From the Fundamental Theorem of Algebra to Astrophysics: A "Harmonious" Path

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The fundamental theorem of algebra (FTA) tells us that every complex polynomial of degree n has precisely n complex roots. The first published proofs (including those of J. d'Alembert in 1746 and C. F. Gauss in 1799) of this conjecture from the seventeenth century had flaws, though Gauss's proof was generally accepted as correct at the time. Gauss later published three correct proofs of the FTA (two in 1816 and the last presented in 1849). It has subsequently been proved in a multitude of ways, using techniques from analysis, topology, and algebra; see [Bur 07], [FR 97], [Re 91], [KP 02], and the references therein for discussions of the history of FTA and various proofs. In the 1990s T. Sheil-Small and A. Wilmshurst proposed to extend FTA to a larger class of polynomials, namely, harmonic polynomials. (A complex polynomial $h(x, y)$ is called harmonic if it satisfies the Laplace equation $\triangle h = 0$, where $\triangle := \partial^2/dx^2 + \partial^2/dy^2$.)

A simple complex-linear change of variables $z = x + iy, \overline{z} = x - iy$ allows us to write any complex valued harmonic polynomial of two variables in the complex form

$$h(z) := p(z) - \overline{q(z)}$$

where p, q are analytic polynomials. While including terms in $\overline{z}$ looks harmless, the combination of these terms with terms in z can have drastic effects.

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Indeed, the harmonic polynomial $h(z) = z^n - \overline{z}^n$ has an infinite number of zeros (the zero set consists of n equally spaced lines through the origin). In 1992 Sheil-Small conjectured that if $n := deg\ p > m := deg\ q$, then h has at most n^2 zeros. In 1994 Wilmshurst found a more general sufficient condition for h to have a finite number of zeros and settled this conjecture using Bézout's theorem from algebraic geometry. While Wilmshurst's bound on the number of zeros is sharp, he also conjectured a smaller bound when the degrees of p and q differ by more than one.

In 2001 the first author and G. Świątek [KS 03] proved that the bound in Wilmshurst's conjecture held for the case of $f(z) = p(z) - \overline{z}$. Because this proof involves complex dynamics, it is natural to wonder whether this approach can be extended to find a bound on the number of zeros of the rational harmonic function $f(z) = p(z)/q(z) - \overline{z}$. The authors explored this question in 2003 [KN 06]. After posting a preprint, we learned that this bound settles a conjecture of S. H. Rhie concerning gravitational lensing. More precisely, it gives the maximal possible number of images of a light source (such as a star or a galaxy) that may occur due to the deflection of light rays by some massive obstacles. Even more surprising, Rhie had already shown that this bound is attained.

In this expository article we give a brief introduction to gravitational lensing. We also describe necessary background concerning harmonic polynomials and Wilmshurst's conjecture, as well as related results. We then discuss the ideas behind the proofs and also look at the question of sharpness. We close with a discussion of several possible directions for further study.
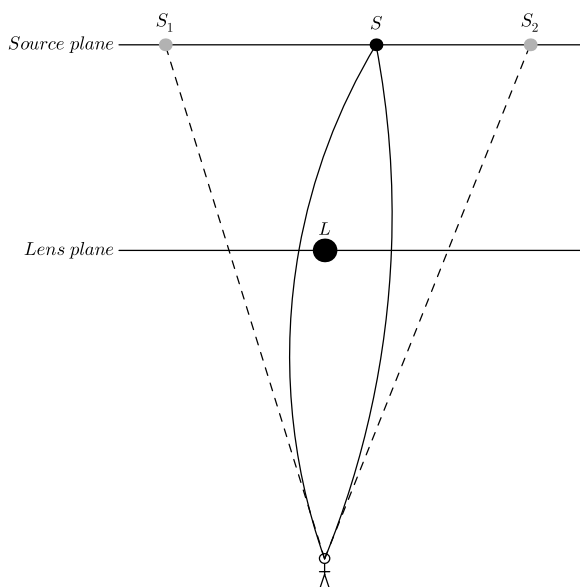
Figure 1. Basics of gravitational lensing (adapted from [Wa 98]**).**

Astrophysics: Gravitational Microlensing

Imagine that you are stargazing some clear, dark night and looking for a star S (see Figure 1). Light rays emanate from S in all directions and you would expect to "see" S along the shortest path between you and S. However, there is a massive object L between you and S. Light rays traveling past L are bent. Instead of seeing S, you "see" images of S at S_1 and S_2. This is the basic idea of gravitational lensing. Figure 2 is an example of gravitational lensing observed using NASA's Hubble Space Telescope.

