Model Theory and Complex Geometry

Rahim Moosa

Model theory is a branch of mathematical logic whose techniques have proven to be useful in several disciplines, including algebra, algebraic geometry, and number theory. The last fifteen years have also seen the application of model theory to *bimeromorphic geometry*, which is the study of compact complex manifolds up to bimeromorphic equivalence. In this article I will try to explain why logic should have anything to say about compact complex manifolds. My primary focus will be on the results in bimeromorphic geometry obtained by model-theoretic methods and the questions about compact complex manifolds that model theory poses.

Structures and Definable Sets

Besides being a discipline in its own right, model theory is also a *way* of doing mathematics. Given a mathematical object, such as a ring or a manifold, we begin by stating explicitly what structure on that object we wish to investigate. We then study those sets that can be described using formal expressions that refer only to the declared structure and whose syntax is dictated by first-order logic. Let me give a few details.

A *structure* consists of an underlying set M together with a set of distinguished subsets of various cartesian powers of M called the *basic relations*. It is assumed that equality is a basic (binary) relation in every structure. One could also allow *basic functions* from various cartesian powers of M to M, but by replacing them with their graphs I will, without loss of generality, restrict myself to relational structures. For example, a ring can be viewed as a structure where the underlying set is the set of elements of the ring and there are, besides equality, two basic relations: the ternary relations given by the graphs of addition and multiplication. If the ring also admits an ordering that we are interested in, then we can consider the new structure where we add the ordering as another basic binary relation. The *definable sets* of a structure are those subsets of cartesian powers of M that are obtained from the basic relations in finitely many steps using the following operations:

- intersection,
- union,
- complement,
- cartesian product,
- image under a coordinate projection, and
- fibre of a coordinate projection.

I am avoiding talking about the logical syntax here, but the reader familiar with some first-order logic will recognize that the basic relations form the *language* of the structure and the various operations correspond to *logical symbols* such as conjunction, disjunction, and negation. The operation of taking the image under a coordinate projection corresponds to existential quantification, while that of taking a fibre of a coordinate projection corresponds to substituting parameters for variables (that is, *specialisation*). The definable sets are then the sets described by *first-order formulae.* This way of viewing definable sets is an essential feature of model theory, even though many expositions (including this one) avoid formulae by introducing definability set-theoretically, just as I have done here.

Rahim Moosa is professor of mathematics at the University of Waterloo, Canada. His email address is `rmoosa@math.uwaterloo.ca`*.*

The author was supported by an NSERC Discovery Grant.

In any case, given a structure we have an associated collection of definable sets. When $(R, +, \times)$ is a commutative unitary ring, for example, it is not hard to see that if $f_1, \ldots, f_\ell$ are polynomials in $R[x_1, \ldots, x_n]$, then the *algebraic set* they define, namely their set of common zeros in R^n, is definable. Hence the finite boolean combinations of such sets, that is, the *Zariski constructible* sets, are all definable. It is an important fact that if R is an algebraically closed field, then these are the only definable sets.[1]

But we are interested in a somewhat different sort of example. Fix a compact complex manifold X and consider the structure $\mathcal{A}(X)$ where the basic relations are the complex analytic subsets of X^n, for all $n > 0$. By a *complex analytic subset*, or just *analytic* subset for short, I mean a subset A such that for all $p \in X^n$ there is a neighbourhood U of p and finitely many holomorphic functions $f_1, \ldots, f_\ell$ on U such that $A \cap U$ is the common zero set of $\{f_1, \ldots, f_\ell\}$. Note that the local data of U and $f_1, \ldots, f_\ell$ are not part of our structure; only the global set A is named as a basic relation. The model theory of compact complex manifolds was begun by Zilber's [14] observation in the early 1990s that $\mathcal{A}(X)$ is "tame". In particular, as a consequence of Remmert's proper mapping theorem, Zilber shows that every definable set in $\mathcal{A}(X)$ is a finite boolean combination of analytic subsets. But the tameness goes much further, making the geometry of analytic sets susceptible to a vast array of model-theoretic techniques.

