Phase Plots of Complex Functions: A Journey in Illustration

Elias Wegert and Gunter Semmler

This work was inspired by the article "Möbius transformations revealed" by Douglas Arnold and Jonathan Rogness [3]. There the authors write:

"*Among the most insightful tools that mathematics has developed is the representation of a function of a real variable by its graph. … The situation is quite different for a function of a complex variable. The graph is then a surface in four-dimensional space, and not so easily drawn. Many texts in complex analysis are without a single depiction of a function. Nor is it unusual for average students to complete a course in the subject with little idea of what even simple functions, say trigonometric functions, 'look like'.*"

In the printed literature there are a few laudable exceptions to this rule, such as the prize-winning *Visual Complex Analysis* by Tristan Needham [28], Steven Krantz's textbook [21] with a chapter on computer packages for studying complex variables, and the MAPLE-based (German) introduction to complex function theory [15] by Wilhelm Forst and Dieter Hoffmann.

But looking behind the curtain, one encounters a different situation that is evolving very quickly. Some of us have developed our own techniques for visualizing complex functions in teaching and research, and one can find many beautiful illustrations of complex functions on the Internet.

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Figure 1. The phase plot of an analytic function in the unit disk.

This article is devoted to "phase plots" or "phase portraits", a special tool for visualizing and exploring analytic functions. Figure 1 shows such a fingerprint of a function in the complex unit disk. The explanation of this illustration is deferred to a later section, where it is investigated in detail.

Phase plots have been invented independently by a number of people, and it is impossible to give credit to someone for being the first. Originally, they were mainly used in teaching as simple and effective methods for visualizing complex functions. Over the years, and in particular during the process of writing and rewriting this manuscript, the topic developed its own dynamic, and gradually these

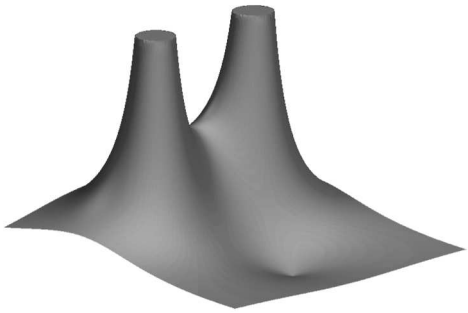

Figure 2. The analytic landscape of $f(z) = (z-1)/(z^2+z+1)$.

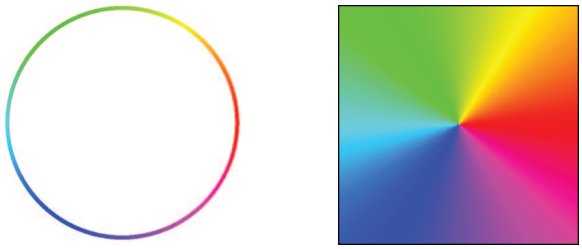

Figure 3. The color circle and the color-coded phase of points close to the origin.

innocent illustrations transmuted to sharp tools for dissecting complex functions.

So the main purpose of this article is not only to present nice pictures that allow one to recognize complex functions by their individual faces, but also to develop the mathematical background and demonstrate the utility and creative uses of phase plots. That they sometimes also facilitate a new view on known results and may open up new perspectives is illustrated by a universality property of the Riemann zeta function that, in the setting of phase plots, can be explained to (almost) anyone.

The final section is somewhat special. It resulted from a self-experiment carried out to demonstrate that phase plots are sources of inspiration that can help to establish new results. The main finding is that any meromorphic function is associated with a *dynamical system* that generates a *phase flow* on its domain and converts the phase plot into a *phase diagram*. These diagrams will be useful tools for exploring complex functions, especially for those who prefer thinking geometrically.

Visualization of Functions

The graph of a function $f : D \subset \mathbb{C} \to \mathbb{C}$ lives in four real dimensions, and since our imagination is trained in three-dimensional space, most of us have difficulties in "seeing" such an object.[1]

Some books on complex function theory have nice illustrations showing the *analytic landscape* of a function, which is the graph of its modulus (see Figure 2). The concept was not introduced by Johann Jensen in 1912, as is sometimes claimed, but probably earlier by Edmond Maillet [27] in 1903 (see also Otto Reimerdes's paper [32] of 1911).

Differential geometric properties of analytic landscapes have been studied in quite a number of early papers (see Ernst Ullrich [37] and the references therein). Jensen [20] and others also considered the graph of $|f|^2$, which is a smooth surface. The second edition of "Jahnke-Emde" [19] made analytic landscapes popular in applied mathematics.

Analytic landscapes involve only one part of the function f, its *modulus* $|f|$; the *argument* $\arg f$ is lost. In the era of black-and-white illustrations our predecessors sometimes compensated for this shortcoming by complementing the analytic landscape with lines of constant argument. Today we can achieve this much better using colors. Since coloring is an essential ingredient of phase plots, we consider it in some detail.

Recall that the argument $\arg z$ of a complex number z is unique up to an additive multiple of 2π. In order to make the argument well defined its values are often restricted to the interval $(-\pi, \pi]$, or, even worse, to $[0, 2\pi)$. This ambiguity disappears if we replace $\arg z$ with the *phase* $z/|z|$ of z. Though one usually does not distinguish between the notions of "argument" and "phase", it is essential here to keep these concepts separate.

