Connected Sets and the AMS, 1901–1921

David E. Zitarelli

Chapter 1 of Kelley's famous book *General Topology* introduces the most fundamental concepts of a topological space. One such notion is defined as follows [1]:

> A topological space (X, τ) is **connected** if and ony if X is not the union of two nonvoid separated subsets, where A and B are **separated** in X if and only if $\overline{A} \cap B = \emptyset$ and $\overline{A} \cap B = \emptyset$.

As usual, $\overline{Y}$ denotes the closure of a subset Y of X.

Kelley's book has been a staple for several generations of graduate students, many of whom must have wondered what this formal definition had to do with their intuitive notion of a connected set. Frequently, such queries can be answered by an historical investigation, and the aim here is to trace the development of the formal concept of a connected set from its origins in 1901 until its ultimate ascension into the ranks of mathematical concepts worthy of study for their own sake twenty years later. Much of this development took place at the University of Chicago under E. H. Moore, and the evolution of connected sets exemplifies one specific way in which ideas that germinated there would be promulgated by his descendants. Although Moore exerted little direct influence, his department's core of outstanding graduate students, like the well-known Oswald Veblen, and its cadre of small-college instructors and high school teachers from across the country who pursued degrees during summer sessions, like the lesser-known N. J. Lennes, would play a decisive role.

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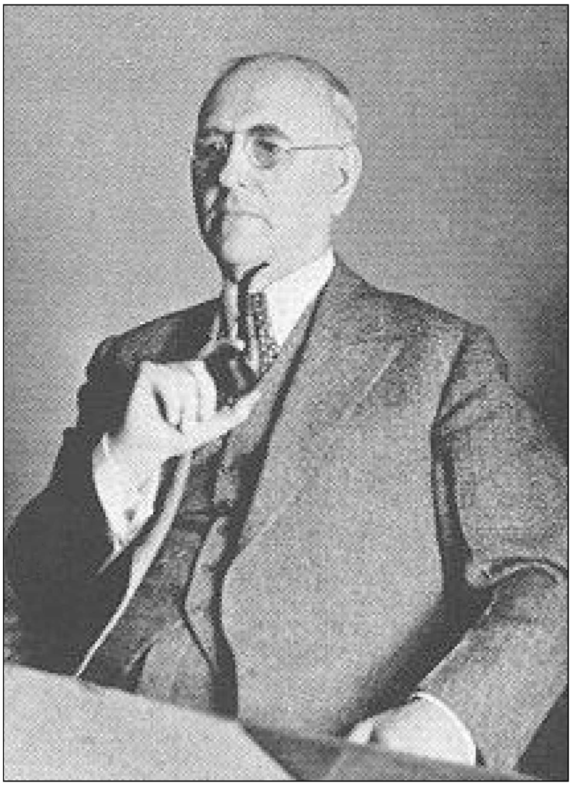
An overarching theme is how the rapidly evolving AMS abetted this development in two ways. For one, local and national meetings provided venues where researchers could present their work and keep abreast of the progress of others. For another, the two AMS publications, the *Bulletin* and the *Transactions*, provided outlets for publishing these findings. Neither journal had widespread readership across the Atlantic so, as we will see, the study of connected sets would proceed in Europe independently of advances in America. Initially, European initiatives appeared in a Polish journal and were based on a classic book by Hausdorff, but ultimately the two schools of topology would interact symbiotically over connected sets. One of the contributors who engendered the ensuing international collaboration was Anna Mullikin, a second-generation Moore descendant who became the first American to publish a paper devoted to connected sets. We end our account by examining one of her chief examples, Mullikin's nautilus, to illustrate the power of the general definition first proposed by N. J. Lennes. We begin by introducing the latter's life and work, emphasizing the formulation of connected sets he announced fifty years before the appearance of Kelley's classic.

The Pioneer N. J. Lennes

It seems appropriate that the person who pioneered the modern definition of a connected set would be a pioneer himself in a geographic sense. Nels Johann Lennes (1874–1951) came to the U.S. from his native Norway at age sixteen and, like many Scandinavians, settled in Chicago. Recall that the University of Chicago opened its doors in 1892;



N. J. Lennes

Lennes enrolled four years later and earned a bachelor's degree in just two years. Upon graduation he became a high school teacher (1898–1907), taking graduate courses at Chicago all the while. With his Ph.D. in hand he then taught at the Massachusetts Institute of Technology and Columbia University for three years each before his pioneering move to distant Missoula, Montana, where he was professor of mathematics and head of the department from 1913 until retirement in 1944. ([2] provides an overview of his life and work; he is described as "one of the precursors of modern abstract mathematics".)