Zilber's analysis of individual compact complex manifolds extends to the *many-sorted* structure $\mathcal{A}$, which includes all compact complex manifolds at once, and where all complex analytic subsets are named as basic relations. Note that algebraic geometry is part of this structure; amongst the compact complex manifolds in $\mathcal{A}$ are the complex projective spaces, and so all quasi-projective algebraic varieties are definable in $\mathcal{A}$. But there are also nonalgebraic compact complex manifolds, and in some sense the model-theoretic analysis focuses on those. Let me discuss two examples that we will see again later.

Suppose $\{\alpha_1, \ldots, \alpha_{2n}\}$ is an $\mathbb{R}$-basis for $\mathbb{C}^n$, and $\Lambda = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{2n}$ is the real $2n$-dimensional lattice it generates. Then the quotient $T = \mathbb{C}^n/\Lambda$ is a compact complex manifold of dimension n, called a *complex torus*, that inherits a holomorphic group structure from $(\mathbb{C}^n, +)$. While some complex tori can be embedded into projective space, if Λ is chosen sufficiently generally, then the resulting torus is nonalgebraic. For example, if the real and imaginary parts of $\{\alpha_1, \ldots, \alpha_{2n}\}$ form an algebraically independent set over $\mathbb{Q}$, then T will not contain any proper infinite analytic subsets. In particular, such tori, which are called *generic complex tori*, cannot be projective varieties if $n > 1$. Another example of a nonalgebraic compact complex manifold is the following *Hopf surface*: fix a pair of complex numbers $\alpha = (\alpha_1, \alpha_2)$ with $0 < |\alpha_1| \leq |\alpha_2| < 1$, and consider the infinite cyclic group Γ of automorphisms of $\mathbb{C}^2 \setminus \{(0,0)\}$ generated by $(u, v) \mapsto (\alpha_1 u, \alpha_2 v)$. The quotient $H_\alpha = \mathbb{C}^2 \setminus \{(0,0)\}/\Gamma$ is a compact complex surface that is never algebraic. Indeed, this can already be deduced from its underlying differentiable structure; H_α is diffeomorphic to $S^1 \times S^3$, something that is never the case for a projective surface. Unlike tori, Hopf surfaces always contain proper infinite analytic subsets: the images of the punctured axes in $\mathbb{C}^2 \setminus \{(0,0)\}$ give us at least two irreducible curves on H_α. If α is chosen sufficiently generally, then these will be the only curves. In that case, like generic complex tori, H_α will have no nonconstant meromorphic functions.

In model theory there are several notions, of varying resolution, that describe the possible interaction between two definable sets. These notions, specialised to the structure $\mathcal{A}$, have relevant counterparts in bimeromorphic geometry. Suppose that X and Y are irreducible complex analytic sets in $\mathcal{A}$. The strongest possible interaction is if X and Y are biholomorphic. This can be weakened to *bimeromorphic equivalence*: there exists an irreducible analytic subset $G \subset X \times Y$ such that both projections $G \to X$ and $G \to Y$ are generically bijective. By "generically bijective" I mean that off a countable union of proper analytic subsets of X the fibres of $G \to X$ are singletons, and similarly for $G \to Y$. If $G \to X$ and $G \to Y$ are only assumed to be generically finite-to-one, then we say that G is a *generically finite-to-finite correspondence* between X and Y. Finally, we can weaken the condition much further by merely asking that there exist some proper analytic subset of $X \times Y$ that projects onto both X and Y. In that case we say that X and Y are *nonorthogonal*. For example, any two projective varieties are nonorthogonal. On the other hand, any compact complex manifold without nonconstant meromorphic functions—such as a generic complex torus or a generic Hopf surface—is *orthogonal* to any projective variety. Indeed, nonorthogonality to some projective variety implies nonorthogonality to the projective line $\mathbb{P}_1$, and if $G \subset X \times \mathbb{P}_1$ witnesses this, then G has nonconstant meromorphic functions, and as $G \to X$ will necessarily be generically finite-to-one, so does X.