Let's return to Figure 1. One can think of the images at S_1 and S_2 in terms of the angle between the original light ray from S and the "observed" ray to the observed image and calculate this angle in terms of a potential. J. Soldner is credited with the first published (1804) calculation of the deflection angle. Soldner's calculations were based on Newtonian mechanics. A. Einstein arrived at a similar answer in 1911, but revisited his assumptions in 1915 using general relativity. This calculation predicted a deflection angle that was twice as large as that predicted by Newtonian mechanics. Measurements of the deflection angle of starlight during a solar eclipse in 1919 provided early experimental support for general relativity. Theoretical work on gravitational lensing continued until the late 1930s. The discovery of quasars in the 1960s reawakened interest in gravitational lensing and it entered the mainstream in astrophysics after the discovery of a gravitational lensing system in 1979. Nowadays, it is used to find distant objects in the universe and determine the masses of these

objects. General references on gravitational lensing, and its history include [SEF 92] (written for physicists), [PLW 01] (written for both mathematicians and physicists), [Wa 98] (an online survey article), and the many references therein.

What does this have to do with counting zeros of rational harmonic functions? It's time for us to discuss how the positions of the point masses in our lens, our source, and the images are related.

Lens equation

Imagine n point masses (for example, condensed galaxies or black holes) and imagine a light source S (for example, a star or a quasar) farther away from the observer than these point masses. Due to deflection of light from S by these point masses, multiple images $S_1, S_2, \ldots$ of S can be formed. This phenomenon is known as *gravitational microlensing*. The point masses are said to form a *gravitational lens L*. We will look at the case where the point masses are sufficiently close together that they can be treated as co-planar (i.e., the distance between the point masses in L is small compared to the distance between the observer and the point masses and also small as compared to the distance between the source and the masses). We construct a plane through the center of mass of these point masses that is orthogonal to the line of sight between the observer and the center of mass of the point masses. This plane is called the *lens plane* or *deflector plane*. We then construct a plane through S that is parallel to the lens plane (*source plane*). Note that the lens plane lies between the observer and the light source. Figure 1 illustrates this for $n = 1$.

Since we're working with planes, we can represent the position of each item using complex numbers. We will choose the origin of each plane to lie on the line between the observer and the center of mass of our point masses. Project each point mass of our lens to the lens plane; for example, the jth point mass of our lens L is projected to position z_j in the lens plane. The *lens equation* for L is a mapping from the lens plane to the source plane and is given by

$$w = z - \sum_{j=1}^{n} \sigma_j / (\overline{z} - \overline{z_j}) \, ,$$

where σ_j is a nonzero real constant related to the mass of the jth point mass in our lens. For further details about this form of the lens equation, see [Wit 90] and [St 97].

If z satisfies the lens equation for a given value of w, our gravitational lens will map z to the position w in the source plane. When we let w be the position of S, each solution of the lens equation corresponds to a lensed image of S. Note that the right hand side of the equation is often called the

Figure 2. The bluish bright spots towards the center are the lensed images of a quasar (a QUASi-stellAR radio source), which has been lensed by the bright galaxy in the center. There are actually five images (four are bright and one dim), but one cannot really see the dim image here. (Photo credit: ESA, NASA, K. Sharon (Tel Aviv University) and E. Ofek (Caltech).)

lensing map. To model the effect caused by an extra ("tidal") gravitational pull by a distant object (a galaxy "far, far away"), a shear term (linear term in $\overline{z}$) is added to the lensing map. Sticking to the very basics, we shall omit this term in our discussion.

Consequences of the lens equation

What can we say about the number of lensed images using the lens equation given above? We first note that the number of lensed images depends on the relative positions of the observer, lens, and source. Notice that the observer, lens L, and source S do not lie on a straight line in Figure 1. If the source is directly behind the lens (from the observer's viewpoint), something very remarkable occurs when $n = 1$. Putting $w = 0 = z_1$ in the lens equation, we see that the lens equation becomes that of a circle centered about the lens. In other words, instead of seeing S, the observer would see a circle with center L. This effect was predicted by O. Chwolson in 1924 and is usually called an Einstein ring. Figure 3 shows some Einstein rings observed using the Hubble Space Telescope.

What happens if $n > 1$? H. Witt [Wit 90] showed by a direct calculation (not involving Bézout's theorem) that the maximum number of observed images is at most $n^2 + 1$. S. Mao, A. Petters, H. Witt [MPW 97] showed that the maximum possible number of images produced by an n-lens is at least $3n + 1$. S. H. Rhie [Rh 01] conjectured that the upper bound for the number of lensed images for an n-lens is $5n - 5$. Moreover, she showed in [Rh 03] that the conjectured bound is attained for every $n > 1$. If we let

$$r(z) = \sum_{j=1}^{n} \sigma_j / (z - z_j) + \overline{w},$$

finding the number of "lensed" images of the source is identical to finding the number of zeros of a rational harmonic function.