This biographical vignette indicates that while Lennes was a student at Chicago he rubbed elbows with two other students who would ultimately make important contributions to this development, Oswald Veblen (1880–1960) and R. L. Moore (1882–1974). The mathematics department in Eckhart Hall was a cauldron of activity, and this trio emerged during a ten-year period that produced such legendary figures as L. E. Dickson (1874–1954), G. A. Bliss (1876–1951), and G. D. Birkhoff (1884–1944).

Veblen, recently profiled in the *Notices* [3], enrolled at Chicago in 1900 after having spent a year at Harvard obtaining a second bachelor's degree. It is known that he first impressed E. H. Moore during the fall 1901 seminar "Foundations of Geometry", and that his subsequent proof of the Jordan Curve Theorem (JCT) was inspired by discussions in the seminar. It is not so widely known that Lennes played a decisive role with the JCT too. In fact, Veblen's first paper on topology reported that Lennes, in his 1903 master's thesis, proved the special case of the JCT for a simple polygon [4, p. 83]. Because the idea of a curve dividing a plane into separated subsets suggests the definition of a disconnected set, as implicitly defined by Lennes, we date our history from 1901.

The Moore seminar was pivotal for the careers of Veblen and Lennes, and as usual the AMS played a major role. The Chicago cauldron's mix of topological ingredients simmered at the April 1903 meeting of the Chicago Section, where Moore's advanced graduate students delivered noteworthy presentations of research they had begun in the seminar. The twenty-two-year-old Veblen announced major results that would appear in his dissertation, one of which included a proof independent of Lennes that "The boundary of a simple polygon lying entirely in a plane α decomposes α into two regions" [5, p. 365]. Veblen was in the process of establishing the first rigorous proof of the JCT, which he labeled "The fundamental theorem of Analysis Situs" in a paper read at the AMS meeting that preceded the International Congress at the St. Louis World's Fair in 1904 [4, p. 83]. The JCT remains a benchmark of mathematical rigor today. (See [7] for a modern, computer-oriented account of this phenomenon.)

At the April 1903 meeting, Lennes, six years Veblen's senior, presented results from his master's thesis, "Theorems on the polygon and the polyhedron". In what was to become customary for him, he would not submit this work for publication for another seven years even though he would then assert that only "minor changes and additions have been made since that time" [8, p. 37]. The chief result was that "the polygon and polyhedron separate the plane and the three-space respectively into two mutually exclusive sets" [*Ibid.*], an affirmation of the inspiration for his definition of a connected set in terms of separated domains.

Even though primarily engaged with teaching high school, Lennes continued his activity with the AMS by presenting two papers at the December 1904 meeting of the Chicago Section. One dealt with Hilbert's theory of area and resulted in a *Transactions* publication the next year, reflecting an ongoing interest in geometrical topics. The other was concerned with improper definite integrals, resulting in a paper in the *American Journal*. During this time he also published a paper on uniform continuity in the *Annals*, his first publication *not* read first before an AMS audience. Moreover, Lennes submitted a paper on real function theory that was read by title at the annual AMS summer meeting in September 1905.

The focus of Nels Lennes's research program was analysis, a topic he was pursuing in earnest with Oswald Veblen, who remained at Chicago for two years after receiving his Ph.D. in 1903. It is known that Veblen played a central role in mentoring R. L. Moore and directing his dissertation, but the symbiotic relationship between Veblen and Lennes seems to have escaped attention. Their joint activity can be seen in Veblen's dissertation, where he expressed his "deep gratitude to Professor E. H. Moore…and also to Messrs. N. J. Lennes and R. L. Moore, who have critically read parts of the manuscript" [5, p. 344].

