Invariant Theory of Tensor Product Decompositions of $U(N)$ and Generalized Casimir Operators

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That symmetries, invariance, and conservation laws are related has long been known. For example, Einstein exploited the relationship between the symmetry of Newton's equations and the (relativistic) symmetry of Maxwell's equations to develop a relativistic classical mechanics. But it is with quantum physics that the full power of symmetry, as expressed in the invariance of quantities under group operations, comes to the fore. This is particularly clear in the connection between the representation spaces of groups and the Hilbert spaces used in quantum mechanics on which the representations act. The first group symmetries of importance in quantum mechanics were those related to space-time symmetries; later, internal symmetries were introduced, and finally these symmetries were generalized to gauge symmetries. One of the main points of this paper is to show how the unitary groups have played a key role in all these different applications to quantum physics.

Historically one of the first applications of the unitary groups to quantum physics was with the group $SU(2)$. Rotations in physical three-space can be generated from elements of $SU(2)$ via the Cayley-Klein transformations. In particular the fundamental two-dimensional representation of $SU(2)$ leads to a description of spin 1/2 objects such as electrons and protons. The Lie algebra of $SU(2)$ leads to the angular momentum commutation relations, and shows that both orbital and spin angular momentum is necessarily quantized, in contrast to classical mechanics. Further, when the theory of quantized angular momentum is applied to many-body systems, for example the many electrons in an atom, or the many nucleons (= protons and neutrons) in a nucleus, the relevant spin spaces are tensor products of irreducible representation spaces. If the Hamiltonian, the operator governing the time evolution of the many-body quantum system, commutes with the angular momentum operators (or, is invariant under $SU(2)$), the overall angular momentum of the system is conserved; it is then useful to choose a basis in the many-body representation space which is diagonal in the overall angular momentum. Basis dependent coefficients (called Clebsch-Gordan, or vector coupling, or Wigner coefficients), which transform between a tensor product basis and a direct sum basis, play a key role in the structure of many-body spin quantum systems; see for example reference [1] .

One of the important ingredients in the analysis of spin in quantum systems concerns the notion of multiplicity, wherein an irreducible representation (irrep) occurs more than once in the decomposition of tensor products of single particle systems. $SU(2)$ is unusual among the $U(N)$ groups in that in the two-fold tensor product decomposition an irrep appears at most once. For n-fold tensor products, with $n \geq 3$ multiplicity does appear, and a fundamental issue is how to deal with the repeated appearance of the same irrep. For $SU(2)$ this issue is dealt with by intermediate coupling labels; thus, if system 1 is coupled (tensored) to system 2, 1-2 to 3 and so forth, the multiplicity can be fixed by the value of the intermediate (1-2) value of the angular momentum. Such a solution only works when there is no multiplicity for two-fold

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products. One of the goals of this paper is to show that eigenvalues of generalized Casimir operators can be used to resolve the multiplicity problem, for arbitrary tensor products.

Though the multiplicity problem can be dealt with in a systematic fashion for $SU(2)$ by stepwise coupling, it is clear that there are many different stepwise schemes. For three-fold tensor products electron 1 could be coupled first to electron 3 and then 1-3 coupled to 2. The multiplicity is then resolved by the 1-3 angular momentum rather than the 1-2 angular momentum. Coefficients that transform between different coupling schemes are called Racah (or recoupling or 3j, 6j...) coefficients and also play an important role in the quantum theory of angular momentum for many-body systems. Of particular importance is that Racah coefficients are basis independent; there are tables and computer programs for calculating these coefficients, which are usually obtained by summing over Clebsch-Gordan coefficients [1]. We will show how to compute such coefficients for the general $U(N)$ groups using a procedure not tied to knowledge of the basis dependent Clebsch-Gordan coefficients.

All of the structural features that have been discussed with respect to the group $SU(2)$ generalize to the other unitary groups. Our goal has been to find ways to compute the various coefficients that arise for general representations of the unitary groups, and in particular to generate computer programs that implement these operations. As is discussed in the following paragraphs, the $U(N)$ groups and their representations play an important role in various subfields of physics, going well beyond angular momentum and $SU(2)$.

The first application of the unitary groups not related to angular momentum occurred in the early 1930s when Heisenberg applied the known structure of $SU(2)$ to the strong nuclear force. Early in the development of nuclear physics it was realized that the proton and (at the time the newly discovered) neutron behaved similarly with respect to the strong nuclear force. Where they differed was with respect to the electromagnetic force. For example, the neutron is uncharged whereas the proton is charged. Ignoring the weaker electromagnetic force and focusing on the strong nuclear force, Heisenberg introduced a two-dimensional complex space, the spin $1/2$ space consisting of two basis elements, the proton and neutron. In this case the symmetry is called isospin (or isotopic spin) and has nothing to do with spin angular momentum discussed in previous paragraphs. Only the group is the same, namely $SU(2)$; physical three-space is replaced by an "internal symmetry" space. Thus, to the extent it is possible to isolate the strong nuclear force from the other forces of nature, all strongly interacting particles fit into irreducible representations of isospin $SU(2)$. Well-known examples are the three pi mesons, π^+, π^0, π^- which all have spin 0 (the superscripts refer to the charge) and four Δ resonances, all of spin $3/2$, which sit in the four-dimensional representation of isospin $SU(2)$.

Once it is realized that the strong nuclear force is to a very good approximation invariant under isospin $SU(2)$, it is possible to use the known machinery from spin $SU(2)$ to analyze multiparticle nuclear systems. Nuclei are bound states of nucleons, and they carry an isospin quantum number; for example the alpha particle, the nucleus of the helium atom, consists of two protons and two neutrons and has isospin zero. When solving bound state problems the invariance of the nuclear Hamiltonian under isospin is exploited in calculating the bound state energies and wavefunctions. Moreover, since nucleons also interact electromagnetically, isospin invariance is broken. But it is broken in a systematic fashion, so that the electromagnetic part of the Hamiltonian may transform as a component of a tensor operator under isospin transformations. In such a situation matrix elements of physical interest are related to Clebsch-Gordan coefficients.

Wigner was one of the first to make use of higher dimensional unitary groups. In the so-called supermultiplet theory [2], spin $SU(2)$ is embedded with isospin $SU(2)$ into a larger $SU(4)$ group. Bases for representation spaces are now indexed by isospin times spin multiplets and the strong nuclear Hamiltonian is supposed to have well-defined transformation properties under the larger group action. While such a supermultiplet model has not been particularly useful, as discussed in the following paragraphs, its generalization to particle physics has been quite successful.

