Survey of Non-Desarguesian Planes

Charles Weibel

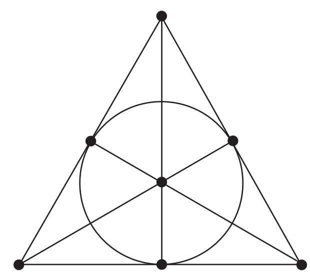
The abstract study of projective geometry first arose in the work of J.-V. Poncelet (1822) and K. von Staudt (1847). About 100 years ago, axiomatic frameworks were developed by several people, including G. Fano, D. Hilbert, E. H. Moore, I. Schur, and O. Veblen. It was a very active branch of mathematics during 1900–1935, and a partial list of people then in this field reads like a "Who's Who of Mathematics": A. Albert, E. Artin, Dickson, Jacobson, Jordan, Moufang, Wedderburn, Zassenhaus, and Zorn. It was reinvigorated by R. Baer and M. Hall about 50 years ago. To my delight, it has many connections to modern mathematics.

Definition. By a *projective plane* we mean a set, whose elements are called *points*, together with a family of subsets called *lines*, satisfying the following axioms:

- (P1) Any two distinct points belong to exactly one line;
- (P2) Any two distinct lines meet in exactly one point;
- (P3) There exists a *quadrilateral*: a set of four points, no three on any line.

Perhaps the most familiar example is the real projective plane $\mathbb{P}^2(\mathbb{R})$, whose "points" are the lines through the origin in Euclidean 3-space and whose "lines" are planes in 3-space. Of course the projective plane $\mathbb{P}^2(F)$ over any field F will also be a projective plane. The smallest projective plane is $\mathbb{P}^2(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field of 2 elements. It has 7 points and 7 lines, and is often called the *Fano plane*, having been discovered in 1892 by Gino Fano [Fano].

Charles Weibel is professor of mathematics at Rutgers University. His email address is `weibel@math.rutgers.edu`.

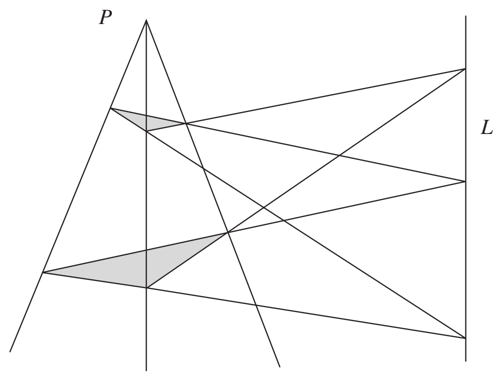


The Fano plane of 7 points.

If there are exactly $q+1$ points on any (hence every) line, we say that the plane has *order* q. A plane of order q has q^2+q+1 points, and also q^2+q+1 lines. Of course, $\mathbb{P}^2(\mathbb{F}_q)$ has order q. It is conjectured that the order q of a finite projective plane must be a prime power; this is known only for $q \le 11$. (Tarry proved in 1901 that $q \ne 6$; $q \ne 10$ was only proven in 1988 by a computer search [Lam] and even the case $q = 12$ is still open.)

A projective plane is the same as a 2-dimensional projective geometry. By a d-dimensional *projective geometry*, we mean a set (of points), together with a family of subsets (lines) satisfying the following axioms, taken from the 1910 book by Veblen and Young [VY]:

- (PG1) Two distinct points lie on exactly one line;
- (PG2) If a line meets two sides of a triangle, not at their intersection, then it also meets the third side;
- (PG3) Every line contains at least 3 points;
- (PG4) The set of all points is spanned by $d+1$ points, and no fewer.



Desargues' Theorem.

The feature that makes projective planes more complicated than higher dimensional projective geometries is that Desargues' Theorem need not hold, an observation made by Hilbert in [Hi]. We say that two triangles are *perspective* from a point P (resp., from a line L) if their corresponding vertices are on lines through P (resp., edges meet on L).

Desargues' Theorem.[1] *Let F be any field (or division ring). Two triangles in $\mathbb{P}^d(F)$ are perspective from a point if and only if they are perspective from a line.*

Definition. A projective geometry is said to be *Desarguesian* if whenever two triangles are perspective from a point, they are perspective from a line, and vice versa. If this property fails, it is said to be *non-Desarguesian.*

This terminology is due to Hilbert [Hi], who proved (see [VB]) that any Desarguesian projective geometry is just a projective space $\mathbb{P}^d(F)$ over a field (or division ring) F. If $d \geq 3$, every d-dimensional projective geometry is Desarguesian. The projective plane over Cayley's Octonions (see below) is non-Desarguesian.

Every finite projective plane of order $q \leq 8$ is Desarguesian, and hence is isomorphic to the plane $\mathbb{P}^2(\mathbb{F}_q)$. There are three distinct non-Desarguesian planes of order 9, each consisting of 91 points. We will describe them below; the first of these was constructed by Veblen and Wedderburn in [VW]; it is coordinatized by the Quaternionic near-field (see below).

Collineations

Automorphisms of a projective plane must preserve lines, so they are called *collineations.* The collineations form a group, and the geometry of the plane is reflected by the structure of this group.

A collineation $\alpha \neq 1$ is called a *perspective* if it fixes every point on a line L (the *axis* of α). Every perspective has a unique fixed point C (the *center* of α) such that α fixes every line through C [Hall, 20.4.1]; α is called a *C-L collineation* and has no fixed points other than C and the points of L. If $C \in L$, α is also called a *translation* with axis L; there are no fixed points except those on L.