[1] *This is* quantifier elimination *for algebraically closed fields, or equivalently Chevellay's theorem that over an algebraically closed field the projection of a constructible set is constructible.*

Classifying Simple Compact Complex Manifolds

One way for a structure to be considered "tame" is if the definable sets have a *dimension theory*, that is, if a certain intrinsic model-theoretically defined dimension function, called the *rank*, takes on ordinal values on all definable sets. Zilber's initial analysis of $\mathcal{A}$ showed that every analytic subset of a compact complex manifold is of *finite* rank. In fact, this rank is bounded by, but typically not equal to, the complex dimension. I will not define rank here, but I will in a moment explain what it means for a compact complex manifold to have rank 1. This reticence is partially justified by the fact that there is general model-theoretic machinery available, due mostly to Shelah, that analyses arbitrary finite-rank definable sets in terms of rank 1 sets. In particular, and this is only the starting point of such an analysis in $\mathcal{A}$, every analytic set will be nonorthogonal to one of rank 1. It follows that the study of rank 1 sets is central to understanding compact complex manifolds in general. Note that this is not true of complex dimension: understanding compact complex curves tells us essentially nothing about compact complex surfaces that contain no curves.

So what does it mean for a compact complex manifold X to be of "rank 1"? Here is a geometric characterisation: X is not covered by a definable family of proper infinite analytic subsets. Pillay [10] observed that such manifolds were already of interest to complex geometers and were called *simple*. More precisely:

Definition. A compact complex manifold X is said to be *simple* if $\dim X > 0$, and whenever Y is a compact complex manifold, $A \subset Y \times X$ is an analytic subset, and $E \subset Y$ is a proper analytic subset such that the fibres of A above $Y \setminus E$ are proper infinite subsets of X, then the union of all the fibres of A above $Y \setminus E$ is contained in a proper analytic subset of X.

Projective curves are simple; they are the only simple projective varieties. But so is a generic complex torus, or indeed any compact complex manifold without proper infinite analytic subsets. The generic Hopf surfaces, having precisely two curves on them, are also simple. It is not hard to see that simplicity is a bimeromorphic invariant; in fact, it is preserved by generically finite-to-finite correspondences. In a sense that I have hinted at above and will return to again later, simple manifolds are the building blocks for all compact complex manifolds.

The contribution of model theory to bimeromorphic geometry begins with the following dichotomy for simple compact complex manifolds, which is a consequence of the deep results of Hrushovski and Zilber from [5]. It says that a simple compact complex manifold is either algebraic or its cartesian powers have no "rich" definable families of analytic subsets. More precisely, if X is a simple compact complex manifold, then exactly one of the following holds:

I. X is a projective curve, or
II. X is *modular:* whenever Y is a compact complex manifold with $\dim Y > 0$, and $A \subseteq Y \times X^2$ is an analytic subset whose generic fibres over Y are distinct proper infinite irreducible analytic subsets of X^2 that project onto each coordinate, then Y is simple.

While the condition of modularity only explicitly mentions X^2, it actually restricts the rank of families of analytic subsets of X^n, for all n. All projective curves are nonmodular. For example, the family of lines $y = ax + b$ is a two-parameter algebraic family; it gives rise to a family of subvarieties of $\mathbb{P}_1 \times \mathbb{P}_1$ that is parameterised by a two-dimensional projective variety, and two-dimensional projective varieties are not simple.