[Lennes's definition] represented a dramatic shift from the geometric, constructive approach…

Veblen and Lennes also collaborated on a textbook aimed to "be used as a basis for a rather short theoretical course on real functions" [9, p. iii]. Three years before the text appeared in 1907, Veblen revealed that "The equivalence [of the Heine-Borel theorem with the Dedekind cut proposition] in question suggested itself to Mr. N. J. Lennes and myself while we were working over some elementary propositions in real function theory" [6, p. 436]. The central feature of the book was the "Heine-Borel property", today called compactness, which Lennes had discussed at an AMS meeting in December 1905. Moreover, the final chapter, described as "more advanced in character than the other chapters and intended as an introduction to the study of a special subject" [9, p. iii], elaborated upon Lennes's paper on improper integrals.

Before embarking on Lennes's formulation of connected sets, it is instructive to examine the intuitive approach that preceded his more general advance. This can be seen in the doctoral dissertation of L. D. Ames that was published in the October 1905 issue of the *American Journal*. One of the "preliminary fundamental conceptions" he presented was the following definition, whose wording shows that set theory was still in its formative stage [10, p. 366]:

> An assemblage is *connected* … if $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are any two points of the assemblage, then it is possible to draw a simple curve
>
> $$x = \lambda(t),\ y = \mu(t),\ z = \upsilon(t),$$
> $$t_0 \le t \le t_1$$
>
> having P_0 and P_1 as end points and such that all points of the curve are points of the assemblage.

Such a property is called arcwise (or pathwise) connected today.

Lewis Darwin Ames (b. 1869) was no slouch. While teaching at a normal school from 1890 to 1900, he spent the summers of 1897 and 1898 taking courses at the University of Chicago. He earned bachelor's degrees from Missouri in 1899 and Harvard in 1901. Ames remained at Harvard for another two years, the first as an instructor and second as a graduate scholar, before returning to the University of Missouri in the fall of 1903. During that academic year he completed his dissertation under the well-known analyst William Fogg Osgood (1864–1943), thus becoming the first of Osgood's four doctoral students and showing that research using the notion of a connected set was taking place outside Chicago. It is conceivable that Lennes and Ames crossed paths during their summer studies but no evidence supports such a link.

Just two months after Ames's paper appeared, Lennes delivered three lectures at a Chicago Section meeting of the AMS that exhibit his depth and versatility as well as the breadth of offerings within Moore's department. One was the work cited above on the Heine-Borel Theorem. Another elaborated a fundamental theorem in the calculus of variations. From the present vantage point, however, the third was the most important. We turn to it now.

The Genesis of Connected Sets

The remaining paper that Nels Lennes delivered at that December 1905 meeting, "Curves in nonmetrical analysis situs", announced the earliest formulation of a connected set [11, pp. 284–5]:

> A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set.

The abstract, including the specific wording, was published in the March 1906 issue of the *Bulletin*. This marked the first time such a formulation appeared in print, but initially it elicited little interest. Though expressed only for subsets of a Euclidean space, the definition extends unchanged to more general spaces. Importantly, it represented a dramatic shift from the geometric, constructive approach championed in Ames's paper to an abstract, nonconstructive formulation requiring proof by contradiction.

Over the next year Lennes developed his paper into a doctoral dissertation with the expansive title "Curves in non-metrical analysis situs with an application in the calculus of variations". It was written under the direction of E. H. Moore and resulted in Lennes's Ph.D. in 1907. Yet once again Lennes did not rush into print, publishing his thesis only in 1911. He emphasized, however, that no critical advances had taken place in the meantime, writing, "Changes made since then are entirely unimportant" [12, p. 287]. Therefore, our analysis will center on this paper. Here one can also see the distinction between the older, geometric definition of connectedness and the abstract formulation. Initially Lennes supplied an arcwise connected definition only slightly more general than the one adopted by Ames [12, p. 293]:

> An entirely open set of points is said to be connected if for any two points of the set there is a broken line connecting them which lies entirely in the set.

But then he presented the standard definition equivalent to the one in Kelley [12, p. 303]:

> A set of points is a "connected set" if at least one of any two complementary subsets contains a limit-point of points in the other set.

It is germane to point out that the paper itself was not devoted to connected sets *per se*, but to simple arcs (curves) in nonmetric spaces. The formal definition of a connected set was given along with several other terms about midway through the paper. He then supplied the following critical definition [12, p. 308]:

> A continuous simple arc connecting two points A and B, $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing A and B such that no connected proper subset of $[A]$ contains A and B.