Not every collineation fixes the points on a line. For example, if α is an automorphism of a field F then $(x:y:z) \mapsto (\alpha x:\alpha y:\alpha z)$ is a collineation of $\mathbb{P}^2(F)$ whose fixed points form the sub-plane $\mathbb{P}^2(F^\alpha)$. If $a, b, c \in F$ are distinct, the collineation $(x:y:z) \mapsto (ax:by:cz)$ fixes only $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$.

The collineation group of $\mathbb{P}^2(F)$ is the semi-direct product of $PGL_2(F)$ and $\mathrm{Aut}(F)$. (Cf. [Hall, 20.9.4].) If L_∞ is the line at infinity in $\mathbb{P}^2(F)$, and O is the origin, the O-L_∞ collineations are the dilations $(x,y) \mapsto (mx, my)$ in $PGL_2(F)$. If $C \in L_\infty$, the C-L_∞ collineations are just the translations $(x,y) \mapsto (x+a, y+b)$ such that (a,b) is on the line OC.

A plane is said to be (P,Q)*-transitive* if it is (P,L)-transitive for every line L through Q. This condition is related to near-fields, as we shall see below.

The following conditions, due to Baer [Baer], are related to the linearity and distributivity of the corresponding ternary rings.

Definition. The plane is said to be *C-L transitive* if, for every line $L' \neq L$ through C, the group of C-L collineations acts transitively on the points of L' (with the obvious exception of C and $L \cap L'$). This condition is equivalent to the "little Desargues property", also called the (C,L)-Desarguesian condition, that two triangles that are perspective from C are perspective from L.

We say that a projective plane is a *translation plane* with respect to a line L if it is C-L transitive for every $C \in L$. That is, for every $C \in L$ the group of C-L collineations acts transitively on the points (other than C) on every line L' through C. Instead of "for every $C \in L$", it suffices to check two points on L; see [Hall, 20.4.4]. Translation planes are related to quasi-fields, as we shall see below.

Example. The Quaternionic non-Desarguesian plane of order 9, described on page 1298, is a translation plane with respect to a distinguished line L, which is fixed (as a set) by every collineation. The collineation group of this plane has order 311,040, far less than the order (42,456,960) of $PGL_2(\mathbb{F}_9)$. See [Ha43].

[1] *Girard Desargues (1591–1661) discovered this property for projective spaces over $\mathbb{R}$.*

Modular lattices

We briefly mention the connection between projective geometry and modular lattices. Readers interested in this connection may want to read the recent article [Gr] by G. Grätzer in the *Notices*.

A lattice is said to be *modular* if $x \vee (y \wedge z) = (x \vee y) \wedge z$ for every x, y, z with $x \leq z$. Finite-dimensional modular lattices are graded by height; height-one elements are called *points*, and height-two elements are called *lines*. A lattice is *complemented* if for every x there is an x' so that $x \vee x' = 1$, $x \wedge x' = 0$. In a finite-dimensional complemented modular lattice, every element is a $\vee$ of points. A lattice is *simple* if it has no quotient lattices. The following result is proven in IV of Birkhoff's 1940 book [Bff].

Theorem. *There is a 1–1 correspondence between d-dimensional projective geometries and simple complemented modular lattices of dimension $d+1$, $d \neq 0$. Under this correspondence, the projective geometry is the set of points and lines of the lattice.*

One of Dilworth's theorems states that every finite-dimensional complemented modular lattice is a product of a Boolean algebra and projective geometries.

The Lenz-Barlotti classification

There is a classification of projective planes by Lenz [Lenz], refined by Barlotti, according to the possible central collineation groups. This classification contains 53 possible classes, all but one of which exists as a group; 36 of them exist as finite groups. Between 7 and 12 exist as finite projective planes, and either 14 or 15 exist as infinite projective planes. The list is given on pp. 123–126 of [Dem]. Rather than attempt any kind of exhaustive description of this incomplete listing, I shall focus on the classes of projective planes that I find most interesting.

Moufang Planes and Alternative Division Rings

I shall begin with Moufang planes, a class of (infinite) projective planes with many collineations that was studied in the 1930s by Ruth Moufang [Mou].

Definition. A *Moufang plane* is a projective plane Π with the property that, for every line L, the group of automorphisms fixing L pointwise acts transitively on the "affine plane" $\Pi - L$. In other words, Π is a translation plane for every line.

Moufang related these planes to alternative algebras; the algebra allows us to give a complete classification. In particular, every finite Moufang plane is a classical plane $\mathbb{P}^2(\mathbb{F})$ over some finite field $\mathbb{F}$. To explain this, we need some algebraic definitions.

An *alternative ring* A is an abelian group equipped with a multiplication that is left and right distributive and that satisfies the two laws $(xx)y = x(xy)$, $y(xx) = (yx)x$. This implies that the symmetric group Σ_3 acts on the associator $\langle x, y, z\rangle = (xy)z - x(yz)$ via the sign representation; the name (due to Zorn) comes from the fact that the alternating group A_3 acts trivially on the associator.

Definition. An *alternative division ring A* is an alternative ring with a 2-sided identity 1, such that every $a \neq 0$ has a two-sided inverse a^{-1}. (It follows that the nonzero elements form a "loop," i.e., $a^{-1}(ab) = b = (ba)a^{-1}$ for all b.) Of course, an associative alternative division ring is just an (associative) division ring.

If A is an alternative division ring, we can form a projective plane $\mathbb{P}^2(A)$ following the classical formulas: points are lines through the origin in A^3, and lines are planes through the origin.

Theorem. ([Hall, 20.5.3]) *There is a 1–1 correspondence between alternative division rings and Moufang planes, with A corresponding to $\mathbb{P}^2(A)$.*

Example. The classical Octonions form an 8-dimensional alternative division algebra over $\mathbb{R}$ (the reals). This gives a very interesting non-Desarguesian plane.