So, by the above dichotomy, every simple manifold of dimension at least two is modular. As we have seen, examples can be found among the complex tori and the Hopf surfaces. In fact the model-theoretic analysis provides us with a further dichotomy which distinguishes sharply between these two examples. Very roughly speaking, a modular manifold that is not a complex torus admits only binary definable relations. More precisely, if X is a simple modular compact complex manifold, then exactly one of the following holds:

I. X is in generically finite-to-finite correspondence with a complex torus, or
II. X is *relationally trivial:* if $A \subseteq X^n$ is an irreducible analytic subset that projects onto X in each coordinate, and if $\{\pi_1, \dots, \pi_\ell\}$ is an enumeration of all the coordinate projections from X^n to X^2, then A is an irreducible component of $\bigcap_{i=1}^{\ell} \pi_i^{-1}(\pi_i(A))$.

Note that complex tori are not relationally trivial: being complex Lie groups they admit a truly ternary analytic relation, namely the graph of the group law. Simple Hopf surfaces are relationally trivial. Actually one can be more precise in case I: there exists a generically finite-to-one meromorphic surjection from X to a *Kummer manifold*, a manifold of the form T/Γ where T is a complex torus and Γ is a finite group of holomorphic automorphisms of T.

The above characterisation of simple modular compact complex manifolds was suggested by Hrushovski in his 1998 address to the International Congress of Mathematicians [4]. It was proved by Pillay and Scanlon in [12], building on some unpublished work of Scanlon. The result is obtained as a corollary to the main theorem in that paper which I would at least like to state in brief. *Meromorphic groups* are a natural generalisation of algebraic groups to the complex analytic category; they are complex Lie groups that are "compactifiable" in an appropriate sense. What Pillay and Scanlon actually prove, using model-theoretic methods and building on work of Fujiki, is that every meromorphic group is the extension of a complex torus by a linear algebraic group.

In any case, putting the two dichotomies together, we get:

Theorem 1. *If X is a simple compact complex manifold, then*

- *X is a projective curve, or*
- *X is in generically finite-to-finite correspondence with a simple complex torus of dimension > 1, or*
- *X is relationally trivial.*

It remains then to understand the relationally trivial compact complex manifolds. In dimension one there are none because all compact curves are algebraic (this is by the Riemann existence theorem), and it is not hard to see that projective curves are not relationally trivial. Besides Hopf surfaces, examples of relationally trivial surfaces can also be found among the $K3$ and Inoue surfaces. Relationally trivial manifolds remain quite elusive, and, except for the case of surfaces, there are only conjectural results. These conjectures are restricted to Kähler manifolds, which also play a special role from the model-theoretic point of view, and to which I now turn.

Compact Cycle Spaces and Kähler Manifolds

One of the obstacles to the full application of model-theoretic techniques to compact complex manifolds is the size of the language, the fact that *every* analytic set is a basic relation. Let me point out that in the algebraic case there is a more economical choice of structure. Consider, for example, the compact complex manifold $X = \mathbb{P}_m$, projective m-space over the complex numbers. Instead of $\mathcal{A}(X)$ we can consider the structure $\mathcal{A}_{\mathbb{Q}}(X)$, where we only include as basic relations the algebraic subsets of X^n that are defined by polynomials with *rational* coefficients. Then $\mathcal{A}_{\mathbb{Q}}(X)$ has only countably many basic relations. But all analytic subsets of X^n are still *definable* in $\mathcal{A}_{\mathbb{Q}}(X)$. This is because every complex analytic subset of projective space is algebraic (Chow's theorem) and every algebraic set is obtained by specialisation from an algebraic set over $\mathbb{Q}$. Hence $\mathcal{A}(X)$ and $\mathcal{A}_{\mathbb{Q}}(X)$ have the same definable sets, even though the latter is equipped with a much reduced collection of basic relations.

This motivates the following definition: A compact complex manifold X is called *essentially saturated* if there exists a countable collection $\mathcal{L}_0$ of analytic subsets of cartesian powers of X such that every analytic set is definable in the reduct where only the sets in $\mathcal{L}_0$ are named as basic relations. Essentially saturated compact complex manifolds are significantly more amenable to a model-theoretic analysis. Exactly why is described in [7] and is somewhat beyond the scope of this article. Instead, I would like to focus on a very suggestive geometric characterisation of essential saturation.