The phase lives on the complex unit circle $\mathbb{T}$, and points on a circle can naturally be encoded by *colors*. We thus let color serve as the lacking fourth dimension when representing graphs of complex-valued functions (see Figure 3). The *colored analytic landscape* is the graph of $|f|$ colored according to the phase of f. Since the modulus of analytic functions typically varies over a wide range, one does better to use a logarithmic scaling of the vertical axis. This representation is also more natural since $\log|f|$ and $\arg f$ are conjugate harmonic functions (see Figure 4). Colored analytic landscapes came to life with easy access to computer graphics and by now quite a number of people have developed software for their visualization. Andrew Bennett [7] has an easy-to-use Java implementation, and an executable Windows program can be downloaded from Donald Marshall's website [26]. We further refer to Chapter 12 of Steven Krantz's book [21], as well as to the websites run by Hans Lundmark [25] and Tristan Needham [29]. Very beautiful pictures of (uncolored) analytic landscapes can be found on the "The Wolfram Special Function Site" [43].[2]

[1] *One exception is Thomas Banchoff, who visualized four-dimensional graphs of complex functions in* [6].

[2] *As of March 2011 Wolfram's tool visualizes the modulus and the principal value of the argument, but not phase.*

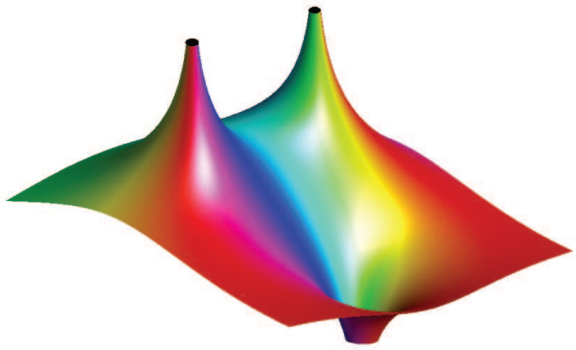

Figure 4. The colored logarithmic analytic landscape of $f(z) = (z-1)/(z^2+z+1)$.

Though color printing is still expensive, colored analytic landscapes also appear in the printed literature (see, for example, the outstanding mathematics textbook [1] by Arens et al. for engineering students).

With colored analytic landscapes the problem of visualizing complex functions could be considered solved. However, there is yet another approach that is not only simpler but even more general.

Instead of drawing a graph, one can depict a function directly on its domain by color-coding its values, thus converting it to an *image*. Such *color graphs* of functions f live in the product of the domain of f with a color space.

Coloring techniques for visualizing functions have been familiar for many decades, for example in depicting altitudes on maps, but mostly they represent *real-valued* functions using a one-dimensional color scheme. It is reported that two-dimensional color schemes for visualizing complex valued functions have been in use for more than twenty years by now (Larry Crone [9], Hans Lundmark [25]), but they became popular only with Frank Farris's review [13] of Needham's book and its complement [14]. Farris also coined the name "domain coloring".

Domain coloring is a natural and universal substitute for the graph of a function. Moreover, it easily extends to functions on Riemann surfaces or on surfaces embedded in $\mathbb{R}^3$ (see Konstantin Poelke and Konrad Polthier [31], for instance).

It is worth mentioning that we human beings are somewhat limited with respect to the available color spaces. Since our visual system has three different color receptors, we can only recognize colors from a three-dimensional space. Mathematicians of the species *Gonodactylus oerstedii*[3] could use domain coloring techniques to even visualize functions with values in a twelve-dimensional space (Welsch and Liebmann [41], p. 268; for details see Cronin and King [10]).

[3] *The Mantis shrimp (Gonodactylus oerstedii) has twelve separate visual pigments.*

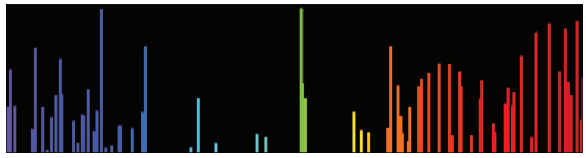

Figure 5. A typical spectrum of polar light.

Indeed, many people are not aware that natural colors in fact provide us with an *infinite dimensional space*—at least theoretically. In reality "color" always needs a carrier. "Colored light" is an electromagnetic wave that is a mixture of monochromatic components with different wavelengths and intensities. A simple prismatic piece of glass reveals how light is composed from its *spectral components*. Readers interested in further information are recommended to visit the fascinating Internet site of Dieter Zawischa [44]. The wavelengths of visible light fill an interval between 375 nm and 750 nm approximately, and hence color spectra form an infinite-dimensional space (see Figure 5).

How many color dimensions are distinguishable in reality depends on the resolution of the measuring device. A simple model of the human eye, which can be traced back to Thomas Young in 1802, assumes that our color recognition is based on three types of receptors that are sensitive to red, green, and blue, respectively.

