A Mathematician Writes for High Schools

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A Brief Personal History
I started out my graduate school education fully intending to be a research mathematician. As a Harvard undergraduate (A.B. 1966), I had been inspired by Richard Brauer’s courses in finite group theory and, at his recommendation, went to Yale to work with Walter Feit. I received my Ph.D. in 1970, and my dissertation ("A characterization of Conway’s group .3") appeared in Journal of Algebra, January 1973.

But, in early 1969, while still in graduate school, I decided that my skills could be put to greater public service if I focused my career on helping to improve public school education. During and right after graduate school, I worked in public school systems in New Haven, Connecticut, and in Compton and Oakland, California. In these settings, I was a mathematics specialist working with full classes of elementary or middle school students. Although I enjoyed this work, I grew to see the need for regular teachers to have a deeper understanding of basic mathematics, and I decided that my impact would be greater if I were working directly in the preparation of teachers. In 1973 I joined the mathematics department at San Francisco State University, where I worked for more than thirty years with preservice teachers and current classroom teachers at all levels, meanwhile continuing to teach service courses such as calculus, upper division courses for mathematics majors, and graduate courses in our master’s program.

In 1989 I got the opportunity of a lifetime. I was invited to join a project, then in its very early stages, whose goal was to create a problem-based curriculum for high schools that would embody the ideas and recommendations of a recent series of reports on the need for reform.

I accepted that invitation, and, along with Diane Resek (Ph.D., 1975, U.C. Berkeley), became one of the principal authors of the Interactive Mathematics Program (IMP).\(^1\) Over the next decade or so, I worked with a wonderful team of mathematics educators and teachers, in an intensive process of writing, testing, and revising, revising, revising, to produce a four-year program. In 1999 the IMP curriculum was one of a handful of mathematics programs designated by the U.S. Department of Education as “exemplary” (the highest rating). The curriculum has been used in over 1,000 schools throughout the United States, as well as in many schools outside the United States.

Writing Curriculum
One of the great challenges of mathematics curriculum development—at any level—is to make complex ideas meaningful and comprehensible to students while maintaining the integrity of the ideas. Anyone who has seen eyes glaze over as mathematics is presented with great rigor and elegance will agree that mathematical knowledge and correctness do not, by themselves, make for good teaching. Curriculum development requires both a deep understanding of mathematics and a realistic view of how students think. And so, no matter how mathematically elegant or aesthetically satisfying an approach may be, a curriculum writer must be willing to discard it if it doesn’t work with students. He or she must then struggle to find something else that is more effective.

I offer here two examples of our curriculum development process. I hope that the description of how I was able to contribute may aid other mathematicians to develop similar curricula.

Expected Value
We want our citizens to be able to make intelligent decisions on issues that involve chance and data,

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\(^1\)Interactive Mathematics Program (Years 1 to 4), by Dan Fendel and Diane Resek, with Lynne Alper and Sherry Fraser; published by Key Curriculum Press, Emeryville, CA (first edition © 1997 through 2000; second edition © 2009 through 2012 [Years 1 through 3 currently available; Year 4 available spring 2011]).
yet most students even today leave high school with only the vaguest ideas about probability and statistics. Therefore our project leaders decided that we would give these topics much greater emphasis than they’d had in the traditional high school curriculum. In particular, we identified expected value as a key concept in probability. 2

One important step was to limit consideration to situations with finite sample spaces. We recognized that, in this context, it’s easy to give a formal definition of expected value as a sum of products of values and probabilities. For example, the expected value for the roll of a single (balanced) die is the sum $1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$, which comes to $21/6$, or 3.5. And, for other situations, even where probabilities are not all the same, the definition is similar. A textbook could define expected value simply as $\sum_{i=1}^{n} x_i \cdot p(x_i)$, where $\{x_1, x_2, \ldots, x_n\}$ is the sample space of possible outcomes, and $p(x_i)$ is the probability of outcome $x_i$.

But, though it might have been aesthetically pleasing to me as a mathematician to use that definition, doing so would have doomed our work to failure with students (and here I mean the vast majority of high school students—not the rare abstract thinker who might become a mathematics Ph.D.).

As a mathematician, I know that many definitions can be equivalent to one another. As a person with experience in high school classrooms, I know that the phrase “sample space” and the use of summation notation, subscripts, and other formal symbolism will lead to “glaze-over” among students.

So, instead of the formal definition described earlier, we chose an equivalent definition based on the idea of “average in the long run”. Before using the formal phrase “expected value”, the IMP curriculum thus gives students concrete experiences, such as asking them to imagine rolling a die many, many times and to compute what they might expect for the average of those rolls.

