (Canada), A. Schmitt (Germany), P. Velmisov (Russia).
Information: http://www.tu-sofia.bg/fpmi/amee/index.html or contact: mtod@tu-sofia.bg.

* 10-12 **Leonhard Euler Festival**, Euler IMI, St. Petersburg, Russia.
Description: Russian Academy of Sciences, St. Petersburg Scientific Center of RAS, St. Petersburg Department of the V. A. Steklov Mathematical Institute, Euler International Mathematical Institute, St. Petersburg State University, and Euler Foundation organize a special congress on the occasion of the 300th anniversary of Leonhard Euler's birth. This main event will comprise a celebration meeting on the 10th of June, and several invited talks related to Euler's tremendous scientific activity.
Organizing Committee: L. D. Faddeev, E. A. Tropp, I. A. Ibragimov, S. V. Kislyakov, G. A. Leonov, Yu. V. Matiyasevich, Yu. D. Burago, V. A. Gritsenko, G. A. Seregin, V. N. Tolstykh, A. M. Vershik, S. V. Vostokov, P. G. Zograf.
Information: Preliminary registrations should be made at: http://www.pdmi.ras.ru/EIMI/2007/Euler300/; email: euler300@imi.ras.ru.

10-12 **Special Congress on the occasion of the 300th anniversary of Leonhard Euler's birth**, St. Petersburg, Russia. (Feb. 2007, p. 305)

11-15 **An Algebraic Geometry Conference**, IHP, Paris, France. (Sept. 2006, p. 960)

11-15 **Barcelona Conference on C*- Algebras and Their Invariants**, Centre de Recerca Matemàtica, Barcelona, Spain. (Jan. 2007, p. 64)

* 11-15 **Conference on Complex Function Theory and Geometry**, Warsaw, Poland.
Description: This conference will explore the interface between the complex function theory of several complex variables and the modern theory of geometric analysis. We particularly welcome young mathematicians and members of under-represented groups. The conference is supported in part by the National Science Foundation of the USA, the BK21 Research Organization in Korea and the Banach Center.
Organizers: Aline Bonami, Bo Berndtsson, Joaquim Bruna, Friedrich Haslinger, Kang-Tae Kim, Jurgen Leiterer, Ewa Ligocka, Jeff McNeal, Marco Peloso, Steven G. Krantz (head organizer).
Information: http://www.math.wustl.edu/~sk/.

11-22 **Workshop: Real, Tropical, and Complex Enumerative Geometry**, Centre de recherches mathématiques, Montréal, Québec, Canada. (Feb. 2007, p. 305)

11-July 6 **Clay Mathematics Institute 2007 Summer School on "Homogeneous flows, moduli spaces, and arithmetic"**, Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy. (Feb. 2007, p. 305)

12-14 **Conference on Ordered Statistical Data and Inequalities: Theory & Applications**, The University of Jordan, Amman, Jordan. (Dec. 2006, p. 1378)

* 13-19 **International Conference on Arithmetic Geometry**, Euler IMI, St. Petersburg, Russia.
Description: The conference on number theory in framework of Mathematical Congress is dedicated to the 300th anniversary of Leonhard Euler.
Topic: The subject of the conference is main trends of modern study of zeta and L functions.
Organizer: Russian Academy of Sciences and the Euler Foundation.
Scientific Commitee: Ch. Deninger (Muenster), I. Fesenko (Nottingham), N. Kurokawa (Tokyo), A. Parshin (Moscow), S. Vostokov (St. Petersburg).
Information: info@eulerfoundation.com.

* 14-16 **Model Theory and Algebra**, Department of Mathematics and Computer Science, Camerino, Italy.
Description: This event is a part of the European Research and Training Network MODNET. It is devoted to Model Theory of Modules, Model Theory of Groups, o-minimality and other matters linking Model Theory and Algebra.
Speakers: H. Adler, A. Berarducci, A. Berkman, P. D'Aquino, O. Frécon, G. Garkusha, I. Herzog, G. Jones, M. Kamensky, A. Macintyre, M. Otero, G. Puninski, P. Rothmaler, C. Rivière.
Programme Committee: Z. Chatzidakis, D. Evans, A. Pillay, F. Point, C. Toffalori, C. Wagner. Local organizers are S. Leonesi, S. L'Innocente, C. Toffalori.
Information: Visit http://modnet07.cs.unicam.it.

14–16 **The XVIth International Colloquium on Integrable Systems and Quantum symmetries**, Czech Technical University, Prague, Czech Republic. (Nov. 2006, p. 1253)

* 15–17 **International Conference on Rings and Things: Dedicated to Carl Faith on his 80th birthday and Barbara Osofsky on her 70th**, Ohio University, Zanesville, Ohio.
Topics: The conference will also have contributed talks. The main topics covered at the conference include Structure Theory of Rings and Modules, Representation Theory, and Group Rings, among others.
Support: National Security Agency, Center of Ring Theory and Applications at Ohio University, and Ohio University regional campuses at Chillicothe, Lancaster, and Zanesville.
Invited Speakers include: Victor P. Camillo (University of Iowa, USA), Alberto Facchini (University of Padova, Italy), Kent R. Fuller (University of Iowa, USA), Jose Luis Gomez Pardo (University de Santiago de Compostela, Spain), Charudatta Hajarnavis (University of Warwick, UK), Birge Huisgen-Zimmermann (University of California Santa Barbara, USA), T. Y. Lam (University of California Berkeley, USA), Patrick F. Smith (University of Glasgow, UK), Agata Smoktunowicz (The University of Edinburgh, UK), Jan Trlifaj (Charles University, Czech Republic), Robert Wisbauer (University of Dusseldorf, Germany).
Registration: For registration, submission of an abstract, and other information, please go to: `http://math.ohiou.edu/~jain/ou-z_conference/`.
Deadline: For submitting abstract: April 30, 2007.
Information: For more information: Contact Pramod Kanwar at email: `pkanwar@math.ohiou.edu`.

15–24 **NEEDS 2007 (Nonlinear Evolution Equations and Dynamical Systems)**, L'Ametlla de Mar, Spain. (Feb. 2007, p. 305)

16–17 **NEEDS 2007 School**, L'Ametlla de Mar, Spain. (Feb. 2007, p. 305)

17–23 **International Conference: Skorokhod Space. 50 Years On**, Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv, Ukraine. (Feb. 2007, p. 305)

17–23 **Trends in Harmonic Analysis**, Strobl, Salzburg, Austria. (Feb. 2007, p. 305)

* 18–21 **Summer School in Model Theory**, Department of Mathematics and Computer Science, Camerino, Italy.
Description: This event is a part of the European Research and Training Network MODNET. It will consist of 3 short courses in Model theory of Groups (A. Baudisch), Model theory of Modules (P. Rothmaler), Introduction to o-minimality (K. Peterzil). It is intended for Ph.D. students, but other participants are welcome.
Information: Visit `http://modnet07.cs.unicam.it`.

18–23 **Combinatorics and Optimization 40th Anniversary Conference**, University of Waterloo, Waterloo, Ontario, Canada. (Sept. 2006, p. 960)

18–23 **Computability in Europe 2007 (CiE 2007)**, Siena, Italy. (Feb. 2007, p. 305)

* 18–23 **Russian-German Geometry Meeting dedicated to the 95th anniversary of A. D. Alexandrov**, Euler IMI, St. Petersburg, Russia.
Organizers: W. Ballmann (Bonn), Yu. D. Burago (St. Petersburg), Yu. G. Reshetnyak (Novosibirsk).
Co-organizers: S. V. Buyalo (St. Petersburg), A. Lytchak (Bonn), I. A. Taimanov (Novosibirsk).
Topics: Topics of the meeting: differential geometry, nonlinear differential equations in geometry, geometric analysis, Alexandrov geometry, applications of geometry to mathematical physics.
Information: email: `geometry@imi.ras.ru`; `http://www.pdmi.ras.ru/EIMI/2007/geo3/`.

18–24 **Algebraic Topology: Old and New (M. M. Postnikov Memorial Conference)**, Stefan Banach International Mathematical Center (Bedlewo, Poland). (Jan. 2007, p. 64)

18–29 **Flow in Porous Media with Emphasis on Modeling Oil Reservoirs**, University of Wyoming, Laramie, Wyoming. (Jan. 2007, p. 64)

18–29 **Hamiltonian Dynamical Systems and Applications Systèmes Dynamiques Hamiltoniens et Applications**, Université de Montréal, Montréal, Québec, Canada. (Feb. 2007, p. 306)

19–22 **Computational Algebraic Geometry**, Oakland University, Rochester, Michigan. (Feb. 2007, p. 306)

20–23 **International Conference "Trends and Challenges in Applied Mathematics" (ICTCAM 2007)**, Technical University of Civil Engineering, Bucharest, Romania.

* 21–26 **Conference on Riemannian Geometry and Applications: Dedicated to the memory of Professor Radu Rosca**, Brasov, Romania.
Topics: Geometry of Riemannian and Pseudo-Riemannian manifolds, submanifold theory, structures on manifolds, complex geometry, Finsler, Lagrange and Hamilton geometries, applications to other fields.
Organizing and Scientific Committee: Ion Mihai (University of Bucharest), Radu Miron (Romanian Academy), Leopold Verstraelen (Katholieke Universiteit Leuven), Stere Ianus (University of Bucharest), Emil Stoica (Transilvania University of Braşov), Bogdan Suceavă (California State University at Fullerton, Gheorghe Munteanu (Transilvania University of Braşov), Adela Mihai (University of Bucharest), Valentin Ghişoiu (University of Bucharest).
Fee: There is no registration fee.
Information: Those who are interested in attending the conference and/or presenting a communication are kindly invited to send an e-mail to: Ion Mihai; email: `imihai@fmi.unibuc.ro`.

24–30 **Lyapunov Memorial Conference: International Conference on the occasion of the 150th Birthday of Aleksandr Lyapunov**, Karazin Kharkiv National University and Verkin Institute for Low Temperature Physics, Kharkiv, Ukraine. (Sept. 2006, p. 960)

24–30 **Seventh International Conference "Symmetry in Nonlinear Mathematical Physics"**, Institute of Mathematics, Kiev, Ukraine. (Sept. 2006, p. 960)

24–July 1 **45th International Symposium on Functional Equations**, Bielsko-Biala, Poland. (Dec. 2006, p. 1379)

* 25–29 **Braids**, Inst. Math. Sciences, National Univ., Singapore, Singapore.
Organizers: Jon Berrick (National University of Singapore)(co-chair), Fred R. Cohen (University of Rochester)(co-chair), Mitch Berger (University College London), Joan S. Birman (Columbia University), Toshitake Kohno (University of Tokyo), Yan-Loi Wong (National University of Singapore), Jie Wu (National University of Singapore).
Confirmed Principal Speakers: M. Berger (London), J. Birman (Columbia), T. Brendle (Cornell), R. Budney (MPIM Bonn), F. Cohen (Rochester), R. Ghrist (UIUC), J. Gonzalez-Meneses (Seville), T. Kohno (Tokyo), D. Margalit (Utah), S. Morita (Tokyo), L. Paris (Bourgogne), D. Rolfsen (Canada), N. Wahl (Chicago), B. Wiest (Rennes).
Information: `http://www.ims.nus.edu.sg/Programs/braids/index.htm`.

25–29 **Conference on Enumeration and Probabilistic Methods in Combinatorics**, Centre de Recerca Matemàtica, Barcelona, Spain. (Jan. 2007, p. 64)

* 25–29 **Workshop on Modeling, Analysis and Simulation of Multiscale Nonlinear Systems**, Oregon State University, Corvallis, Oregon.
Program: The workshop will bring together an interdisciplinary group of scientists working on various aspects of nonlinear coupled phenomena occuring at multiple spatial and temporal scales in

natural and man-made environments. The program includes general overview talks on methods and applications as well as special topics.
Topics: Modeling of preferential fluid flow and transport and other phenomena in subsurface; experiment-based and computational porescale-to-watershed modeling; analysis of multiscale nonlinear PDEs systems as well as stochastic approaches; adaptive multiscale computational techniques and their implementations; biological and engineering applications with multiscale character.
Organizers and Contact: Local (Oregon State): Malgorzata Peszynska, Ralph Showalter, Son-Young Yi. Program: Seth Oppenheimer (Mississipi State), Alexander Panchenko (Washington State), Anna Spagnuolo (Oakland University), Noel Walkington (Carnegie-Mellon).
Support: There are opportunities for at least partial support for some participants. We encourage women, persons with disabilities, and those from underrepresented groups to participate and to apply for support.
Registration: Registration, submission of abstracts, and requests for support will be accepted online starting 1/15/2007.
Information: http://www.math.oregonstate.edu/multiscale/workshop.

*25–30 **16th Summer St. Petersburg Meeting on Mathematical Analysis**, Euler IMI, St. Petersburg, Russia.
Description: This year our meeting is a satellite conference to the Congress dedicated to the 300th Anniversary of Leonhard Euler's Birth.
Preliminary List of Main Speakers: Alexandru Aleman, Lund University; Nikolai Makarov, CalTech, Pasadena; Vitali Milman, Tel-Aviv University; Boris Paneyakh, Technion, Haifa; Edward B. Saff, Vanderbilt University, Nashville; Kristian Seip, Trondheim University; Stanislav Smirnov, Geneva University; Serguei Treil, MIT, Boston.
Information: email: analysis@pdmi.ras.ru; http://www.pdmi.ras.ru/EIMI/2007/analysis16/.

25–30 **ERLOGOL-2007: Intermediate problems of model theory and universal algebra**, State Technical University–Math Institute, Novosibirsk, Novosibirsk-Altai/Russia. (Oct. 2006, p. 1091)

25–30 **International Conference "Algebraic Analysis and Around"–in honor of Professor Masaki Kashiwara's 60th birthday**, Kyoto University, Kyoto, Japan. (Jan. 2007, p. 64)

26–29 **ALM (Adults learning mathematics) 14th International Conference**, Limerick University, Limerick, Republic of Ireland. (Feb. 2007, p. 306)

26–29 **Nonlinear Modeling and Control, An International Seminar**, Nayanova University, Samara, Russia. (Feb. 2007, p. 306)

*28–30 **Fourth European PKI Workshop: Theory and Practice**, University of Illes Balears, Mallorca, Spain.
Description: The 4th European PKI Workshop: Theory and Practice is focusing on all research aspects of Public Key Applications, Services and Infrastructures. Submitted papers may present theory, applications or practical experiences.
Deadlines: Submission of papers: February 28, 2007. Notification to authors: March 30, 2007. Camera-ready copies: April 16, 2007.
General Chair: Jose L. Ferrer, University of Balearic Islands, Spain.
Programme Committee co-Chairs: Javier Lopez, University of Malaga, Spain; Pierangela Samarati, University of Milan, Italy.
Information: http://dmi.uib.es/europki07/.