Since, according to this assumption, our visual color space has dimension three, different spectra of light induce the same visual impression. Interestingly, a mathematical theory of this effect was developed as early as in 1853 by Hermann Grassmann, the ingenious author of the "Ausdehnungslehre", who found three fundamental laws of this so-called *metamerism* [16] (see Welsch and Liebmann [41]).

Bearing in mind that the world of real colors is infinite dimensional, it becomes obvious that its compression to at most three dimensions cannot lead to completely satisfying results, which explains the variety of color schemes in use for different purposes. The most popular color systems in our computer-dominated world are the RGB, CMYK, and HSV schemes.

In contrast to domain colorings that color-code the complete values $f(z)$ by a two-dimensional color scheme, *phase plots* display only $f(z)/|f(z)|$ and thus require just a *one-dimensional* color space with a *circular topology*. As will be shown in the next section, they nevertheless contain almost all relevant information about the depicted analytic or meromorphic function.

In Figure 6 the Riemann sphere $\hat{\mathbb{C}}$ (with the point at infinity on top) is colored using two typical schemes for phase plots (left) and domain

Figure 6. Color schemes for phase plots and domain coloring on the Riemann sphere.

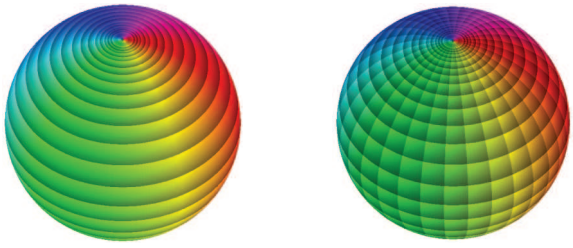

Figure 7. Two color schemes involving sawtooth functions of gray.

coloring (right), respectively. Somewhat surprisingly, the number of people using phase plots seems to be quite small. The website of François Labelle [24] has a nice gallery of nontrivial pictures,[4] including Euler's gamma and Riemann's zeta functions.

Since the phase of a function occupies only one dimension of the color space, there is plenty of room for depicting additional information. It is recommended to encode this information by a gray scale, since color (hue) and brightness are *visually orthogonal*. Figure 7 shows two such color schemes on the Riemann w-sphere. The left scheme is a combination of phase plots and standard domain coloring. Here the brightness b does not depend monotonously on $\log|w|$ but is a sawtooth function thereof, like, for example, $b(w) = \log|w| - \lfloor \log|w| \rfloor$. This coloring works equally well, no matter in which range the values of the function are located.

In the right scheme the brightness is the product of two sawtooth functions depending on $\log|w|$ and $w/|w|$, respectively. The discontinuities of this shading generate a logarithmically scaled polar grid. *Pulling back* the coloring from the w-sphere to the z-domain of f by the mapping $w = f(z)$ resembles a *conformal grid mapping*, another well-known technique for depicting complex functions (see Douglas Arnold [2]). Note that pulling back a grid instead of pushing it forward avoids multiple coverings. Of course all coloring schemes can also be applied to functions on Riemann surfaces.

[4] *Labelle justifies the sole use of phase by reasons of clarity and aesthetics.*

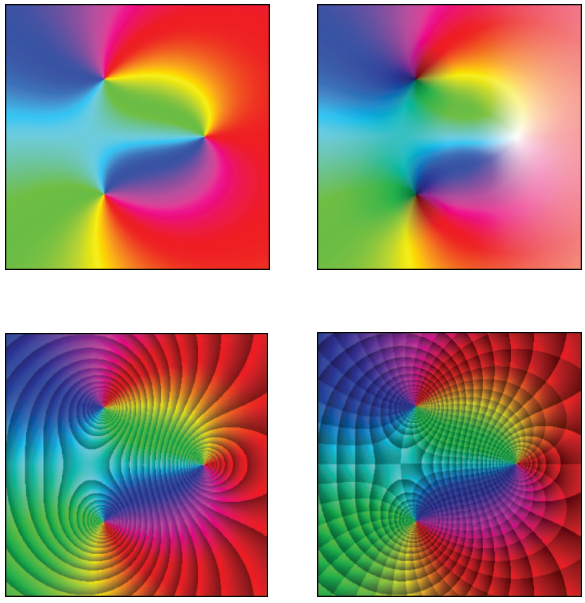

Figure 8. Four representations of the function $f(z) = (z-1)/(z^2+z+1)$.

For comparison, Figure 8 shows the four representations of $f(z) := (z-1)/(z^2+z+1)$ in the square $|\mathrm{Re}\, z| \le 2$, $|\mathrm{Im}\, z| \le 2$ corresponding to the color schemes of Figure 6 and Figure 7, respectively. Though these pictures (in particular the upper two) look quite similar, which makes it simple to use them in parallel, the philosophy and the mathematics behind them is quite different. I shall comment on this issue in the final section.

The Phase Plot

The phase of a complex function $f : D \to \hat{\mathbb{C}}$ is defined on $D_0 := \{z \in D : f(z) \in \mathbb{C}^\times\}$, where $\mathbb{C}^\times$ denotes the complex plane punctured at the origin. Nevertheless we shall speak of phase plots $P : D \to \mathbb{T}, z \mapsto f(z)/|f(z)|$ on D, considering those points where the phase is undefined as singularities. Recall that $\mathbb{T}$ stands for the (colored) unit circle.