Recall that a *k-cycle* of a compact complex manifold X is a finite linear combination $Z = \sum_i n_i Z_i$ where the Z_i are distinct k-dimensional irreducible complex analytic subsets of X and each n_i is a positive integer. In particular, every irreducible analytic subset is itself a cycle. The set of all k-cycles is denoted by $\mathcal{B}_k(X)$, and $\mathcal{B}(X)$ denotes the disjoint union of all the $\mathcal{B}_k(X)$. In the 1970s Barlet endowed $\mathcal{B}(X)$ with a natural structure of a complex analytic space. If X is algebraic, then $\mathcal{B}(X)$ coincides with the Chow scheme. With this terminology in place, we can characterise essential saturation as follows: *X is essentially saturated if and only if all the irreducible components of $\mathcal{B}(X^n)$ are compact, for all $n \geq 0$.* This follows from [7], in which the universal family of complex analytic subspaces (the *Douady spaces*) were used instead of Barlet's cycle spaces. In any case, it is important here that the condition is on all the cartesian powers; if H is a Hopf surface, then $\mathcal{B}(H)$ has only compact components, but $\mathcal{B}(H \times H)$ has a noncompact component. In particular, Hopf surfaces are not essentially saturated. While the property of $\mathcal{B}(X)$ having only compact components is one that appears often in bimeromorphic geometry, it seems that the extension of this condition to all cartesian powers of X has not been studied. For example, is essential saturation preserved under cartesian products and bimeromorphic equivalence?

How does one check if a component of the Barlet space is compact? In the late 1970s Lieberman [6] gave the following differential geometric criterion: a set of cycles is relatively compact in the cycle space if and only if the volume of the cycles in that set (with respect to some hermitian metric) is bounded. This gives us many examples of essentially saturated manifolds. Recall that a *Kähler manifold* is one that admits a (hermitian) metric which locally approximates to order 2 the standard euclidean metric on $\mathbb{C}^n$. With respect to such a metric, the volume of the cycles in any given component of the Barlet space is constant, and this remains true of cartesian powers because kählerianity is preserved under cartesian products. It follows from Lieberman's theorem that all compact Kähler manifolds are essentially saturated. Moroever, any compact complex analytic space that is bimeromorphic to a Kähler manifold—this is Fujiki's class C—is also essentially saturated.

While C does not include all essentially saturated manifolds (Inoue surfaces of type S_M are counterexamples, see [8]), it is a large class of manifolds that contains all projective varieties and complex tori and that is preserved under various operations including meromorphic images and generically finite-to-finite correspondences. Compact Kähler manifolds play a prominent role in bimeromorphic geometry largely because they are susceptible to many of the techniques of algebraic geometry. Because of essential saturation, they are also important to the model-theoretic approach.

Let us return to the classification problem for simple compact complex manifolds. In the previous section I explained how this reduces to understanding the relationally trivial manifolds. We can restrict the problem further and ask: What are the relationally trivial simple *Kähler* manifolds? We have already seen that there are none in dimension one. In dimension two, inspecting the Enriques-Kodaira classification, we see that all relationally trivial simple Kähler surfaces are bimeromorphic to $K3$ surfaces. These surfaces (introduced by Weil and named in honour of Kummer, Kodaira, Kähler, and the mountain K2) are by definition simply-connected compact surfaces with trivial canonical bundle. In higher dimensions the correct generalisation of $K3$ seems to be *irreducible hyperkähler:* simply connected compact Kähler manifolds with the property that their space of holomorphic 2-forms is spanned by an everywhere nondegenerate form. Pillay has conjectured that *all relationally trivial simple Kähler manifolds are in generically finite-to-finite correspondence with an irreducible hyperkähler manifold.* Since irreducible hyperkähler manifolds are always even-dimensional, Pillay's conjecture, coupled with Theorem 1 above, would imply that every odd-dimensional simple Kähler manifold is in generically finite-to-finite correspondence with a complex torus. In dimension three this is essentially a conjecture of Campana and Peternell, namely that every simple Kähler threefold is Kummer.