28–July 4 **6th Congress of Romanian Mathematicians**, Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania. (Sept. 2006, p. 960)

29–July 1 **SIAM Conference on Control and Its Applications (CT07)**, Hyatt Regency San Francisco Airport, San Francisco, California. (Feb. 2007, p. 306)

*30–July 7 **14th International Conference on Waves and Stability in Continuous Media (WASCOM07)**, Hotel Village Baia Samuele, Scicli, Ragusa, Italy.
Dedication: To Professor T. Ruggeri on the occasion of his 60th birthday.
Information: http://mat520.unime.it/wascom07/.

July 2007

*1–4 **Fifth Mathematics & Design International Conference (M&D-2007)**, Universidade Regional de Blumenau, Blumenau, SC - Brazil.
Topics: Computer Design, Mathematical Modeling, Visualization of System Media Design, Computational Geometry, Projects and Communication Design, Mathematical Human Sustainable Development, Art and Mathematics.
Organizers: Brazilian: Emilia Mello, Nelson Hein, Marias Adélia B. Schmitt, Maria Roseli Bertoldi, Rosinete Gaertner. Argentine: Susana Toscano, Leonardo Pablo Diez, Guillermo Rolón.
Deadlines: Submission of Abstracts: January 31, 2007. Abstracts Acceptance: March 15, 2007. Registration: Starting February 15, 2007.
Information: http://www.maydi.org.ar/myd07; email: myd2007@maydi.org.ar; salett@furb.br; vspinade@fibertel.com.ar.

1–7 **The VI International Algebraic Conference in Ukraine, dedicated to the 100th anniversary of Professor D. K. Faddeev**, Kamenets-Podol'sky State University, Kamenets-Podol'sky, Ukraine. (Dec. 2006, p. 1379)

1–13 **Cohomology of groups and Algebraic K-theory**, Center of Math Sciences, Zhejiang University, China. (Dec. 2006, p. 1379)

2–4 **The 2007 International Conference of Applied and Engineering Mathematics**, Imperial College London, London, U.K. (Nov. 2006, p. 1253)

2–6 **19th International Conference on Formal Power Series and Algebraic Combinatorics**, Nankai University, Tianjin, China. (Feb. 2007, p. 306)

2–6 **25th Journees Arithmetiques**, University of Edinburgh, Scotland, UK. (May 2006, p. 612)

2–6 **Des équations aux dérivées partielles au calcul scientifique: Congrès en l'honneur de Luc Tartar à l'occasion de son soixantième anniversaire**, Carré des Sciences, Ancienne Ecole Polytechnique, Paris, France. (Jan. 2007, p. 64)

2–6 **Design Theory of Alex Rosa, a meeting in celebration of Alex Rosa's 70th Birthday**, Bratislava, Slovakia. (Sept. 2006, p. 961)

*2–7 **International Conference: Modular Forms and Moduli Spaces**, Euler IMI, St. Petersburg, Russia.
Description: The conference on modular forms and moduli spaces in framework of Mathematical Congress dedicated to the 300th anniversary of Leonhard Euler organized by the Russian Academy of Sciences and the Euler Foundation.
Organizers: V. Gritsenko, D. Orlov, P. Zograf.
Information: http://www.pdmi.ras.ru/EIMI/2007/mfms/; email: mfms2007@imi.ras.ru.

2–13 **Geometry and Lie Theory**, Australian National University, Canberra, and University of Sydney, Australia. (Feb. 2007, p. 306)

3–6 **18th International Workshop on Operator Theory and its Applications (IWOTA-2007)**, North-West University, Potchefstroom, South Africa. (Dec. 2006, p. 1379)

*3–7 **Symposium on the differential geometry of submanifolds**, University of Valenciennes, Valenciennes, France.
Workshop Topics: The main theme of the conference is the differential geometry of submanifolds. Special emphasis is on the following topics: Lagrangian immersions, Minimal immersions and constant mean curvature immersions, Harmonic maps and harmonic morphisms, Variational problems, Affine differential geometry.

Main Speakers: P. Baird, Université de Brest, France, J. Bolton, Durham University, UK, B.Y. Chen, Michigan State University, U.S.., V. Cortès, University of Hamburg, Germany, S. Gudmundsson, Lund University, Sweden, M. Haskins, Imperial College, UK, M. Koiso, Nara Women's University, Japan, H. Li, Tsinghua University, China, I. Mcintosh, York University, UK, B. Opozda, Cracow University, Pologne, B. Palmer, Idaho University, U.S.., P. Romon, Université de Marne la Vallée, France, W. Rossmann, Kobe University, Japan, F. Urbano, Universidad de Granada, Spain.
Information: email: luc.vrancken@univ-valenciennes.fr; http://symposium.geometry@web.de.

4–7 **APFA 6: Applications of Physics in Financial Analysis Conference**, ISCTE Business School, Lisbon, Portugal. (Feb. 2007, p. 307)

4–8 **International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA 2007)**, Babes-Bolyai University, Cluj-Napoca, Romania. (Oct. 2006, p. 1091)

6–12 **Sixth Summer School on Potential Theory and Applications**, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria. (Feb. 2007, p. 307)

8–12 **2007 von Neumann Symposium**, Snowbird, Utah. (Dec. 2006, p. 1379)

* 8–12 **International Conference on Analytical Methods of Celestial Mechanics**, Euler IMI, St. Petersburg, Russia.
Dedication: Dedicated to the 300th anniversary of Leonhard Euler.
Organizer: By the Russian Academy of Sciences and the Euler Foundation.
Chairs: K. Kholshevnikov, N. Vassiliev.
Organizing Committee: V. Orlov, E. Novikova (secretary), I. Shevchenko, N. Vassiliev.
Information: email: amcm@imi.ras.ru; http://www.pdmi.ras.ru/EIMI/2007/AMCM/.

9–12 **International Conference on Artificial Intelligence and Pattern Recognition**, Orlando, Florida. (Aug. 2006, p. 824)

9–12 **International Conference on Enterprise Information Systems and Web Technologies**, Orlando, Florida. (Aug. 2006, p. 824)

9–12 **International Conference on High Performance Computing, Networking and Communication Systems**, Orlando, Florida. (Aug. 2006, p. 824)

9–12 **International Conference on Software Engineering Theory and Practice (SETP-07)**, Orlando, Florida. (Aug. 2006, p. 824)

9–13 **Conference on Applied Mathematics and Scientific Computing**, Brijuni Island, Croatia. (Dec. 2006, p. 1379)

9–13 **Dynamics Days Europe 2007**, Loughborough University, United Kingdom. (Jan. 2006, p. 70)

* 9–August 3 **Program on Computational Methods in Biomolecular Structures and Interaction Networks**, Institute for Mathematical Sciences, National University of Singapore, Singapore.
Program: The program will be structured around two workshops and two tutorials designed to bring together researchers from a wide spectrum of mathematical and computational biology. The main themes to be covered include: Computational algorithms for large-scale analysis, classification and predictions of structures, motifs, modules, and biomolecular interactions; Mathematical models and computer simulation of structural and evolution dynamics of macromolecules and their interactions; Deterministic, probabilistic, and cellular automata models of biomolecular interaction networks and pathways; Algorithms for visualization of complex data and networks; as well as other unsolved problems arising from macromolecular imaging research.
Information: Please visit http://www.ims.nus.edu.sg/Programs/biomolecular07/index.htm for more information and registration. For general enquiries, please email to imssec@nus.edu.sg. For enquiries on scientific aspects of the program, please email Vladimir A. Kuznetsov at kuznetsov@gis.a-star.edu.sg.

12–14 **Sixth International Conference on Lattice Path Combinatorics and Applications**, East Tennessee State University, Johnson City, Tennessee. (Oct. 2006, p. 1091)

13–19 **1007 ASL European Summer Meeting (Logic Colloquium '07)**, Wroclaw, Poland. (Jan. 2007, p. 64)

16–20 **International Congress on Industrial and Applied Mathematics, ICIAM07**, ETH and University Zurich, Zurich, Switzerland. (Dec. 2006, p. 1379)

16–22 **The 8th International Conference on Fixed Point Theory and Its Applications**, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand. (Sept. 2006, p. 961)

17–30 **The Second International Conference on Optimization and Optimal Control**, National University of Mongolia, Ulaanbaatar, Mongolia. (Dec. 2006, p. 1380)

* 20–22 **Ross Program Fiftieth Anniversary Reunion-Conference**, Ohio State University, Columbus, Ohio.
Description: Founded by Arnold Ross in 1957, the Ross Mathematics Program is a residential summer session for high school students talented in mathematics. This 50th anniversary event will celebrate the successes of this program with a general reunion (and banquet). The celebration will feature several mathematical lectures of general interest. Alumni and friends of the Ross Program are invited to attend.
Information: http://www.math.osu.edu/ross/Reunions/reunion07;email: ross@math.ohio-state.edu.

22–25 **OPTIMIZATION 2007**, Faculty of Economics, University of Porto, Porto, Portugal. (Dec. 2006, p. 1380)

22–27 **CIM/UC Summer School: Topics in Nonlinear PDEs**, Centro Internacional de Matemática (CIM), Coimbra, Portugal. (Nov. 2006, p. 1253)

22–28 **Topological Theory of Fixed and Periodic Points (TTFPP 2007)**, Conference Center of the Mathematical Institute of the Polish Academy of Sciences, Bedlewo near Poznan, Poland. (Feb. 2007, p. 307)

* 23–27 **The Twelfth International Conference on Difference Equations and Applications (ICDEA07)**, The Technical University of Lisbon, Portugal.
Description: The purpose of the conference is to bring together both experts and novices in the theory and application of Difference Equations and Discrete Dynamical Systems. The main theme of the meeting will be "Discrete Dynamical Systems and Nonlinear Science".
Organizer: The Center of Mathematical Analysis Geometry and Dynamical Systems of Instituto Superior Técnico (IST), Technical University of Lisbon.
Information: More details may be found at http://www.math.ist.utl.pt/icdea2007/. All Inquiries should be sent to icdea@math.ist.utl.pt.

23–27 **VII Americas School in Differential Equations**, University of Cartagena, Cartagena, Colombia (South America). (Feb. 2007, p. 307)

23–December 21 **Strong Fields, Integrability and Strings**, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK. (Nov. 2005, p. 1264)

* 24–27 (REVISED) **3rd International workshop "Reliable Methods of Mathematical Modelling" (RMMM 2007)**, Euler IMI, Saint Petersburg, Russia.
Aims and Scope: Workshop is organized to bring together specialists developing mathematical and computational methods intended

to increase the reliability of the results obtained in various mathematical modeling methods.
Topics: Mesh-adaptive numerical methods in various applied problems, a posteriori error control and verification of numerical solutions, validation of mathematical models used in computer simulation.
Important Dates: Deadline for submission of applications: March 1, 2007. Confirmation of acceptance: April 1, 2007. Arrival: July 22 2007. Departure: July 28, 2007.
Information: http://www.rmmm2007.org.ru; email: rmmm2007@pdmi.ras.ru.

30–August 3 **L-functions and modular forms**, AIM Research Conference Center, Palo Alto, California. (Dec. 2006, p. 1380)

* 30–August 3 **The 15th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications**, Media Center and Library (Sugimoto Campus), Osaka City University, Osaka, Japan.
Workshop Topics: Applied Complex Analysis, Clifford Analysis, Complex Dynamical Systems, Complex Function Spaces and Operator Theory, Complex Numerical Analysis, Functional Analysis Methods in Complex Analysis and Applications to Partial Differential Equations, Quasiconformal Mapping, Riemann Surfaces, Teichmüller Theory and Kleinian Groups, Complex Manifolds, Several Complex Variables, Value Distribution Theory, and Related Topics.
Deadlines: Registration: April 15, 2007. Submission of Abstracts: May 31, 2007.
Information: Yoichi Imayoshi, Department of Mathematics, Osaka City University 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585 Japan; email: imayoshi@sci.osaka-cu.ac.jp; http://15ficajpn.org/.

31–August 3 **First Joint International Meeting between the AMS and the Polish Mathematical Society**, Warsaw, Poland. (Jun/Jul. 2006, p. 714)

August 2007

3–6 **First Announcement ACA'2007: 13th International Conference on Applications of Computer Algebra**, Oakland University, Rochester, Michigan. (Feb. 2006, p. 287)

6–8 **Joint SOCR CAUSEway Continuing Education Workshop 2007**, UCAL, Los Angeles, California. (Oct. 2006, p. 1091)

6–10 **Security '07: 16th USENIX Security Symposium**, Sheraton Boston Hotel, 39 Dalton Street, Boston, Mass. (Dec. 2006, p. 1380)

* 6–17 **19th European Summer School in Logic, Language and Information**, Dublin, Ireland.
Aim: Student Session exists to bring together young researchers to present and discuss their work in progress with a possibility to get feedback from senior researchers.
Call for Papers: We invite submission of papers in the areas of Logic, Language and Computation for presentation at the Student Session and for appearance in the proceedings. At least one of the authors of the paper must register as a participant of ESSLLI.
Deadline: Submission deadline: February 11, 2007 Notification of authors: April 20, 2007 Full paper deadline: May 20, 2007
Information: The Student Session webpage is the place for relevant information: http://www.loria.fr/~sustreto/stus07/. Feel free to contact the chairs for any questions about the submissions or the Student Session in general: Ville Nurmi, Phone: +358 9 191 51497, Fax: +358 9 191 51400, email: ville.v.nurmi@helsinki.fi; Dmitry Sustretov Phone: +33 3 83 59 20 35, Fax: +33 3 83 41 30 79, email: dmitry.sustretov@loria.fr.