To begin with we remark that meromorphic functions are characterized almost uniquely by their phase plot.

Theorem 1. *If two nonzero meromorphic functions f and g on a connected domain D have the same phase, then f is a positive scalar multiple of g.*

Proof. Removing from D all zeros and poles of f and g, we get a connected domain D_0. Since, by assumption, $f(z)/|f(z)| = g(z)/|g(z)|$ for all $z \in D_0$, the function f/g is holomorphic and real-valued in D_0, and so it must be a (positive) constant.

It is obvious that the result extends to the case in which the phases of f and g coincide merely on an open subset of D.

In order to check whether two functions f and g with the same phase are equal, it suffices to compare their values at a single point that is neither a

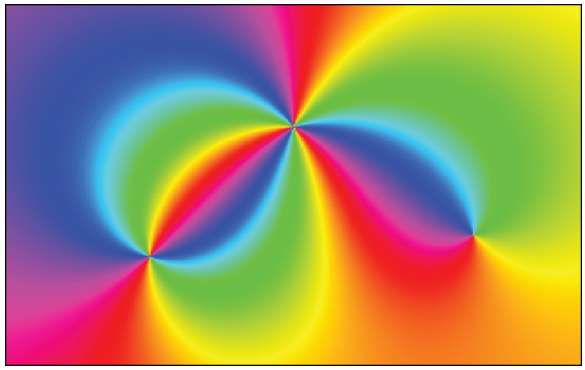

Figure 9. A function with a simple zero, a double zero, and a triple pole.

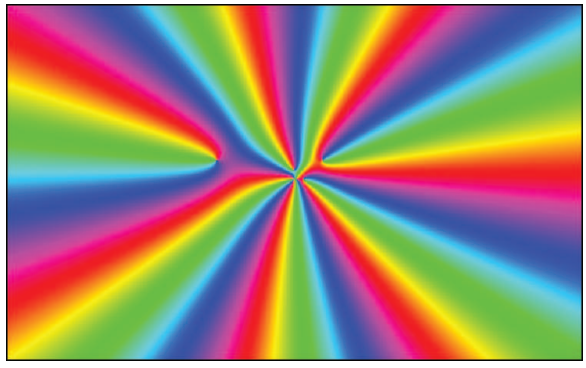

Figure 10. This function has no poles. How many zeros are in the displayed rectangle?

zero nor a pole. For purists there is also an intrinsic test that works with phases alone: Assume that the nonconstant meromorphic functions f and g have the same phase plot. Then it follows from the open mapping principle that $f \neq g$ if and only if the phase plots of $f + c$ and $g + c$ are different for one, and then for all, complex constants $c \neq 0$.

Zeros and Poles

Since the phases of zero and infinity are undefined, zeros and poles of a function are singularities of its phase plot. What does the plot look like in a neighborhood of such points?

If a meromorphic function f has a zero of order n at z_0 it can be represented as

$$f(z) = (z - z_0)^n g(z),$$

where g is meromorphic and $g(z_0) \in \mathbb{C}^\times$. It follows that the phase plot of f close to z_0 resembles the phase plot of z^n at 0, rotated by the angle $-(1/n)\arg g(z_0)$. The same reasoning, with a negative integer n, applies to poles.

Note in Figure 9 that the colors are arranged in opposite orders for zeros and poles. It is now clear that the phase plot not only shows the location of zeros and poles but also reveals their *multiplicity*.

A useful tool for locating zeros is the *argument principle*. In order to formulate it in the context of phase plots we translate the definition of winding number into the language of colors: Let $\gamma : \mathbb{T} \to D_0$ be a closed oriented path in the domain D_0 of a phase plot $P : D_0 \to \mathbb{T}$. Then the usual winding number of the mapping $P \circ \gamma : \mathbb{T} \to \mathbb{T}$ is called the *chromatic number* of γ with respect to the phase plot P and is denoted by $\mathrm{chrom}_P \gamma$ or simply by $\mathrm{chrom}\,\gamma$.

Less formally, the chromatic number counts how many times the color of the point $\gamma(t)$ moves around the complete color circle when $\gamma(t)$ traverses γ once in the positive direction.

Now the argument principle can be rephrased as follows: *Let D be a Jordan domain with positively oriented boundary ∂D and assume that f is meromorphic in a neighborhood of D. If f has n zeros and p poles in D (counted with multiplicity), and none of them lies on ∂D, then*

$$n - p = \mathrm{chrom}\,\partial D.$$

Looking at Figure 10 in search of zeros immediately brings forth new questions, for example: Where do the isochromatic lines end up? Can they connect two zeros? If so, do these lines have a special meaning? What about "basins of attraction"? Is there always a natural (cyclic) ordering of zeros? What can be said about the global structure of phase plots? We shall return to these issues later.