The Artin-Zorn theorem ([Z]) states that every finite alternative division ring is a field; it follows that every finite Moufang plane is just the classical $\mathbb{P}^2(\mathbb{F}_q)$.

The Octonions are an example of a *Cayley-Dickson algebra.* A Cayley-Dickson algebra is an 8-dimensional algebra A whose maximal subfields are quadratic over F, with any two elements not in a subfield generating a quaternion algebra.

Remark. Cayley-Dickson algebras over F are classified by the étale cohomology group $H^3_{et}(F, \mathbb{Z}/2)$; the Cayley-Dickson division algebras over F correspond to the nonzero elements; the Cayley-Dickson algebra corresponding to zero is "split".

This follows from the fact that the split Cayley-Dickson algebra contains the matrix algebra $M_2(F)$ as a quaternion subalgebra, and its automorphism group is the algebraic group G_2. Thus the set of isomorphism classes of Cayley-Dickson algebras over F is the same as the nonabelian cohomology set $H^1(F, G_2)$. It is a deep theorem that $H^1(F, G_2) \cong H^3_{et}(F, \mathbb{Z}/2)$.

In fact (see [SpV, 1.7]) the norm form of a Cayley-Dickson algebra A is a Pfister form $\langle\langle a, b, c\rangle\rangle$. Now 3-Pfister forms are a special class of 8-dimensional quadratic forms and are also classified by $H^1(F, G_2)$. Thus there is a 1–1 correspondence between Cayley-Dickson algebras, elements of $H^3_{et}(F, \mathbb{Z}/2)$, and 3-Pfister forms.

Theorem. *Every nonassociative alternative division ring is a Cayley-Dickson algebra over an infinite field F. Moreover, if $\frac{1}{2} \in F$ then Cayley-Dickson algebras over F are in 1–1 correspondence with the elements of $H^3_{et}(F, \mathbb{Z}/2)$.*

The first part of this theorem was proven by Bruck and Kleinfield [BK] and Skornyakov in 1950. The second part follows easily from the above remark. Note that $H^3_{et}(\mathbb{F}_q, \mathbb{Z}/2) = 0$ for every finite field, which provides another proof of the Artin-Zorn theorem.

Remark. P. Jordan [J49] and Freudenthal [Fr51] gave a "projective" description of the projective plane $\mathbb{P}^2(A)$ over any Cayley-Dickson algebra A (they used the Octonions). Their construction uses the exceptional simple Jordan algebra J over A. The points of $\mathbb{P}^2(A)$ are the irreducible idempotents in J, and the lines are the annihilators of these idempotents. Note that J is the 27-dimensional algebra of Hermitian 3×3 matrices over A with multiplication $X \circ Y = (XY + YX)/2$.

Ternary Rings

Any field (or division ring, or alternative division ring) F has a ternary operation $T(a, b, c) = ab + c$, making it a "ring" in the following sense.

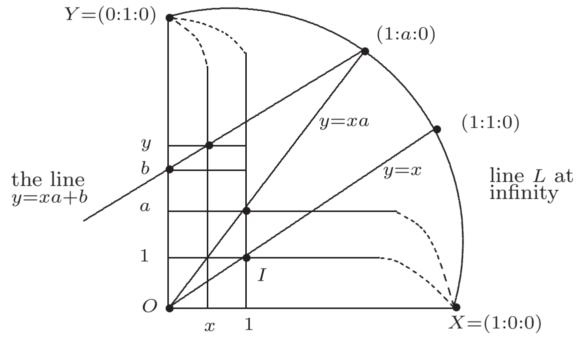
Definition. A *ternary ring R* is a set R with two distinguished elements $0, 1$ and a ternary operation $T : R^3 \to R$ satisfying the following conditions:

- (T1) $T(1, a, 0) = T(a, 1, 0) = a$ for all $a \in R$;
- (T2) $T(a, 0, c) = T(0, a, c) = c$ for all $a, c \in R$;
- (T3) If $a, b, c \in R$, the equation $T(a, b, y) = c$ has a unique solution y;
- (T4) If $a, a', b, b' \in R$ and $a \neq a'$, the equations $T(x, a, b) = T(x, a', b')$ have a unique solution x in R;
- (T5) If $a, a', b, b' \in R$ and $a \neq a'$, the equations $T(a, x, y) = b$, $T(a', x, y) = b'$ have a unique solution x, y in R.

If R is finite, the condition (T5) is redundant, since it can fail only if the evident self-map of R^2, $(x, y) \mapsto (b, b')$, is not a bijection, i.e., if $T(a, x, y) = T(a, x', y')$ and $T(a', x, y) = T(a', x', y')$ for some $(x, y) \neq (x', y')$—and this contradicts either (T3) if $x = x'$ or (T4) if $x \neq x'$.

Theorem. ([AS, 4.2]) *Every ternary ring R determines a projective plane $\mathbb{P}^2(R)$ and a distinguished quadrilateral $Q = (X, Y, O, I)$ in that plane.*

The construction of $\mathbb{P}^2(R)$ follows a familiar construction of $\mathbb{P}^2(F)$. There is one line L "at infinity" consisting of $Y = (0 : 1 : 0)$ together with special points $(1 : a : 0)$, $a \in R$; we set $X = (1 : 0 : 0)$. The other points are the elements in R^2, with $O = (0, 0)$ and $I = (1, 1)$. For each x, there is a "vertical" line consisting of Y together with the points (x, y). For each (a, b) there is a line consisting of $(1 : a : 0)$ and the set of solutions



The ternary operation $y = T(x, a, b) = xa + b$.

(x, y) to $y = T(x, a, b)$; in slope-intercept form, this is the line $y = xa + b$.

Example. Let F be a field, or a division ring, or an alternative division ring. The projective plane determined by its associated ternary ring is just $\mathbb{P}^2(F)$.