* 12–15 **The 13th International Conference of Knowledge Discovery and Data Mining: ACM/SIGKDD**, San Jose, California.
The SIGKDD conference will feature keynote presentations, oral paper presentations, poster sessions, workshops, tutorials, and panels, as well as the KDD Cup competition. Papers on all aspects of knowledge discovery and data mining are solicited.
Topics: Areas of interest include, but are not limited to: Applications of data mining (biomedicine, business, e-commerce, defense), Data mining for community generation, social network analysis, and graph-structured data, Foundations of data mining, high performance and parallel/distributed data mining, integration of data warehousing and data mining, interactive and online data mining, KDD framework and process, mining data streams, mining high-dimensional data, mining sensor data, mining text and semi-structured data, mining multi-media data, novel data mining algorithms, robust and scalable statistical methods, pre-processing and post-processing for data mining, security, privacy and social impact of data mining, spatial and temporal data mining, visual data mining and data visualization.
Deadlines: Abstracts: February 23, 2007. Paper submissions: February 28, 2007 (9 pages).
Information: For submission details and organizers, see http://www.kdd2007.com.

13–17 **Generic Case Complexity**, AIM Research Conference Center, Palo Alto, California. (Feb. 2007, p. 307)

19–25 **The Eighth International Workshop on Differential Geometry and its Applications**, Cluj-Napoca, Romania. (Feb. 2007, p. 307)

20–24 **Geometric Aspects of Analysis and Mechanics: A Conference in Honor of the 65th Birthday of Hans Duistermaat**, Utrecht University, Utrecht, The Netherlands. (Dec. 2006, p. 1380)

* 24–26 **32nd Sapporo Symposium on Partial Differential Equations**, Department of Mathematics, Hokkaido University, Sapporo, Japan.
Description: The Sapporo Symposium on Partial Differential Equations has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school.
Financial support: Limited amount of financial support is available to non-resident visitors who are interested in the meeting. Please make inquiry to cri@math.sci.hokudai.ac.jp by March 31, 2007.

* 26–September 1 **Conference "Algebras, Representations and Applications" Lie and Jordan Algebras, their Representations and Applications, III: In Honour of Professor Ivan Shestakov's 60th Birthday**, Maresias Beach Hotel, Maresias, São Paulo, Brazil.
Registration/Fee: Registration can be made online on the webpage. The registration fee is 120 Euro or US $150 payable in cash upon the arrival; student registration fee is 60 Euro or US $75. Registration fee includes: publication of abstracts, transfers to/from the hotel, conference banquet and a tour.
Deadline: Abstract submission: June 1, 2007.
Information: http://www.ime.usp.br/~liejor/2007/ConferenceShestakov/index.html.

27–29 **International Conference on Biomathematics 2007**, ITB, Bandung, Indonesia. (Feb. 2007, p. 307)

September 2007

3–December 21 **Phylogenetics**, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK. (Nov. 2006, p. 1264)

4–6 **International Conference on Mathematical Biology 2007 (ICMB07)**, Universiti Putra Malaysia, Serdang, Malaysia. (Nov. 2006, p. 1253)

7–13 **9th International Conference of The Mathematics Education into the 21st Century Project**, Charlotte, North Carolina. (Apr. 2006, p. 498)

* 9–10 **SHARCS'07: Special-purpose Hardware for Attacking Cryptographic Systems**, Vienna Marriott Hotel, Vienna, Austria.
Deadline: Submission deadline: June 15.

Information: Please consult the workshop's webpage http://www.sharcs.org for details.

10–14 **High-order methods for computational wave propagation and scattering**, AIM Research Conference Center, Palo Alto, California. (Aug. 2006, p. 824)

10–15 **International Conference on Nonlinear Partial Differential Equations dedicated to the memory of Igor V. Skrypnik**, NPDE2007, Yalta, Crimea, Ukraine. (Feb. 2007, p. 307)

11–15 **CSL07: 16th EACSL Annual Conference on Computer Science and Logic**, Lausanne, Switzerland. (Feb. 2007, p. 308)

13–14 **IMA Tutorial: Mathematics of Nucleic Acids**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1380)

*16–20 **International Conference of Numerical Analysis and Applied Mathematics 2007 (ICNAAM 2007)**, Corfu, Greece.
Invited Speakers: Carl R. de Boor, Department of Computer Sciences and Department of Mathematics, University of Wisconsin-Madison, USA; C. W. Gear (Bill), Senior Scientist, Chemical Engineering, Princeton University (zero-time appointment), Emeritus President, NEC Research Institute, Emeritus Professor, Department of Computer Science, University of Illinois at Urbana-Champaign, USA; Mariano Gasca, Depto. Matematica Aplicada, Fac. Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain; G. Alistair Watson, University of Dundee, Division of Mathematics, Dundee DD1 4HN, Scotland.
Leaflets and Posters: If you want leaflets and posters for ICNAAM 2007, please send your request to: tsimos@mail.ariadne-t.gr (with a carbon copy to: tsimos.conf@gmail.com).
Information: http://www.icnaam.org/.

*17–19 **Indian International Conference on Artificial Intelligence (IICAI-07)**, Pune, India.
Description: The conference consists of invited talks, keynote speeches, local tours, tutorials, etc.
Information: http://www.iiconference.org.

*17–20 **Fourth International Workshop on Meshfree Methods for Partial Differential Equations**, Universitaet Bonn, Bonn, Germany.
Aim: To promote collaboration among engineers, mathematicians, and computer scientists and industrial researchers to address the development, mathematical analysis, and application of meshfree and particle methods especially to multiscale phenomena.
Topics: While contributions in all aspects of meshfree methods are invited, some of the key topics to be featured are Coupling of meshfree methods, finite element methods, particle methods, and finite difference methods; Coupling of multiple scales, e.g. continuum models to discrete models; Application of meshfree, generalized finite element methods; Parallel computation in meshfree methods; Mathematical theory of meshfree, generalized finite element, and particle methods; Fast and stable domain integration methods; Enhanced treatment of boundary conditions; Identification and characterization of problems where meshfree methods have clear advantage over classical approaches.
Deadlines: If you are interested in contributing to this workshop, please submit an abstract of about 300 words (preferably in LaTeX format) by e-mail to the contact address meshfree@ins.uni-bonn.de by May 1, 2007. Confirmation and Program: August 1, 2007.
Information: http://wissrech.ins.uni-bonn.de/meshfree.

17–21 **13th Czech-French-German Conference on Optimization**, University of Heidelberg, Heidelberg, Germany. (Feb. 2007, p. 308)

17–21 **IMA Workshop: Mathematics and Biology of Nucleic Acids**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1380)

23–28 **14th Workshop on Stochastic Geometry, Stereology and Image Analysis**, Friedrich-Schiller-University Jena, Department of Stochastics, Jena, Germany. (Feb. 2007, p. 308)

*24–27 **International Algebraic Conference dedicated to the 100th anniversary of D. K. Faddeev**, Euler IMI, St. Petersburg, Russia.
Organizers: St. Petersburg State University, St. Petersburg Department of the V. A. Steklov Institute of Mathematics of the Russian Academy of Science, Euler International Mathematical Institute, Euler Foundation.
Program Committee: A. V. Yakovlev, S. V. Vostokov, N. A. Vavilov, A. I. Generalov, N. L. Gordeev.
Information: Elena Novikova (novikova@pdmi.ras.ru); http://www.pdmi.ras.ru/EIMI/2007/DKF/.

25–28 **The 2nd International Conference on Nonlinear Dynamics: KhPI 2007 in honor of Alexander Lyapunov 150th Anniversary**, National Technical University, Kharkov Polytechnical Institute, Kharkov, Ukraine. (Feb. 2007, p. 308)

October 2007

5–6 **AMS Central Section Meeting**, DePaul University, Chicago, Illinois. (Dec. 2006, p. 1380)

6–7 **AMS Eastern Section Meeting**, Rutger University-New Brunswick, Busch Campus, New Brunswick, New Jersey. (Dec. 2006, p. 1380)

8–12 **Dichotomy Amenable/Nonamenable in Combinatorial Group Theory**, AIM Research Conference Center, Palo Alto, California. (Jun/Jul. 2006, p. 714)

*9–11 **SIAM Conference on Mathematics for Industry: Challenges and Frontiers (MI07)**, Hyatt Regency Philadelphia, Philadelphia, Pennsylvania.
Description: SIAM's conference on Mathematics for Industry focuses attention on the many and varied opportunities to promote applications of mathematics to industrial problems.
Deadlines: Minisymposium proposals: March 9, 2007. Abstracts for contributed and minisymposium speakers: April 9, 2007.
Information: http://www.siam.org/meetings/mi07/.

13–14 **AMS Western Section Meeting**, University of New Mexico, Albuquerque, New Mexico. (Jun/Jul. 2006, p. 714)

*24–26 **2007 International Conference in Modeling Health Advances**, Clark Kerr Campus, UC Berkeley, California.
Deadlines: Last Date for Submission of Manuscripts: July 6, 2007. Last Date for Submission of Final/Camera Ready Paper: July 30, 2007. Pre-registration Due: July 30, 2007.
Information: For further information, please contact: Prof. B. D. Aggarwala, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada aggarwal@math.ucalgary.ca.

29–November 2 **IMA Workshop: RNA in Biology, Bioengineering and Nanotechnology**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

November 2007

3–4 **AMS Southeastern Section Meeting**, Middle Tennessee State University, Murfreesboro, Tennessee. (Jun/Jul. 2006, p. 714)

*5–9 **Algorithmic Convex Geometry**, AIM Research Conference Center, Palo Alto, California.
Description: This workshop, sponsored by AIM and the NSF, is motivated by algorithms and draws heavily on probability (especially the theory of stochastic processes), convex geometry and functional analysis. The workshop will bring together leading researchers in these areas.
Organizers: Assaf Naor and Santosh Vempala.
Deadline: July 1, 2007.
Information: http://aimath.org/ARCC/workshops/convexgeometry.html.

December 2007

7–11 **Fourth Pacific Rim Conference on Mathematics: Celebrating the Tenth Anniversary of the Liu Bie Ju Centre for Mathematical Sciences**, City University of Hong Kong, Hong Kong. (Jan. 2007, p. 64)

12–15 **First Joint International Meeting between the AMS and the New Zealand Mathematical Society (NZMS)**, Wellington, New Zealand. (Jun/Jul. 2006, p. 714)

* 16–20 **The Twelfth Asian Technology Conference in Mathematics (ATCM2007)**, Taipei, Taiwan.
Description: The ATCM 2007 is an international conference that will continue addressing technology-based issues in all Mathematical Sciences. Thanks to advanced technological tools such as computer algebra systems (CAS), interactive and dynamic geometry, and handheld devices, the effectiveness of our teaching and learning, and the horizon of our research in mathematics and its applications continue to grow rapidly. The aim of this conference is to provide a forum for educators, researchers, teachers and experts in exchanging information regarding enhancing technology to enrich mathematics learning, teaching and research at all levels.
Theme: Making mathematics fun, accessible and challenging through technology.
Language: English.
Deadlines: Submission of Abstracts: June 15, 2007. Submission of Full Papers: July 15, 2007.
Information: `http://www.atcm.mathandtech.org`.

January 2008

7–June 27 **Statistical Theory and Methods for Complex, High-Dimensional Data**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Jun/Jul. 2006, p. 714)

10–11 **IMA Tutorial: Mathematics of Proteins**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

14–18 **IMA Workshop: Protein Folding**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

14–July 4 **Combinatorics and Statistical Mechanics**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Dec. 2006, p. 1381)

March 2008

3–7 **IMA Workshop: Organization of Biological Networks**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

April 2008

17–18 **IMA Workshop: Network Dynamics and Cell Physiology**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

21–25 **IMA Workshop: Design Principles in Biological Systems**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

May 2008

26–30 **IMA Workshop: Quantitative Approaches to Cell Motility and Chemotaxis**, University of Minnesota, Minneapolis, Minnesota. (Dec. 2006, p. 1381)

June 2008

30–July 4 **ICMI/IASE Study: Statistics Education in School Mathematics: Challenges for Teaching and Teacher Education**, ITESM, Monterrey, Mexico. (Dec. 2006, p. 1381)

July 2008

14–18 **5th European Congress of Mathematics**, Amsterdam, the Netherlands. (Feb. 2007, p. 308)

14–December 19 **Mathematics and Physics of Anderson Localization: 50 Years After**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Jun/Jul. 2006, p. 714)

22–26 **International Workshop on Operator Theory and its Applications (IWOTA)**, College of William and Mary, Williamsburg, Virginia. (Feb. 2007, p. 308)

August 2008

26–December 19 **The Nature of High Reynolds Number Turbulence**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom. (Jun/Jul. 2006, p. 714)

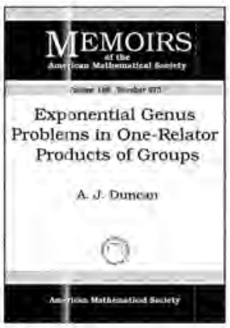
September 2008

12–18 **Models in Developing Mathematics Education**, Dresden University of Applied Sciences, Dresden, Germany. (Apr. 2006, p. 498)

24–27 **Vector Measures, Integration and Applications**, Katholische Universitaet Eichstaett-Ingolstadt, Eichstaett, Germany. (Feb. 2007 p. 308)

New Publications Offered by the AMS

Algebra and Algebraic Geometry



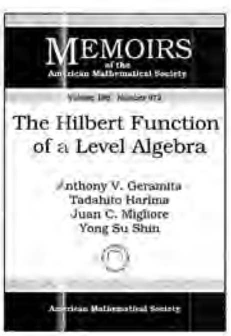
Exponential Genus Problems in One-Relator Products of Groups

A. J. Duncan, *University of Newcastle, Newcastle upon Tyne, England*

Contents: Introduction; Quadratic words; Quadratic exponential equations and $\mathcal{L}$-genus; Resolutions of quadratic equations; Decision problems; Pictures; Corridors; Angle assignment; Curvature; Configurations C; Configurations D; Final angle adjustment; Isoperimetry; Proof of Theorem 5.9; Bibliography.