The Logarithmic Derivative

Along the *isochromatic lines* of a phase plot the argument of f is constant. The Cauchy-Riemann equations for a continuous branch of the logarithm $\log f = \log|f| + \mathrm{i}\arg f$ imply that these lines are orthogonal to the level lines of $|f|$, i.e., the isochromatic lines are parallel to the gradient of $|f|$. According to the chosen color scheme, we have red on the right and green on the left when walking on a yellow line in the ascending direction.

To go a little beyond this qualitative result, we denote by s the unit vector parallel to the gradient of $|f|$ and set $n := \mathrm{i}s$. With $\varphi := \arg f$ and $\psi := \log|f|$ the Cauchy-Riemann equations for $\log f$ imply that the directional derivatives of φ and ψ satisfy

$$\partial_s \psi = \partial_n \varphi > 0, \quad \partial_n \psi = -\partial_s \varphi = 0,$$

at all points z of the phase plot where $f(z) \neq 0$ and $f'(z) \neq 0$. Since the absolute value of $\partial_n \varphi$ measures the density of the isochromatic lines, we can visually estimate the growth of $\log|f|$ along these lines from their density. Because the phase plot delivers no information on the absolute value, this does not say much about the growth of $|f|$. But taking into account the second Cauchy-Riemann equation and

$$|(\log f)'|^2 = (\partial_n \varphi)^2 + (\partial_s \varphi)^2,$$

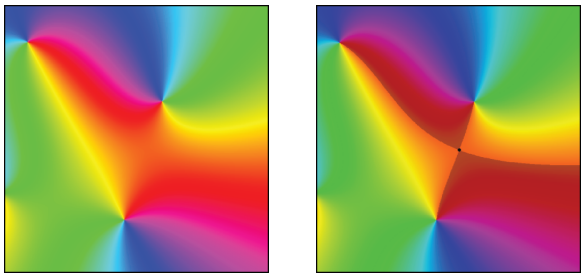

Figure 11. Zeros of f' are color saddles.

we obtain the correct interpretation of the density $\partial_n \varphi$: it is the modulus of the *logarithmic derivative*,

$$\partial_n \varphi = |f'/f|\,. \tag{1}$$

So, finally, we need not worry about branches of the logarithm. It is worth mentioning that $\partial_n \varphi(z)$ behaves asymptotically like $k/|z-z_0|$ if z approaches a zero or pole of order k at z_0.

But this is not yet the end of the story. What about zeros of f'? Equation (1) indicates that something should be visible in the phase plot. Indeed, points z_0 where $f'(z_0) = 0$ and $f(z_0) \neq 0$ are "*color saddles*", i.e., intersections of isochromatic lines.

If f' has a zero of order k at z_0, then $z \mapsto f(z) - f(z_0)$ has a zero of order $k+1$ at z_0. Consequently, f can be represented as

$$f(z) = f(z_0) + (z - z_0)^{k+1}\, g(z),$$

where $g(z_0) \neq 0$. It follows that $f(z)$ travels $k+1$ times around $f(z_0)$ when z moves once around z_0 along a small circle. In conjunction with $f(z_0) \neq 0$ this can be used to show that there are exactly $2k+2$ isochromatic lines emanating from z_0 where the phase of f is equal to the phase of $f(z_0)$. Alternatively, one can also think of $k+1$ smooth isochromatic lines intersecting each other at z_0. Color saddles appear as diffuse spots such as at the center of the left picture in Figure 11. To locate them precisely it is helpful to modify the color scheme so that it gets a jump at some point t of the unit circle. If $t := f(z_0)/|f(z_0)|$ is chosen, then the phase plot shows a sharp saddle at the zero z_0 of f' as in the right picture.

Essential Singularities

Have you ever seen an essential singularity? Figure 12 is the picture that usually illustrates this situation.

Despite the massive tower, this is not very impressive, and with regard to the Casorati-Weierstrass theorem or the Great Picard theorem one would expect something much wilder. Why does the analytic landscape not reflect this behavior? For the example the answer is easy: the function has a tame modulus, every contour line is a single circle through the origin. Now look at the phase plot in Figure 13. But must there not be a symmetry between modulus and phase? In fact

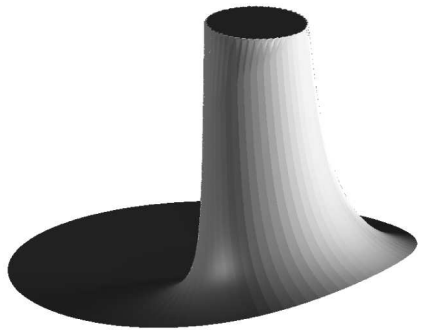

Figure 12. The analytic landscape of $f(z) = e^{1/z}$.