Plane coordinates. Conversely, any quadrilateral (X, Y, O, I) in a projective plane gives rise to a ternary ring R, via a coordinate system. Here is a sketch of the construction, due to Von Staudt (1856-1860 for $\mathbb{R}$) and Hilbert (1899).

If R denotes the set of points on the line OI, except the intersection $(1 : 1 : 0)$ of OI with $L = XY$, then there is a standard labeling of the points not on L (the "affine points") by the set R^2; O is $(0, 0)$ and I is $(1, 1)$. (It is useful to think of $L = XY$ as the line "at infinity", the line OX as the X-axis and the line OY as the Y-axis.) We say that a line has slope m if it meets L in the same point as the line through $(0, 0)$ and $(1, m)$. Each line not through Y has a standard slope-intercept description, which we may symbolically write as $y = xm + b$. The formula $y = T(x, m, b)$ makes R into a ternary ring.

It is clear that ternary rings are in 1–1 correspondence with isomorphism classes of projective planes and distinguished quadrilaterals.

Theorem. ([AS, 4.4]) *If R and R' are two ternary rings for the same projective plane, arising from quadrilaterals Q and Q', then $R \cong R'$ if and only if there is an automorphism of the plane sending Q to Q'.*

Corollary. *Every Desarguesian plane has a unique associated ternary ring, which is an associative division ring.*

Indeed, $PGL_2(F)$ acts transitively on quadrilaterals in $\mathbb{P}^2(F)$.

Isotopisms. An *isotopism* between ternary rings (R, T) and (R', T') is a set of three bijections (F, G, H) from R to R' such that $H(0) = 0$ and $HT(a, b, c) = T'(Fa, Gb, Hc)$. It defines an isomorphism $\alpha : \mathbb{P}^2(R) \cong \mathbb{P}^2(R')$ fixing O, X, and Y by $\alpha(x, y) = (F(x), H(y))$; lines of slope m

map to lines of slope $G(m)$. Conversely, every isomorphism of projective planes fixing O, X, and Y comes from an isotopism of the coordinate ternary rings. (See [Kn65].)

Near-fields

Near-fields constitute a beautiful class of ternary rings with $T(a,b,c) = ab + c$. They were introduced in 1905 by Dickson [D05] and named in [Z36]. Although Dickson found all finite near-fields in [D05], his list was shown to be exhaustive only in Zassenhaus' 1936 thesis [Z36]. We describe them now, as motivation for linear ternary rings.

Definition. A (right) *near-field* is an associative ring K with 1 whose non-zero elements $K^\times = K - \{0\}$ form a group under multiplication, such that:

(a) multiplication is right distributive: $(a + b)c = ab + ac$; and

(b) If $a, a', b \in K$ and $a \neq a'$, the equation $xa - xa' = b$ has a (unique) solution x. Uniqueness in (b) is automatic; if K is finite, all of axiom (b) is redundant.

Clearly, the center F of a near-field K is a field, and (by right distributivity) K is a subalgebra of $\mathrm{End}_F(K)$; a finite near-field is an algebra over a finite field $\mathbb{F}_q$.

Geometric interpretation. A ternary ring R is a near-field if and only if it is (X,Y)-transitive. Associativity is equivalent to being X-OY transitive; in this case the map $(x,y) \mapsto (ax,y)$ is a collineation sending the line $y = xm+b$ to the line $y = (ma)x + b$. (See [St, 12.3.3].)

The quaternionic near-field J_9. Additively, J_9 is a vector space over $\mathbb{F}_3$ on basis $\{1, i\}$. Multiplicatively J_9 is the subset $J = \{0, \pm 1, \pm i, \pm j, \pm k\}$ of the quaternions, and $(J_9 - \{0\}, \cdot)$ is the quaternionic group of order 8. The identification between the additive and multiplicative descriptions is given by setting $j = 1 + i$ and $k = 1 - i$. (See the addition table below.) The verification that J_9 is a near-field is an easy exercise.

Veblen and Wedderburn observed in 1907 that the projective plane $\mathbb{P}^2(J_9)$ cannot be Desarguesian since J_9 is not a division ring.

+	0	1	-1
0	0	1	-1
i	i	j	-k
-i	-i	k	-j

Addition table for the quaternionic near-field.

Dickson near-fields. Here is Dickson's construction of a finite near-field K with center $\mathbb{F}_q$; see [Hall, 20.7.2]. Let ν be an integer whose prime factors all divide $q - 1$; if $q \equiv 3 \pmod 4$ we require $4 \nmid \nu$. The abelian group underlying K will be $\mathbb{F}_{q^\nu}$ but the product $\circ$ in K is defined by $w \circ u = u \cdot w^{q^i}$, where $q^i = 1 + j(q-1) \mod \nu(q-1)$, and $u = \zeta^{k\nu+j}$ for a fixed primitive root of unity $\zeta \in \mathbb{F}_{q^\nu}$. It is a subalgebra of $M_\nu(\mathbb{F}_q)$; if $\nu = 1$ then $K = \mathbb{F}_q$.

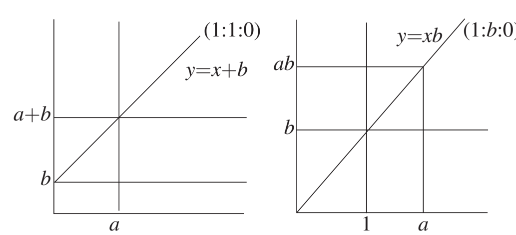
To illustrate, note that $w \circ \zeta = \zeta w^q$, and $\zeta \circ \zeta = \zeta^{q+1}$. If $\nu = 2$, ζ generates a cyclic subgroup of $K^\times$ of order $2(q-1)$ containing $\mathbb{F}_q^\times$. When $q = 3$ and $\nu = 2$, K is J_9, the unique near-field of order 9 described above.