The origins of such a generalization go back to the 1960s when a symmetry now called flavor $SU(3)$ was introduced. In interactions involving pi mesons colliding with protons, new (or so-called strange) particles were observed, and to account for their production and decay properties, a new quantum number, called variously strangeness or hypercharge, was introduced. Combining the known isospin invariance with strangeness necessitated analyzing rank two compact groups. After investigating the representation structure of various rank two compact groups, it was seen that the group $SU(3)$ was best able to accommodate the newly discovered strange particles. The two most important representations turned out to be the eight- and ten-dimensional representations. Actually the group of interest is $U(3)$, but the $U(1)$ subgroup, corresponding to the exact conservation of baryon number, is factored off. Then, for baryon number one, the eight-dimensional representation contains the proton and neutron, and six other newly discovered strange particles. For baryon number zero the eight-dimensional multiplet is a meson multiplet which includes the three previously known pi mesons, along with five other strange mesons, including the four K mesons. The ten-dimensional representation for baryon number one contains the four Δ particles, along with six other strange particles. In fact at the time the flavor $SU(3)$ model was being developed, the last particle in the ten-dimensional representation had not been discovered experimentally. Gell-Mann used tensor transformation properties of the mass operator under $SU(3)$ transformations to predict the mass and other properties of the unknown particle. The discovery of the Ω^- was part of the reason the Nobel Prize was awarded to Gell-Mann in 1969.

A further development of flavor $SU(3)$ deals with scattering, which in turn deals with tensor products of, in particular, the eight-dimensional representation with itself. It is well-known that the decomposition of two eight-dimensional representations involves multiplicity, in which the eight-dimensional representation occurs twice in the decomposition. The way in which the multiplicity was broken in the original applications was to make use of the fact that there is an underlying permutation symmetry on two numbers, the representations of which are either symmetric or antisymmetric.

In the 1960s when the flavor $SU(3)$ model was being developed, it was known that unlike the electron, which seems to be a fundamental particle, not made out of more fundamental particles, the same was not the case for the proton and neutron. The reason for believing that the nucleons were not fundamental came from a variety of experiments, including the anomalous magnetic moments of the proton and neutron, the existence of excited states of the proton and neutron, and electron scattering experiments on the proton. A way to account for the nonfundamental character of the nucleons (and their associated strange counterparts) was to postulate the existence of quarks, entities that transformed under the three-dimensional (fundamental) representation of flavor $SU(3)$ (and correspondingly, antiquarks transforming under the complex conjugate representation, inequivalent to the fundamental representation). The three-fold tensor product of the fundamental representation with itself gave the eight- (with multiplicity 2) and ten-dimensional representations, along with a one-dimensional representation. That is, the physically observed particles occupying the eight- and ten-dimensional representations were thought to be "made out of" the fundamental (quark) representations [3]. Similarly the mesons in the eight-dimensional representation, with baryon number zero, were thought of as a two-fold tensor product of the fundamental with the conjugate representation. Or put differently, mesons were thought to be bound states of quark-antiquark pairs.

The flavor $SU(3)$ model was developed in a number of different ways, but for the purposes of this survey, it suffices to note that the Wigner supermultiplet theory was generalized to a supermultiplet $SU(6)$ model, in which spin $SU(2)$ times flavor $SU(3)$ is embedded in $SU(6)$ [4]. The main point to note is that while the particles in the various $SU(3)$ representations do not all have the same mass (which would be the case if flavor $SU(3)$ were an exact symmetry) the spin (and parity) of all the particles in a flavor irrep are the same. Therefore it makes sense to combine spin and flavor into a larger group. The irreps of $SU(6)$ are used to select out those multiplets with the correct spin-flavor structure. In particular the fifty-six-dimensional representation of $SU(6)$ contains the eight-dimensional representation of nucleons (since the spin is $1/2$, this gives sixteen dimensions) along with the ten-dimensional resonances (the spins of which are $3/2$, so that 4x10 = 40), for a total of fifty-six dimensions. A number of physicists continue to develop the $SU(6)$ model, in conjunction with a nonrelativistic or relativistic quantum mechanics which incorporates the spatial parts of the quark wavefunctions.

To conclude this brief overview of applications of the $U(N)$ groups to quantum physics, we consider their use in quantum field theory. The starting point is fields defined over Minkowski space, which can be thought of as the manifold $\mathcal{P}/SO(1,3)$, the Poincaré group modulo the Lorentz group. Along with transformations under the Poincaré group, the fields also carry indices which transform under compact internal symmetry groups. Gauge groups are map groups from Minkowski space to the internal symmetry group, and interactions are generated from the requirement that the field theory be invariant under gauge transformations. The two most important internal symmetry groups in this context are color $SU(3)$, which generates the quantum field theory for the strong nuclear force (the quantum field theory so generated is called quantum-chromodynamics, QCD) and $SU(2)$ x $U(1)$, the internal symmetry group for the electroweak interactions; see for example reference [5]. It is interesting to note that flavor $SU(3)$ also appears as an internal symmetry in QCD, however not as a gauge symmetry. Finally it should be pointed out that attempts have been made to unify the strong and electroweak interactions, using among other possibilities the group $SU(5)$.

After this brief introduction to applications of the $U(N)$ groups in quantum physics, we can state mathematically the problems we wish to investigate. Let G denote the unitary group $U(N)$ and $V^{(m)}$ a unitary irreducible G-module of signature (m). Form the r-fold tensor product $V^{(m)_1} \otimes \ldots \otimes V^{(m)_r}$ and give an explicit decomposition of this tensor

product G-module. This involves the following steps:
1. Give concrete realizations of $V^{(m)}$ and $V^{(m)_1} \otimes ... \otimes V^{(m)_r}$ as subspaces of a common Hilbert space on which G acts unitarily.
2. Give a computationally effective formula to calculate the multiplicity $\mu(M) = \mu((M); (m)_1 \otimes ... \otimes (m)_r)$ of the equivalence class of irreps of signature (M) in the orthogonal direct sum decomposition of the tensor product representation.
3. Construct intertwining operators that map the G-modules $V^{(M)}$ into the G-module $V^{(m)_1} \otimes ... \otimes V^{(m)_r}$ for all (M) that occur in the orthogonal direct sum decomposition of the tensor product G-module.
Most importantly, realize the steps above in the most general and "canonical" way; that is the method should work for all signatures (m), all ranks N, and arbitrary r-fold tensor products.

Here it should be noted that the analysis of tensor product decompositions of compact groups, especially the $U(N)$ groups, has a long history, and a number of different methods have been deployed to deal with the problem; expressions for multiplicities in two-fold tensor products and Clebsch-Gordan and Racah coefficients of $U(N)$ are investigated in great detail in reference [6] and references cited therein.

U(N) Representation Theory

Let $C^{n\times N}$ denote the vector space of all $n \times N$ complex matrices. If $Z = (Z_{ij})$ is an element of $C^{n\times N}$, let Z^* denote its complex conjugate and write $Z_{ij} = X_{ij} + \sqrt{-1}\, Y_{ij}$; $1 \le i \le n$, $1 \le j \le N$. If dX_{ij} (resp. dY_{ij}) denotes Lebesgue measure on $\mathcal{R}$, we let dZ denote the Lebesgue product measure on $\mathcal{R}^{2nN}$. Define a Gaussian measure $d\mu$ on $C^{n\times N}$ by

$$(1) \qquad d\mu(Z) = \pi^{-nN} \exp[-tr(ZZ^\dagger)]dZ,$$

where tr denote the trace of a matrix and $Z^\dagger$ is the transpose of Z^*.