Memoirs of the American Mathematical Society, Volume 186, Number 873

February 2007, 156 pages, Softcover, ISBN-10: 0-8218-3945-4, ISBN-13: 978-0-8218-3945-4, LC 2006047919, 2000 *Mathematics Subject Classification:* 20F65, 20F05, 20F10; 20F06, 57M07, **Individual member US$40**, List US$66, Institutional member US$53, Order code MEMO/186/873



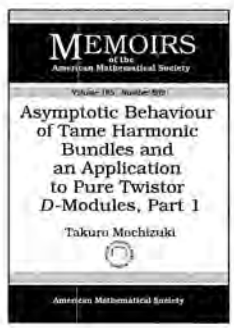
The Hilbert Function of a Level Algebra

Anthony V. Geramita, *Queen's University, Kingston, ON, Canada*, **Tadahito Harima**, *Hokkaido University of Education, Kushiro, Hokkaido, Japan*, **Juan C. Migliore**, *University of Notre Dame, IN*, and **Yong Su Shin**, *Sungshin Women's University, Seoul, Korea*

Contents: *Part 1. Nonexistence and Existence:* Introduction; Numerical conditions; Homological methods; Some refinements; Constructing Artinian level algebras; Constructing level sets of points; Expected behavior; *Part 2. Appendix: A Classification of Codimension Three Level Algebras of Low Socle Degree:* Appendix A. Introduction and notation; Appendix B. Socle degree 6 and Type 2; Appendix C. Socle degree 5; Appendix D. Socle degree 4; Appendix E. Socle degree 3; Appendix F. Summary; Appendix. Bibliography.

Memoirs of the American Mathematical Society, Volume 186, Number 872

February 2007, 139 pages, Softcover, ISBN-10: 0-8218-3940-3, ISBN-13: 978-0-8218-3940-9, LC 2006047920, 2000 *Mathematics Subject Classification:* 13D40, 13D02; 13C13, 13C40, 14C20, **Individual member US$38**, List US$64, Institutional member US$51, Order code MEMO/186/872



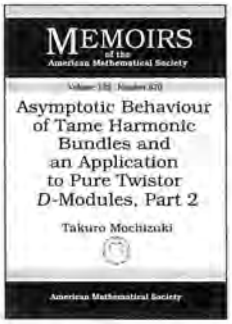
Asymptotic Behaviour of Tame Harmonic Bundles and an Application to Pure Twistor *D*-Modules, Part 1

Takuro Mochizuki, *Kyoto University, Japan*

Contents: Introduction; *Part 1. Preliminary:* Preliminary; Preliminary for mixed twistor structure; Preliminary for filtrations; Some lemmas for generically splitted case; Model bundles; *Part 2. Prolongation of Deformed Holomorphic Bundles:* Harmonic bundles on a punctured disc; Harmonic bundles on a product of punctured discs; The KMS-structure of the space of the multi-valued flat sections; The induced regular λ-connection on $\Delta^n \times C^*$; *Part 3. Limiting Mixed Twistor theorem and Some Consequence:* The induced vector bundle over $\mathbb{P}^1$; Limiting mixed twistor theorem; Norm estimate; Bibliography; Index.

Memoirs of the American Mathematical Society, Volume 185, Number 869

January 2007, 324 pages, Softcover, ISBN-10: 0-8218-3942-X, ISBN-13: 978-0-8218-3942-3, LC 2006047813, 2000 *Mathematics Subject Classification:* 14C30, 32S40, 53C07, 53C43, **Individual member US$51**, List US$85, Institutional member US$68, Order code MEMO/185/869



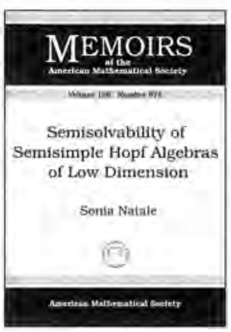
Asymptotic Behaviour of Tame Harmonic Bundles and an Application to Pure Twistor D-Modules, Part 2

Takuro Mochizuki, *Kyoto University, Japan*

Contents: *Part 4. An Application to the theory of Pure Twistor D-modules:* Pure twistor D-module; Prolongation of $\mathcal{R}$-module $\mathcal{E}$; The filtrations of $\mathfrak{E}[\eth_t]$; The weight filtration on $\psi_{t,u}\mathfrak{E}$ and the induced $\mathcal{R}$-triple; The sesqui-linear pairings; Polarized pure twistor D-module and tame harmonic bundles; The pure twistor D-modules on a smooth curve (Appendix); *Part 5. Characterization of Semisimplicity by Tame Pure Imaginary Pluri-harmonic Metric:* Preliminary; Tame pure imaginary harmonic bundle; The Dirichlet problem in the punctured disc case; Control of the energy of twisted maps on a Kahler surface; The existence of tame pure imaginary pluri-harmonic metric; Bibliography; Index.

Memoirs of the American Mathematical Society, Volume 185, Number 870

January 2007, 240 pages, Softcover, ISBN-10: 0-8218-3943-8, ISBN-13: 978-0-8218-3943-0, LC 2006047813, 2000 *Mathematics Subject Classification:* 14C30, 32S40, 53C07, 53C43, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/185/870



Semisolvability of Semisimple Hopf Algebras of Low Dimension

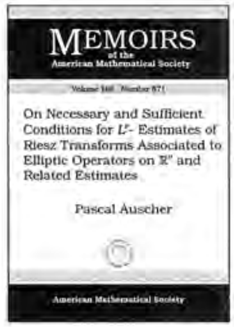
Sonia Natale, *Universidad Nacional de Córdoba, Argentina*

Contents: Introduction and main results; Conventions and notation; Semisimple Hopf algebras; The Nichols-Richmond theorem; Quotient coalgebras; Braided Hopf algebras; Cocycle deformations of some Hopf algebras; Dimension 24; Dimension 30; Dimension 36; Dimension 40; Dimension 42; Dimension 48; Dimension 54; Dimension 56; Appendix A. Drinfeld double of H_8; Appendix. Bibliography.

Memoirs of the American Mathematical Society, Volume 186, Number 874

February 2007, 123 pages, Softcover, ISBN-10: 0-8218-3948-9, ISBN-13: 978-0-8218-3948-5, LC 2006047928, 2000 *Mathematics Subject Classification:* 16W30; 17B37, **Individual member US$37**, List US$62, Institutional member US$50, Order code MEMO/186/874

Analysis



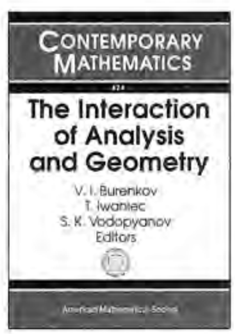
On Necessary and Sufficient Conditions for L^p-Estimates of Riesz Transforms Associated to Elliptic Operators on $\mathbb{R}^n$ and Related Estimates

Pascal Auscher, *Université Paris-Sud, Orsay, France*

Contents: Beyond Calderón-Zygmund operators; Basic L^2 theory for elliptic operators; L^p theory for the semigroup; L^p theory for square roots; Riesz transforms and functional calculi; Square function estimates; Miscellani; Appendix A. Calderón-Zygmund decomposition for Sobolev functions; Appendix. Bibliography.

Memoirs of the American Mathematical Society, Volume 186, Number 871

February 2007, 75 pages, Softcover, ISBN-10: 0-8218-3941-1, ISBN-13: 978-0-8218-3941-6, 2000 *Mathematics Subject Classification:* 42B20, 42B25, 47F05, 47B44, 35J15, 35J30, 35J45, **Individual member US$36**, List US$60, Institutional member US$48, Order code MEMO/186/871



The Interaction of Analysis and Geometry

V. I. Burenkov, *Cardiff University, United Kingdom*, **T. Iwaniec**, *Syracuse University, NY*, and **S. K. Vodopyanov**, *Sobolev Institute of Mathematics, Novosibirsk, Russia*, Editors

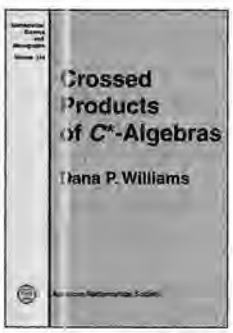
The papers in this volume are based on talks given at the International Conference on Analysis and Geometry in honor of the 75th birthday of Yuriĭ Reshetnyak (Novosibirsk, 2004). The topics include geometry of spaces with bounded curvature in the sense of Alexandrov, quasiconformal mappings and mappings with bounded distortion (quasiregular mappings), nonlinear potential theory, Sobolev spaces, spaces with fractional and generalized smoothness, variational problems, and other modern trends in these areas. Most articles are related to Reshetnyak's original works and demonstrate the vitality of his fundamental contribution in some important fields of mathematics such as the geometry in the "large", quasiconformal analysis, Sobolev spaces, potential theory and variational calculus.

This item will also be of interest to those working in geometry and topology.

Contents: **I. D. Berg** and **I. G. Nikolaev**, On an extremal property of quadrilaterals in an Aleksandrov space of curvature $\leq K$; **V. I. Burenkov**, **H. V. Guliyev**, and **V. S. Guliyev**, On boundedness of the fractional maximal operator from complementary Morrey-type spaces to Morrey-type spaces; **V. N. Dubinin** and **D. B. Karp**, Generalized condensers and distortion theorems for conformal mappings of planar domains; **M. L. Goldman**, Rearrangement invariant envelopes of generalized Besov, Sobolev, and Calderon spaces; **T. Iwaniec**, Null Lagrangians, the art of integration by parts; **M. Karmanova**, Geometric measure theory formulas on rectifiable metric spaces; **A. P. Kopylov**, Stability and regularity of solutions to elliptic systems of partial differential equations; **V. M. Miklyukov**, Removable singularities of differential forms and A-solutions; **H. Murakami**, Various generalizations of the volume conjecture; **P. Pedregal**, Gradient Young measures and applications to optimal design; **H. M. Riemann**, Wavelets for the cochlea; **Y. G. Reshetnyak**, Sobolev-type classes of mappings with values in metric spaces; **L. Székelyhidi, Jr.**, Counterexamples to elliptic regularity and convex integration; **S. K. Vodopyanov**, Geometry of Carnot-Carathéodory spaces and differentiability of mappings; **S. K. Vodopyanov**, Foundations of the theory of mappings with bounded distortion on Carnot groups.

Contemporary Mathematics, Volume 424

April 2007, approximately 342 pages, Softcover, ISBN-10: 0-8218-4060-6, ISBN-13: 978-0-8218-4060-3, LC 2006052735, 2000 *Mathematics Subject Classification:* 26-XX, 28-XX, 30Cxx, 35-XX, 46Exx, 49-XX, 53Cxx, 57Mxx, 58-XX, **All AMS members US$79**, List US$99, Order code CONM/424



Crossed Products of C^*-Algebras

Dana P. Williams, *Dartmouth College, Hanover, NH*

The theory of crossed products is extremely rich and intriguing. There are applications not only to operator algebras, but to subjects as varied as noncommutative geometry and mathematical physics. This book provides a detailed introduction to this vast subject suitable for graduate students and others whose research has contact with crossed product C^*-algebras. In addition to providing the basic definitions and results, the main focus of this book is the fine ideal structure of crossed products as revealed by the study of induced representations via the Green-Mackey-Rieffel machine. In particular, there is an in-depth analysis of the imprimitivity theorems on which Rieffel's theory of induced representations and Morita equivalence of C^*-algebras are based. There is also a detailed treatment of the generalized Effros-Hahn conjecture and its proof due to Gootman, Rosenberg, and Sauvageot.

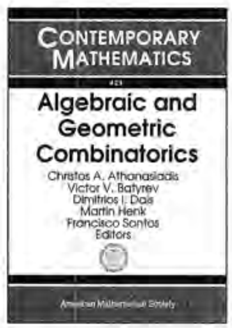
This book is meant to be self-contained and accessible to any graduate student coming out of a first course on operator algebras. There are appendices that deal with ancillary subjects, which while not central to the subject, are nevertheless crucial for a complete understanding of the material. Some of the appendices will be of independent interest.

Contents: Locally compact groups; Dynamical systems and crossed products; Special cases and basic constructions; Imprimitivity theorems; Induced representations and induced ideals; Orbits and quasi-orbits; Properties of crossed products; Ideal structure; The proof of the Gootman-Rosenberg-Sauvageot theorem; Amenable groups; The Banach *-algebra $L^1(G, A)$; Bundles of C^*-algebras; Groups; Representations of C^*-algebras; Direct integrals; Effros's ideal center decomposition; The Fell topology; Miscellany; Notation and Symbol Index; Index; Bibliography.

Mathematical Surveys and Monographs, Volume 134

March 2007, 528 pages, Hardcover, ISBN-10: 0-8218-4242-0, ISBN-13: 978-0-8218-4242-3, LC 2006047931, 2000 *Mathematics Subject Classification:* 46L55, 46L05, 22D25, 22D30, 46L45, 54H15, **All AMS members US$87**, List US$109, Order code SURV/134

Discrete Mathematics and Combinatorics



Algebraic and Geometric Combinatorics

Christos A. Athanasiadis, *University of Athens, Hellas, Greece*, **Victor V. Batyrev**, *Universität Tübingen, Germany*, **Dimitrios I. Dais**, *University of Crete, Hellas, Greece*, **Martin Henk**, *Otto von Guericke University, Magdeburg, Germany*, and **Francisco Santos**, *University of Cantabria, Santander, Spain*, Editors

This volume contains original research and survey articles stemming from the Euroconference "Algebraic and Geometric Combinatorics". The papers discuss a wide range of problems that illustrate interactions of combinatorics with other branches of mathematics, such as commutative algebra, algebraic geometry, convex and discrete geometry, enumerative geometry, and topology of complexes and partially ordered sets. Among the topics covered are combinatorics of polytopes, lattice polytopes, triangulations and subdivisions, Cohen-Macaulay cell complexes, monomial ideals, geometry of toric surfaces, groupoids in combinatorics, Kazhdan-Lusztig combinatorics, and graph colorings. This book is aimed at researchers and graduate students interested in various aspects of modern combinatorial theories.

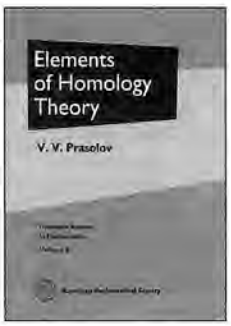
Contents: **V. V. Batyrev**, Lattice polytopes with a given $h*$-polynomial; **A. Conca**, **S. Hoten**, and **R. R. Thomas**, Nice initial complexes of some classical ideals; **V. C. Quiñonez**, Ratliff-Rush monomial ideals; **P. Csorba** and **F. H. Lutz**, Graph coloring manifolds; **D. I. Dais**, Geometric combinatorics in the study of compact toric surfaces; **D. I. Dais**, **M. Henk**, and **G. M. Ziegler**, On the existence of Crepant resolutions of Gorenstein abelian quotient singularities in dimensions ≥ 4; **P. Fiebig**, Kazhdan-Lusztig combinatorics via sheaves on Bruhat graphs; **G. Fløystad**, Cohen-Macaulay cell complexes; **D. N. Kozlov**, Homology tests for graph colorings; **P. McMullen**, Polyhedra and polytopes: Algebra and combinatorics; **B. Nill**, Classification of pseudo-symmetric simplicial reflexive polytopes; **A. Paffenholz** and **A. Werner**, Constructions for 4-polytopes and the cone of flag vectors; **R. T. Živaljević**, Groupoids in combinatorics–Applications of a theory of local symmetries.