Variation in Families

In justifying our focus on simple manifolds I have already mentioned the fact that every compact complex manifold is nonorthogonal to a simple one. From this one can deduce that for every compact complex manifold X there exists a meromorphic surjection $f : X \to Y$, where $\dim Y > 0$ and Y is in generically finite-to-finite correspondence with some cartesian power of a simple manifold. The same is then also true of each generic fibre X_y of f. At least in the Kähler case, using essential saturation, one can show that the corresponding meromorphic surjections on X_y vary uniformly in the parameter y. Since the dimension of X_y is strictly less than that of X, such an analysis must stop after finitely many iterations. It is in this sense, via sequences of meromorphic fibrations, that simple manifolds control the structure $\mathcal{A}$. Of course this does not reduce the classification problem to the case of simple manifolds; at the very least one needs to also understand how such manifolds fit into meromorphic fibrations and how they interact with each other. I want to state one conjecture that is central to this question.

First of all, from the definition of simplicity it follows that two simple manifolds are nonorthogonal to each other if and only if they are in generically finite-to-finite correspondence. The three classes of compact complex manifolds appearing in Theorem 1—projective curves, simple complex tori of dimension > 1, and simple relationally trivial manifolds—are all mutually orthogonal in the sense that any two manifolds coming from different classes will be orthogonal. Moreover, while all curves are nonorthogonal to each other, there exist orthogonal pairs within each of the other two classes.

One fundamental question is whether or not there exist entire definable families of simple manifolds any two of which are orthogonal. Actually, it follows from observations of Pillay and Scanlon that such families do exist, but not, conjecturally, among the Kähler manifolds. More precisely, the conjecture is that *if $f : X \to Y$ is a meromorphic surjection between compact Kähler manifolds with simple generic fibres, then any two generic fibres are in generically finite-to-finite correspondence.* In 2005 Campana [2] proved an isotriviality result which proves the conjecture, with the stronger conclusion of isomorphism rather than correspondence, in the following cases: when the generic fibres are surfaces, when the generic fibres are "general" complex tori, and when the generic fibres are irreducible hyperkähler.

Analogies

By way of conclusion, I want to discuss the special role of model theory as a medium between different geometric contexts. Certain results from bimeromorphic geometry have informed advances in abstract model theory which can then be reapplied in other areas. In this way model theory acts as a conduit between bimeromorphic geometry and, in the case I want to discuss, *differential-algebraic geometry*. I need to say a few words about differential-algebraic geometry.

Like bimeromorphic geometry, differential-algebraic geometry is an "expansion" of algebraic geometry. A *differential field* is a field K equipped with a *derivation*; an additive map $\delta : K \to K$ such that $\delta(xy) = x\delta(y) + y\delta(x)$. For example, $(\mathbb{C}(t), \frac{d}{dt})$ is a differential field. *Differential algebra* is then commutative algebra in the presence of this derivation. The role of polynomials is played by *differential polynomials*: functions of the form $P(x, \delta(x), \ldots, \delta^r(x))$, where $P \in K[X_0, \ldots, X_r]$ is an ordinary polynomial. A differential field (K, δ) is *differentially closed* if any system of differential polynomial equations and inequations having a solution in some differential field extension already has a solution in K. The model theory of differentially closed fields of characteristic zero is also tame. In particular, every definable set in $(K, +, \times, \delta)$ is a finite boolean combination of *differential-algebraic sets*: zero sets of systems of differential polynomial equations. These are the objects of study in differential-algebraic geometry.