Students understand, based on experiments and intuition, that if the number of rolls is “large enough”, then the fraction of rolls giving each result will be “pretty close” to the value given by the probability. (Indeed, students using IMP come to see that as, essentially, the meaning of probability.) For example, if they use 600 rolls, they can expect about 100 rolls for each possible outcome of the die. This leads IMP students to a computation such as $1 \cdot 100 + 2 \cdot 100 + 3 \cdot 100 + 4 \cdot 100 + 5 \cdot 100 + 6 \cdot 100 = 2100$ for the total value of all the rolls, giving $2100/600 = 3.5$ for the average.

In this context, students’ intuition about probability serves them well. They see that if the number of rolls for each outcome were off a bit from the “perfect 1/6”, this would change the total value of the rolls, but it would not change the average value much, because the total value is being divided by “a big number”. After working with several such computations, students with sufficient understanding of the distributive property can see that the result of such a computation is independent of the actual number of rolls.

Gravitational Fall

One of my favorite units in the IMP curriculum involves the following circus scenario:

A performer is on a Ferris wheel that is turning at a constant rate. A cart with a tub of water is moving along a straight track at a constant speed. The track passes under the Ferris wheel, and the performer is to be dropped from the moving Ferris wheel so that he lands in the tub.

Based on specific parameters provided (such as the rates of motion, the dimensions of the Ferris wheel, and starting positions of the cart and the performer), when should the performer be dropped? 3

The problem involves many mathematical considerations. It serves as the IMP curriculum’s vehicle for generalization from right-triangle trigonometry to circular functions. It also involves the idea of vector decomposition, as students take into account the initial “airborne” velocity of the performer—that is, the performer’s velocity at the moment of release, due to the motion of the Ferris wheel itself.

Here I want to focus on the simplified version of the problem that students do first, in which they disregard the performer’s initial velocity. (If the Ferris wheel is moving slowly enough, this initial velocity has only a small effect on the performer’s fall.) As part of the analysis, students must determine how long it will take for the performer to fall a given distance.

Although some high school students are familiar with the formula $s = \frac{1}{2}gt^2$ from their science courses, few have any understanding of where this formula comes from. In particular, even if they know that gravitational fall involves a constant rate of acceleration, they don’t understand how this is connected to the formula.

We decided to take the principle of constant acceleration as a given—as an axiom, to put it in mathematical terms. As a mathematician and curriculum developer, I was faced with the challenge of finding a way to get from that principle to the formula, and I needed to do so within the restrictions of what would be meaningful to high school students. Therefore our project leaders decided that we would give these topics much greater emphasis than they’d had in the traditional high school curriculum. In particular, we identified expected value as a key concept in probability. 2

The introduction of this concept appears in the Year 1 IMP unit The Game of Pig.

3 This scenario was the central problem in the Year 4 unit High Dive in IMP’s first edition. In the second edition, the discussion of this scenario is in two separate units, one at the end of Year 3 and one at the start of Year 4.
school students, based on assumptions of what they knew from earlier elements of the IMP curriculum.

We tried several approaches, with both high school teachers and students, before hitting on one that worked. On a sophisticated mathematical level, the definition of velocity involves the derivative. But this also means that one can find distance traveled by finding the integral of the velocity function.

This last insight suggested to me a way of using students’ understanding of area to develop an expression for position in terms of time. The first step was to establish an intuitive connection between area under the graph of the velocity function and total distance traveled. As part of this development, we opted to have IMP students work on an activity with the following as its first part:

1. Curt drove from 1 p.m. to 3 p.m. at an average speed of 50 miles per hour, and then drove from 3 p.m. until 6 p.m. at an average speed of 60 mph.
   a. Draw a graph showing Curt’s speed as a function of time for the entire period from 1 p.m. to 6 p.m., treating his speed as constant for each of the two time periods—from 1 p.m. to 3 p.m. and from 3 p.m. until 6 p.m.
   b. Describe how to use areas in this graph to represent the total distance he traveled.

The graphs students create look like the one below, and they see that, in using the familiar "rate \cdot time = distance" idea to find the distance traveled, they are doing the same computation that they would use to find the areas of the two rectangles.

Through their discussion of Question 1, IMP students generally see intuitively that this connection between area and distance traveled should remain valid if the velocity is not constant. (In bringing out this insight, their teachers are laying a foundation for students who may later study the idea of defining area via approximating rectangles.)