Exceptional near-fields. Zassenhaus [Z36] classified all finite near-fields in 1936; in addition to the Dickson near-fields described above, there are exactly 7 exceptional near-fields—of orders 5^2, 7^2, 11^2, 11^2, 23^2, 29^2 and 59^2. They are described on page 391 of [Hall]. For example, the multiplicative group of the exceptional 25-element near-field is $SL_2(\mathbb{F}_3)$, embedded in $GL_2(\mathbb{F}_5)$ with generators the two matrices of order 4 and 3, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix}$. The two 11-element exceptional near-fields have multiplicative groups $SL_2(\mathbb{F}_5)$ and $SL_2(\mathbb{F}_3) \times C_5$.

Zassenhaus' classification is related to some interesting group theory that lurks behind the structure of near-fields. A *Frobenius group* is a semidirect product $G = K \rtimes H$ such that $H \cap gHg^{-1} = 1$ for every $g \notin H$. Elementary considerations (see [AB], pp. 172–174) show that G acts 2-transitively on $K \cong G/H$, with K acting freely, and that only the identity fixes two elements.

For example, if K is a near-field then the group G of "affine" transformations $g(x) = xm + b$ is such a Frobenius group with $H = K^\times$. The following result was proven in [Ha43]; see [Hall, 20.7.1].

Theorem. (Hall) *Let $G = K \rtimes H$ be a Frobenius group. If H acts transitively on K then K has the structure of a near-field and G is the group of affine transformations $g(x) = xm + b$ of K.*



Addition and multiplication in R. Horizontal lines meet in X, and vertical lines meet in Y.

Linear Ternary Rings

In a ternary ring R, it is convenient to write $a+b$ and ab for $T(a,1,b)$ and $T(a,b,0)$, respectively. Axioms (T1) and (T2) imply the familiar identities $a+0=a=0+a$, $1a=a=a1$ and $a0=0=0a$ for all a. We may think of $(R,+,\cdot)$ as the underlying binary ring of R.

In fact, both $(R,+,0)$ and $(R-\{0\},\cdot,1)$ are *loops*, or nonassociative groups, meaning that there is a unique solution x to each equation $xa=b$, and also to each equation $ax=b$. This follows from (T4) and (T5), by setting $a'=0$.

Definition. A ternary ring is called *linear* if $T(a,b,c)=ab+c$ and $(R,+,0)$ is a group. We can also describe a linear ternary ring as a group $(R,+,0)$ equipped with a multiplication and an identity 1, satisfying $a0=0=0a$, and such that both $xa=xa'+b$ and $ay=a'y+b$ have unique solutions for every $a\neq a'$ and b.

Clearly, if R is a linear ternary ring then so is the opposite ring R^{op}. Any near-field K is a linear ternary ring, and if K is not a division ring then K^{op} cannot be a near-field because it is left distributive but not right distributive. In particular, J_9 and J_9^{op} are distinct linear ternary rings (and yield non-isomorphic projective planes of order 9).

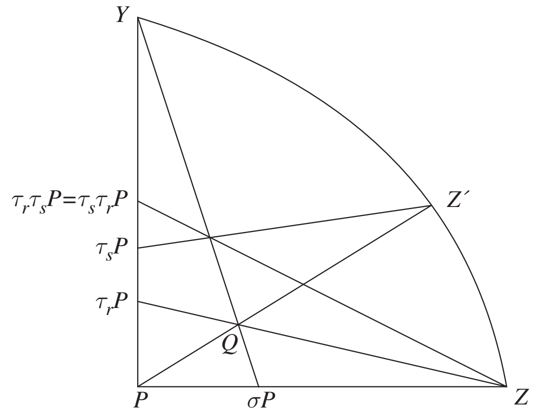
Theorem. *A ternary ring R is linear if and only if its projective plane is Y-L transitive. That is, if and only if the group of Y-L collineations acts transitively on the affine points on any vertical line in the plane.*

This is proven in [Hall, 20.4.5]. The constructive half of the proof is elementary: for each $r\in R$, the mapping $\tau_r:(x,y)\mapsto(x,y+r)$ determines a collineation of the corresponding plane, with center Y and axis L. Hence the group $G(Y,L)$ of Y-L collineations acts transitively on the set $\{(a,y):y\in R\}$ of affine points on any vertical line in the plane, and we may identify $G(Y,L)$ with $(R,+)$.

Examples exist showing that $(R,+)$ need not be abelian; in this case, every collineation with axis L has center Y; see [Hall, p. 359].

Lemma. *Let R be a ternary ring, and let V be the y-axis OY. Then the plane $\mathbb{P}(R)$ is X-V transitive if and only if (a) $T(x,m,b)=xm+b$, and (b) $(R-\{0\},\cdot)$ is a group.*

Proof. If multiplication is a group then each $(x,y)\mapsto(xm,y)$ is a collineation fixing the y-axis, the point X, and the lines $y=b$; these are enough to make the plane (X,V) transitive. Conversely, the X-V collineation σ_m sending $(1:m:0)$ to $(1:1:0)$ on L must map the line $y=T(x,m,b)$ to $y=T(x,1,b)$ and map (x,y) to (xm,y). Hence if $y=T(x,m,b)$ then $y=T(mx,1,b)=xm+b$. Now $\sigma_n\sigma_m$ and σ_{mn} send $(x,1)$ to $((xm)n,1)$ and $(x(mn),1)$; as both send $(1,1)$ to $(mn,1)$ they must agree and hence $(xm)n=x(mn)$. □



Proof that addition is commutative, using σ.