A function $f : C^{n\times N} \to C$ is holomorphic square integrable if it is holomorphic on the entire domain $C^{n\times N}$, and if

$$(2) \qquad \int_{C^{n\times N}} |f(Z)|^2 d\mu(Z) < \infty.$$

Clearly the holomorphic square-integrable functions form a Hilbert space, the Bargmann-Segal-Fock space, with respect to the inner product

$$(3) \qquad < f_1|f_2 >= \int_{C^{n\times N}} \overline{f_1(Z)} f_2(Z) d\mu(Z).$$

Let $\mathcal{F} \equiv \mathcal{F}(C^{n\times N})$ denote this Hilbert space. From [7] this inner product also can be defined by the following formula:

$$(4) \qquad < f_1|f_2 >= f_1^*(D) f_2(Z)|_{Z=0}.$$

Thus if $f \in \mathcal{F}(C^{n\times N})$, then $f(Z) = \sum_{|(\alpha)|=0}^{\infty} C_{(\alpha)} Z^{(\alpha)}$, where $(\alpha) = (\alpha_{11}, \dots, \alpha_{nN})$, is an $n \times N$-tuple of integers ≥ 0, $|(\alpha)| = \alpha_{11} + \cdots + \alpha_{nN}$, $C_{(\alpha)} \in C$, and $Z^{(\alpha)} = Z_{11}^{\alpha_{11}} \dots Z_{nN}^{\alpha_{nN}}$. Moreover, $C_{(\alpha)}$ must satisfy $\sum_{|(\alpha)|=0}^{\infty} (\alpha)! |C_{(\alpha)}|^2 < \infty$, where $(\alpha)! = \alpha_{11}! \dots \alpha_{nN}!$. For $f \in \mathcal{F}(C^{n\times N})$ define f^* by

$$(5) \qquad f^*(Z) = \sum_{|(\alpha)|=0}^{\infty} C^*_{(\alpha)} Z^{(\alpha)}.$$

Then $f^*(D)$ is the differential operator obtained by formally replacing $Z_{\gamma j}$ by the partial derivative $\partial/\partial z_{\gamma j}$ $(1 \le \gamma \le n, 1 \le j \le N)$. If $f \in \mathcal{F}(C^{n\times N})$, then obviously $(f^*)^* = f$ and $f^* \in \mathcal{F}(C^{n\times N})$. Moreover, for all $f_1, f_2 \in \mathcal{F}(C^{n\times N})$

$$(6) \qquad < f_1^*, f_2^* > = f_1(D) f_2^*(Z)|_{Z=0} = \sum_{|(\alpha)|=0}^{\infty} (\alpha)! C^1_{(\alpha)} \overline{C}^2_{(\alpha)} = < \overline{f_1, f_2} > = < f_2, f_1 > .$$

Therefore, $||f^*|| = ||f||$ for all $f \in \mathcal{F}(C^{n\times N})$. If $\mathcal{P}(C^{n\times N})$ denotes the subspace of $\mathcal{F}(C^{n\times N})$ of all polynomial functions in Z, then $\mathcal{P}(C^{n\times N})$ is dense in $\mathcal{F}(C^{n\times N})$. It is straightforward to show that the representation R of $U(N)$ on $\mathcal{F}$ defined by

$$(7) \qquad (R(g)f)(Z) = f(Zg)\ , \quad g \in U(N)$$

is unitary.

Irreducible representations of $GL(N, C)$ are realized on subspaces of $\mathcal{F}$ defined by

$$(8) \qquad V^{(M)} := \{f \in \mathcal{F}(C^{n\times N})\ ,\ f(bZ) = \pi^{(M)}(b) f(Z)\}$$

where $b \in B_n$, the subgroup of $GL(n, C)$ of lower triangular matrices, and $\pi^{(M)}(b) \in C$ is a representation of B_n defined by

$$(9) \qquad \pi^{(M)}(b) := d_1^{(M_1)} \cdots d_n^{M_n}$$

where $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ is an element of the diagonal subgroup of B_n and (M) is an n-tuple of integers, $M_1, \dots, M_n$ satisfying the dominant condition, $M_1 \ge \cdots \ge M_n$, $n \le N$. We restrict ourselves to the case when the integers are nonnegative. In general $V^{(M)}$ can be realized as a subspace of $\mathcal{F}$ with an additional condition ([8]). Then the Borel-Weil theorem implies that the representation of $GL(N, C)$ obtained by right translation on $V^{(M)}$ is irreducible with signature (highest weight) (M). It follows from Weyl's "unitarian trick" that the restriction to $U(N)$ remains irreducible.

Irreducible representations of $GL(N, C)$ can also be realized on $\mathcal{F}(C^{N\times N})$ as follows: Let $W^{(M)} := \{\phi \in \mathcal{F}(C^{N\times N}) | \phi(wb^T) = \pi^{(M)}(b)\phi(w)\}$, where $b \in B_N$ and $w \in C^{N\times N}$, and define the representation L of $GL(N, C)$ on $W^{(M)}$ by $(L(g)\phi)(w) = \phi(g^T w)$. Then the map $\Phi : W^{(M)} \to V^{(M)}$ defined by $(\Phi\phi)(Z) := \phi(Z^T)$ is a $GL(N, C)$ module isomorphism. Hence the $U(N)$-modules $V^{(M)}$ and $W^{(M)}$ are unitarily equivalent and have the same highest weight vector. In reference [9], Theorem 3, section 5.6, Zelobenko gives an orthogonal direct sum decomposition of $\mathcal{F}(C^{n\times N})$ into irreducible $GL(n, C) \times GL(N, C)$-modules of signature (M, M), where $(M) = (M_1, ..., M_k, 0, ..., 0), k = min(n, N)$.

The r-fold tensor products of irreps of $U(N)$ are also subspaces of an appropriate $\mathcal{F}$; define

$$\mathcal{H}^{(m)} = V^{(m_1)} \otimes \cdots \otimes V^{(m_r)}, \tag{10}$$

the subspace of $\mathcal{F}(C^{p\times N})$, as

$$\begin{aligned} \mathcal{H}^{(m)} &= \{f \in \mathcal{F}(C^{p\times N}) | \\ f(\beta Z) &= \pi^{(m)}(\beta) f(Z) \\ &= \pi^{(m_1)}(b_1)\cdots \pi^{(m_r)}(b_r) f(Z)\}, \end{aligned} \tag{11}$$

where $p = \sum_{i=1}^{r} p_i$ and β is an element of the product Borel group,

$$\beta = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_r \end{pmatrix} \tag{12}$$

with $b_i \in B_i$, the $p_i \times p_i$ lower triangular matrix. It follows that the outer product group $U(N) \times \cdots \times U(N)$, consisting of elements $(g_1, \ldots, g_r)$, $g_i \in U(N)$ is irreducible on $\mathcal{H}^{(m)}$, with irrep

$$(R_{(g_1 \ldots g_r)} f) \begin{pmatrix} Z_1 \\ \vdots \\ Z_r \end{pmatrix} = f \begin{pmatrix} Z_1 g_1 \\ \vdots \\ Z_r g_r \end{pmatrix}, \quad f \in \mathcal{H}^{(m)} \tag{13}$$

$(m) := (m_{11} \ldots m_{1p_1}, m_{21} \ldots m_{2p_2}, \ldots, m_{rp_r})$, that is, all the zeros in (m_i) have been deleted.