Counting Zeros

Let us return to the question of extending the FTA. We begin with Wilmshurst's approach to the Sheil-Small conjecture concerning the number of zeros of harmonic polynomials ([SS 02] and [Wil 98]). By writing $z = x + iy$, finding the zeros of a complex valued harmonic polynomial h is equivalent to finding the zeros of a system of two real polynomials

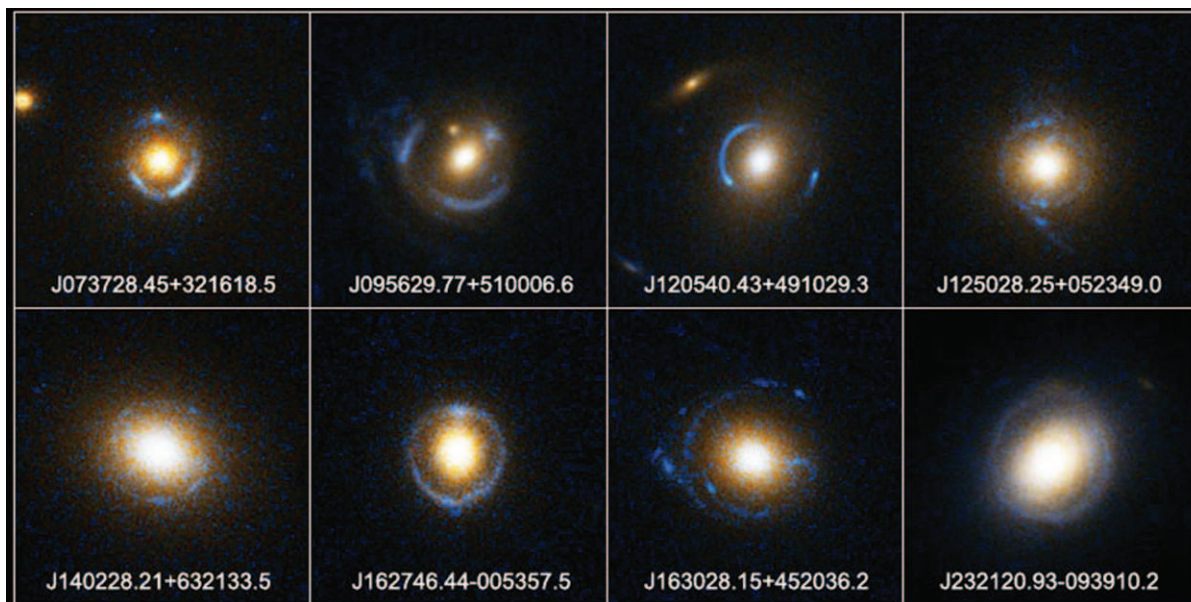$$\begin{cases} A(x, y) := Re\, h(z) \\ B(x, y) := Im\, h(z). \end{cases}$$

Figure 3. Einstein rings produced by a galaxy behind the lensing galaxy. The sources are actually extended and that is why one sometimes sees arcs rather than complete rings. (Photo credit: NASA, ESA, and the SLACS Survey team: A. Bolton (Harvard/Smithsonian), S. Burles (MIT), L. Koopmans (Kapteyn), T. Treu (UCSB), and L. Moustakas (JPL/Caltech).)

Recall that Bézout's theorem essentially says that if A and B are relatively prime with $deg\,A = m$ and $deg\,B = n$, then the number of common solutions for $A = 0$ and $B = 0$ does not exceed mn. Moreover, if $deg\,h = n$ and if $h(z) = p(z) - \overline{q(z)}$ has a finite number of zeros, then it has at most n^2 zeros. Further, if $deg\,p \neq deg\,q$, then $\lim_{z \to \infty} |h(z)| = \infty$. Wilmshurst showed that if a complex harmonic function $f(z)$ has a sequence of distinct zeros converging to a point in the domain of f, then it is constant on an analytic arc (notice that f is not required to be a harmonic polynomial). Suppose also that f is entire and has an infinite number of zeros in a bounded set. Wilmshurst then showed that f must be constant on a closed loop and is therefore constant by the maximum principle. He combined these ideas to prove Sheil-Small's conjecture and also constructed an example to show that the n^2 bound is sharp.

Theorem 1 ([Wil 98]). *If $h(z) = p(z) - \overline{q(z)}$ is a harmonic polynomial of degree n such that $\lim_{z \to \infty} |h(z)| = \infty$, then h has at most n^2 zeros. Moreover, there exist complex polynomials p and q where $deg\,q = n - 1$ such that the upper bound n^2 is attained.*

When the degrees of p and q are different, $\lim_{z \to \infty} |h(z)| = \infty$; thus, h can have only finitely many zeros and Sheil-Small's conjecture follows. Note that this conjecture was also proven independently by R. Peretz and J. Schmid [PS 98], while the sharpness result was also obtained independently by D. Bshouty, W. Hengartner, and T. Suez [BHS 95].