In motivating this definition Lennes served notice of his complete understanding of the role he was about to play, asserting, "This definition seems to be very near the obvious intuitional meaning of the term 'arc' or 'curve' " [12, p. 289]. Although not referring to connected sets, his comment on arcs could apply equally well to his formulation of this important concept. Next Lennes moved to his primary goal of proving properties of arcs in the plane, such as the following: for every interior point C of an arc AB, the arcs AC and BC are closed, are connected, and have only C in common.

As the expanded title of the paper indicates, the origin of connected sets lay in the calculus of variations, a subject that was one of the specialties at the University of Chicago, notably with Oskar Bolza and Gilbert Bliss. Concerning the concluding section, Lennes wrote that "the general theory of the paper is applied to the problem of proving the existence of minimizing curves in an important class of problems in the calculus of variations" [12, p. 290]. Moreover, he supplied a very useful history of the need for a proper definition of the concept of connectedness going back to G. Cantor and W. H. Young, concluding with a Veblen paper from their Chicago days together.

Lennes had an impressive résumé by the time of his doctorate, having published six papers in every American outlet available to him (the *Annals*, *Transactions*, *Bulletin* (2), *American Journal*, and *Monthly*), as well as the book with Veblen. But it was his definition of connected sets over the long haul, and his definition of a continuous simple arc for a shorter period, that represent his biggest contributions to mathematics. (As a curious aside, Kelley's résumé included papers in the *American Journal*, *Duke Journal*, and *Proceedings of the National Academy of Sciences* when he sought employment after receiving his 1940 doctorate at Virginia under G. T. Whyburn.)

After receiving his Ph.D., Lennes left his high school post for an instructorship at MIT, where he remained from 1907–10 but did not publish one work. At that time MIT was fifteen years away from becoming the top-notch research institution it is today, beginning with the hiring of Norbert Wiener and Dirk Struik. But then Lennes moved to Columbia (1910–13), where he experienced a spurt of nine publications, including the forty-page paper based on his dissertation.

In February 1911 Lennes delivered three lectures at an AMS meeting in New York, each of which resulted in a paper published before the year was out. Two months later he returned to Chicago for an historic AMS meeting. Founded in 1888, the Society remained mainly a local organization in New York City until the enterprising faculty at the upstart University of Chicago inspired national expansion six years later and established a western outpost called the Chicago Section in 1897. Yet it was April 1911 when, at the invitation of the Chicago Section, the Society became truly national by holding its first meeting outside New York City (with the exception of summer affairs). AMS secretary F. N. Cole gushed, "This was in many ways a remarkable occasion…arranged [so] that this reunion of the eastern and western members should be especially marked by the delivery of President Bôcher's retiring address.… As was under these circumstances to be expected, the meeting was in every way a most successful and inspiring one" [13, p. 505]. The meeting was successful and inspiring too for Lennes, who gave two lectures. The first, "Curves and surfaces in analysis situs", showed that his definition of a simple continuous arc (shortened here, as then, to *arc*) "applies without any change whatever to arcs in space" [13, p. 525]. The primary aim of this work, however, was to provide a rigorous proof that a closed continuous surface separates space into two connected sets.

…the origin of connected sets lay in the calculus of variations…

That October, six months later, Lennes delivered yet another paper at an AMS meeting back in New York that extended results from the April meeting [14, p. 165]. This turned out to be the last time he referred to connected sets in print, but it did not signal the end of his research, as he presented four more papers at AMS meetings, two in April 1912 and two in February 1913. It is telling from this activity, however, that his career would take a dramatic turn when he left New York that fall, as only one of those presentations resulted in a published paper. The other three were perhaps consigned to piles in his office that would remain fixed points during the thirty-one years he headed the department, because once Lennes left Columbia for the University of Montana later in 1913, his research output virtually ground to a halt, numbering only one minor paper (in the *Monthly*) on the fundamental theorem of calculus and another on nonmathematical logic (in the *Mathematics Teacher*). Nonetheless, he resumed writing textbooks at all levels, from high school through college, an activity he had begun in collaboration with H. E. Slaught while still at Chicago. Ultimately Lennes's numerous textbooks became so successful commercially that the house he built on campus provides the residence for the president of the University of Montana to this day. The mathematics department at the university has held the Lennes Exam for undergraduate students since shortly after his death.