If the elements of $U(N) \times \cdots \times U(N)$ are restricted to the diagonal subgroup of all elements $(g, g, \ldots, g), (g \in U(N))$ which is identified with $U(N)$, the representation $R_{(g,g,\ldots,g)}$ of $U(N)$ on $\mathcal{H}^{(m)}$ becomes reducible and decomposes into a direct sum of irreducible representations of $U(N)$, with multiplicity $\mu(M)$:

$$\mathcal{H}^{(m)} = \sum_{(M)} \oplus \mu(M) V^{(M)}. \tag{14}$$

Rather than decomposing $\mathcal{H}^{(m)}$ directly, the strategy will be to adjoin the contragredient representation of (M), denoted by $(M)^{\surd}$ to $\mathcal{H}^{(m)}$ and find the invariant subspace of $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$, that is, the space of identity representations of $U(N)$. This is possible since the multiplicity $\mu(M)$ is equal to the dimension of the $U(N)$-invariant subspace of $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$. (See [10].) References [11] and [14] show that the contragredient representation—defined with respect to linear functionals of the representation space $V^{(M)}$—can be written in the following way; consider the irrep space defined in Eq.(8) and set

$$\begin{aligned} (R^{\surd}(g) f)(Z) &= f(Z g^{\surd}), \\ f \in V^{(M)}, g \in GL(N,C), & \; g^{\surd} := (g^{-1})^T. \end{aligned} \tag{15}$$

Then $R^{\surd}(g)$ is equivalent to the contragredient representation.

Now let $\underbrace{GL(N,C) \times \cdots \times GL(N,C)}_{r} \times GL(N,C)$ act on $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ via the outer tensor product. If the signature (M) is $\underbrace{(M_1, \ldots, M_q, 0 \ldots, 0)}_{N}$, set

$$n = p + q, Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \in C^{p\times N}$$

and $W \in C^{q\times N}$, then the inner (or Kronecker) tensor product representation of $GL(N,C)$ on $\mathcal{H}^{(m)} \otimes V^{(M)^{\surd}}$ can be defined as

$$\left[R^{(m)} \otimes R^{(M)^{\surd}}(g) f\right] \left(\begin{bmatrix} Z \\ W \end{bmatrix} \right) = f\left(\begin{bmatrix} Zg \\ W g^{\surd} \end{bmatrix} \right) \tag{16}$$

for all $f \in \mathcal{H}^{(m)} \otimes V^{(M)^{\surd}} \subset \mathcal{F}(C^{n\times N})$ and $g \in GL(N,C)$. Then the restriction of $R^{(m)\otimes(M)^{\surd}}$ to $U(N)$ is unitary.

In general, $GL(N,C)$ acts on $\mathcal{P}(C^{n\times N}) \subset \mathcal{F}(C^{n\times N})$ via the representation

$$[R(g) f] \left(\begin{bmatrix} Z \\ W \end{bmatrix} \right) = f\left(\begin{bmatrix} Zg \\ W g^{\surd} \end{bmatrix} \right), \quad \forall f \in \mathcal{P}(C^{n\times N}). \tag{17}$$

Then it follows from [12] that the ring of all polynomials in $\begin{bmatrix} Z \\ W \end{bmatrix}$ that are invariant under this action is generated by the constants and the pq algebraically independent polynomials $P_{a\alpha}$ defined by

$$P_{a\alpha}\left(\begin{bmatrix} Z \\ W \end{bmatrix} \right) = (ZW^T)_{a\alpha} = \sum_{i=1}^{N} Z_{ai} W_{\alpha i}, 1 \le a \le p, 1 \le \alpha \le q. \tag{18}$$

Set $X_{a\alpha} = P_{a\alpha}\left(\begin{bmatrix} Z \\ W \end{bmatrix} \right)$ and let X denote the $p \times q$ matrix with entries $X_{a\alpha}$. If $\mathcal{J}$ denotes the ring of all $GL(N,C)$-invariants, it follows that an element of $\mathcal{J}$ is a polynomial in the variable X, i.e., $f \in \mathcal{J}$ if and only if

$$f\left(\begin{bmatrix} Z \\ W \end{bmatrix} \right) = \phi_f(X), \quad X = ZW^T \tag{19}$$

for some polynomial $\phi_f \in \mathcal{P}(C^{p\times q})$. Note that by construction $q \le \min(p, N)$ ([13]), and by abuse of language if $(M) = \underbrace{(M_1, \ldots, M_q, 0, \ldots, 0)}_{p}$ let $(M)_p$ (or simply (M) if there is no possible confusion) denote the signature of the equivalent class of irreducible representations of $GL(p,C)$ with highest weight $\underbrace{(M_1, \ldots, M_q, 0 \ldots, 0)}_{p}$. Let $\mathbf{W}^{(M)_p}$ denote the vector space of all polynomial functions ϕ in X which also satisfy the covariant condition

$$\phi(Xb^T) = \pi^{(M)}(b)\phi(X), \quad \forall b \in B_q. \tag{20}$$

Define the representation $L^{(M)_p}$ of $GL(p,C)$ on $\mathcal{P}(C^{p\times q})$ by the equation

(21) $L^{(M)_p}(\gamma)\phi(X) = \phi(\gamma^T X)\,, \quad \gamma \in GL(p,C).$

Then the Borel-Weil theorem together with Weyl's "unitarian trick" imply that the representation $L^{(M)_p}$ is irreducible with signature $(M)_p$ and its restriction to $U(p)$ is an irreducible unitary representation of the same signature. The proof of the following theorem can be found in [14]:

Theorem 1. *If $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ denotes the subspace of all $GL(N,C)$-invariant polynomials in $\mathcal{H}^{(m)\otimes(M)^{\surd}}$, then every element f in $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ can be uniquely identified with an element ϕ_f in $\mathbf{W}^{(\mathbf{M})_\mathbf{p}}$ which also satisfies the covariant condition.*

$$L^{(M)_p}(\beta^T)\phi_f = \pi^{(m)}(\beta)\phi_f,$$

where β and $\pi^{(m)}(\beta)$ are defined by Eqs.(11) and (12). In other words the ϕ_f's constitute the subspace $(\mathbf{W}^{(\mathbf{M})_\mathbf{p}};\pi^{(\mathbf{m})})$ of $\mathbf{W}^{(\mathbf{M})_\mathbf{p}}$ of all highest weight vectors of the restriction

$$L^{(M)_p}|GL(p_1,C)\times\cdots\times GL(p_r,C).$$

Corollary 1. *Let $G = U(N)$ and let $(R^{(M)},\mathbf{V}^{(\mathbf{M})})$ denote the irreducible unitary G-module with signature $(M) = \underbrace{(M_1,\ldots,M_q,0,\ldots,0)}_{N}$. Then the multiplicity of $R^{(M)}$ in $\mathcal{H}^{(m)}$ is equal to the dimension of the subspace $(\mathbf{W}^{(\mathbf{M})_\mathbf{p}};\pi^{(\mathbf{m})})$ defined in the Theorem.*

Remark 1. *The conditions in Eq.(21) can be broken into two parts: if β is unipotent, then $L^{(M)_p}(\beta^T)\phi_f = \phi_f$, and if β is a diagonal matrix $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p\end{pmatrix}$, then $L^{(M)_p}(d)\phi_f = d_1^{m_1,p_1}\ldots d_p^{m_r,p_r}\phi_f$. This means that ϕ_f are weight vectors of $(\mathbf{W}^{(\mathbf{M})_\mathbf{p}})$. Now the Gelfand-Cetlin tableaux provide a set of labels that can be used to get the dimension of the subspace of $(\mathbf{W}^{(M)_p})$ with a definite weight. It follows that a bound on the dimension of $(\mathbf{W}^{(\mathbf{M})_\mathbf{p}};\pi^{(\mathbf{m})})$ is given by the number of Gelfand-Cetlin tableaux associated with irreducible representations of $GL(p,C)$ of signature $(M)_p$ and with weight (m). A special case occurs when $\mathcal{H}^{(m)}$ is an r-fold tensor product of "symmetric" representations (a representation of $GL(N,C)$ is called symmetric if its signature is of the form $\underbrace{(m,0,\ldots,0)}_{N}$, so-called because it is the space of symmetric tensors that occur in the m-fold tensor product of the vector representation $\underbrace{(1,0,\ldots,0)}_{N}$ in the Schur-Weyl duality theorem, see ([12], Th. 4AD)). In this special case $r = p$ and the elements β are reduced to the diagonal elements d. Thus we have also proven the following:*

Corollary 2. *If $\mathcal{H}^{(m)}$ is a p-fold tensor product of symmetric representations of $GL(N,C)$, then $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ admits an orthogonal basis $\{f_\xi\}$ where f_ξ corresponds to a Gelfand-Cetlin basis element φ_ξ of $\mathcal{P}(C^{p\times q})$, and ξ ranges over all Gelfand-Cetlin tableaux of $(M)_p$ with weight (m), i.e.,*

$$f_\xi\left(\begin{bmatrix} Z \\ W\end{bmatrix}\right) = \varphi_\xi(ZW^T).$$

To explicitly construct a basis of $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ we construct a basis of $(\mathbf{W}^{(M)_p};\pi^{(m)})$. For this let $\{L_{\alpha\gamma}\}$ denote the basis of the infinitesimal operators of the left representation of $GL(p,C)$ on $\mathcal{F}(C^{p\times q})$ given by $(L(h)f)(X) = f(h^T X)$. Then

(22) $$L_{\alpha\gamma} = \sum_{i=1}^{q} X_{\alpha i}\frac{\partial}{\partial X_{\gamma i}}\,,\quad 1\le\alpha,\gamma\le p$$

and the $L_{\alpha\gamma}$ generate a Lie algebra isomorphic to $\mathcal{G}l_p(C)$. Moreover $L^{\dagger}_{\alpha\gamma} = L_{\gamma\alpha}$, and the $L_{\alpha\gamma}$ with $\alpha<\gamma$ are raising operators while the $L_{\alpha\gamma}$ with $\alpha>\gamma$ are lowering operators.

If ϕ is a weight vector of $(\mathbf{W}^{(M)_p})$ of weight (m) then

(23)
$$L(d)\phi(X) = \phi(dX) = d_{11}^{m_1}\ldots d_{pp}^{m_p}\phi(X)\,,\ \forall d\in D_p.$$

It follows that

(24) $$L(d)(L_{\alpha\beta}\phi) = d_{11}^{m_1}\ldots d_{\alpha\alpha}^{m_\alpha+1}\ldots d_{\beta\beta}^{m_\beta-1}\phi,$$

that is, $L_{\alpha\beta}\varphi$ is also a weight vector of weight $(m_1,\ldots,m_\alpha+1,\ldots,m_\beta-1,\ldots,m_p)$ if $\alpha<\beta$ and $(m_1,\ldots,m_\beta-1,\ldots,m_\alpha+1,\ldots,m_p)$ if $\alpha>\beta$. And in our ordering of the weights this justifies the claim that $L_{\alpha\beta}$ is a lowering operator if $\alpha>\beta$ and is a raising operator if $\alpha<\beta$. Among these infinitesimal operators we have the particular operators $L_{\alpha_p\beta_p}$, where $p = p_1,\ldots,p_r$, which correspond to the infinitesimal operators of the $GL(p_i,C)$ subgroup actions, $1\le i\le r$. Thus the condition $L^{(M)_p}(\beta^T)\phi = \phi$, $\phi\in V^{(M)_p}$, β unipotent, is equivalent to the condition

(25) $L_{\alpha_p\beta_p}\phi = 0\,,\quad \forall\alpha_p<\beta_p\,,\quad p = p_1,\ldots,p_r.$

By exploiting the weight changing properties of the $L_{\alpha\beta}$ we construct a set of operators $\{\widetilde{\Phi}_\nu\}$, where ν ranges from 1 to the number of Gelfand-Cetlin tableaux associated with $(M)_p$ of weight (m). Each operator $\widetilde{\Phi}_\nu$ is a product of lowering operators $L_{\alpha\beta}$, $\alpha<\beta$. By applying $\widetilde{\Phi}_\nu$ to the highest weight vector $\phi^{(M)_p}_{\max}$ in $\mathbf{W}^{(M)_p}$, where

(26) $$\phi^{(M)}_{\max}(X) = \Delta_1(X)^{M_1-M_2}\cdots\Delta_q^{M_q}(X)$$

we send $\phi^{(M)}_{\max}$ into

(27) $$\mathcal{P}(C^{p\times q})^{(m)} = \{f\in P(C^{p\times q}: f(dX) = \pi^{(m)}(d)f(X),\quad \forall d\in D_p\}.$$

The systematic procedure for doing this, which can be implemented on a computer, makes use of the Gelfand-Cetlin tableaux for irreps $(M)_p$ and weight (m) of $U(p)$ (see [14] for details.)

