WHAT IS...
a Rota-Baxter Algebra?

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A **Rota-Baxter algebra**, also called a **Baxter algebra**, is an associative algebra with a linear operator that generalizes the algebra of continuous functions with the integral operator. More precisely, for a given commutative ring $\mathbf{k}$ and $\lambda \in \mathbf{k}$, a **Rota-Baxter k-algebra (of weight λ)** is a $\mathbf{k}$-algebra R together with a $\mathbf{k}$-linear map $P : R \to R$ such that

$$(1) \quad P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy)$$

for all $x, y \in R$. Such a linear operator is called a **Rota-Baxter operator (of weight λ)**. Note that the relation (1) still makes sense when the associative algebra R is replaced by a k-module with a bilinear binary operation, such as the Lie bracket. Despite its simple form, the Rota-Baxter operator has appeared in a wide range of areas in pure and applied mathematics, providing a unified framework to study these different areas. Advances in one of these areas often stimulated developments in Rota-Baxter algebra, which, in turn, inspired progress in other related areas.

Let R be the $\mathbb{R}$-algebra of continuous functions on $\mathbb{R}$ and P the integral operator sending a function $f(x)$ in R to the function $P(f)(x) := \int_0^x f(t)\,dt$. Then the integration by parts formula

$$\int_0^x P(f)'(t)P(g)(t)dt = P(f)(x)P(g)(x) - \int_0^x P(f)(t)P(g)'(t)dt$$

is just (1) with $\lambda = 0$.

In the discrete context, consider the algebra of sequences in a $\mathbf{k}$-algebra A, with componentwise addition and multiplication. Define an operator P that sends a sequence $(a_1, a_2, a_3, \cdots, a_n, \cdots)$ in A to the sequence of partial sums $(0, a_1, a_1 + a_2, \cdots, \sum_{k<n} a_k, \cdots)$. Then it is easy to check that P is a Rota-Baxter operator of weight 1.

Despite its natural connection with integral analysis, the Rota-Baxter algebra was not introduced as an abstraction of integral analysis, as in the well-known differential case, but was introduced in 1960 by Glenn Baxter [1] in his probability study of fluctuation theory, in particular the Spitzer identity with the algebraic formulation

$$(2) \qquad b = \exp(-P(\log(1 - ax)))$$

for the solution of the fixed point equation

$$b = 1 + P(bax)$$

in the power series ring $A[[x]]$, where (A, P) is any commutative Rota-Baxter algebra of weight -1. It was then studied in the 1960s and 1970s by Cartier and the school of Rota [3] in connection with combinatorics. For example, they showed that the well-known Waring's formula,

$$\exp\left(-\sum_{k=1}^{\infty}(-1)^k p_k(x_1, \ldots, x_m)t^k/k\right) = \sum_{n=0}^{\infty} e_n(x_1, \ldots, x_m)t^n, \ \forall\ m \geq 1$$

between the power sum symmetric functions $p_k(x_1, \cdots, x_m)$ and the elementary symmetric functions $e_n(x_1, \cdots, x_m)$, is equivalent to Spitzer's identity in a free Rota-Baxter algebra.

In part to acknowledge Rota's contribution in Rota-Baxter algebra and in part to distinguish this algebraic structure from the well-known Yang-Baxter equation, named after the distinguished physicists, the term *Rota-Baxter algebra* has been used recently in place of Baxter algebra. Quite remarkably, even though the two Baxters are not

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related genealogically, they are mathematically—the operator form of a skew-symmetric solution of the classical Yang-Baxter equation in a Lie algebra is just a Rota-Baxter operator of weight zero on this Lie algebra. An analogous relationship has been established for associative algebras through the work of Aguiar and several other authors.