Contemporary Mathematics, Volume 423

February 2007, approximately 453 pages, Softcover, ISBN-10: 0-8218-4080-0, ISBN-13: 978-0-8218-4080-1, LC 2006043048, 2000 *Mathematics Subject Classification:* 05-06, 05Exx, 52-06; 05Axx, 05Cxx, 14Mxx, 20Cxx, 52Axx, 52Bxx, 52Cxx, **All AMS members US$71**, List US$89, Order code CONM/423

Geometry and Topology

COURSE ADOPTION



Elements of Homology Theory

V. V. Prasolov, *Independent University of Moscow, Russia*

The book is a continuation of the previous book by the author (*Elements of Combinatorial and Differential Topology*, Graduate Studies in Mathematics, Volume 74, American Mathematical Society, 2006). It starts with the definition of simplicial homology and cohomology, with many examples and applications. Then the Kolmogorov–Alexander multiplication in cohomology is introduced. A significant part of the book is devoted to applications of simplicial homology and cohomology to obstruction theory, in particular, to characteristic classes of vector bundles. The later chapters are concerned with singular homology and cohomology, and Čech and de Rham cohomology. The book ends with various applications of homology to the topology of manifolds, some of which might be of interest to experts in the area.

The book contains many problems; almost all of them are provided with hints or complete solutions.

Contents: Simplicial homology; Cohomology rings; Applications of simplicial homology; Singular homology; Čech cohomology and de Rham cohomology; Miscellaneous; Hints and solutions; Bibliography; Index.

Graduate Studies in Mathematics, Volume 81

March 2007, approximately 424 pages, Hardcover, ISBN-10: 0-8218-3812-1, ISBN-13: 978-0-8218-3812-9, LC 2006047074, 2000 *Mathematics Subject Classification:* 55-01, **All AMS members US$55**, List US$69, Order code GSM/81

Number Theory

Hypergéométrie et Fonction Zêta de Riemann

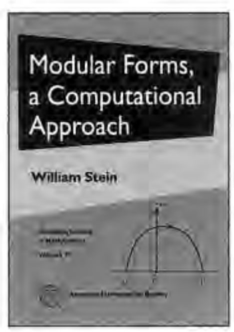
C. Krattenthaler, *Université Claude Bernard, Villeurbanne, France*, and **T. Rivoal**, *Université de Grenoble I, Saint-Martin d'Héres, France*

Contents: Introduction et plan de l'article; Arrière plan; Les résultats principaux; Conséquences diophantiennes du Théorème 1; Le principe des démonstrations des Théorèmes 1 à 6; Deux identités entre une somme simple et une somme multiple; Quelques explications; Des identités hypergéométrico-harmoniques; Corollaires au Théorème 8; Corollaires au Théorème 9; Lemmes arithmétiques; Démonstration du Théorème 1, partie i); Démonstration du Théorème 1, partie ii); Démonstration du Théorème 3, partie i) et des Théorèmes 4 et 5; Démonstration du Théorème 3, partie ii) et du Théorème 6; Encore un peu d'hypergéométrie; Perspectives; Bibliographie.

Memoirs of the American Mathematical Society, Volume 186, Number 875

February 2007, 87 pages, Softcover, ISBN-10: 0-8218-3961-6, ISBN-13: 978-0-8218-3961-4, LC 2006047930, 2000 *Mathematics Subject Classification:* 11J72; 11J82, 33C20, **Individual member US$36**, List US$60, Institutional member US$48, Order code MEMO/186/875

COURSE ADOPTION



Modular Forms, a Computational Approach

William Stein, *University of Washington, Seattle, WA*
with an appendix by Paul E. Gunnells

This marvellous and highly original book fills a significant gap in the extensive literature on classical modular forms. This is not just yet another introductory text to this theory, though it could certainly be used as such in conjunction with more traditional treatments. Its novelty lies in its computational emphasis throughout: Stein not only defines what modular forms are, but shows in illuminating detail how one can compute everything about them in practice. This is illustrated throughout the book with examples from his own (entirely free) software package SAGE, which really bring the subject to life while not detracting in any way from its theoretical beauty. The author is the leading expert in computations with modular forms, and what he says on this subject is all tried and tested and based on his extensive experience. As well as being an invaluable companion to those learning the theory in a more traditional way, this book will be a great help to those who wish to use modular forms in applications, such as in the explicit solution *of Diophantine equations. There is also a useful Appendix by Gunnells on extensions to more general modular forms, which has enough in it to inspire many PhD theses for years to come. While the book's main readership will be graduate students in number theory, it will also be accessible to advanced undergraduates and useful to both specialists and non-specialists in number theory.*

—John E. Cremona, University of Nottingham

William Stein is an associate professor of mathematics at the University of Washington at Seattle. He earned a PhD in mathematics from UC Berkeley and has held positions at Harvard University and UC San Diego. His current research interests lie in modular forms, elliptic curves, and computational mathematics.

Contents: Modular forms; Modular forms of level 1; Modular forms of weight 2; Dirichlet characters; Eisenstein series and Bernoulli numbers; Dimension formulas; Linear algebra; General modular symbols; Computing with newforms; Computing periods; Solutions to selected exercises; Appendix A: Computing in higher rank; Bibliography; Index.

Graduate Studies in Mathematics, Volume 79

March 2007, 268 pages, Hardcover, ISBN-10: 0-8218-3960-8, ISBN-13: 978-0-8218-3960-7, LC 2006047950, 2000 *Mathematics Subject Classification:* 11F11, 11Y16, 11F67, 11F55, 11F75, **All AMS members US$44**, List US$55, Order code GSM/79

New AMS-Distributed Publications

Algebra and Algebraic Geometry

Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires

Cédric Bonnafé, *Université de Franche-Comté, Besancon, France*

A first aim of this paper is to present an overview of results obtained by several authors on the characters of finite reductive groups with non-connected centre. The author is particularly interested in problems directly linked to the non-connectedness of the centre. He emphasises Gelfand-Graev and semisimple characters.

A second aim is to study the influence of the non-connectedness of the centre on the theory of character sheaves. The author studies more precisely the family of character sheaves whose support meets the regular unipotent class: these are analogues of the semisimple characters.

The last aim is the application of these results to finite reductive groups of type A, split or not (as for instance the special linear or special unitary groups). Whenever the cardinality of the finite field is large enough, the author obtains a parametrization of the irreducible characters, a parametrization of the character sheaves, and he shows that the characteristic functions of character sheaves are Fourier transforms of the irreducible characters (Lusztig's conjecture). This gives a theoretical algorithm for computing the character table of these groups.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Introduction; Préliminaires, notations, définitions; Le groupe $Z(\mathbf{G})$; Induction et restriction de Lusztig, séries de Lusztig; Théorie de Harish-Chandra; Autour des caractères de Gelfand-Graev; Faisceaux-caractères; Groupes de type A; Le groupe spécial linéaire; A. Produits en couronne; B. Sommes de Gauss; Bibliographie; Index.

Astérisque, Number 306

November 2006, 165 pages, Softcover, ISBN-10: 2-85629-190-2, ISBN-13: 978-2-85629-190-0, 2000 *Mathematics Subject Classification:* 20G05, 20G40, **Individual member US$47**, List US$52, Order code AST/306

Analysis

Surprises and Counterexamples in Real Function Theory

A. R. Rajwade *and* **A. K. Bhandari**, *Panjab University, Chandigarh, India*

This book presents a variety of intriguing, surprising and appealing topics and nonroutine proofs of several theorems in real function theory. It is a reference book to which one can turn for finding answers to curiosities that arise while studying or teaching analysis.

Chapter 1 is an introduction to algebraic, irrational and transcendental numbers and contains the construction of the Cantor ternary set. Chapter 2 contains functions with extraordinary properties. Chapter 3 discusses functions that are continuous at each point but differentiable at no point. Chapters 4 and 5 include the intermediate value property, periodic functions, Rolle's theorem, Taylor's theorem, points of inflexion and tangents. Chapter 6 discusses sequences and series. It includes the restricted harmonic series, rearrangements of alternating harmonic series and some number theoretic aspects. In Chapter 7, the infinite exponential x with its peculiar range of convergence is studied. Appendix I deals with some specialized topics. Exercises are included at the end of chapters and their solutions are provided in Appendix II.

This book will be useful for students and teachers alike.

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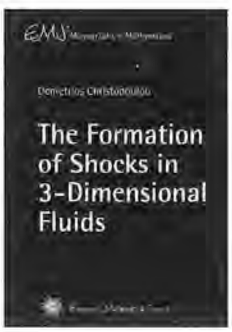
Contents: Introduction to the real line R and some of its subsets; Functions: Pathological, peculiar and extraordinary; Famous everywhere continuous, nowhere differentiable functions: van der Waerden's and others; Functions: Continuous, periodic, locally recurrent and others; The derivative and higher derivatives; Sequences, harmonic series, alternating series and related result; The infinite exponential $x x x$ ::: and related results; A.1. Stirling's formula and the trapezoidal rule; A.2. Schwarz differentiability; A.3. Cauchy's functional equation $f(x+y) = f(x) + f(y)$; Appendix II: Hints and solutions to exercises.

Hindustan Book Agency

January 2007, 298 pages, Hardcover, ISBN-10: 81-85931-71-2, ISBN-13: 978-81-85931-71-5, 2000 *Mathematics Subject Classification:* 26A06, **All AMS members US$34**, List US$42, Order code HIN/32

Applications



The Formation of Shocks in 3-Dimensional Fluids

Demetrios Christodoulou, *Eidgen Technische Hochschule, Zurich, Switzerland*

The equations describing the motion of a perfect fluid were first formulated by Euler in 1752. These equations were among the first partial differential equations to be written down, but, after a lapse of two and a half centuries, we are still far from adequately understanding the observed phenomena which are supposed to lie within their domain of validity.

These phenomena include the formation and evolution of shocks in compressible fluids, the subject of the present monograph. The first work on shock formation was done by Riemann in 1858. However, his analysis was limited to the simplified case of one space dimension. Since then, several deep physical insights have been attained and new methods of mathematical analysis invented. Nevertheless, the theory of the formation and evolution of shocks in real three-dimensional fluids has remained up to this day fundamentally incomplete.

This monograph considers the relativistic Euler equations in three space dimensions for a perfect fluid with an arbitrary equation of state. The author considers initial data for these equations which outside a sphere coincide with the data corresponding to a constant state. Under suitable restriction on the size of the initial departure from the constant state, he establishes theorems that give a complete description of the maximal classical development. In particular, it is shown that the boundary of the domain of the maximal classical development has a singular part where the inverse density of the wave fronts vanishes, signalling shock formation. The theorems give a detailed description of the geometry of this singular boundary and a detailed analysis of the behavior of the solution there. A complete picture of shock formation in three-dimensional fluids is thereby obtained. The approach is geometric, the central concept being that of the acoustical spacetime manifold.

This item will also be of interest to those working in differential equations.

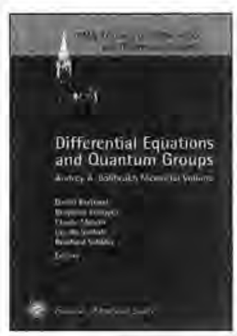
A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: Prologue and summary; Relativistic fluids and nonlinear wave equations. The equations of variations; The basic geometric construction; The acoustical structure equations; The acoustical curvature; The fundamental energy estimate; Construction of the commutation vectorfields; Outline of the derived estimates of each order; Regularization of the propagation equation for $\not{d}\mathrm{tr}\chi$. Estimates for the top order spatial derivatives of χ; Regularization of the propagation equation for $\Delta\mu$. Estimates for the top order spatial derivatives of μ; Control of the angular derivatives of the first derivatives of the x^i. Assumptions and estimates in regard to χ; Control of the spatial derivatives of the first derivatives of the x^i. Assumptions and estimates in regard to μ; Recovery of the acoustical assumptions. Estimates for up to the next to the top order angular derivatives of χ and spatial derivatives of μ; The error estimates involving the top order spatial derivatives of the acoustical entities. The energy estimates. Recovery of the bootstrap assumptions. Statement and proof of the main Theorem: Existence up to shock formation; Sufficient conditions on the initial data for the formation of a shock in the evolution; The nature of the singular hypersurface. The invariant curves. The trichotomy theorem. The structure of the boundary of the domain of the maximal solution; Epilogue; Bibliography; Index.

EMS Monographs in Mathematics

January 2007, 1000 pages, Hardcover, ISBN-10: 3-03719-031-0, ISBN-13: 978-3-03719-031-9, 2000 *Mathematics Subject Classification:* 35L67, 35L65, 35L70, 58J45, 76L05, 76N15, 76Y05, **All AMS members US$158**, List US$198, Order code EMSMONO/2

Differential Equations



Differential Equations and Quantum Groups

Andrey A. Bolibrukh Memorial Volume

Daniel Bertrand, *Université Pierre et Marie Curie, Paris, France*, **Benjamin Enriquez** and **Claude Mitschi**, *Université Louis Pasteur et CNRS, Strasbourg, France*, **Claude Sabbah**, *Ecole Polytechnique, Palaiseau, France*, and **Reinhard Schäfke**, *Université*

Louis Pasteur et CNRS, Strasbourg, France, Editors

This special volume is dedicated to the memory of Andrey A. Bolibrukh. It contains two expository articles devoted to some aspects of Bolibrukh's work, followed by ten refereed research articles.

Topics cover complex linear and nonlinear differential equations and quantum groups: monodromy, Fuchsian linear systems, Riemann–Hilbert problem, differential Galois theory, differential algebraic groups, multisummability, isomonodromy, Painlevé equations, Schlesinger equations, integrable systems, KZ equations, complex reflection groups, and root systems.