We thus have constructed a linearly independent subspace of $\mathcal{P}(C^{p\times q})$. In order that elements of this subspace belong to $(\mathbf{W}^{(m)};\pi^{(m)})$, it must also satisfy the condition given in Eq.(27). This gives a set of basis elements of $(\mathbf{W}^{(m)};\pi^{(m)})$ as well as the multiplicity $\mu(M)$. And this also gives us a basis for $\mathcal{J}^{(m)\otimes(M)^{\surd}}$. The problem of constructing an orthogonal basis for $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ is considered in the next section.

Orthogonal Bases in $\mathcal{J}^{(m)\otimes(M)^{\surd}}$

In the previous section we have shown that the space of invariants $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ corresponds to the subspace $(\mathbf{W}^{(M)_p};\pi^{(m)})$ of the irreducible $U(p)$-module $W^{(M)_p}$. We also showed how to construct a (nonorthogonal) basis of $(\mathbf{W}^{(M)_p};\pi^{(m)})$, and hence of $\mathcal{J}^{(m)\otimes(M)^{\surd}}$, by exploiting properties of the Gelfand-Cetlin tableaux associated with the weight (m). The goal of this section is to generate orthogonal bases for $(\mathbf{W}^{(M)_p};\pi^{(m)})$, or equivalently for $\mathcal{J}^{(m)\otimes(M)^{\surd}}$ by introducing generalized Casimir operators whose eigenvalues can be used as labels of orthogonal basis vectors.

First, we make the following observation. According to our theory of dual representations (see [7], [15]), the spectral decompositions of the pairs $(U(p),U(q))$ on $\mathcal{F}(C^{p\times q})$ and $(U(p),U(N))$ on $\mathcal{F}(C^{p\times N})$ are identical if $p \geq N$; for $p < N$ there is a one-to-one correspondence between the isotypic components with signature $(M_1,\ldots,M_p)$ in $\mathcal{F}(C^{p\times p})$ and those with signature $\underbrace{(M_1,\ldots,M_p,0,\ldots,0)}_{N}$ in $\mathcal{F}(C^{p\times N})$. This observation applied to the pairs $(U(p),U(q))$ acting on $\mathcal{F}(C^{p\times q})$, $(U(p),U(N))$ acting on $\mathcal{F}(C^{p\times N})$ (recall that $q \leq min(p,N)$) implies that there is a correspondence between the dual modules $W^{(M)_p}\otimes V^{(M)_p}$, $W^{(M)_p}\otimes V^{(M)_q}$, and $W^{(M)_p}\otimes V^{(M)_N}$, which are the isotypic components with signature (M) in the corresponding Bargmann-Segal-Fock spaces. In particular, the highest weight vectors of the irreducible dual modules are identical if expressed in terms of the same dummy variable. It follows that the effect of the operators $\tilde{\Phi}_\nu$ on ϕ_{Max}, whether $\tilde{\Phi}_\nu$ are expressed in terms of the infinitesimal operators

$$L^q_{\alpha\beta} = \sum_{i=1}^{q} Z_{\alpha i}\frac{\partial}{\partial Z_{\beta i}} \quad \text{or} \quad L^N_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i}\frac{\partial}{\partial Z_{\beta i}}$$

is identical (in fact the global action $L(h)$, $h \in U(p)$, is always the same on $\mathcal{F}(C^{p\times q})$, $\mathcal{F}(C^{p\times p})$, or $\mathcal{F}(C^{p\times N})$). But the operators $\tilde{\Phi}_\nu$, if expressed in terms of the $L^N_{\alpha\beta}$, are exactly the linearly independent intertwining operators that map the $U(N)$ irreducible module $\mathbf{V}^{(M)}$ into the tensor product $\mathcal{H}^{(m)}(C^{n\times N})$. This is exactly the problem we considered in [7].

The procedure by which generalized Casimir operators are used to break the multiplicity is quite general. Let (G',G) and (H',H) be two pairs of dual (representation) modules acting on $\mathcal{F}(C^{n\times N})$ in such a way that G is a closed subgroup of H and H' is a closed subgroup of G'. Let $\mathcal{W}_{n\times N}$ denote the Weyl algebra of all differential operators with polynomial coefficients on $C^{n\times N}$. Let U_G, $U_{G'}$, U_H, and $U_{H'}$ denote the universal algebras of (the representations) of G, G', H, and H', respectively. Then all these algebras are subalgebras of $\mathcal{W}_{n\times N}$. If $\mathcal{Z}(\mathcal{U}_G;\mathcal{W}_{n\times N})$, $\mathcal{Z}(\mathcal{U}_{G'};\mathcal{W}_{n\times N})$, $\mathcal{Z}(\mathcal{U}_H;\mathcal{W}_{n\times N})$, and $\mathcal{Z}(\mathcal{U}_{H'};\mathcal{W}_{n\times N})$ denote the centralizers of $\mathcal{U}_G$, $\mathcal{U}_{G'}$, $\mathcal{U}_H$, and $\mathcal{U}_{H'}$ in $\mathcal{W}_{n\times N}$, then for many dual representations $\mathcal{Z}(\mathcal{U}_G;\mathcal{W}_{n\times N}) = \mathcal{U}_{G'}$, $\mathcal{Z}(\mathcal{U}_{G'};\mathcal{W}_{n\times N}) = \mathcal{U}_G$,$\mathcal{Z}(\mathcal{U}_H;\mathcal{W}_{n\times N}) = \mathcal{U}_{H'}$, and $\mathcal{Z}(\mathcal{U}_{H'};\mathcal{W}_{n\times N}) = \mathcal{U}_H$.

Definition 1. *Let ρ_H be a unitary representation of a Lie group H on a Hilbert space $\mathcal{H}$, let G be a closed subgroup of H. Let $\mathcal{U}_H$ (resp. $\mathcal{U}_G$) denote the universal enveloping algebra generated by the infinitesimal action of ρ_H (resp. $\rho_G = \rho_{H|G}$). An element $C \in \mathcal{U}_H$ that commutes with $\mathcal{U}_G$ is called a generalized Casimir operator for the pair (ρ_H,ρ_G) or (simply (H,G)).*

Such operators are useful not only for compact groups but also more general classes of groups, including semidirect product groups such as the Poincaré or Galilei groups, where it is known how to construct sets of generalized commuting operators whose eigenvalues label the invariant subspaces.

Theorem 2. *Under the assumption that (H',H) and (G',G) are two dual (representations) modules acting on $\mathcal{F}(C^{n\times N})$ such that G is a closed subgroup of H and H' is a closed subgroup of G', if $C_H(G)$ (resp. $C_{G'}(H')$) denotes the set of generalized Casimir operators for (H,G) (resp. (G',H')) then $C_H(G) = C_{G'}(H')$.*

Now if λ_i denotes an equivalence class of the irreducible representation of the group G on the space V^{λ_i}, $1 \leq i \leq n$, then $V^{\lambda_1}\otimes\cdots\otimes V^{\lambda_n}$ is an irreducible $\underbrace{G\times\cdots\times G}_{n} = H$-module. On the restriction to the diagonal subgroup which is identified with G, the Kronecker tensor product G-module $V^{\lambda_1}\otimes\cdots\otimes V^{\lambda_n}$ becomes reducible and in general multiplicity occurs. Generalized Casimir operators may then be used to break this multiplicity.