Like Lennes, the four Pennsylvanians benefited enormously from AMS meetings and journals.

Lennes's record illustrates the vital role the AMS played in three distinct ways. First, the periods when he published papers occurred while he was in Chicago and New York, the two foci of the Society. Columbia had served as AMS headquarters since its founding in 1888, while Chicago inspired the transformation from a local to a national organization and became the first official section. Secondly, Lennes normally vetted his results before AMS audiences prior to submitting them for publication. Finally, his papers often appeared in the two AMS journals. But the time was ripe for his general definition to take hold, and although he played no role in its advance, various mathematicians made use of it over the next ten years, most of them allied with the University of Chicago. We turn to this development next.

The Pennsylvanians

Shortly after Nels Lennes went into research hibernation in Montana, another product of E. H. Moore awoke from a prolonged slumber in an entirely different part of the country to feast on the savory pickings that Lennes had left behind. R. L. Moore had been a graduate student at Chicago from 1903–05, and Veblen's expressed gratitude to Lennes and Moore for critically reading Veblen's dissertation suggests that this trio of companions collaborated quite closely. However, after receiving his Ph.D. in 1905, R. L. Moore endured six years of academic thaw, publishing only two papers, both based on work done in graduate school. Whereas Lennes did little research after 1913, R. L. Moore tilled fertile soil at the University of Pennsylvania, resulting in seventeen papers during 1911–20, several of which dealt with connected sets. As well, all three Ph.D. students he mentored at Penn made use of Lennes's pioneering work. Like Lennes, the four Pennsylvanians benefited enormously from AMS meetings and journals.

R. L. Moore's connection to connected sets began with a paper delivered at an AMS meeting in April 1914 in New York and published later that year [15]. However, his investigation was not concerned with connected sets *per se*. Rather he sought to characterize linear continua in terms of point and limit by extending a set of axioms given by F. Riesz at the 1908 International Congress of Mathematicians in Rome regarding postulates enunciated by David Hilbert in his classic book on the foundations of geometry. Moore began by stating Lennes's definition of a connected set, thereby becoming the first person to make use of the power and generality of the reformulation almost nine years after its initial pronouncement. Then he listed four axioms that extended Riesz's three, one of which read, "If P is a point of S, then $S - P$ is composed of two connected subsets neither of which contains a limit point of the other" [15, p. 124]. This idea of expressing the complement of a set as the union of two separated, connected sets would bear fruit in subsequent investigations by Moore and his students.

Frigyes (Frederick in English) Riesz himself played a curious role in the development of connected sets, having stated independently of Lennes an equivalent definition of a connected set just a month after Lennes presented his at the December 1905 AMS meeting. While Lennes's influence was restricted to American mathematicians, Riesz's definition seems to have remained unknown until an investigation by R. L. Wilder on the evolution of connected sets seventy-two years later [16]. (See [17] for an analysis of Riesz's contributions to topology.)

One year after Moore's initial foray, he presented a paper at another AMS meeting in which connected sets were defined among axioms for the plane, leading to a paper in the *Bulletin* later that year [18]. It is of interest to note that his first Ph.D. student,

J. R. Kline, presented his dissertation at that April 1915 meeting, continuing E. H. Moore's policy of involving students in AMS affairs early in their careers. There is no need to examine this work of R. L. Moore because he soon extended its major result appreciably in perhaps his most important paper, "On the foundations of plane analysis situs" [19]. This long and detailed work which we, like Moore, abbreviate F. A., firmly solidified his reputation as a first-rate researcher. Furthermore, he used its sequence of theorems in his classes over the next fifteen years to form the basis for what would come to be known as the Moore Method of teaching.