Building from that insight, students move on to the second part of the activity:

2. Consider a runner who is going at a steady twenty feet per second. At exactly noon, he starts to increase his speed. His speed increases at a constant rate so that twenty seconds later, he is going thirty feet per second.
   a. Graph the runner’s speed as a function of time for this twenty-second time interval.
   b. What is his average speed for this twenty-second interval?
   c. Explain how to use area to find the total distance he runs during this twenty-second interval.

On Question 2b, IMP students generally take a purely intuitive approach, saying that the speed increases at a constant rate from 20 feet per second to 30 feet per second, so the average speed is simply 25 feet per second. But they also recognize that Question 2c involves a variation on the earlier Question 1b—here the area is a trapezoid, as shown below, instead of the combination of rectangles in Question 1.

Working from the idea that the total distance traveled is again the area, they can confirm their insight that the average speed is the “midpoint” between the initial speed and the final speed. Moreover, they can see the role of the “constant acceleration” assumption. Putting these ideas together leads to this conclusion:

If an object is traveling with constant acceleration, then its average speed over any time interval is the average of its beginning speed and its final speed during that time interval.

Just a few small steps lead from this conclusion to the formula for gravitational fall:

- If velocity starts at 0 and increases at a rate $g$, then after $t$ seconds, the velocity is $gt$.
- Therefore, the average velocity over $t$ seconds is $\frac{0 + gt}{2}$.
- Therefore, the total distance traveled over $t$ seconds is $\frac{gt^2}{2} \cdot t$.

By first going through these steps for some specific examples, IMP students are able to develop the general formula.
Some Lessons

The above examples illustrate some important points that may be of use to future curriculum developers:

- A flexible and deep understanding of a mathematical concept can provide insights into how to present that concept to students. Such an understanding profoundly informed our development of the IMP curriculum.
  
  — In the case of expected value, it was important to recognize that there was another definition of expected value that is mathematically equivalent to the standard definition.
  
  — In the case of gravitational fall, the key was recognizing how distance traveled could be represented via area.

- Creating curricula for high school students requires a clear picture of what they know, what they don't know, and the depth of their understanding of what they do know, as well as a clear picture of what their intuitions are likely to tell them.
  
  — For expected value, IMP students knew how to find totals and averages. They also appreciated intuitively that if the denominator of a fraction is "big", then a small change in the numerator won't affect the fraction very much. But, since most high school students are not comfortable with symbolic formalism, an appeal to the distributive property to prove that the size of the sample doesn't matter would have had little meaning for them.
  
  — For gravitational fall, it was important to know that students were comfortable with the "rate \cdot time = distance" idea and that they would also know how to find the relevant areas. (Note: If this material had not yet been a part of students' background, the curriculum writer would have needed to think about how to introduce it prior to its use here.)

- If we want students to apply a definition or formula with understanding, we need to build gradually, using concrete situations.
  
  — With expected value, the definition was preceded by concrete work with dice and examples involving averages. A deep understanding of the meaning of probability was crucial for building the concept of expected value.
  
  — For gravitational fall, we started with students' intuition about situations involving constant speed and their ideas about area. We combined this with the use of specific examples to strengthen and expand those intuitions and to build the necessary connections.

- The true test of a curriculum element is its effectiveness in classrooms.
  
  — For expected value, some teachers wanted to define expected value using "the fraction method". [This was their term for a definition along the lines of the expression $\sum_{i=1}^{n} x_i \cdot p(x_i)$. Generally, they wanted to use this approach because it was the definition they had learned in their own college courses.] Teachers who tried "the fraction method" reported considerable confusion among students, so they switched to the "in the long run" approach and got better student understanding.
  
  — For gravitational fall, we had tried some other approaches before coming up with the one described here. Teachers reported that students accepted the principle of "averaging the endpoints" but had no intuitive understanding of why that was legitimate. They indicated that the approach described here was successful because it both appealed to students' intuition and made meaningful use of their prior knowledge.

Carrying out these guidelines made different demands on me. As a mathematician, I already had a "flexible and deep understanding" of most of the mathematical concepts, but I soon realized that I needed to learn much more about statistics, which is an important part of the curriculum. Determining what the students knew and what their intuitions were telling them involved many hours spent in high school classrooms and talking with students (and that was fun). The principle of building ideas gradually and concretely had been driven home to me through many years of experience as a college-level teacher but needed constant attention in this work, especially because I was working at a different level of mathematics learning.

As to whether something really worked in the classroom, that dimension involved a more personal challenge. I needed to be willing to tear up something I'd spent months developing and start over with a new approach. I had to do that more often than I liked, but the final rewards made it worthwhile.

Overall, the work was as challenging and satisfying to me as a mathematician as any theorem or proof I ever developed, and certainly gave me a greater sense of contributing to society than I ever hoped to do through mathematical research.