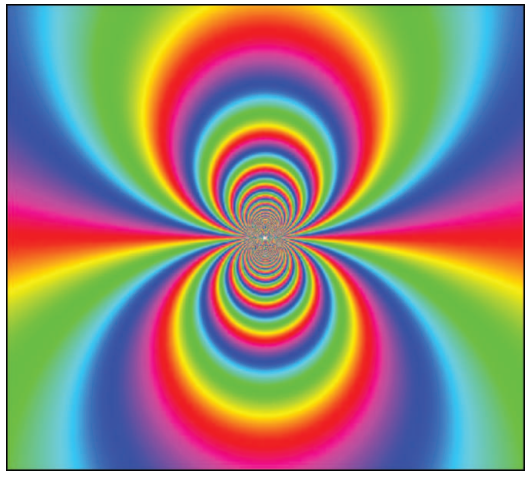

Figure 13. A phase plot depicting the essential singularity of $f(z) = e^{1/z}$.

not. There is such a symmetry of modulus and *argument* (for nonvanishing functions), but phase plots depict the phase and not the argument—and this makes a difference.

So much for the example, but what about the general case? Perhaps there are also functions that conceal their essential singularities in the phase plot?

In order to show that this cannot happen, we assume that $f : D\{z_0\} \to \mathbb{C}$ is analytic and has an essential singularity at z_0.

By the Great Picard theorem, there exists a color $c \in \mathbb{T}$ such that any punctured neighborhood U of z_0 contains infinitely many points $z_k \in U$ with $f(z_k) = c$. Moreover, the set of zeros of f' in D is at most countable, and hence we can choose c such that no saddle point $z \in U$ with $f'(z) = 0$ and $f(z) \neq 0$ lies on an isochromatic line with color c. As was shown in the preceding section, the modulus of f is strictly monotone along such lines, so that two distinct points z_k cannot lie on the same isochromatic line.

Consequently any neighborhood of an essential singularity contains a countable set of pairwise disjoint isochromatic lines with color c. Combining

Figure 14. Phase plot of $f(z) = e^z$.

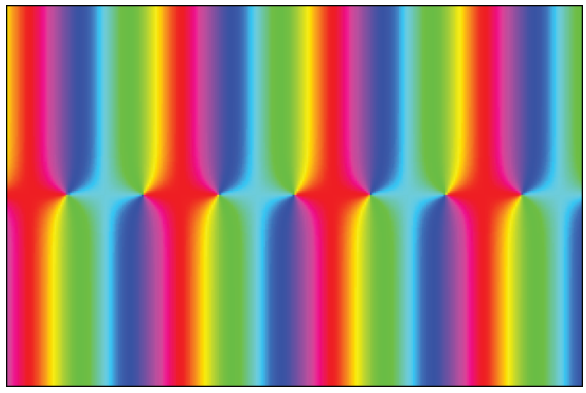

Figure 15. Phase plot of $f(z) = \sin z$.

this observation with the characterization of phase plots near poles and removable singularities, we obtain the following result.

Theorem 2. *An isolated singularity z_0 of an analytic function f is an essential singularity if and only if any neighborhood of z_0 intersects infinitely many isochromatic lines of the phase plot with one and the same color.*

Note that a related result does not hold for the argument, since then, in general, the values of $\arg f(z_k)$ are different. For example, any two isochromatic lines of the function $f(z) = \exp(1/z)$ have a different argument.

Periodic Functions

Obviously, the phase of a periodic function is periodic, but what about the converse?

Though there are only *two* classes (simply and doubly periodic) of nonconstant periodic meromorphic functions on $\mathbb{C}$, we can observe *three* different types of periodic phase plots.

"Striped" phase plots such as Figure 14 always depict exponential functions $f(z) = e^{az+b}$ with $a \neq 0$. Functions with simply p-periodic phase (see Figure 15, for example) need not be periodic, but have the more general form $e^{\alpha z/p}\, g(z)$ with $\alpha \in \mathbb{R}$ and a p-periodic function g.

The first result basically follows from the fact that the function $\arg f$ is harmonic and has parallel straight contour lines, which implies that $\arg f(x + iy) = \alpha x + \beta y + \gamma$. Since $\log |f|$ is conjugate harmonic to $\arg f$, it necessarily has the form $\log |f(x + iy)| = -\alpha y + \beta x + \delta$.

If the phase of f is p-periodic, then we have

$$h(z) := \frac{f(z+p)}{f(z)} = \frac{|f(z+p)|}{|f(z)|} \in \mathbb{R}_+,$$

and since h is meromorphic on $\mathbb{C}$, it must be a positive constant e^{α}. Now it follows easily that $g(z) := f(z) \cdot e^{-\alpha z/p}$ is periodic with period p.

Finally, if the phase plot of f is doubly periodic (see Figure 16), i.e., $f/|f|$ has periods p_1 and p_2 with $p_1/p_2 \notin \mathbb{R}$, then it does not necessarily follow

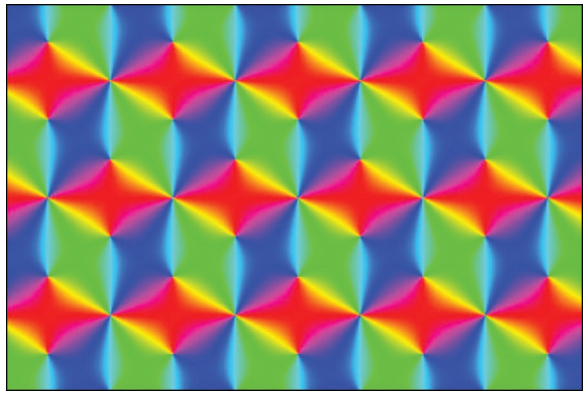

Figure 16. Phase plot of a Weierstrass $\wp$-function.

that f is an elliptic function. A counterexample is $f(z) := e^{\alpha z} g(z)$, where $g \neq$ const has periods p_1 and p_2, and $\alpha \in \mathbb{C}\backslash\{0\}$ is a solution of the linear system

$$\operatorname{Im}(\alpha p_j) = 2k_j\pi, \quad j = 1, 2,$$

with $k_1, k_2 \in \mathbb{Z}$. It may be an interesting problem to prove that all functions with doubly periodic phase plots are indeed of this form.

