W H A T I S . . .

Boy's Surface

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Boy's surface is an immersion of the real projective plane in 3-dimensional space found by Werner Boy in 1901 (he discovered it on assignment from Hilbert to prove that the projective plane could not be immersed in 3-space) [1]. Many beautiful pictures of it can be found on the Internet, but here we will build it from the inside out, so as to see clearly the features of Boy's surface.

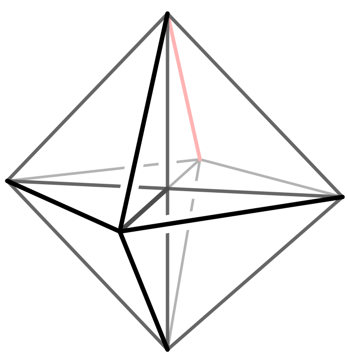

Figure 1.

To begin, there must be a triple point where three planes intersect as with the coordinate planes in R^3. In fact, the number of triple points of an immersed surface S in R^3 must be congruent, modulo 2, to the square of the first Stiefel-Whitney class of S in $H^2(S; Z/2)$.

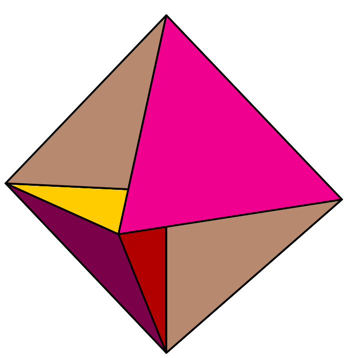

Figure 2.

Take a square in the xy-plane with vertexes at $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$, and similar squares in the xz-plane and yz-plane, as drawn in Figure 1. The 1-skeleton of this polyhedron is the logo for the Park City Mathematics Institute; the logo inspired a general-audience talk I gave on this subject in July 2006 at Park City. The construction given here is not original, though I know of no written account. I learned about it at the PCMI from Bob Edwards, whose memory of the construction was triggered by the logo.

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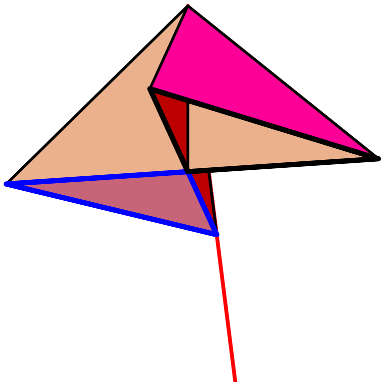
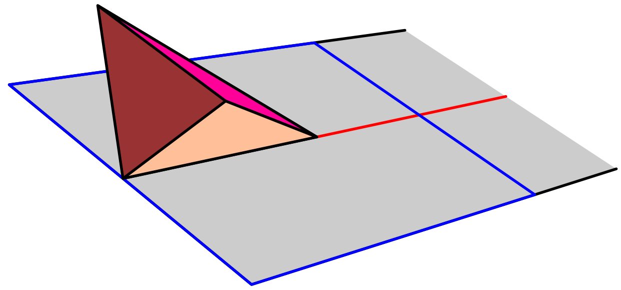
Now add to Figure 1 four 2-simplexes to "opposite" triangles; two, dark pink and purple, are drawn in Figure 2. "Opposite" means that no two of the four 2-simplexes have an edge in common. This polyhedron P is a 2-manifold, for each edge lies on the boundary of a square and a triangle; at any of the six symmetric vertexes we have the cone on a figure-8 (a circle immersed in the plane with a double point), which is abstractly a 2-disk. The squares and triangles form RP^2 because their Euler characteristic is $1 = 6 - 12 + 7$.

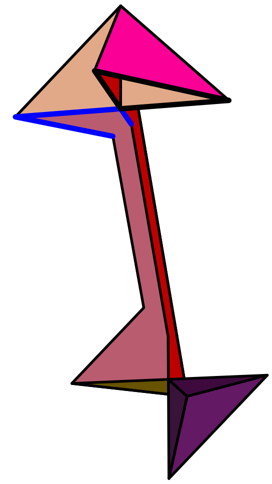
Note that if we take a cube and cut off each of its vertexes in a maximal way, then we have a solid with 8 triangular faces (corresponding to the original 8 vertexes) and 6 squares, one each in the middle of an original side. If antipodal points are identified, then we get RP^2 and the polyhedron P constructed above. Note also the symmetries of this object.