Proposition. (Baer) *Let R be a linear ternary ring. If there is a nontrivial Z-L collation σ for some point $Z\neq Y$ on L, then $(R,+)$ is abelian. Also, the abelian group R is either torsion-free or an elementary p-group for some prime p.*

Proof. (See figure.) For $r\in R$, let τ_r be the vertical translation $\tau_r(x,y)=(x,y+r)$. If P is any point not on L, we must have $\sigma\tau_r(P)=\tau_r\sigma(P)$ because both operations take P to the intersection Q of the lines $\sigma(PY)$ and $\tau(PZ)$. Hence σ commutes with every τ_r. Since $\sigma\tau_r$ fixes L and no point off L, it must be a Z'-L collineation for some Z' distinct from Y and Z. Hence $\sigma\tau_r$ and σ must both commute with τ_s for every $s\in R$, which implies that τ_r and τ_s commute, i.e., $r+s=s+r$.

If $(R,+)$ has torsion, there is an element r with $pr=0$ for some prime p. But then $(\sigma\tau_r)^p=\sigma^p$; this collineation must be the identity because it fixes the distinct lines PZ and PZ'. In turn this yields $\tau_s^p=(\sigma\tau_s)^p$ and hence $\tau_s^p=1$ for all $s\in R$. □

Hughes planes. (See [Hu] [Hall, 20.9.13].) This is an infinite family of projective planes that are not transitive; their ternary rings are not quasi-fields (see below). Let K be a near-field of odd order q^2 whose center is $\mathbb{F}_q$. There is a 3×3 matrix α over $\mathbb{F}_q$ of order q^2+q+1 that cyclically permutes the points and also the lines of $\mathbb{P}^2(\mathbb{F}_q)$, when regarded as a collineation. The Hughes plane is given by extending α to a collineation of the plane coordinatized by K. The lines in the Hughes plane are just the iterates under α of the q^2-q+1 lines $y=xm+b$, where $b=1$ or $b\notin\mathbb{F}_q$.

Hughes has shown in [Hu] that the ternary ring R associated to this plane has the opposite near-field K^{op} as its underlying binary ring, but that R is not a linear ternary ring.

Quasi-fields and Translation Planes

Definition. A (right) *quasi-field* R is a linear ternary ring in which $+$ is abelian and multiplication is right distributive: $(a+b)c = ac + bc$.

In other words, a quasi-field is an abelian group $(R, +, 0)$, with a right distributive multiplication (with 1) forming a loop on $R - \{0\}$, with the additional condition that, for every $a \neq a'$ and b, there is a unique solution x to $xa - xa' = b$.

Quasi-fields were called *Veblen-Wedderburn systems* in the literature before 1975, since they were first studied in the 1907 paper [VW]. A quasi-field R with associative multiplication is just a (right) *near-field*.

Although we do not yet have a satisfactory classification of quasi-fields, their importance stems from their geometric interpretation as the coordinate rings of translation planes. This interpretation was given in [VW].

Veblen-Wedderburn Theorem. *A ternary ring is a quasi-field if and only if its projective plane is a translation plane with respect to the line L at infinity.*

The required C-L collineations with center $C = (1:m:0)$ are just the translation operations $(x,y) \mapsto (x+a, y+am)$; the group of these acts transitively on every line through C.

Warning. Different quadrilaterals in a translation plane may induce non-isomorphic quasi-fields. For example, different quadrilaterals in the unique non-Desarguesian translation plane of order 9 induce four non-isomorphic quasi-fields of order 9. Given nonzero $r_1, r_2 \in R$, we can form a new quasi-field $(R, +, \circ)$ by defining $u = x \circ y$ when there is a z so that $ur_1 = xz$ and $yr_2 = r_1 z$; see [Ha43].

9-element ternary rings. There are five non-isomorphic quasi-fields of order 9, all linear. Two of course are $\mathbb{F}_9$ and J_9. Two others are Hall algebras associated to the polynomials $z^2 \pm z - 1$ (see below). The last one is the strange quasi-field U in our next example. Except for $\mathbb{F}_9$, all of them arise from systems of coordinates in the unique non-Desarguesian translation plane of order 9, and are described in the Appendix to [Ha43].

Example. (Hall [Ha43, p. 274].) Here is a strange quasi-field of order 9. Its center is $\{0,1\}$ instead of $\mathbb{F}_3$. Let U be a 2-dimensional left vector space over $\mathbb{F}_3$ on basis $\{1, a\}$ equipped with right action $(a+i)(-1) = -a + (-i+1)$, $(-a+i)(-1) = -a + (-i-1)$, $i \in \mathbb{F}_3$. Since we have $(-x)y = -(xy)$, the multiplication is given by the table:

row· col	-1	a	a+1	a-1	-a	-a+1	-a-1
a	-a+1	a-1	1	-a	a+1	-a-1	-1
a+1	-a	-a-1	a-1	-1	1	a	-a+1
a-1	-a+1	-1	-a	a+1	-a+1	1	a

The abelian group $(R, +)$ underlying a quasi-field is a vector space over a division ring F. Thus if R is finite its order must be p^n for some prime p. To see this, let E denote the ring of endomorphisms of the abelian group $(R, +)$, and let Σ denote the set of all nonzero automorphisms $x \mapsto xa$ in E. Since Σ operates irreducibly on $(R, +)$, Schur's lemma implies that $F = \mathrm{End}_E(\Sigma)$ is a division ring and that R is a vector space over F.

Left quasi-fields. A *left quasi-field R* is a linear ternary ring that is left distributive. That is, R^{op} is a right quasi-field.