Partial Sums of Power Series

Figure 17 shows a strange image that, in similar form, occurred in a numerical experiment as the phase plot of a Taylor polynomial. Since it looks so special, one could attribute it to a programming error. A moment's thought reveals what is going on here. This example demonstrates again that looking at phase plots can immediately provoke new questions.

Indeed the figure illustrates a general result (see Titchmarsh [36], Section 7.8) that was proven by Robert Jentzsch in 1914:

If a power series $a_0 + a_1 z + a_2 z^2 + \cdots$ has a positive finite convergence radius R, then the zeros of its partial sums cluster at every point z with $|z| = R$.

The reader interested in the life and personality of Robert Jentzsch is referred to the recent

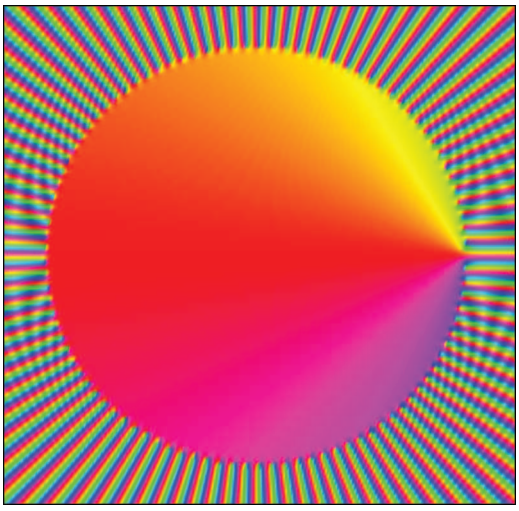

Figure 17. A Taylor polynomial of $f(z) = 1/(1-z)$.

paper [11] by Peter Duren, Anne-Katrin Herbig, and Dmitry Khavinson.

Boundary Value Problems

Experimenting with phase plots raises a number of new questions. One such problem is to find a criterion for deciding which color images are *analytic phase plots*, i.e., phase plots of analytic functions.

Since phase plots are painted with the restricted palette of *saturated* colors from the color circle, Leonardo's Mona Lisa will certainly never appear. But for analytic phase plots there are much stronger restrictions: By the uniqueness theorem for harmonic functions an arbitrarily small open piece determines the plot entirely.

So let us pose the question a little differently: What are appropriate data that can be prescribed to construct an analytic phase plot, say, in a Jordan domain D? Can we start, for instance, with given colors on the boundary ∂D? If so, can the boundary colors be prescribed arbitrarily, or are they subject to constraints?

In order to state these questions more precisely we introduce the concept of a *colored set* K_C, which is a subset K of the complex plane together with a mapping $C : K \to \mathbb{T}$. Any such mapping is referred to as a *coloring* of K.

For simplicity we consider here only the following setting of boundary value problems for phase plots with continuous colorings:

Let D be a Jordan domain and let B be a continuous coloring of its boundary ∂D. Find all continuous colorings C of $\overline{D}$ such that the restriction of C to ∂D coincides with B and the restriction of C to D is the phase plot of an analytic function f in D.

If such a coloring C exists, we say that the coloring B *admits a continuous analytic extension to* $\overline{D}$.

The restriction to continuous colorings automatically excludes zeros of f in D. It does, however, *not* imply that f must extend continuously onto $\overline{D}$—and in fact it is essential not to require the continuity of f on $\overline{D}$ in order to get a nice result.

Theorem 3. *Let D be a Jordan domain with a continuous coloring B of its boundary ∂D. Then B admits a continuous analytic extension to $\overline{D}$ if and only if the chromatic number of B is zero. If such an extension exists, then it is unique.*

Proof. If $C : \overline{D} \to \mathbb{T}$ is a continuous coloring, then a simple homotopy argument (contract ∂D inside D to a point) shows that the chromatic number of its restriction to ∂D must vanish.

Conversely, any continuous coloring B of ∂D with chromatic number zero can be represented as $B = e^{i\varphi}$ with a continuous function $\varphi : \partial D \to \mathbb{R}$. This function admits a unique continuous harmonic extension Φ to $\overline{D}$. If Ψ denotes a harmonic conjugate of Φ, then $f = e^{i\Phi - \Psi}$ is analytic in D. Its phase $C := e^{i\Phi}$ is continuous on $\overline{D}$ and coincides with B on ∂D. Recall that f need not be continuous on $\overline{D}$.