The defect in P is that this polyhedron is not smoothly immersed. The edges can be rounded, but the 6 cone points cannot. To remedy this, pair off the 6 cone points by 3 "opposite" edges, (in Figure 1 the red edge is such an edge). Each edge may be used to "cancel" the figure-8 cones at the ends of the edge, as illustrated in Figures 3-7.

A neighborhood of the top vertex is drawn in Figure 3. This neighborhood can be flattened out to look like the polyhedron in Figure 4; it is still a cone on a figure-8. A neighborhood of the red edge is homeomorphic to the polyhedron in Figure 5, having cones at both ends. Flattening then gives the polyhedron Q in Figure 6. Q is the image of a rectangle, immersed except at the cone points. This can be changed, relative to the boundary, to the immersed image of a rectangle as in Figure 7, where the two cone points have been canceled.







Figures 3 (left), 4 (above), and 5 (right).

When this process is done to each of the three pairs of vertexes (cone points), then we have obtained Boy's surface, an immersion of the projective plane. It is a piecewise linear immersion, but its edges and corners can easily be rounded to get a smooth immersion.

Note that there is an immersed circle of double points that passes through the triple point three times. This immersed circle consists of the original coordinate axes in R^3 together with the three edges that were used in canceling pairs of cone points. A neighborhood of that circle in RP^2 is of course a Möbius band, and its complement a disk that is embedded.

This smooth immersion (or any other) may be used to see Smale's eversion of the 2-sphere [3]. This is an arc of immersions that turns the 2-sphere inside out; its existence was proved by Smale and various constructions ([2]) have been carried out since. The normal 0-sphere bundle (the endpoints of the normal $[-1, 1]$-bundle) is an immersed 2-sphere, and it may be turned inside out by taking the endpoints and moving them through each other to their opposites along $[-1, 1]$. This is not generic, for at half time, the 2-sphere is immersed as Boy's surface, whereas a generic arc of immersions will not have a 2-dimensional multiple point set.

To evert a round 2-sphere, one has to see how to move the round 2-sphere through immersions to the 0-sphere bundle, then pass it through itself, and then go back to the round 2-sphere by the inverse of the first step.

The first (and third) step is known to be possible. An immersed 2-sphere has a Gauss map defined by taking a point on the 2-sphere to a point on the standard unit 2-sphere in R^3 that is the end point of a unit normal vector pointing out of the immersed 2-sphere. (This requires orienting the 2-sphere, and then everting will move the outward normal to the inward normal.) The normal 0-sphere bundle to Boy's surface has a degree-one Gauss map because the outward normal on one of the triangular faces (bowed slightly out) is the only one to hit that point on the round 2-sphere.

Smale's classification of immersions in this dimension states that, if two immersed 2-spheres have Gauss maps with the same degree, then they are connected through an arc of immersions. Thus the eversion exists, although Boy's surface only shows how to turn the immersed 0-sphere bundle inside out.

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References

[1] Werner Boy, *Über die Curvatura Integra und die Topologie der Geschlossener Flächen*, Dissertation, Göttingen, 1901; *Math. Ann.* **57** (1903), 151–184.

[2] Anthony Phillips, Turning a surface inside out, *Scientific American* **214** (1966), no. 5, 112–120.

[3] Stephen Smale, A classification of immersions of the two-sphere, *Trans. Amer. Math. Soc.* **90** (1958), 281–290.

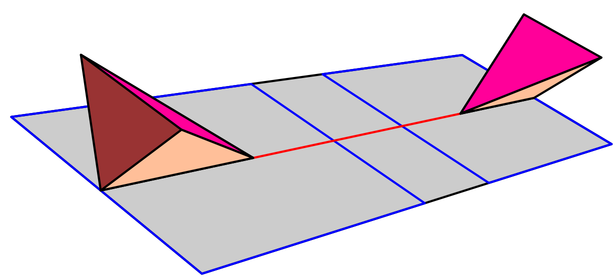

Figure 6.

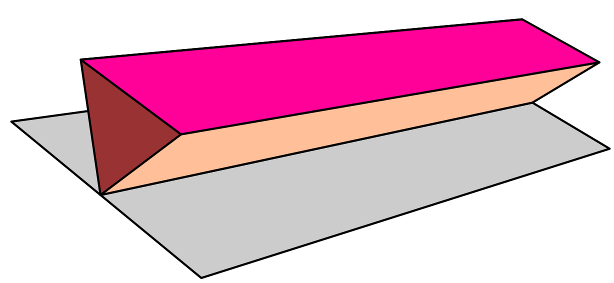

Figure 7.