Many of the model-theoretic techniques that apply to the structure $\mathcal{A}$ also apply to $(K, +, \times, \delta)$. In fact, there is a fruitful analogy between these structures whereby complex analytic sets correspond to finite-rank differential-algebraic sets (see [10]). This analogy can lead to transferring results between the two disciplines that these structures represent. The example I have in mind is based on the following theorem of Campana [1], due also independently to Fujiki [3], from the early 1980s. Suppose X is a compact complex manifold and C is a compact analytic subset of the cycle space $\mathcal{B}(X)$. Then for any $a \in X$, the set of cycles in C that pass through a is (up to bimeromorphism) an algebraic set. Moreover this happens uniformly as a varies. In 2001 Pillay [11] observed that this theorem could be used to give a more direct proof of the first dichotomy; the fact that every simple manifold is either a curve or modular. Pillay's argument involves formulating an abstract model-theoretic counterpart to Campana's theorem, which replaces the difficult and much more general results of Hrushovski and Zilber [5]. Pillay and Ziegler [13] then show that this model-theoretic counterpart also holds in differentially closed fields. As a result one has an analogue of Campana's theorem in differential-algebraic geometry, as well as a direct proof of the corresponding dichotomy for rank 1 differential-algebraic sets. Another related example of this phenomenon can be found in [9], in which Campana's "algebraic connectedness" is abstracted from bimeromorphic geometry to model theory and then applied to differentially closed fields. The outcome in that case is a criterion for when a finite-rank differential-algebraic set is in generically finite-to-finite correspondence with the C_K-points of an algebraic variety, where $C_K = \{x \in K : \delta(x) = 0\}$.

The kind of transfer of ideas that we see in the above examples, and the role that model theory plays here of recognising, formalising, and facilitating analogies between different geometric settings, is not something new or unique to its interaction with bimeromorphic geometry. This has been a defining feature of model theory since the 1970s and continues to fuel the internal development of the subject.

References

[1] F. Campana, Algébricité et compacité dans l'espace des cycles d'un espace analytique complexe, *Mathematische Annalen* **251**(1) (1980), 7–18.

[2] ———, Isotrivialité de certaines familles Kählériennes de varietés non projectives, *Mathematisches Zeitschrift* **252**(1) (2006), 147–56.

[3] A. Fujiki, On the Douady space of a compact complex space in the category C, *Nagoya Mathematical Journal* **85** (1982), 189–211.

[4] E. Hrushovski, Geometric model theory, in *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 281–302, Documenta Mathematica, 1998.

[5] E. Hrushovski and B. Zilber, Zariski geometries, *J. Amer. Math. Soc.* **9**(1) (1996), 1–56.

[6] D. Lieberman, Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds, in *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, Springer, 1978, 140–86.

[7] R. Moosa, On saturation and the model theory of compact Kähler manifolds, *Journal für die reine und angewandte Mathematik* **586** (2005), 1–20.

[8] R. Moosa, R. Moraru, and M. Toma, An essentially saturated surface not of Kähler-type, *Bull. London Mathematical Society* **40**(5) (2008), 845–54.

[9] R. Moosa and A. Pillay, On canonical bases and internality criteria, to appear in the *Illinois Journal of Mathematics.*

[10] A. Pillay, Some model theory of compact complex spaces, in *Hilbert's Tenth Problem: Relations with Arithmetic and Algebraic Geometry (Ghent, 1999)*, American Mathematical Society, Providence, RI, 2000, 323–38.

[11] ———, Model-theoretic consequences of a theorem of Campana and Fujiki, *Fundamenta Mathematicae* **174**(2) (2002), 187–92.

[12] A. Pillay and T. Scanlon, Meromorphic groups, *Trans. Amer. Math. Soc.* **355**(10) (2003), 3843–59.

[13] A. Pillay and M. Ziegler, Jet spaces of varieties over differential and difference fields, *Selecta Math. (N.S.)* **9**(4) (2003), 579–99.

[14] B. Zilber, Model theory and algebraic geometry, in *Proceedings of the 10th Easter Conference on Model Theory (Wendisch Reitz, 1993)*, Humboldt Universität, 1993, 93–117.