We now discuss Wilmshurst's elegant example [Wil 98] that implies sharpness:

Example. Consider

$$h(z) := Im(e^{-\frac{i\pi}{4}} z^n) + iIm(e^{\frac{i\pi}{4}} (z - 1)^n).$$

An elementary calculation shows that the zero set of $Re\,h(z)$ forms a set of n equally spaced lines through the origin. Moreover, applying a rotation and a translation to each of these lines gives the zero set of $Im\,h(z)$. Figure 4 shows the zero sets of $Re\,h(z)$ and $Im\,h(z)$ for the case $n = 2$.

A moment's thought shows that each line from the zero set of $Re\,h(z)$ intersects each line in the zero set of $Im\,h(z)$ at exactly one point. Thus, the cardinality of the zero set of $h(z)$ is given by the total number of intersections of these lines and is precisely $n \times n = n^2$. Let

$$k(z) := -2e^{i\pi/4} h(\overline{z}) = z^n + (z-1)^n + i\overline{z}^n - i(\overline{z}-1)^n.$$

$k(z)$ has the same number of zeros as $h(z)$. Writing $k(z) = p(z) - \overline{q(z)}$, we see that $deg\,q = n - 1 = deg\,p - 1$ as in Theorem 1. □

In contrast to this example where $deg\,q = n - 1$, Wilmshurst noted that the upper bound of n^2 distinct zeros is probably too large when $deg\,q < n - 1$ and proposed the following

Conjecture 1 ([Wil 98]). *If $n := deg\,p > m := deg\,q$, then*

$$\sharp\{z : p(z) - \overline{q(z)} = 0\} \le m(m - 1) + 3n - 2.$$

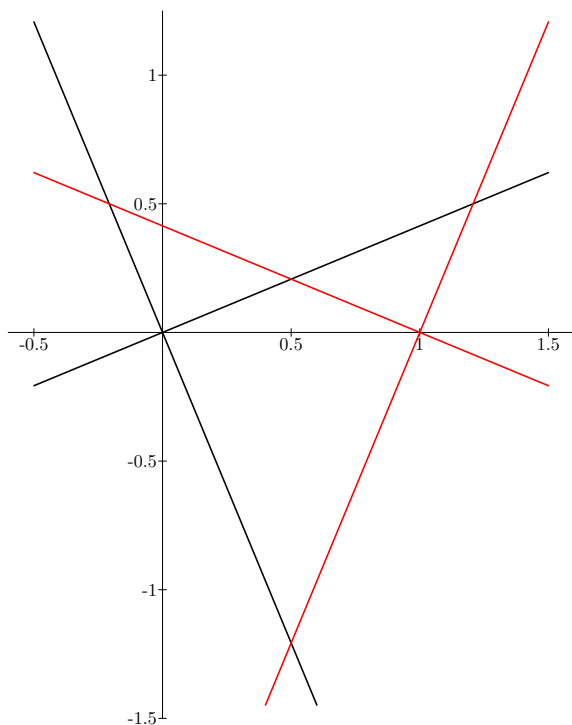**Figure 4. Wilmshurst's example for $n = 2$ (the zero set of $Re\, h(z) = 0$ is in black and the zero set of $Im\, h(z)$ is in red).**

It can be found in Remark 2 of [Wil 98] and is discussed in more detail in [SS 02, pages 50-55]. For $m = n - 1$, the example above shows that Conjecture 1 holds and is sharp. The simplest of the remaining cases of Conjecture 1 is the case $m = 1$. Let us explicitly state this case as

Conjecture 2 ([Wil 98])**.**
$$\sharp\{z : p(z) - \overline{z} = 0, n > 1\} \leq 3n - 2.$$

In the late 1990s D. Sarason and B. Crofoot [Sa 99] verified Conjecture 2 for $n = 2, 3$ and for several examples when $n = 4$. In 2001, using elementary complex dynamics and the argument principle for harmonic mappings, G. Świątek and the first author [KS 03] proved Conjecture 2 for all $n > 1$.

Theorem 2 ([KS 03])**.**
$$\sharp\{z : p(z) - \overline{z} = 0, n > 1\} \leq 3n - 2.$$

In a 2004 paper D. Bshouty and A. Lyzzaik [BL 04] showed that the $3n - 2$ bound in Conjecture 2 is sharp for $n = 4, 5, 6, 8$. More recently, L. Geyer [Ge 08] used complex dynamics to show that the $3n - 2$ bound is sharp for all n.

What happens if we extend the class of functions under consideration to include rational harmonic functions and extend the methods in [KS 03] to this case? Let

$$r(z) := \frac{p(z)}{q(z)}$$

be a rational function, where $p(z)$ and $q(z)$ are relatively prime polynomials in z. The degree of r is given by

$$deg\, r := max\{deg\, p, deg\, q\}.$$

The authors studied this case.