This item will also be of interest to those working in mathematical physics.

A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

Contents: Y. Ilyashenko, Realization of irreducible monodromy by Fuchsian systems and reduction to the Birkhoff standard form (by Andrey Bolibrukh); **C. Sabbah**, The work of Andrey Bolibrukh on isomonodromic deformations; **M. Audin**, Two notions of integrability; **W. Balser**, Formal power series solutions of the heat equation in one spatial variable; **P. Belkale, E. Mukhin**, and **A. Varchenko**, Multiplicity of critical points of master functions and Schubert calculus; **P. Boalch**, Some explicit solutions to the Riemann-Hilbert problem; **P. J. Cassidy** and **M. F. Singer**, Galois theory of parameterized differential equations and linear differential algebraic groups; **B. Dubrovin** and **M. Mazzocco**, On the reductions and classical solutions of the Schlesinger equations; **V. A. Golubeva**, On the Riemann-Hilbert correspondence for generalized Knizhnik–Zamolodchikov equations for different root systems; **V. P. Kostov**, Monodromy groups of regular systems on the Riemann sphere; **V. P. Leksin**, Monodromy of Cherednik–Kohno–Veselov connections; **H. Umemura**, Invitation to Galois theory; List of Participants.

IRMA Lectures in Mathematics and Theoretical Physics, Volume 9

December 2006, 302 pages, Softcover, ISBN-10: 3-03719-020-5, ISBN-13: 978-3-03719-020-3, 2000 *Mathematics Subject Classification:* 14N15, 20F55, 32G34, 32S22, 33E17, 34Mxx, 35C10, 35Q15, 70H05, **All AMS members US$43**, List US$54, Order code EMSILMTP/9

Dynamique des difféomorphismes conservatifs des surfaces: un point de vue topologique

Sylvain Crovisier, *Université Paris XIII, Villetaneuse, France,* **John Franks**, *Northwestern University, Evanston, IL,* **Jean-Marc Gambaudo**, *Universidad de Chile, Santiago, Chile,* and **Patrice Le Calvez**, *Université Paris XIII, Villetaneuse, France*

This volume deals with the dynamics of area-preserving surface diffeomorphisms. In dimension 2, some specific mathematical tools are available. In addition, the authors present several approaches and some applications. In particular, they try to show how the geometrical theory of dynamical systems, the group theory, the hydrodynamics and the plane topology interact.

This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: S. Crovisier, Perturbation of C^1-diffeomorphisms and generic conservative dynamics on surfaces; **J. Franks**, Distortion in groups of circle and surface diffeomorphisms; **J.-M. Gambaudo**, Knots, flows, and fluids; **P. Le Calvez**, Identity isotopies on surfaces.

Panoramas et Synthèses, Number 21

November 2006, 142 pages, Softcover, ISBN-10: 2-85629-220-8, ISBN-13: 978-2-85629-220-4, 2000 *Mathematics Subject Classification:* 37-01, 37A05, 37C25, 37E30, 37E45, **Individual member US$47**, List US$52, Order code PASY/21

Probability

Lectures on Empirical Processes

Theory and Statistical Applications

Eustasio del Barrio, *Universidad de Valladolid, Valladolid, Spain,* **Paul Deheuvels**, *Université de Paris VI, Bourg-la-Reine, France,* and **Sara van de Geer**, *ETH Zentrum, Zurich, Switzerland*

The theory of empirical processes constitutes the mathematical toolbox of asymptotic statistics. Its growth was accelerated by the 1950s work on the Functional Central Limit Theorem and the Invariance Principle. The theory has developed in parallel with statistical methodologies, and has been successfully applied to a large diversity of problems related to the asymptotic behaviour of statistical procedures.

The three sets of lecture notes in the book offer a wide panorama of contemporary empirical processes theory. Techniques are developed in the framework of probability in Banach spaces, Hungarian-style strong approximations, using tools from general stochastic process theory. Other tools appear in this text in connection with historical as well as modern applications, such as goodness-of-fit tests, density estimation or general M-estimators.

This book gives an excellent overview of the broad scope of the theory of empirical processes. It will be an invaluable aid for students and researchers interested in an advanced and well-documented approach to the selected topics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: E. del Barrio, Empirical and quantile processes in the asymptotic theory of goodness-of-fit tests; **P. Deheuvels**, Topics on empirical process; **S. van de Geer**, Oracle inequalities and regularization; Index.

EMS Series of Lectures in Mathematics

January 2007, 264 pages, Softcover, ISBN-10: 3-03719-027-2, ISBN-13: 978-3-03719-027-2, 2000 *Mathematics Subject Classification:* 60F05, 60K35, 62C12, 62F05, 62F12, 62G30, 62Jxx, **All AMS members US$38**, List US$48, Order code EMSSERLEC/6

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For more information about the department, visit our website at http://www.math.uah.edu/.
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Department of Mathematics
Endowed Chair in Computational Mathematics

Applications are invited for the position of Endowed Chair of Computational Mathematics in the Mathematics Department of the University of Louisiana at Lafayette. Duties will include teaching primarily at the graduate level, conducting research, directing graduate students, and engaging in departmental and professional service. Applicants should have a Ph.D. in mathematics or a closely related field, with a strong research program and a commitment to excellence. Preference will be given to candidates in the area of biological mathematics who have had experience in working with doctoral students and have successfully generated external funding.

The Department of Mathematics has an established Ph.D. program and is one of the largest academic units on campus. Current faculty research interests include algebra, analysis, applied mathematics, statistics, and topology, with particular emphasis on the modeling of biological and physical phenomena.

The University of Louisiana at Lafayette has a Carnegie Foundation classification of Research University (High Research Activity). It is a public institution with an enrollment of approximately 16,300 students and a faculty of about 550. Located midway between New Orleans and Houston, Lafayette is the heart of Louisiana's Acadian-Creole region. The city of over 120,000 is one of Louisiana's fastest-growing and is the hub of numerous music and cultural festivals and celebrations. Lafayette serves as the base of Louisiana's off-shore oil industry, as well as the financial, retail, and medical center for South-Central Louisiana. Further information about the university and the Mathematics Department is available on the university's webpage at http://www.louisiana.edu.

The appointment will commence August 15, 2007, or at the beginning of a subsequent term. Salary is negotiable, depending on qualifications.

We will review applications as they are received, and continue until the position

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2007 rate is $110 per inch or fraction thereof on a single column (one-inch minimum), calculated from top of headline. Any fractional text of 1/2 inch or more will be charged at the next inch rate. No discounts for multiple ads or the same ad in consecutive issues. For an additional $10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: December 2006 issue–September 28, 2006; January 2007 issue–October 27, 2006; February 2007 issue–November 28, 2006; March 2007 issue–December 29, 2006; April 2007 issue–January 29, 2007; May 2007 issue–February 28, 2007.

U.S. laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to classads@ams.org. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 20904. Advertisers will be billed upon publication.

is filled. Applicants should provide a cover letter detailing their vision for the department and how their background could support this vision, a complete curriculum vitae, and five letters of reference. UL Lafayette is an Equal Opportunity Employer dedicated to increasing diversity among its ranks. Submit applications via regular mail or email to:

Roger Waggoner
Department of Mathematics
University of Louisiana at Lafayette
P.O. Box 41010
Lafayette, LA 70504-1010
rwag@louisiana.edu

000026

NEW YORK

THE COLLEGE OF STATEN ISLAND (CSI)
Department of Mathematics
Assistant Professor of Pure Mathematics

The College of Staten Island (CSI), a senior college of The City of New York (CUNY), invites applications for an anticipated tenure-track position as assistant professor in pure mathematics to start September 2007. Required: Ph.D. in mathematics and a demonstrated commitment to research, publication, and teaching; postdoctoral experience preferred. Candidates working in the research areas of algebra, analysis, geometry, logic, number theory, probability theory, or topology are especially encouraged to apply. Responsibilities include teaching, department and college service, and the advancement of an active, productive research program. The candidate's teaching will include courses in the liberal arts and sciences major for future elementary school teachers; experience working with this student population is an asset. Salary range: $51,344–$66,292 commensurate with qualifications. Send AMS coversheet, cover letter, curriculum vitae, list of publications, a short description of current and planned research, a short statement on teaching experience and philosophy, and three letters of recommendations to: Professor Deborah Franzblau, Chair, Pure Mathematics Search Committee, College of Staten Island, 2800 Victory Boulevard, Room 1S-215, Staten Island, NY 10314. For more information about the Mathematics Program see website: http://www.math.csi.cuny.edu.

000027

COLOMBIA

UNIVERSIDAD DE LOS ANDES
Department of Mathematics
Regular and visiting positions

The Department of Mathematics invites applications for positions at the tenure-track assistant professor level and visiting professor to begin in August 2007. All areas of pure and applied mathematics will be considered but preference will be given to analysis, algebra, differential and algebraic geometry, probability and statistics. Applicants are required to have a Ph.D. in the mathematical sciences and be able to develop a significant research program. A strong commitment with undergraduate and graduate teaching is also required. Duties include courses for undergraduate students in natural sciences, engineering, and economics; graduate courses in mathematics; and the eventual supervising of undergraduate, master, or Ph.D. theses. The department offers internationally competitive salaries with start-up grants for research. Proficiency in Spanish is desirable. Please send an AMS standard cover sheet, curriculum vitae, research plan, teaching statement, and three letters of recommendation to:

Faculty Hiring
Department of Mathematics
Universidad de los Andes
A.A. 4976
Bogotá, Colombia

Electronic submission can also be sent to: matema@uniandes.edu.co. Applicants interested in any further information regarding the Mathematics Department at Los Andes please visit the website: http://matematicas.uniandes.edu.co/. Preference will be given to applicants whose applications are submitted by March 8, 2007. Review of applications will continue until positions are filled.

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ISRAEL

BAR ILAN UNIVERSITY
Department of Mathematics

Several postdoctoral positions in geometry and topology are available at: Department of Mathematics, Bar Ilan University, Israel. For further information, the interested candidates may contact Prof. Mikhail Katz at (972) (3) 531 8194, or via email at katzmik@math.biu.ac.il.

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SINGAPORE

NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

The Department of Mathematics at the National University of Singapore (NUS) invites applications for tenure-track and visiting (including postdoctoral) positions beginning from August 2007.

NUS is a research intensive university that provides quality undergraduate and graduate education. The Department of Mathematics, which is one of the largest in the university, will continue to build upon its strength in pure and applied mathematics and to develop mathematical expertise in emerging areas of applications. We seek promising young scholars or candidates with outstanding track records in any field of pure and applied mathematics. The department offers internationally competitive salaries with start-up grants for research, attractive teaching load for young scholars, a conducive research environment and opportunities for development.

Research areas which the department plans to expand in the near future include (but are not limited to): analysis biomedical imaging, cryptography, financial mathematics partial differential equations, probability.

Application materials should be sent to:

Search Committee
Department of Mathematics
National University of Singapore
2 Science Drive 2, Singapore 117543
Republic of Singapore
Fax: +65 6779 5452

and in addition electronically in a pdf-file to search@math.nus.edu.sg, where inquires may be sent as well.

Please include the following supporting documentation in your application: (1) an American Mathematical Society Standard Cover Sheet, (2) a detailed CV including publications list, (3) a statement of research accomplishments and plan, (4) a statement (max. of 2 pages) of teaching philosophy and methodology. Please attach evaluation on teaching from faculty members or students of your current institution, where applicable, (5) at least three letters of recommendation including one which indicates the candidate's effectiveness and commitment in teaching.

Applicants are requested to complete their applications by April 15. Review of applications will continue until positions are filled. For further information about the department, please see http://www.math.nus.edu.sg.

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Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

Davidson, North Carolina

Davidson College

March 3–4, 2007

Saturday – Sunday

Meeting #1024

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: January 2007
Program first available on AMS website: January 18, 2007
Program issue of electronic *Notices*: March 2007
Issue of *Abstracts*: Volume 28, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

Invited Addresses

Nigel Boston, University of South Carolina and University of Wisconsin, Madison, *Novel applications of algebra to engineering*.

Chaim Goodman-Strauss, University of Arkansas at Fayetteville, *Growth, aperiodicity, and undecidability*.

Andrew J. Granville, University of Montreal, *Erdős's dream and pretentious characters* (Erdős Memorial Lecture).

Alex Iosevich, University of Missouri-Columbia, *Analysis, combinatorics, and arithmetic of incidence theory*.

Shrawan Kumar, University of North Carolina, *Eigenvalue problem for Hermitian matrices and its generalization to arbitrary reductive groups*.

Special Sessions

Algebraic and Extremal Combinatorics, **Gábor Hetyei**, University of North Carolina-Charlotte, and **László A. Székely**, University of South Carolina.

Applicable Algebra, **Nigel Boston**, University of South Carolina, and **Hiren Maharaj**, Clemson University.

Between Harmonic Analysis, Number Theory, and Combinatorics, **Alex Iosevich**, University of Missouri-Columbia, **Michael T. Lacey**, Georgia Institute of Technology, and **Konstantin Oskolkov**, University of South Carolina.

Commutative Algebra and Algebraic Geometry, **Florian Enescu**, Georgia State University, and **Andrew R. Kustin** and **Adela N. Vraciu**, University of South Carolina.

Commutative Rings and Monoids, **Evan G. Houston** and **Thomas G. Lucas**, University of North Carolina, Charlotte.

Computational Group Theory, **Arturo Magidin**, University of Louisiana at Lafayette, **Luise Charlotte Kappe**, Binghamton University, and **Robert F. Morse**, University of Evansville.

Computational and Combinatorial Aspects of Tiling and Substitutions, **Chaim Goodman-Strauss**, University of Arkansas, **Casey Mann**, University of Texas at Tyler, and **Edmund O. Harriss**, Queen Mary University of London.

Dynamical Systems, **Emily B. Gamber**, Santa Fe Institute, **Donna K. Molinek**, Davidson College, and **James S. Wiseman**, Agnes Scott College.

Geometric and Combinatorial Methods in Representation Theory, **Brian Boe** and **William A. Graham**, University of Georgia, and **Kailash C. Misra**, North Carolina State University.

Microlocal Analysis and Partial Differential Equations (in honor of Michael E. Taylor's 60th Birthday), **Anna L. Mazzucato**, Pennsylvania State University, and **Martin Dindos**, University of Edinburgh.