Pickert proved that the plane $\Pi = \mathbb{P}^2(R)$ associated to a left quasi-field R is the dual plane of the plane associated to R^{op}. (The point (a,b) of $\mathbb{P}^2(R)$ corresponds to the line $y = ax - b$ of $\mathbb{P}^2(R^{op})$, Y corresponds to L_∞, and the point $(1:b:0)$ at infinity corresponds to the vertical line $x = b$; see [St, 11.2.4].)

It follows that Π is the dual of a translation plane (with respect to Y): for every line L through Y, the group of Y-L collineations acts transitively on the lines (other than L) through every point on L (except Y). (See [Dem, 3.1.36].) These planes are sometimes called *shear planes*.

Semi-fields

We now turn to semi-fields, a class of linear ternary rings that complements near-fields, first studied by Dickson in 1906. The name dates to 1965 and is due to Knuth (see [Kn65]); they are sometimes called "nonassociative division rings" or "distributive quasi-fields".

Definition. A *semi-field S* is an abelian group $(S, +, 0)$, with a bilinear multiplication (with 1) with the additional condition that for every a and b there are unique solutions x, y to $xa = b$, $ay = b$. That is, S is a linear ternary ring in which $+$ is abelian and multiplication is left and right distributive.

It is easy to see that a semi-field contains a field. There are non-associative semi-fields of every prime power order p^n with $n \geq 3$, $n \neq 8$. None are alternative.

I believe that it is possible to give a classification of all finite semi-fields in terms of descent data. The families of semi-fields in this section provide supporting evidence for this belief. We have already encountered two classes of semi-fields: associative division algebras and alternative algebras (Cayley-Dickson algebras).

Example. The smallest non-associative semi-fields have order 16; there are 23 of these, 18 isotopic to S_0 and 5 isotopic to S_ω; see [Kn65]. Here S_0 is the 2-dimensional algebra over $\mathbb{F}_4$ on generator λ with multiplication $(a+\lambda b)(c+\lambda d) = (ac + b^2 d) + \lambda(bc + a^2 d + b^2 d^2)$. This is a semi-field with 6 automorphisms.

Another 16-element semi-field S_ω is defined by the product $(a+\lambda b)(c+\lambda d) = (ac+\omega b^2 d) + \lambda(bc+a^2 d)$, where $\omega \in \mathbb{F}_4 - \{0,1\}$. It has only 3 automorphisms.

Example. More generally, suppose that $q = p^n$ and $q > p$ (if $p = 2$ we also require n even). Then there exists an $\omega \in \mathbb{F}_q$ that is not a $(p+1)^{st}$ power. We define S_ω to be the 2-dimensional ring over $\mathbb{F}_q^2$ with basis $\{1, \lambda\}$, and product

$$(a+\lambda b) \circ (c+\lambda d) = (ac+\omega b^p d) + \lambda(bc+a^p d).$$

This is a semi-field of order q^2 with exactly $p+1$ automorphisms: $\lambda \mapsto \zeta\lambda$, $\zeta^{p+1} = 1$; see [AS, 7.2]. S_ω is a twisted form of the algebra S_1, and the forms of S_1 over $\mathbb{F}_q$ are classified by $\mathbb{F}_q^\times / \mathbb{F}_q^{\times(p+1)}$.

Jordan division algebras. Let A be an alternative division algebra over an (infinite) field F. Then not only is A a semi-field, but it has a canonical involution. Consequently, we can form the 27-dimensional exceptional Jordan algebra J of Hermitian 3×3 matrices over A. It is known [P81] that J is a semi-field.

Similarly, if D is any division algebra then the associated Jordan algebra (i.e., D with product $(xy+yx)/2$) is a semi-field if and only if no subfield E of D has a Galois group of order 2. This is the case, for example, when $\dim D$ is odd.

Non-unital trick. A *non-unital semi-field* S is a ring with bilinear product for which the equations $xa = b$, $ay = b$ have unique solutions (if it had a unit it would be a semi-field). For each $0 \neq u \in S$, the following trick produces a product $\circ$ with unit u^2, making S into a unital semi-field. The maps $s \mapsto su$ and $t \mapsto ut$ are linear automorphisms of S, so $\circ$ is determined by the formula $(su) \circ (ut) = st$.

Albert's twisted semi-fields. (See [AA][Kn65].) Let p be a prime and $q = p^m$ with $q > 2$. Then from $\mathbb{F}_{q^n}$ we may construct a semi-field S with q^n elements, depending on an element c not a $(q-1)^{\text{st}}$ power in $\mathbb{F}_{q^n}$; c exists because $q > 2$.

The ($\mathbb{F}_q$-bilinear) product $\langle x, y\rangle = xy^q - cx^q y$ on $S = \mathbb{F}_{q^n}$ makes S into a non-unital semi-field. The non-unital trick above (for $u = 1$) turns S into a semi-field.

It is easy to see that $x \circ y = xy$ for $x \in \mathbb{F}_q$, and that S is a commutative $\mathbb{F}_p$-algebra. If $n > 2$, Albert has shown that the powers of any element not in $\mathbb{F}_p$ do not associate; this also implies that S is not an alternative algebra.

Cubic semi-fields. Dickson discovered the following class of 3-dimensional commutative semi-fields in [D06]. Suppose that $1/2 \in F$ and that $x^3 + ax^2 + bx + c$ is an irreducible cubic over F. Then the vector space S with basis $\{1, i, j\}$ and commutative product $i^2 = j$, $ij = c + bi + aj$, $j^2 = (4ac - b^2) - 8ci - 2bj$ is a semi-field.

Kaplansky studied these algebras in [Kap], showing that they all arise as irreducible twisted forms of the algebra D presented with basis $\{u_1, u_2, u_3\}$ and multiplication $u_i^2 = 0$, $u_i u_j = (u_i + u_j - u_k)/2$; $1 = \sum u_i$.