Theorem 3 ([KN 06])**.** *Let $r(z) = p(z)/q(z)$, where p and q are relatively prime polynomials in z, and let $n := deg\, r$. If $n > 1$, then*

$$\sharp\{z : r(z) - \overline{z} = 0\} \leq 5n - 5.$$

In contrast to the polynomial case, the question of sharpness for rational functions had been resolved previously by an example of S. H. Rhie [Rh 03], an astrophysicist, in her work on gravitational lensing.

Back to gravitational lensing

If we let $r(z) = \sum_{j=1}^{n} \sigma_j/(z - z_j) + \overline{w}$, finding the number of "lensed" images of the source is identical to the situation described in Theorem 3. In other words, Theorem 3 settles Rhie's conjecture and Rhie's results settle the question of sharpness of the bound in Theorem 3.

Surprisingly, when the lensing is done by a smoothly-distributed mass, rather than point masses, the following result (well known to astronomers) of W. Burke [Bu 81] yields

Theorem 4 ([Bu 81])**.** *If the number of lensed images produced by a smooth mass distribution is finite, it is always odd.*

Rhie [Rh 01] extended this result to the case of point masses and argued that the number of images will be even when n is odd and odd when n is even. This also follows from the proof of Theorem 3 ([KN 06], Corollary 2). Let's summarize our discussion of counting zeros with

Corollary 1 ([Rh 01], [Rh 03], [KN 06])**.** *Let $n > 1$. The number of lensed images by an n-mass planar lens with zero shear cannot exceed $5n - 5$, and this bound is sharp. Moreover, the number of images is even when n is odd and odd when n is even.*

Main Ideas Behind the Proofs

We shall sketch the proof of Theorem 2 for harmonic polynomials [KS 03]. The proof of Theorem 3 for rational harmonic functions is similar, with the added complication that finite poles must also be considered.

Returning to Theorem 2 and the polynomial case, we let

$$h(z) := z - \overline{p(z)}, \text{ where } n := deg\, p > 1.$$

Treating h as a mapping of $\hat{\mathbb{C}}$, it is natural to consider the domains in $\hat{\mathbb{C}}$ in which h is respectively
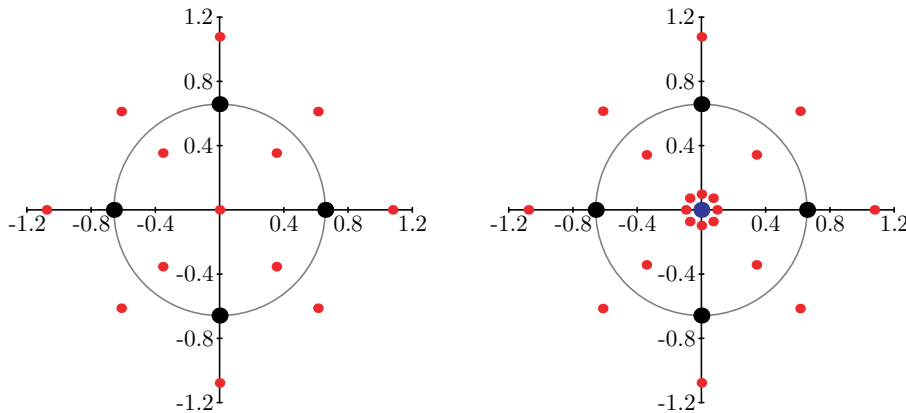
Figure 5. An example of Rhie's construction of an $(n+1)$**-point gravitational lens, producing** $5(n+1) - 5$ **images. Start with an** n**-point lens having** $3n + 1$ **images; each point has mass** $1/n$**. The system is perturbed by removing a mass of** ϵ/n **from each of the** n **point masses and then adding a small point mass of mass** ϵ **at the origin. In the graphs above,** $n = 4$ **and** $\epsilon = 1/100$**. The light source is located at the origin and is not shown in the graphs. Solutions to the lens equation (images of the source) are shown in red. The graph on the left shows the unperturbed system, with the four point masses in black. The graph on the right shows the perturbed system, with the small additional point mass at the origin shown in blue.**

sense-preserving and sense-reversing. These domains are separated by the *critical set of h* defined by

$$L := \{z : \text{Jacobian}(h(z)) = 1 - |p'(z)|^2 = 0\}.$$

We note that L is a lemniscate with at most $n-1$ connected components. A moment's thought indicates that h is sense-preserving (Jacobian$(h) > 0$) inside each component of L and sense-reversing (Jacobian$(h) < 0$) outside of L.