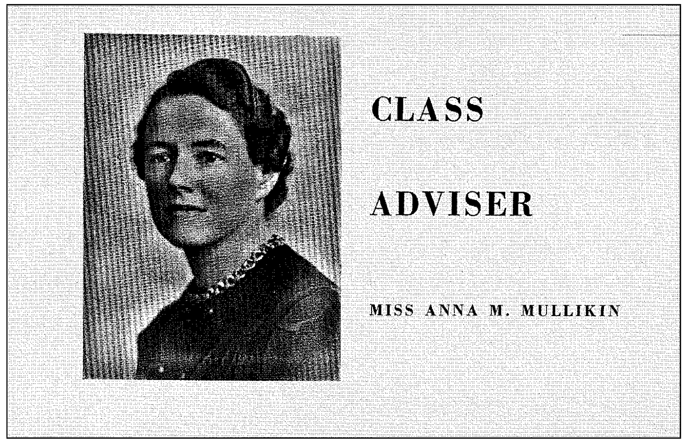
The use of connected sets abounds in F. A., starting with their definition in the section following the introduction and continuing throughout the paper. In one part Moore quoted Lennes's definition of an arc and defined a *domain* as a connected set of points M such that if P is a point of M, then there exists a region that contains P and lies in M. These definitions form the basis for Moore's intricate proof of a major theorem [19, p. 136]:

> If A and B are distinct points of a domain M, there exists an arc from A to B that lies wholly in M.

Although the proofs of almost all 52 theorems in F. A. involve connected sets, such sets were not the primary object of study. Rather, connectedness remained a tool for characterizing other kinds of sets.

The AMS role would repeat itself at an October 1916 meeting where both Moore and Kline found Lennes's formulation of connected sets to be highly profitable. Moore's paper was aimed at proving the property that any two points on a continuous curve C in any number of dimensions form the extremities of an arc lying entirely in C. Once again he found it necessary to supply the definition of a connected set beforehand [20, p. 233]. This would not mark Moore's swan song with connected sets, but his student John Robert Kline (1891–1955) became more active in this regard over the next few years. J. R. Kline had come under R. L. Moore's spell shortly after entering Penn in the fall of 1913 and, as we have seen, presented the results of his dissertation at an AMS meeting in his second year of graduate study. He remained at Penn for two years after receiving his Ph.D. in June 1916. At the AMS meeting that October he proved the converse of the following theorem on open curves that Moore had proved in F. A. [21, p. 178]:

> If ℓ is an open curve in a universal set S, then $S - \ell = S_1 \cup S_2$, where S_1 and S_2 are connected sets such that every arc from a point of S_1 to a point of S_2 contains at least one point of ℓ.



Anna Mullikin in the 1941 yearbook from Germantown High School.

As an indication of the extent to which connected sets had become a primary tool, Kline presented another paper at the annual AMS meeting held in New York just two months later. Its aim was to generalize a result from Hausdorff's classic *Grundzüge der Mengenlehre* that the complement of a countable set in n-dimensional space is always connected. Kline, like R. L. Moore before him, still felt obliged to state Lennes's definition of a connected set, though without attribution this time. He did not find it necessary, however, to state that Lennes's formulation was equivalent to the one Hausdorff supplied in his 1914 book. Apparently Hausdorff had come upon his definition in ignorance of Lennes.

J. R. Kline taught at Yale from 1918–19 and Illinois from 1919–20 before returning to Penn for the rest of his life. Before leaving Penn in 1918 he wrote a joint paper with R. L. Moore, the only one that Moore ever co-authored, providing necessary and sufficient conditions for a closed and bounded set M to be a subset of an arc in terms of closed, connected subsets of M [22]. Thus the central structures of interest remained arcs and open curves; connected sets were relegated to auxiliary status.

Two other former students of E. H. Moore also made use of connected sets at this time. Arthur Dunn Pitcher (1880–1923) arrived in Chicago in 1907 right after Lennes had left. Pitcher received his Ph.D. three years later and then embarked on a research program in Moore's general analysis, resulting in the paper "Biextremal connected sets", read at the same annual AMS meeting as Kline in December 1916. In spite of the title, connected sets were still used only as a tool. Actually, Pitcher had worked with Edward Wilson Chittenden (1885–1977), a 1912 Chicago Ph.D., several years earlier when the latter read their joint paper at a Chicago Section meeting in March 1913. Their joint work would ultimately be published in two papers in the *Transactions*, with [23] containing results announced at meetings in 1913 and 1916. Incidentally, A. D. Pitcher is the father of longtime AMS secretary (1967–88) Arthur Everett Pitcher (1912–2006).