Noncommutative Algebra, **Ellen E. Kirkman** and **James J. Kuzmanovich**, Wake Forest University, and **James Zhang**, University of Washington.

Recent Applications of Numerical Linear Algebra, **Timothy P. Chartier**, Davidson College, and **Amy Langville**, College of Charleston.

Representation Theory and Galois Cohomology in Number Theory, **Ján Mináč**, University of Western Ontario, and **John R. Swallow**, Davidson College.

Stochastic Analysis and Applications, **Armando Arciniéga**, University of Texas at San Antonio.

Oxford, Ohio

Miami University

March 16–17, 2007
Friday – Saturday

Meeting #1025
Central Section
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: January 2007
Program first available on AMS website: February 1, 2007
Program issue of electronic *Notices*: March 2007
Issue of *Abstracts*: Volume 28, Issue 2

Deadlines
For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

Invited Addresses
Sergey Fomin, University of Michigan, *Cluster algebras*.

Naichung Conan Leung, University of Minnesota, *Title to be announced*.

Emil J. Straube, Texas A&M University, *Title to be announced*.

Shouhong Wang, Indiana University, *Title to be announced*.

Special Sessions
Combinatorial and Geometric Group Theory, **John Donnelly**, Mount Union College, and **Daniel Farley**, Mathematisches Institut Einsteinstrasse and Miami University.

Complex Dynamics and Complex Function Theory, **Stephanie Edwards**, University of Dayton, and **Rich Lawrence Stankewitz**, Ball State University.

Finite Geometry and Combinatorics, **Mark A. Miller**, Marietta College.

Geometric Topology, **Jean-Francois LaFont**, Ohio State University, and **Ivonne J. Ortiz**, Miami University.

Graph Theory, **Tao Jiang**, **Zevi Miller**, and **Dan Pritikin**, Miami University.

Large Cardinals in Set Theory, **Paul B. Larson**, Miami University, **Justin Tatch Moore**, Boise State University, and **Ernest Schimmerling**, Carnegie Mellon University.

Noncommutative Algebraic Geometry, **Dennis S. Keeler**, Miami University, **Rajesh Shrikrishna Kulkarni**, Michigan State University, and **Daniel S. Rogalski**, University of California San Diego.

Optimization Theory and Applications, **Olga Brezhneva** and **Doug E. Ward**, Miami University.

PDE Methods in Several Complex Variables, **Jeffery D. McNeal**, Ohio State University, and **Emil J. Straube**, Texas A&M University.

Quantum Topology, **Sergei Chmutov** and **Thomas Kerler**, Ohio State University.

Random Matrices and Non-commutative Probability, **Wlodzimierz Bryc**, University of Cincinnati, and **Narcisse J. Randrianantoanina**, Miami University.

Spectral Theory, Orbifolds, Symplectic Reduction and Quantization, **William Kirwin**, University of Notre Dame, and **Christopher Seaton**, Rhodes College.

Theoretical and Numerical Issues in Fluid Dynamics, **Jie Shen**, Purdue University, and **Shouhong Wang**, Indiana University.

Time Scales: Theory and Applications, **Ferhan M. Atici**, Western Kentucky University, and **Paul W. Eloe**, University of Dayton.

Vector Measures, Banach Spaces and Applications, **Patrick N. Dowling**, Miami University, and **Christopher J. Lennard**, University of Pittsburgh.

Hoboken, New Jersey

Stevens Institute of Technology

April 14–15, 2007
Saturday – Sunday

Meeting #1026
Eastern Section
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: February 2007
Program first available on AMS website: March 8, 2007
Program issue of electronic *Notices*: April 2007
Issue of *Abstracts*: Volume 28, Issue 2

Deadlines
For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: February 27, 2007

Invited Addresses
Neal Koblitz, University of Washington, *Stormy marriage—A periodization of the history of the relationship between mathematics and cryptography*.

Florian Luca, Universidad Nacional Autónoma de México, *Values of arithmetic functions*.

Natasa Pavlovic, Princeton University, *The enigma of the equations of fluid motion: A survey of existence and regularity results*.

Elisabeth Werner, Case Western Reserve University, *Convex bodies: Best and random approximation*.

Special Sessions
Affine Invariants, Randomness, and Approximation in Convex Geometry (Code: SS 2A), **Elisabeth Werner**, Case Western Reserve University, and **Artem Zvavitch**, Kent State University.

Automorphic Forms and Arithmetic Geometry (Code: SS 5A), **Gautam Chinta**, City College of New York, and **Paul E. Gunnells**, University of Massachusetts, Amherst.

Combinatorial Algebraic Geometry (Code: SS 9A), **Angela C. Gibney**, University of Pennsylvania, and **Diane Maclagan**, Rutgers University.

Convex Sets (Code: SS 1A), **David Larman**, University College London, and **Valeriu Soltan**, George Mason University.

Differential Algebra (Code: SS 4A), **Phyllis J. Cassidy**, Smith College and The City College of CUNY, **Richard C. Churchill**, Hunter College and The Graduate Center of CUNY, **Li Guo** and **William F. Keigher**, Rutgers University at Newark, and **Jerald J. Kovacic** and **William Sit**, The City College of CUNY.

Fourier Analysis and Convexity (Code: SS 3A), **Alexander Koldobsky**, University of Missouri Columbia, and **Dmitry Ryabogin**, Kansas State University.

Graph Theory and Combinatorics (Code: SS 11A), **Daniel J. Gross**, **Nathan W. Kahl**, and **John T. Saccoman**, Seton Hall University, and **Charles L. Suffel**, Stevens Institute of Technology.

History of Mathematics on Leonhard Euler's Tercentenary (Code: SS 8A), **Patricia R. Allaire**, Queensborough Community College, CUNY, and **Robert E. Bradley** and **Lee J. Stemkoski**, Adelphi University.

Languages and Groups (Code: SS 6A), **Sean Cleary**, The City College of New York and CUNY Graduate Center, **Murray J. Elder**, Stevens Institute of Technology, and **Gretchen Ostheimer**, Hofstra University.

Mathematical Aspects of Cryptography (Code: SS 7A), **Robert H. Gilman**, Stevens Institute of Technology, **Neal I. Koblitz**, University of Washington, and **Susanne Wetzel**, Stevens Institute of Technology.

Nonlinear Waves in Dissipative/Dispersive Media (Code: SS 12A), **Keith S. Promislow**, Michigan State University, and **Yi Li**, Stevens Institute of Technology.

Number Theory (Code: SS 10A), **Florian Luca**, Universidad Nacional Autónoma de México, and **Allison M. Pacelli**, Williams College.

Optimization of Stochastic Systems (Code: SS 13A), **Darinka Dentcheva**, Stevens Institute of Technology, and **Andrzej Ruszczynski**, Rutgers University.

Tucson, Arizona

University of Arizona

April 21–22, 2007

Saturday – Sunday

Meeting #1027

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: February 2007
Program first available on AMS website: March 8, 2007
Program issue of electronic *Notices*: April 2007
Issue of *Abstracts*: Volume 28, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: February 27, 2007

Invited Addresses

Liliana Borcea, Rice University, *Array Imaging in Random Media*.

James Cushing, University of Arizona, Tucson, *Matrix population models & semelparity*.

Hans Lindblad, University of California, San Diego, *The weak null condition and global existence for Einstein's equations*.

Vinayak Vatsal, University of British Columbia, Vancouver, *Local splitting of ordinary Galois representations*.

Special Sessions

Advances in Spectral Theory of Operators (Code: SS 12A), **Roger Roybal**, California State University, Channel Islands, and **Michael D. Wills**, Weber State University.

Algebraic Combinatorics (Code: SS 14A), **Helene Barcelo** and **Susanna Fishel**, Arizona State University.

Automorphisms of Curves (Code: SS 4A), **Aaron D. Wootton**, University of Portland, **Anthony Weaver**, Bronx Community College, and **S. Allen Broughton**, Rose-Hulman Institute of Technology.

Graph Theory and Combinatorics (Code: SS 9A), **Sebastian M. Cioaba**, University of California at San Diego, and **Joshua Cooper**, University of South Carolina.

Inverse Problems for Wave Propagation (Code: SS 2A), **Liliana Borcea**, Rice University.

Mathematical Modeling in Biology and Medicine (Code: SS 3A), **Carlos Castillo-Chavez**, **Yang Kuang**, **Hal L. Smith**, and **Horst R. Thieme**, Arizona State University.

Moduli Spaces and Invariant Theory (Code: SS 7A), **Philip Foth** and **Yi Hu**, University of Arizona.

New Developments and Directions in Random Matrix Theory (Code: SS 13A), **Peter David Miller**, University of Michigan, and **Estelle Basor**, California Polytechnic State University.

Number Theory in the Southwest (Code: SS 10A), **Dinesh S. Thakur** and **Douglas L. Ulmer**, University of Arizona.

Operator Algebras (Code: SS 6A), **Steven P. Kaliszewski**, **Jack Spielberg**, and **John C. Quigg**, Arizona State University.

Partial Differential Equations and Geometric Analysis (Code: SS 11A), **Sunhi Choi**, **Lennie Friedlander**, and **David Alan Glickenstein**, University of Arizona.

Representations of Algebras (Code: SS 1A), **Frauke Maria Bleher**, University of Iowa, **Birge K. Huisgen-Zimmermann**, University of California Santa Barbara, and **Dan Zacharia**, Syracuse University.

Special Functions and Orthogonal Polynomials (Code: SS 15A), **Diego Dominici**, State University of New York at New Paltz, and **Robert S. Maier**, University of Arizona.

Spectral Analysis on Singular and Noncompact Manifolds (Code: SS 8A), **Juan Bautista Gil**, Pennsylvania State University, and **Thomas Krainer**, Pennsylvania State University.

Subjects in and around Fluid Dynamics (Code: SS 5A), **Robert Owczarek**, Los Alamos National Laboratory, and **Mikhail Stepanov**, University of Arizona.

Zacatecas, Mexico

Universidad Autónoma de Zacatecas

May 23-26, 2007

Wednesday - Saturday

Meeting #1028

Seventh Joint International Meeting of the AMS and the Sociedad Matematica Mexicana.
Associate secretary: Matthew Miller
Announcement issue of *Notices*: April 2007
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

AMS Invited Addresses

Monica Clapp, Universidad Nacional Autónoma de México, *Title to be announced.*

Edward L. Green, Virginia Polytechnic Institute & State University, *Title to be announced.*

Kiran S. Kedlaya, Massachusetts Institute of Technology, *Title to be announced.*

John Lott, University of Michigan, Ann Arbor, *Title to be announced.*

Gelasio Salazar, Universidad Autónoma de San Luis Potosi, *Title to be announced.*

Petr Zhevandrov, Universidad Michoacana de San Nicolás Hidalgo, *Title to be announced.*

Warsaw, Poland

University of Warsaw

July 31 - August 3, 2007

Tuesday - Friday

Meeting #1029

First Joint International Meeting between the AMS and the Polish Mathematical Society
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: April 2007
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Invited Addresses

Henryk Iwaniec, Rutgers University, *Title to be announced.*

Tomasz J. Luczak, Adam Mickiewicz University, *Title to be announced.*

Tomasz Mrowka, Massachusetts Institute of Technology, *Title to be announced.*

Ludomir Newelski, University of Wroclaw, *Title to be announced.*

Madhu Sudan, Massachusetts Institute of Technology, *Title to be announced.*

Anna Zdunik, Warsaw University, *Title to be announced.*

Special Sessions

Complex Analysis, **Zeljko Cuckovic**, University of Toledo, **Zbigniew Blocki**, Jagiellonian University, and **Marek Ptak**, University of Agriculture.

Complex Dynamics, **Robert Devaney**, Boston University, **Jane N. Hawkins**, University of North Carolina, and **Janina Kotus**, Warsaw University of Technology.

Complexity of Multivariate Problems, **Joseph F. Traub**, Columbia University, **Grzegoorz W. Wasilkowski**, University of Kentucky, and **Henryk Wozniakowski**, Columbia University.

Control and Optimization of Non-linear PDE Systems, **Irena Lasiecka**, University of Virginia, and **Jan Sokolowski**, Systems Research institute.

Dynamical Systems, **Steven Hurder**, University of Illinois at Chicago, **Michal Misiurewicz**, Indiana University-Purdue University Indianapolis, and **Pawel Walczak**, University of Lodz.

Ergodic Theory and Topological Dynamics, **Dan Rudolph**, Colorado State University, and **Mariusz Lemanczyk**, Nicholas Copernicus University.

Extremal and Probabilistic Combinatorics, **Joel Spencer**, New York University-Courant Institute, and **Michal Karonski** and **Andrzej Rucinski**, Adam Mickiewicz University.

Geometric Applications of Homotopy Theory, **Yuli B. Rudyak**, University of Florida, **Boguslaw Hajduk**, Warsaw University, **Jaroslaw Kedra**, University of Aberdeen, and **Aleksy Tralle**, The College of Economics & Computer Science.

Geometric Function Theory, **Michael Dorff**, Brigham Young University, **Piotr Liczberski**, University of Lodz, **Maria Nowak**, Biblioteka Instytutu Matematyki, and **Ted Suffridge**, University of Kentucky.

Geometric Group Theory, **Mladen Bestvina**, University of Utah, **Tadeusz Januszkiewicz**, Ohio State University, and **Jacek Swiatkowski**, University of Wroclaw.

Invariants of Links and 3-manifolds, **Mieczyslaw Dabkowski**, University of Texas at Dallas, **Jozef H. Przytycki**, George Washington University, **Adam S. Siroka**, State University of New York at Buffalo, and **Pawel Traczyk**, Warsaw University.

Mathematics of Large Quantum Systems, **Michael Loss**, Georgia Institute of Technology, **Jan Philip Solovej**, University of Copenhagen, and **Jan Derezinski**, University of Warsaw.

Partial Differential Equations of Evolution Type, **Susan J. Friedlander**, University of Illinois at Chicago, and **Grzegorz A. Karch**, University of Wroclaw.

Quantum Information Theory, **Robert Alicki**, University of Gdansk, and **Mary Beth Ruskai**, Tufts University.