Let n_+ denote the number of sense-preserving zeros of h and n_- the number of sense-reversing zeros. Outside of L, h is sense-reversing. Since $|h| \to \infty$ at ∞, all of the sense-reversing zeros are finite. Moreover, we may choose R sufficiently large such that $C(0, R)$, the circle of radius R centered at the origin, contains all of the zeros of h and does not intersect the critical set. We now consider the region Ω bounded by $C(0, R)$ (in the positive sense) and L (in the negative sense).

Provided that the boundary of a finitely connected region is sufficiently nice (piecewise smooth, with no zeros on the boundary) and no zeros lie in the critical set, the argument principle can be used to count the isolated zeros of harmonic mappings in almost exactly the same way it applies to analytic mappings; the only difference is that sense-reversing zeros are counted with a minus sign (see [Du 04] and [SS 02]). We note that near infinity, h behaves like $\overline{z}^n$; hence for sufficiently large R, the argument increment of h along $C(0, R)$ modulo 2π is $-n$.

We now employ some wishful thinking by supposing that (i) no zeros of h lie on the critical set and (ii) h is univalent inside each component of

L. Hence, $n_+ \leq n - 1$ and the change in argument on L modulo 2π is $-n_+ \geq -(n-1)$. Applying the argument principle to Ω and noting that Ω contains only sense-reversing zeros of h, we see that

$$-n_- = \text{ change in argument modulo } 2\pi$$
$$\geq -n - (n-1),$$

giving $n_- \leq 2n - 1$. The total number of zeros of h is $n_+ + n_-$; therefore there are at most

$$(n-1) + (2n-1) = 3n - 2$$

zeros of h. Modulo the wishful thinking, we are done.

The following example inspires one to believe that the above argument is right on the money:

Example. Consider

$$h(z) = z - \overline{\tfrac{1}{2}(3z - z^3)}.$$

Here $n = 3$. Our function has $3n - 2 = 3 \times 3 - 2 = 7$ zeros:

$$0, \pm 1, \tfrac{1}{2}(\pm\sqrt{7} \pm i),$$

where all combinations of plus and minus signs are considered in the last term. There are $2n - 1 = 2 \times 3 - 1 = 5$ sense-reversing zeros, namely, $0, \tfrac{1}{2}(\pm\sqrt{7}\pm i)$ and $n-1 = 3-1 = 2$ sense-preserving zeros, namely, ± 1. $\square$

Our assumption requiring that no zeros of h lie on the critical set can be removed by showing that the set of harmonic polynomials satisfying this assumption is dense (see [KS 03]). Unfortunately there are examples showing that the assumption that h must be univalent in each component of
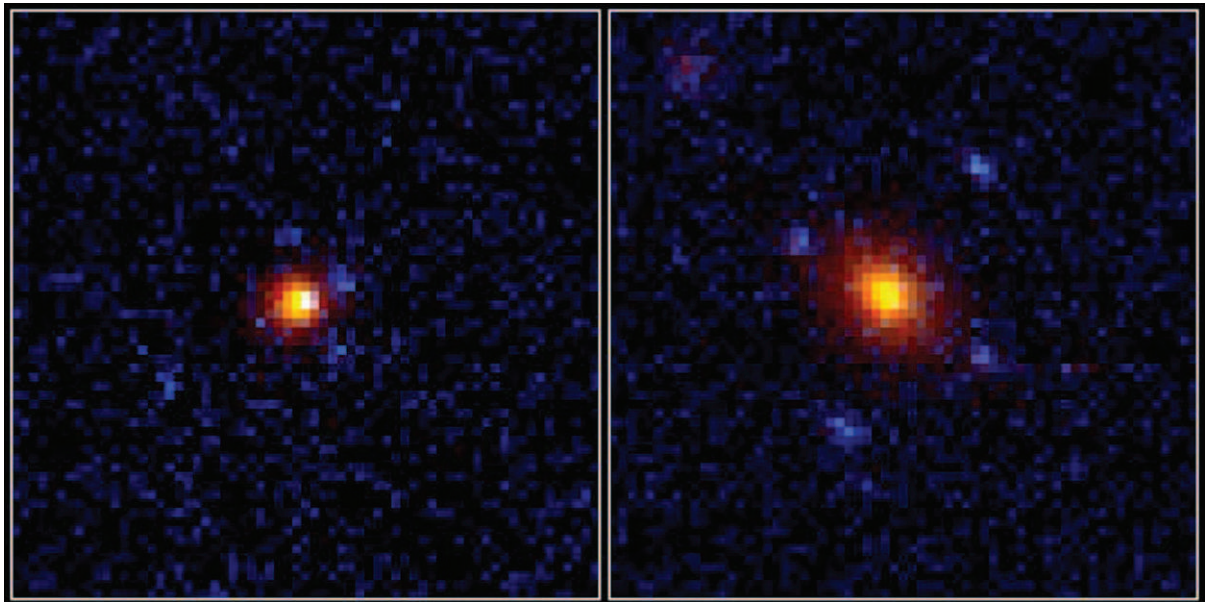
Figure 6. Four images of a distant light source (larger blue dots) lensed by an elliptical galaxy (red). This picture serves as an illustration only. Realistically, one cannot expect a galaxy to have a uniform mass distribution. (Photo credit: Kavan Ratnatunga (Johns Hopkins University, Baltimore, MD) and NASA.)