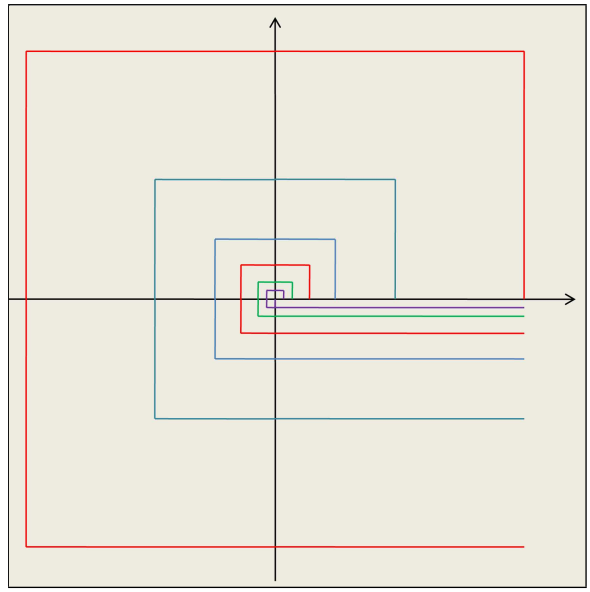
R. L. Moore must have been a very proud *Doktorvater* at the April 1918 AMS meeting in New York. Even though he did not present a paper, his former student Kline delivered one characterizing simple curves in terms of connected domains, and his second student, G. H. Hallett, announced theorems that appeared in his dissertation for the Ph.D. he earned two months later. George H. Hallett Jr. (1895–1985), like Kline, wrote his thesis in geometry inspired by Moore's early mentor G. B. Halsted. Hallett would ultimately pursue a career in government, but before embarking on that path he too initiated a research program in topology, resulting in a paper whose sole purpose was to prove that the boundedness assumption in Lennes's characterization of an arc was superfluous. When this result appeared in print, Hallett added, "Since I wrote this paper it has been pointed out to me by Professor R. L. Moore that a modification of [his argument in F. A.] …would accomplish the same result" [24, p. 325]. This paper was cited as late as 1927 by one of R. L. Moore's initial successes at the University of Texas, Gordon Thomas Whyburn (1904–69), thus suggesting that Hallett could have become a successful research mathematician should he have chosen to remain in the field [25].

Before leaving Philadelphia, R. L. Moore mentored a third doctoral student, Anna Margaret Mullikin (1893–1975), who enrolled in the fall of 1918, right after George Hallett graduated. It is not known what motivated Mullikin to enroll at Penn but we offer a connection to Clara Bacon, who attended the October 1911 meeting where Lennes spoke about connected sets. Bacon was a long-time professor of mathematics at Goucher College, including Mullikin's undergraduate years, 1911–15. Earlier she had pursued graduate studies during summer sessions at the University of Chicago (1901–04), when she likely came in contact with fellow graduate students Lennes, Veblen, and R. L. Moore. Moreover, Bacon kept abreast of research developments, becoming the first woman to receive a Ph.D. in mathematics at Johns Hopkins in 1911. Thus it is conceivable that she knew about the success of R. L. Moore at Penn and the achievements of his students Kline and Hallett. Unfortunately we have no firm evidence to support this contention.[1]

Miss Mullikin, as she came to be known, progressed quickly under R. L. Moore's special tutelage during her first year in his class. A theorem that had been only recently proved by W. Sierpiński read [26]:

> A closed, bounded, connected set M in $\mathfrak{R}^n$ cannot be expressed as a countable union of disjoint closed sets.

Moore, well known for meticulous examinations of axioms, challenged Miss Mullikin to discover what would happen if any of the conditions in this theorem were relaxed. By the following October, at the start of only her second year in graduate school, she was prepared to announce her discovery at an AMS meeting in New York [27]. The published account states, "It will be shown in the present paper that for the case where $n = 2$, this theorem does not remain true if the stipulation that M is closed be removed" [28, p. 144]. In summarizing her paper, F. N. Cole reported, "In one dimension no countably infinite collection of mutually exclusive closed point sets ever has a connected sum [union]. One might rather naturally be inclined to believe that this proposition holds true also in two dimensions. Miss Mullikin shows by an example that this is, however, not the case" [29, p. 147].