In the context of our problem let

$$\underbrace{U(N)\times\cdots\times U(N)}_{r},$$

or equivalently, $GL(N,C)\times\cdots\times GL(N,C) = H$ act on $\mathcal{H}^{(m)}$. Let $G = GL(N,C)$ and let $\mathcal{U}_H$ (resp.$\mathcal{U}_G$) denote the universal enveloping algebra of the infinitesimal action, then $\mathcal{U}_H = \mathcal{U}(\mathcal{G}\times\cdots\times\mathcal{G}) \cong \mathcal{U}(\mathcal{G})\otimes\cdots\otimes\mathcal{U}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra generated by the infinitesimal action of G on $\mathcal{H}^{(m)}$. The set of generalized Casimir operators $C_H(G)$ is generated by the differential operators of the form

$$tr[[R^{(p_1)}]^{d_1}\cdots[R^{(p_r)}]^{d_r}], \tag{28}$$

where the matrices $R^{(p_i)}$, $1 \le i \le r$, have (j,k) entry

$$R_{jk} = \sum_{\alpha=1}^{p} Z_{\alpha j} \frac{\partial}{\partial Z_{\alpha k}}, \quad 1 \le j,k, \le N; \tag{29}$$

the d_i are integers ≥ 0 (see [7] Prop. 3.3), and "tr" denotes the noncommutative trace operator. Moreover, as shown in [7], Prop. 3.5, these generalized Casimir operators are Hermitian.

To see how these generalized Casimir operators act on $\mathcal{J}^{(m)\otimes(M)^\surd}$, and also for computational purposes, it is more convenient to use the dual representation and the above Theorem to compute $C_H(G) = C'_G(H')$ in terms of the dual actions of H and G on $\mathcal{F}(C^{p\times N})$. The dual action of H on $\mathcal{F}(C^{p\times N})$ is defined by

$$L \begin{pmatrix} g_1' & & 0 \\ & \ddots & \\ 0 & & g_r' \end{pmatrix} f(X) = f(\begin{pmatrix} g_1'^T & & 0 \\ & \ddots & \\ 0 & & g_r'^T \end{pmatrix} X) \tag{30}$$

for all $g_i' \in GL(p_i C)$, $1 \le i \le r$, and for all $f \in \mathcal{F}(C^{p\times N})$. The dual action of G on $\mathcal{F}(C^{p\times N})$, $p = p_1 + \cdots + p_r$, is given by

$$[L(g')f](X) = f((g')^T X), \; g' \in GL(p, C) \tag{31}$$

and thus $H' = GL(p_1, C) \times \cdots \times GL(p_r, C)$. The Lie algebra of the infinitesimal action of G' is generated by the vector fields

$$L_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} \quad 1 \le \alpha, \beta \le p_j$$

and the universal enveloping algebra $\mathcal{U}_{G'}$ is particularly simple. If we write the matrix $[L] = (L_{\alpha\beta})$, $1 \le \alpha, \beta \le p_j$, in block form as

$$[L] = \begin{bmatrix} [L]_{11} & \dots & [L]_{1r} \\ \vdots & & \\ [L]_{r1} & \dots & [L]_{rr} \end{bmatrix}, \tag{32}$$

then, as was shown in [16], $C_{G'}(H')$ is generated by the generalized Casimir operators of the form

$$tr([L]_{u_1u_2}[L]_{u_2u_3} \dots [L]_{u_ku_1}), 1 \le u_j \le r, 1 \le j \le k. \tag{33}$$

The Hermitian operators formed from these generalized Casimir operators were used in [7] to break the multiplicity in the tensor product decomposition of $\mathcal{H}^{(m)}$. But as remarked earlier in this section, in the construction of a nonorthogonal basis $(\mathbf{W}^{(M)_p}; \pi^{(n)})$, this basis is obtained by applying the maps $\widetilde{\Phi}_\nu$ to $\phi_{\text{Max}}^{(M)}$ and then requiring that they satisfy condition (27). Further, as remarked earlier, $\widetilde{\Phi}_\nu$ can be expressed equivalently in terms of $L_{\alpha\beta}^q$ or $L_{\alpha\beta}^N$. And the condition (27) can be expressed as

$$L_{\alpha_p\beta_p}\varphi = 0, \quad \forall \alpha_p < \beta_p, \quad p = p_1, \dots, p_r \tag{34}$$

where

$$L_{\alpha_p\beta_p} = \sum_{i=1}^{N} Z_{\alpha_p i} \frac{\partial}{\partial Z_{\beta_p i}} \tag{35}$$

instead of $\sum_{i=1}^{q} Z_{\alpha_p i}\, \partial/\partial Z_{\beta_p i}$. But these are part of the infinitesimal operators of the action of H'. It follows that if Φ_μ are obtained from $\widetilde{\Phi}_\nu$ by applying condition (27), then for $C \in C_{G'}(H') = C_H(G)$, C commutes with Φ_μ. Indeed, $\widetilde{\Phi}_\nu$ maps $V^{(M)}$ into $\mathcal{P}^{(m)}$, and C commuting with H' implies that C commutes with Φ_μ. We summarize the results above in the following

Proposition 1. *The generalized Casimir operators given by Eq.(33) leaves the subspace* $(\mathbf{W}^{(\mathbf{M})_\mathbf{p}}; \pi^{(\mathbf{m})})$, *or equivalently,* $\mathcal{J}^{(m)\otimes(M)^\surd}$, *invariant.*

Assume now that a set of generalized commuting Hermitian operators $\{C_t\}$ has been chosen such that

$$C_t \Phi_\mu \phi_{\text{Max}}^{(M)} = \Phi_\mu C_t \phi_{\text{Max}}^{M}; \tag{36}$$

that is, each C_t leaves the space $(\mathbf{W}^{(M)}; \pi^{(m)})$ invariant. Since $\{C_t\}$ is a commuting set of Hermitian operators on $(\mathbf{W}^{(M)}; \pi^{(m)})$ they can be simultaneously diagonalized; call the eigenvalues η, then the set $\{\eta\}$ may be used to label an orthogonal basis of $(\mathbf{W}^{(M)}; \pi^{(m)})$, and hence of $\mathcal{J}^{(m)\otimes(M)^\surd}$. An example will be given in the next section.

Example

In this section we present an example to show the power of our procedures. Other examples are given in references [17] and [18]. We consider $SU(3)$ Racah coefficients, in which we wish to find the embedding of the eight-dimensional representation in the three-fold tensor product of eight-dimensional representations. The eight-dimensional representations and their tensor products arise in applications of flavor and color $SU(3)$ gauge theories of the strong interactions. The eight-dimensional irrep is entered into the computer as [4,3,2,0,0,0], while the three-fold tensor product of eight-dimensional irreps [[2,1,0],[2,1,0],[2,1,0]] is entered into the computer as $[m] = [2, 1, 2, 1, 2, 1]$. The programs then calculate $\vec{\Phi}$, which in this case has multiplicity 8.