L does not hold in general [Wil 94]. Thus, the above argument fails. However, it can be salvaged by applying some elementary facts regarding iteration of polynomials (and iteration of rational functions) that come from complex dynamics.

Help from dynamics

First, some minimal vocabulary. Complex dynamics studies iterations; in other words, compositions of an analytic function with itself. If F is analytic and z_0 a fixed point of F ($F(z_0) = z_0$), z_0 is called an *attracting fixed point* of F if $|F'(z_0)| < 1$.

As the previous discussion shows, the crux of the matter is the following proposition which shows that $n_+ \leq n - 1$:

Proposition 1. *Let $\deg p = n$. Then*

$$\sharp\{z : z - \overline{p(z)} = 0, |p'(z)| < 1\} \leq n - 1.$$

Why is this so? First, we note that $Q(z) := \overline{p(\overline{p(z)})}$ is an analytic polynomial of degree n^2. Moreover, every fixed point of $\overline{p(z)}$ with $|p'(z)| < 1$ is an attracting fixed point of Q, and by Fatou's theorem (see [CG 93], Theorem III.2.2, page 59), it "attracts" at least one critical point of Q (z is a critical point if $Q'(z) = 0$). In other words, at least one critical point of Q "runs" to a fixed point of $\overline{p(z)}$ with $|p'(z)| < 1$ under iteration of the map Q.

The following lemma is elementary and its proof can be found in [KS 03]. It essentially says that the limiting behavior under the iteration of the map Q occurs in clusters of at least $n + 1$ points exhibiting the same limiting behavior under the

iteration of the map $\overline{p(z)}$. This is natural since $\overline{p(z)}$ covers the Riemann sphere n times and Q is obtained by iterating $\overline{p(z)}$ twice.

Lemma 1. *Each fixed point of $\overline{p(z)}$ with $|p'(z)| < 1$ attracts at least a group of $n + 1$ critical points of Q.*

With this lemma, it is not difficult to finish the proof of the proposition. Q has $n^2 - 1$ critical points. Divided into groups of at least $n + 1$ points they "run" to at most $\frac{n^2 - 1}{n + 1} = n - 1$ fixed points of $\overline{p(z)}$. The proposition follows and completes our sketch of the proof of Theorem 2.

Sharpness Results

The following theorem of L. Geyer [Ge 08] rests on fairly deep results in topological dynamics. Yet, it was already conjectured by B. Crofoot and D. Sarason [Sa 99]:

Theorem 5 ([Ge 08]). *For every $n > 1$ there exists a complex analytic polynomial p of degree n and mutually distinct points $z_1, \ldots, z_{n-1}$ with $p'(z_j) = 0$ and $p(z_j) = \overline{z_j}$.*

Theorem 5 implies that for every $n > 1$, there are polynomials with precisely $n - 1$ attracting fixed points for $\overline{p}$; hence, Proposition 1 is indeed sharp for all n and so is the upper bound $3n - 2$ in Theorem 2.

The question of sharpness for rational functions was settled by S. H. Rhie [Rh 03]:

Theorem 6 ([Rh 03]). *For every $n > 1$, there exists a rational harmonic function with exactly $5n - 5$ distinct zeros.*

Rhie established Theorem 6 in the context of gravitational lensing with n point masses. For $n \geq 3$, she used a very elegant perturbation argument to build an example with $5n - 5$ distinct zeros, the bound she had previously conjectured in [Rh 01]. She mentions that the case $n = 2$ is established by other lensing examples; we also note that [KN 06] includes a non-lensing example for this case.

In a nutshell, Rhie's construction for $n \geq 3$ is as follows: Mao, Petters, and Witt [MPW 97] studied a gravitational lens with n equal point masses located at the vertices of a regular n-gon and a light source at the center of the n-gon and showed that $3n + 1$ distinct lensed images can be attained. Rhie analyzed the case where these point masses each have mass $1/n$ and are located at the vertices of a regular n-gon inscribed in a circle of radius $a = (n-1)^{\frac{1}{2}-\frac{1}{n}}/\sqrt{n}$ (one of the point masses lies on the positive x-axis). This configuration produces $3n + 1$ distinct images: one image at the origin, another image on each ray with a point mass (even multiples of π/n), and two additional images on each ray with argument that is an odd multiple of π/n. Now perturb this system by adding a sufficiently small mass at the origin. In particular, suppose that a mass of $\epsilon > 0$ has been added at the origin and the mass of each of the original n point masses has been reduced by ϵ/n. One can then show that an additional $2n-1$ images are produced for ϵ sufficiently small (the perturbed system does not have an image at the origin, has two images on each ray with argument that is an even multiple of π/n, and has three images on each ray with argument that is an odd multiple of π/n). Hence, the perturbed system has $(n + 1)$ point lenses and produces $(3n+1)+(2n-1) = 5(n+1)-5$ distinct lensed images. Figure 5 illustrates this for $n = 4$ and $\epsilon = \frac{1}{100}$.