These cubic semi-fields over F are classified by the irreducible cubic extension fields of F (up to conjugacy). Since $\mathrm{Aut}(D)$ is the symmetric group Σ_3 (permuting the u_i), the claim follows from an analysis of the nonabelian cohomology calculation that $H^1(F, \Sigma_3) = \mathrm{Hom}(\mathrm{Gal}(\overline{F}/F), \Sigma_3)$.

We now turn to the geometric interpretation of semi-fields. The following result characterizes them as being simultaneously shear planes and translation planes.

Theorem (Albert). *Every ternary ring isotopic to a semi-field S is a semi-field, and two semi-fields are isotopic if and only if they coordinatize the same plane. Moreover, S is a semi-field if and only if:*

(1) *$\mathbb{P}^2(S)$ is C-L_∞ transitive for every point C on the line L_∞ at infinity, and*
(2) *for every line L through Y, and every P on L ($P \neq Y$), the group of Y-L collineations acts transitively on the lines though P (excluding L).*

Standard collineations of a semi-field plane. A semi-field plane has lots of collineations. Translation by any pair (h, k) in S^2 is a collineation fixing X and Y, acting on affine points by $(x, y) \mapsto (x+h, y+k)$. For $r \in S$, the *shear* translation $(x, y) \mapsto (x, y+xr)$ fixes the y-axis OY but does not fix the point X.

These standard collineations form a normal subgroup of all collineations. The quotient is the group of collineations fixing O, X and Y—and we have seen that this is isomorphic to the group of autotopisms of S. (See [Kn65].)

Classification of Translation Planes

We conclude our survey by returning to translation planes (and quasi-fields). We have described several types of quasi-fields: division rings, alternative algebras, near-fields, and semi-fields. Each characterizes something about the geometry of its translation plane, so it is not surprising that a geometric taxonomy exists.

A classification of translation planes (with respect to a line L) was given by André in [And], using the set Z of admissible pairs (p, q) of points on the line L. We say that (p, q) is *admissible* if for each line H through q, other than L, the group of p-H collineations acts transitively on the points of $L - \{p, q\}$.

Classification of Translation Planes. *Every translation plane belongs to exactly one of the following 6 classes.*

(1) $Z = \emptyset$*; the quasi-field* R *is not a left quasi-field.*
(2) $Z = \{(p,p)\}$ *for one* p*;* R *is a semi-field but is not alternative.*
(3) Z *consists of all pairs* (p,p)*;* R *is a Cayley-Dickson alternative algebra.*
(4) $Z = \{(p,q),(q,p)\}$*;* R *is a near-field but not a division ring, and* $|R| > 9$*.* [2]
(5) Z *contains exactly one pair* (p,q) *for each* $p \in L$*, and also contains* (q,p)*;* R *is the unique near-field* J_9 *of order 9 (other than* $\mathbb{F}_9$*).*
(6) Z *is all pairs;* R *is a division ring (or a field) and the plane is* $\mathbb{P}^2(R)$*.*

Example. If a plane is a translation plane for two lines, then it is a translation plane for every line through their intersection; see [Hall, 20.5.1]. If the intersection is the distinguished point Y, this implies that (Y,Y) is admissible, so we are in cases (2), (3) or (6) of the classification theorem. In this case, R is a semi-field in which every element $a \neq 0$ has a two-sided inverse a^{-1}, and $a^{-1}(ab) = b$. See [Hall, 20.5.2].

Hall algebras. Here is a family of quasi-fields introduced in [Ha43]. Suppose that $f(x) = x^2 - rx - s$ is an irreducible polynomial (in x) over a field F. Let R denote the vector space F^2 with $(a,b)c = (ac,bc)$ and the following multiplication:

$$\begin{aligned}(a,b)(c,0) &= (a,b)c = (ac,bc);\\ (a,b)(c,d) &= (ac + d^{-1}b(rc + s - c^2), ad - bc + rb)\\ &\qquad\qquad\qquad\qquad \text{if } d \neq 0\\ &= (sv,0) + (c,d)(u + rv),\\ &\qquad \text{if } (a,b) = u + (c,d)v,\ u,v \in F.\end{aligned}$$

This is a quasi-field in which every $x \neq F$ satisfies the equation $f(x) = 0$. It is also an algebra over F. If $F = \mathbb{F}_2$ then $R = \mathbb{F}_4$; if $|F| > 2$, then R is not a division ring, because $x^2 - rx - s = 0$ has at least three solutions.

If $F = \mathbb{F}_3$, then $f(x) = x^2 + 1$ yields the unique near-field J_9 of order 9, described above. The choices $f(x) = x^2 \pm x - 1$ yield two of the other quasi-fields of order 9. (There is another strange quasi-field of order 9 in which the center is $\{0,1\}$ instead of $\mathbb{F}_3$. It is described on page 1300 above.)

André planes. Let Γ be a finite group of automorphisms of a field L with fixed subfield K, and $\beta : L^\times/K^\times \to \Gamma$ a function with $\beta(1) = 1$. For $a,b \neq 0$ in L, define $a \circ b = a^{\beta(b)}b$. Then $(L,+,\circ)$ is a quasi-field. If $\beta(a) = \beta(xa)$ for all x of norm 1, the associated translation plane is called an *André plane.* Lüneburg has proven that André planes are characterized by the fact that there is an abelian collineation group A that fixes two vertical lines L_1, L_2 such that, for any other vertical line H, its stabilizer subgroup A_H acts transitively on $H - \{Y\}$. (See [Lü, II.12].) Little is known about André planes. A typical result is that there are only 3 non-Desarguesian André planes of order 25 [Chen].

[2] *Such a plane may also be coordinatized by quasi-fields that are not near-fields, but all of its coordinatizing near-fields are isomorphic.*

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