In this example rather than focusing on coupling schemes, we find two sets of commuting Casimir operators. A first choice is [[1,2],[2,2],[2,1]] which has two 2-fold degeneracies. A second Casimir that commutes is [[[2,3],[3,3],[3,2]] + [[1,3],[3,3],[3,1]]] which then breaks the degeneracy. The resulting eigenvalues are

$$\eta = \begin{bmatrix} 42 & 30 \\ 30 & 36 \\ 36 & 42 \\ 6 & 66 \\ \frac{39}{2} - 3/2\,\sqrt{5} & \frac{105}{2} + 3/2\,\sqrt{5} \\ \frac{39}{2} + 3/2\,\sqrt{5} & \frac{105}{2} + 3/2\,\sqrt{5} \\ \frac{39}{2} + 3/2\,\sqrt{5} & \frac{105}{2} - 3/2\,\sqrt{5} \\ \frac{39}{2} - 3/2\,\sqrt{5} & \frac{105}{2} - 3/2\,\sqrt{5} \end{bmatrix}$$

A second Casimir operator that does not commute with the previous two is [[[2,3],[3,3],[3,2]] + [[1,2],[2,2],[2,1]]] , for which there are no degenerate eigenvalues:

$$\eta' = \begin{bmatrix} 27.90 \\ 38.30 \\ 39.94 \\ 48.57 \\ 52.16 \\ 53.42 \\ 56.88 \\ 66.84 \end{bmatrix}$$

Then the overlap between these two sets of noncommuting Casimir operators is

$$R_{\eta\eta'} = \begin{bmatrix} -0.094 & 0.019 & 0.054 & -0.019 & 0.019 & 0.019 & 0.019 & -0.019 \\ 0.00000069 & 0.00000055 & -0.00000031 & -0.00000081 & 0.0000013 & 0.0000022 & 0.0000017 & -0.0000013 \\ -0.84 & 0.99 & 0.99 & -0.99 & 0.99 & 0.99 & 0.99 & -0.99 \\ -0.27 & 0.054 & 0.053 & -0.054 & 0.055 & 0.055 & 0.055 & -0.055 \\ 0.40 & -0.080 & -0.079 & 0.080 & -0.081 & -0.082 & -0.081 & 0.081 \\ -0.15 & 0.031 & 0.030 & -0.031 & 0.031 & 0.031 & 0.031 & -0.031 \\ 0.059 & -0.012 & -0.012 & 0.012 & -0.012 & -0.012 & -0.012 & 0.012 \\ -0.15 & 0.031 & 0.030 & -0.031 & 0.031 & 0.031 & 0.031 & -0.031 \end{bmatrix}$$

The time needed for this Racah calculation is about five minutes.

Conclusion

We have solved the following problems:

1. We give the most general (non-inductive) construction of the Gelfand-Cetlin basis of irreps of $U(N)$ (or equivalently of $GL(N,C)$) as polynomial functions.

2. If $(M)^{\surd}$ denotes the signature of the contragredient representation of (M), we show the multiplicity $\mu(M)$ is equal to the dimension of the G-invariant subspace of $V^{(m)_1} \otimes \cdots \otimes V^{(m)_r} \otimes V^{(M)^{\surd}}$. Further we give a method for constructing an orthonormal basis in the G-invariant subspace.

3. We realize Casimir operators, and more importantly, generalized Casimir operators, as invariant differential operators which are intertwining operators of the G-modules $V^{(M)}$ and $V^{(m)_1} \otimes \cdots \otimes V^{(m)_r}$; thus we give a resolution of the important multiplicity problem in physics.

4. We present a general method for computing Clebsch-Gordan and Racah coefficients which are fundamental in quantum physics. A website (`http://www.physics.uiowa.edu/wklink/Racah/index.html`) has been developed which makes it possible for users to compute Gelfand-Cetlin basis elements, Clebsch-Gordan and Racah coefficients by downloading the programs from the website.

References

[1] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum*, World Scientific, Singapore, 1988.

[2] E. P. Wigner, On the consequences of the symmetry of the nuclear Hamiltonian on the spectroscopy of nuclei, *Phys. Rev.* **51** (1937), 106–119; see also the review by O. W. Greenberg, From Wigner's supermultiplet theory to quantum chromodynamics, *Acta Phys. Hung.* **A19** (2004), 353, arXiv:hep-ph/0212174.

[3] M. Gell-Mann, A schematic model of baryons and mesons, *Phys. Lett.* **8** (1964), 214–215.

[4] F. Gursey and L. A. Radicati, Spin and unitary spin independence of strong interactions, *Phys. Rev. Lett.* **13** (1964), 173–175.

[5] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, Cambridge, 1998.

[6] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions, Recent Advances,* Kluwer Academic Publishers, 1995, Chapters 5, 6.

[7] W. H. Klink and T. Ton-That, On resolving the multiplicity of arbitrary tensor products of the $U(N)$ Groups, *J. Phys.* **A21** (1988), 3877–3892.

[8] T. Ton-That, Sur la decomposition des produits tensoriels des representations irreductibles de $GL(k,C)$, *Jour. Math. Pures Appl.* **56** (1977), 251–262.

[9] D. P. Zelobenko, *Compact Lie Groups and Their Representations* (English translation in Transl. Math. Monographs), Amer. Math. Soc., Providence, R.I., 1973.

[10] W. H. Klink and T. Ton-That, Multiplicity, invariants, and tensor products of compact groups, *J. Math. Phys.* **37** (1996), 6468–6485.

[11] T. Ton-That, Dual representations and invariant theory, *Contemp. Math.* **191** (1995), 205–221.

[12] H. Weyl, *The Classical Groups, Their Invariants and Representations*, 2nd ed., Princeton University Press, Princeton, N.J., 1946.

[13] W. H. Klink and T. Ton-That, n-fold tensor products of $GL(N,C)$ and decomposition of Fock spaces, *J. Func. Anal.* **84** (1989), 1–18.

[14] R. T. Aulwes, W. H. Klink, and T. Ton-That, Invariant theory, generalized Casimir operators, and tensor product decompositions of $U(N)$, *J. Phys.* **A34** (2001), 8237–8257.

[15] W. H. Klink and T. Ton-That, Duality in representation theory, *Ulam Quart.* **1** (1992), 41–49.

[16] ———, Invariant theory of the block diagonal subgroups of $GL(N,C)$ and generalized Casimir operators, *J. Algebra* **145** (1992), 187–203.

[17] S. Gliske, W. Klink, and T. Ton-That, Algorithms for computing generalized $U(N)$ Racah coefficients, *Acta Appl. Math.* **88** (2005), 229–249.

[18] ———, Algorithms for computing $U(N)$ Clebsch-Gordan coefficients, *Acta Appl. Math.* **95** (2007), 51–72.