Mullikin nautilus.

The example, now called the Mullikin nautilus, is shown in the figure. The nautilus M is the union of a countably infinite collection of arcs M_n, $n = 1, 2, \ldots$, each composed of four line segments running from $(\frac{1}{2^n}, 0)$ to $(\frac{1}{2^n}, \frac{1}{2^n})$ to $(\frac{-1}{2^n}, \frac{1}{2^n})$ to $(\frac{-1}{2^n}, \frac{-1}{2^n})$ to $(1, \frac{-1}{2^n})$. The fact that M is connected is not obvious in terms of the traditional geometric approach of connecting any two points of M while remaining within M. Yet it can be proved easily with the Lennes definition, as follows. Assume M is the union of two separated sets A and B. Notice that the restrictions A_n and B_n of A and B to each arc M_n separate M_n. Since M_n is connected, either $A_n = M_n$ or $B_n = M_n$. Thus A and B are collections of arcs, one of which must be infinite. It is then

[1] *We are indebted to Thomas L. Bartlow for suggesting this possible link between Mullikin and R. L. Moore.*

easy to see that A and B must have a limit point in common, contradicting the assumption that they are separated. This proves that the Mullikin nautilus M is connected, and hence serves as an example that not every connected set is arcwise connected. Importantly, it illustrates the need for the shift from the earlier constructive, geometric definition.

Moore arranged an instructorship for Miss Mullikin at Texas for 1920–21 so she could complete her thesis there, and during that year two of her papers were read at AMS meetings in New York, both dealing with connected sets. The nautilus in [27] and theorems announced in [30] and [31] formed the basis for her Ph.D. dissertation, titled "Certain theorems relating to plane connected point sets", which was published in the *Transactions* in 1922 [28]. Miss Mullikin thus became the first American to study properties of connected sets in their own right. Her work was scrutinized in a recent paper that discusses its fifty-year mathematical legacy and details its role in subsequent collaborations between the emerging schools of topology in Poland and the U.S. [32]. Overall, then, all three R. L. Moore doctoral students at Penn contributed to the development of connected sets.

Beyond the AMS

Miss Mullikin was not the first person in the world to publish a paper in which connected sets were the primary object of study. In reviewing the history of this topological concept, the famed mathematician/historian R. L. Wilder (1896–1982) observed that "the first paper devoted to the study of connected sets was not published until 1921; we refer here to the classic paper *Sur les ensembles connexes* of B. Knaster and C. Kuratowski [33]" [16, p. 724]. In a strict sense, Wilder is absolutely correct about this fifty-page survey. Nonetheless, *Fundamenta Mathematicae*, the prestigious Polish journal where [33] appeared, included four earlier papers that established various properties of connected sets but received no mention in [16]. We cite them briefly. The second paper ever published in *Fundamenta*, written by Wacław Sierpiński, explored properties of connected sets that contain no subsets that are continua. The other three predecessors of [33] appeared the following year, 1921. In the first, Sierpiński provided proofs of several properties of connected sets; for instance, the complement of a connected set in R^n containing no subsets that are continua is always connected. *Fundamenta* also included two short notes by Stefan Mazurkiewicz solving problems posed by Sierpiński, one of which established the existence of a connected set in the plane having no connected and bounded subsets, while the other introduced the notion of a quasi-connected set. The latter note was followed immediately by [33]. In addition to these four papers, the short note by Sierpiński in 1918 that motivated the Mullikin nautilus seems also to have escaped Wilder's attention.

It was [33] together with [28] that elevated connected sets from secondary tools to primary objects. As we have seen, a twenty-year gestation period in three phases preceded this elevation, beginning with the proof of the Jordan Curve Theorem, centering on the published definition by Lennes in 1911, and continuing with the work of several graduates of E. H. Moore at Chicago and R. L. Moore at Penn. Connected sets did not remain the exclusive dominion of these Moore schools in the U.S., but they certainly accounted for the most lasting contributions. And at the center of this flurry of activity was the increasingly influential AMS with its meetings in New York and Chicago and its two journals, the *Bulletin* and the *Transactions*.

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