Let us summarize the results concerning sharpness:

Corollary 2. *For every $n > 1$ there exists a complex analytic polynomial p of degree n such that $\overline{p(z)}-z$ has precisely $3n - 2$ zeros. Similarly, there exists a rational function $r(z)$ with (finite) poles $z_1, ..., z_n$ such that the $\overline{r(z)} - z$ has precisely $5n - 5$ zeros.*

Where Might One Go from Here?
Extending the FTA further

Unfortunately, the techniques from complex dynamics used in the proofs of Theorems 2 and 3 do not seem to work as well for more general harmonic polynomials or rational functions with the conjugate degree greater than 1. In particular, Wilmshurst's conjecture (Conjecture 1 above) remains open for the cases $1 < m < n - 1$. The following question is a natural next step in extending FTA further:

Question. *Let $p(z)$ be an analytic polynomial of degree n. Let $1 < m < n - 1$ be an integer. What is a sharp upper bound on the number of zeros of the harmonic polynomial $h(z) := \overline{z}^m - p(z)$?*

For example, Wilmshurst's conjecture predicts an upper bound of $3n$ for the case $m = 2$; this is smaller than the bound of n^2 given by Bézout's theorem when $n > 3$. If the conjectured bound holds, is it sharp?

Recently, W. Li and A. Wei [LW 08] proved a probabilistic variant of Wilmshurst's conjecture. They assume that the coefficients are distributed as Gaussian independent variables. Let $h(z) = p(z)-\overline{q(z)}$ and let $\deg q = m$ and $n = \deg p$. When $m = n-a$ where $a > 0$, they show that the expected number of zeros asymptotically approaches $n^{3/2}$ when n is large. This falls between the minimum number of possible zeros (there are at least n zeros) and the maximum of n^2 given by Bézout's theorem. Surprisingly, when $m/n < \alpha < 1$ for all n, the expected number of zeros tends to n [LW 08].

Gravitational microlensing by a mass distribution

Theorem 3 can be applied to the case of n "spherically symmetric" mass distributions in the lens plane, showing that there will be at most $5n - 5$ lensed images outside the support of the mass [KN 06]. In practice, however, other mass distributions are more natural.

Question. *How many lensed images can an arbitrary mass distribution produce?*

Consider, for example, an elliptic mass distribution. C. Fassnacht, Ch. Keeton, and the first author [FKK 07] have shown that an elliptic uniform mass distribution considered as a gravitational lens can produce at most four visible images outside the lens. Note that there are observations of four lensed images (for example, Figure 6). Furthermore, the same upper bound holds for densities that are constant on ellipses confocal with the initial ellipse [FKK 07].

An even more realistic assumption would be using a polynomial in place of the uniform density. A sharp upper estimate for the number of images is unknown in this case. For most models in astronomy, the densities are assumed to be constant on ellipses *homothetic* to the initial ellipse. One such density is called the *isothermal density* and is especially important. It arises by taking a three-dimensional mass density that is inversely proportional to the square of the distance from the origin and projecting it onto the lens plane. (If a spherical galaxy is filled with gas with such

density, it will remain at constant temperature.) However, the right hand side of the lens equation becomes a transcendental function; as far as we know, there is no proof at the present time showing that the number of images is even finite. This situation has been studied extensively through modeling (cf. [KMW 00]) with, or without, external shear and there are models that produce up to 9 images [KMW 00], though there do not seem to be any observations where this number of images is actually seen.

Microlensing by nonplanar lenses

All along, we have been assuming that the masses in a gravitational lens can be treated as if they were coplanar.

Question. *When an n-lens consists of point masses that cannot be treated as coplanar, what is the sharp upper bound on the number of possible images?*

In this case, the problem of estimating the precise number of images is excruciatingly difficult. Petters has used Morse theory to make some preliminary estimates [PLW 01]. Sharper, better focused estimates are still far out of reach. This last question resembles the classical fundamental problem of Maxwell concerning the precise number of points in space where the gradient of the electrostatic potential produced by n point charges vanishes; in other words, find the points of equilibrium where no electrostatic force is in fact present. Maxwell conjectured the maximal number of such points may not exceed $(n-1)^2$; see [GNS 07] and the references therein for the starting point of another exciting tale.

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