Another connection of Rota-Baxter algebra with mathematical physics was found in the seminal work of Connes and Kreimer in the late 1990s in their Hopf algebra approach to renormalization of quantum field theory. There divergent Feynman integrals, through dimensional regularization, acquire Laurent series expansions in the Laurent series ring $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$, which gives back the divergent integrals when $\varepsilon = 0$. The splitting of $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ as a vector space direct sum of the two subrings $\mathbb{C}[[\varepsilon]]$ and $\varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$ means that the projection $\mathbb{C}[\varepsilon^{-1}, \varepsilon]] \to \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$ is a Rota-Baxter operator of weight -1. This operator and the Hopf algebra structure on the Feynman diagrams uncovered by Connes and Kreimer allowed them to give an algebraic formulation for the BPHZ process of QFT renormalization, which is named after Bogoliubov, Parasiuk, Hepp, and Zimmermann for their work in the 1950s and 1960s. In particular, the algebraic Birkhoff decomposition of Connes and Kreimer decomposes a regularized Feynman rule into the renormalized part and the counterterm.

Quite unexpectedly, the algebraic Birkhoff decomposition can be naturally derived from the generalization of a factorization theorem for Rota-Baxter algebras whose original form was discovered by Atkinson in 1963 and was independently established for Lie algebras as a fundamental theorem of integrable systems by Reyman and Semenov-Tian-Shansky in 1979. This generalization of the Atkinson factorization theorem also implies the factorization theorem of Barron, Huang, and Lepowsky in vertex operator algebras; the even-odd decomposition of Aguiar, Bergeron, and Sottile in combinatorial Hopf algebras; and the Lie algebra polar decomposition of Zanna et al. in matrix exponentials of ODEs.

Free commutative Rota-Baxter algebras were first constructed by Rota and Cartier in the 1970s. A third construction was later obtained in terms of the mixable shuffle product that includes both the shuffle product from iterated integrals and its discrete analogue of quasi-shuffle product of Hoffman. The latter two products play a prominent role in the study of multiple zeta values, where stuffle instead of quasi-shuffle had been used. For example, it is conjectured that all algebraic relations among multiple zeta values can be derived by intertwining the shuffle and stuffle relations among these values that come from their definition as iterated sums and from their integral representations shown by Kontsevich. More recently, the algebraic framework of Connes and Kreimer on renormalization of Feynman integrals has been adapted to study divergent multiple zeta values.

In the middle 1990s Loday introduced an operad called dendriform dialgebra with motivation from algebraic K-theory. This operad has two binary operations, $\prec$ and $\succ$, that satisfy certain relations so that the binary operator

$$x \star y := x \prec y + x \succ y \tag{3}$$

is associative. Aguiar showed that for a Rota-Baxter algebra (R, P) of weight 0, the binary operations $x \prec_P y = xP(y)$ and $x \succ_P y = P(x)y$ define a dendriform dialgebra structure on R, giving rise to a functor from the category of Rota-Baxter algebras of weight 0 to the category of dendriform dialgebras analogous to the functor from the category of associative algebras to the category of Lie algebras. Further investigation of this analog led to the study of the adjoint functor that assigns a dendriform dialgebra to its enveloping Rota-Baxter algebra. As in the case of enveloping (associative) algebras of Lie algebras obtained from free associative algebras, the enveloping Rota-Baxter algebras are obtained from free *noncommutative* Rota-Baxter algebras. These free Rota-Baxter algebras carry natural combinatorial structures of trees and Motzkin paths.

A large part of the theoretical study of Rota-Baxter algebras can be summarized in the following relationship diagram:

$$\begin{tikzcd}
\text{Rota-Baxter operator} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Dendriform operad} \\
& \text{Mixable shuffle product} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

As it turns out, the dendriform dialgebra is just the first case of a class of closely related operads that share the property of "splitting of associativity", as in (3). Also, there are several operators, such as the averaging operator and Nijenhuis operator, resembling the Rota-Baxter operator, particularly in their special form of products. The study of these three classes of objects should greatly enrich our understanding of these operators, operads, and products.

References

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[2] L. Guo, `http://newark.rutgers.edu/~liguo/rba.html`.

[3] G.-C. Rota, Baxter algebras and combinatorial identities. I, II, *Bull. Amer. Math. Soc.* **75** (1969), 325–9, 330–4.