Chicago, Illinois

DePaul University

October 5–6, 2007

Friday – Saturday

Meeting #1030

Central Section
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: August 2007
Program first available on AMS website: August 16, 2007
Program issue of electronic *Notices*: October 2007
Issue of *Abstracts*: Volume 28, Issue 3

Deadlines

For organizers: March 6, 2007
For consideration of contributed papers in Special Sessions: June 19, 2007
For abstracts: August 7, 2007

Invited Addresses

Martin Golubitsky, University of Houston, *Title to be announced.*

Matthew J. Gursky, University of Notre Dame, *Title to be announced.*

Alex Iosevich, University of Missouri, *Title to be announced.*

David E. Radford, University of Illinois at Chicago, *Title to be announced.*

Special Sessions

Algebraic Combinatorics: Association Schemes and Related Topics (Code: SS 1A), **Sung Y. Song**, Iowa State University, and **Paul Terwilliger**, University of Wisconsin.

Extremal and Probabilistic Combinatorics (Code: SS 3A), **Jozsef Balogh**, University of Illinois at Urbana-Champaign, and **Dhruv Mubayi**, University of Illinois at Chicago.

The Euler and Navier-Stokes Equations (Code: SS 4A), **Alexy Cheskidov**, University of Michigan, **Susan J. Friedlander** and **Roman Shvydkoy**, University of Illinois at Chicago.

Hopf Algebras and Related Areas (Code: SS 2A), **Yevgenia Kashina** and **Leonid Krop**, DePaul University, **M. Susan Montgomery**, University of Southern California, and **David E. Radford**, University of Illinois at Chicago.

New Brunswick, New Jersey

Rutgers University-New Brunswick, Busch Campus

October 6–7, 2007

Saturday – Sunday

Meeting #1031

Eastern Section
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: August 2007
Program first available on AMS website: August 16, 2007
Program issue of electronic *Notices*: October 2007
Issue of *Abstracts*: Volume 28, Issue 3

Deadlines

For organizers: March 6, 2007
For consideration of contributed papers in Special Sessions: June 19, 2007
For abstracts: August 7, 2007

Invited Addresses

Satyan L. Devadoss, Williams College, *Title to be announced.*

Tara S. Holm, University of Connecticut, *Title to be announced.*

Sir Roger Penrose, University of Oxford, *Title to be announced* (Einstein Public Lecture in Mathematics).

Scott Sheffield, Courant Institute and Institute for Advanced Science, *Title to be announced.*

Mu-Tao Wang, Columbia University, *Title to be announced.*

Special Sessions

Commutative Algebra (Code: SS 4A), **Jooyoun Hong**, University of California Riverside, and **Wolmer V. Vasconcelos**, Rutgers University.

Mathematical and Physical Problems in the Foundations of Quantum Mechanics (in honor of Shelly Goldstein's 60th Birthday) (Code: SS 3A), **Roderich Tumulka** and **Detlef Dürr**, München University, and **Nino Zanghi**, University of Genova.

Partial Differential Equations in Mathematical Physics (in honor of Shelly Goldstein's 60th Birthday) (Code: SS 2A), **Sagun Chanillo**, **Michael K.-H. Kiessling**, and **Avy Soffer**, Rutgers University.

Partial Differential Equations of Mathematical Physics, I (dedicated to the memory of Tom Branson) (Code: SS 7A), **Sagun Chanillo**, **Michael K.-H. Kiessling**, and **Avy Soffer**, Rutgers University.

Probability and Combinatorics (Code: SS 1A), **Jeffry N. Kahn** and **Van Ha Vu**, Rutgers University.

Set Theory of the Continuum (Code: SS 5A), **Simon R. Thomas**, Rutgers University.

Toric Varieties (Code: SS 6A), **Milena S. Hering**, Institute for Mathematics and Its Applications, and **Diane Maclagan**, Rutgers University.

Albuquerque, New Mexico

University of New Mexico

October 13–14, 2007

Saturday – Sunday

Meeting #1032

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2007
Program first available on AMS website: August 30, 2007
Program issue of electronic *Notices*: October 2007
Issue of *Abstracts*: Volume 28, Issue 4

Deadlines

For organizers: March 13, 2007
For consideration of contributed papers in Special Sessions: June 26, 2007
For abstracts: August 21, 2007

Invited Addresses

Emmanuel Candes, California Institute of Technology, *Title to be announced.*

Alexander Polischuk, Univerity of Oregon, *Title to be announced.*

Eric Raines, University of California Davis, *Title to be announced.*

William E. Stein, University of California San Diego, *SAGE: Software for Algebra and Geometry Experimentation.*

Special Sessions

Affine Algebraic Geometry (Code: SS 2A), **David Robert Finston**, New Mexico State University

Computational Methods in Harmonic Analysis and Signal Processing (Code: SS 1A), **Emmanuel Candes**, California Institute of Technology, and **Joseph D. Lakey**, New Mexico State University.

Murfreesboro, Tennessee

Middle Tennessee State University

November 3–4, 2007

Saturday – Sunday

Meeting #1033

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: September 2007
Program first available on AMS website: September 20, 2007
Program issue of electronic *Notices*: November 2007
Issue of *Abstracts*: Volume 28, Issue 4

Deadlines

For organizers: April 3, 2007
For consideration of contributed papers in Special Sessions: July 17, 2007
For abstracts: September 11, 2007

Invited Addresses

Sergey Gavrilets, University of Tennessee, *Title to be announced.*

Daniel K. Nakano, University of Georgia, *Title to be announced.*

Carla D. Savage, North Carolina State University, *Title to be announced.*

Sergei Tabachnikov, Pennsylvania State University, *Title to be announced.*

Special Sessions

Advances in Algorithmic Methods for Algebraic Structures (Code: SS 3A), **James B. Hart**, Middle Tennessee State University.

Applied Partial Differential Equations (Code: SS 4A), **Yuri A. Melnikov**, Middle Tennessee State University, and **Alain J. Kassab**, University of Central Florida.

Differential Equations and Dynamical Systems (Code: SS 1A), **Wenzhang Huang** and **Jia Li**, University of Alabama, Huntsville, and **Zachariah Sinkala**, Middle Tennessee State University.

Graph Theory (Code: SS 2A), **Rong Luo**, **Don Nelson**, **Chris Stephens**, and **Xiaoya Zha**, Middle Tennessee State University.

Wellington, New Zealand

To be announced

December 12–15, 2007

Wednesday – Saturday

Meeting #1034

First Joint International Meeting between the AMS and the New Zealand Mathematical Society (NZMS).
Associate secretary: Matthew Miller
Announcement issue of *Notices*: June/July 2007
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: March 31, 2007

For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

AMS Invited Addresses

Marston Conder, University of Auckland, *Title to be announced.*

Rodney G. Downey, Victoria University of Wellington, *Title to be announced.*

Michael H. Freedman, Microsoft Research/University of California Santa Barbara, *Title to be announced.*

Gaven J. Martin, Massey University, *Title to be announced.*

Assaf Naor, Microsoft Research/Courant Institute, *Title to be announced.*

Theodore A. Slaman, University of California Berkeley, *Title to be announced.*

Matthew J. Visser, Victoria University of Wellington, *Title to be announced.*

AMS Special Sessions

Computability Theory, **Rodney G. Downey** and **Noam Greenberg**, Victoria University of Wellington.

Dynamical Systems: Probabilistic and Semigroup Methods, **Arno Berger**, University of Canterbury, **Rua Murray**, University of Waikato, and **Matthew J. Nicol**, University of Houston.

Hopf Algebras and Quantum Groups, **M. Susan Montgomery**, University of Southern California, and **Yinhuo Zhang**, Victoria University of Wellington.

Infinite-Dimensional Groups and Their Actions, **Christopher Atkin**, Victoria University of Wellington, **Greg Hjorth**, University of California Los Angeles/University of Melbourne, **Alica Miller**, University of Louisville, and **Vladimir Pestov**, University of Ottawa.

Matroids, Graphs, and Complexity, **Dillon Mayhew**, Victoria University of Wellington, and **James G. Oxley**, Louisiana State University.

New Trends in Spectral Analysis and Partial Differential Equations, **Boris P. Belinskiy**, University of Tennessee, Chattanooga, **Anjan Biswas**, Delaware State University, and **Boris Pavlov**, University of Auckland.

Quantum Topology, **David B. Gauld**, University of Auckland, and **Scott E. Morrison**, University of California Berkeley.

Special Functions and Orthogonal Polynomials, **Shaun Cooper**, Massey University, **Diego Dominici**, SUNY New Paltz, and **Sven Ole Warnaar**, University of Melbourne.

San Diego, California

San Diego Convention Center

January 6–9, 2008

Sunday – Wednesday

Meeting #1035

Joint Mathematics Meetings, including the 114th Annual Meeting of the AMS, 91st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2007
Program first available on AMS website: November 1, 2007
Program issue of electronic *Notices*: January 2008
Issue of *Abstracts*: Volume 29, Issue 1

Deadlines

For organizers: April 1, 2007
For consideration of contributed papers in Special Sessions: July 26, 2007
For abstracts: September 20, 2007

New York, New York

Courant Institute of New York University

March 15–16, 2008

Note: Change in Dates!

Saturday – Sunday
Eastern Section
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 15, 2007
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Special Sessions

L-Functions and Automorphic Forms (Code: SS 1A), **Alina Bucur**, Institute for Advanced Study, **Ashay Venkatesh**, Courant Institute of Mathematical Sciences, **Stephen D. Miller**, Rutgers University, and **Steven J. Miller**, Brown University.

Baton Rouge, Louisiana

Louisiana State University, Baton Rouge

March 28–30, 2008

Friday – Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 28, 2007
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Bloomington, Indiana

Indiana University

April 4–6, 2008

Friday – Sunday
Central Section
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 4, 2007
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Claremont, California

Claremont McKenna College

May 3–4, 2008

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: October 4, 2007
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Rio de Janeiro, Brazil

Instituto Nacional de Matemática Pura e Aplicada (IMPA)

June 4–7, 2008

Wednesday – Saturday
First Joint International Meeting between the AMS and the Sociedade Brasileira de Matemática.
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Vancouver, Canada

University of British Columbia and the Pacific Institute of Mathematical Sciences (PIMS)

October 4–5, 2008

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 9, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Middletown, Connecticut

Wesleyan University

October 11–12, 2008

Saturday – Sunday
Eastern Section
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 11, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Kalamazoo, Michigan

Western Michigan University

October 17–19, 2008

Friday – Sunday
Central Section
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 17, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Huntsville, Alabama

University of Alabama, Huntsville

October 24–26, 2008

Friday – Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 24, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Shanghai, People's Republic of China

Fudan University

December 17–21, 2008

Wednesday – Sunday
First Joint Interntional Meeting Between the AMS and the Shanghai Mathematical Society
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Invited Addresses

L. Craig Evans, University of California, Berkeley, *Title to be announced.*

Zhi-Ming Ma, Chinese Academy of Sciences, *Title to be announced.*

Richard Schoen, Stanford University, *Title to be announced.*

Richard Taylor, Harvard University, *Title to be announced.*

Xiaoping Yuan, Fudan University, *Title to be announced.*

Weiping Zhang, Chern Institute, *Title to be announced.*

Washington, District of Columbia

Marriott Wardman Park Hotel and Omni Shoreham Hotel

January 7–10, 2009

Wednesday – Saturday
Joint Mathematics Meetings, including the 115th Annual Meeting of the AMS, 92nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: October 2008

Program first available on AMS website: November 1, 2008
Program issue of electronic *Notices*: January 2009
Issue of *Abstracts*: Volume 30, Issue 1

Deadlines

For organizers: April 1, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Urbana, Illinois

University of Illinois at Urbana-Champaign

March 27–29, 2009

Friday – Sunday
Southeastern Section
Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 29, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Raleigh, North Carolina

North Carolina State University

April 4–5, 2009

Saturday – Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 4, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Francisco, California

San Francisco State University

April 25–26, 2009

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 25, 2008
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Francisco, California

Moscone Center West and the San Francisco Marriott

January 6–9, 2010

Wednesday – Saturday
Joint Mathematics Meetings, including the 116th Annual Meeting of the AMS, 93rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society of Industrial and Applied Mathematics (SIAM).
Associate secretary: Matthew Miller
Announcement issue of *Notices*: October 2009
Program first available on AMS website: November 1, 2009
Program issue of electronic *Notices*: January 2010
Issue of *Abstracts*: Volume 31, Issue 1

Deadlines

For organizers: April 1, 2009
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

New Orleans, Louisiana

New Orleans Marriott and Sheraton New Orleans Hotel

January 5–8, 2011, *Wednesday – Saturday*

Joint Mathematics Meetings, including the 117th Annual Meeting of the AMS, 94th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Susan J. Friedlander
Announcement issue of *Notices*: October 2010
Program first available on AMS website: November 1, 2010
Program issue of electronic *Notices*: January 2011
Issue of *Abstracts*: Volume 32, Issue 1

Deadlines

For organizers: April 1, 2010
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2012, *Wednesday – Saturday*

Joint Mathematics Meetings, including the 118th Annual Meeting of the AMS, 95th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2011
Program first available on AMS website: November 1, 2011
Program issue of electronic *Notices*: January 2012
Issue of *Abstracts*: Volume 33, Issue 1

Deadlines

For organizers: April 1, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 9–12, 2013, *Wednesday – Saturday*

Joint Mathematics Meetings, including the 119th Annual Meeting of the AMS, 96th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Lesley M. Sibner
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center

January 15–18, 2014, *Wednesday – Saturday*

Joint Mathematics Meetings, including the 120th Anual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Sproul Hall, Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

Central Section: Susan-J. Friedlander, Department of Mathematics, University of Illinois at Chicago, 851 S. Morgan (M/C 249), Chicago, IL 60607-7045; e-mail: susan@math.nwu.edu; telephone: 312-996-3041.

Eastern Section: Lesley-M. Sibner, Department of Mathematics, Polytechnic University, Brooklyn, NY 11201-2990; e-mail: lsibner@duke.poly.edu; telephone: 718-260-3505.

Southeastern Section: Matthew Miller, Department of Mathematics, University of South Carolina, Columbia, SC 29208-0001, e-mail: miller@math.sc.edu; telephone: 803-777-3690.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** www.ams.org/meetings/.

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 78 in the the January 2007 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit http://www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences: (see http://www.ams.org/meetings/ for the most up-to-date information on these conferences.)

June 16–July 6, 2007: Joint Summer Research Conferences, Snowbird, Utah.

July 8–July 12, 2007: von Neumann Symposium on Sparse Representation and High-Dimensional Geometry, Snowbird, Utah.