Theorem 3 *parameterizes* analytic phase plots that extend continuously on $\overline{D}$ by their boundary colorings. This result can be generalized to phase plots that are continuous on $\overline{D}$ with the exception of finitely many singularities of zero or pole type in D. Admitting now boundary colorings B with arbitrary chromatic number we get the following result:

For any finite collection of given zeros with orders $n_1, \dots, n_j$ and poles of orders $p_1, \dots, p_k$ the boundary value problem for meromorphic phase plots with prescribed singularities has a (unique) solution if and only if the boundary coloring B satisfies

$$\operatorname{chrom} B = n_1 + \cdots + n_j - p_1 - \cdots - p_k.$$

The Riemann Zeta Function

After these preparations we are ready to pay a visit to "zeta", the mother of all analytic functions. Figure 18 is a phase plot in the square $-40 \le \operatorname{Re} z \le 10$, $-2 \le \operatorname{Im} z \le 48$. We see the pole at $z = 1$, the *trivial zeros* at the points $-2, -4, -6, \dots$, and several zeros on the *critical line* $\operatorname{Re} z = 1/2$. Also we observe that the isochromatic lines are quite regularly distributed in the left half plane.

Saying that zeta is the mother of all functions alludes to its *universality*. Our starting point is the following strong version of Voronin's universality theorem due to Bagchi [4] (see also Karatsuba and Voronin [22], Steuding [35]):

Let D be a Jordan domain such that $\overline{D}$ is contained in the strip

$$R := \{z \in \mathbb{C} : 1/2 < \operatorname{Re} z < 1\},$$

and let f be any function that is analytic in D, continuous on $\overline{D}$, and has no zeros in $\overline{D}$. Then f can be

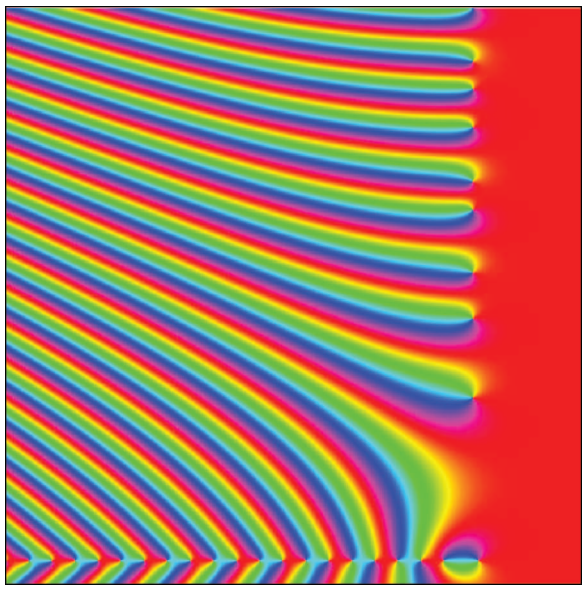

Figure 18. The Riemann zeta function.

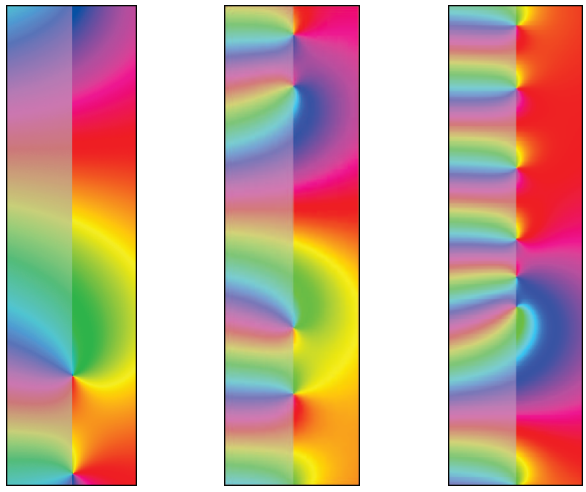

Figure 20. The Riemann zeta function at $\operatorname{Im} z = 171$, 8230 and 121415.

uniformly approximated on $\overline{D}$ *by vertical shifts of zeta,* $\zeta_t(z) := \zeta(z + it)$ *with* $t \in \mathbb{R}$.

Recall that a continuously colored Jordan curve J_C is a continuous mapping $C : J \to \mathbb{T}$ from a simple closed curve J into the color circle $\mathbb{T}$. A *string* S is an equivalence class of all such colored curves with respect to rigid motions of the plane. Like colored Jordan curves, strings fall into different classes according to their *chromatic number*. Figure 19 depicts a representative of a string with chromatic number one.

We say that a string S *lives in a domain* D if it can be represented by a colored Jordan curve J_C with $J \subset D$. A string can *hide itself* in a phase plot $P : D \to \mathbb{T}$ if, for every $\varepsilon > 0$, it has a representative J_C such that $J \subset D$ and

$$\max_{z \in J} |C(z) - P(z)| < \varepsilon.$$

In less technical terms, a string can hide itself if it can move to a place where it is invisible since it blends in almost perfectly with the background.

In conjunction with Theorem 3, the following universality result for the phase plot of the Riemann zeta function can easily be derived from Voronin's theorem.

Figure 19. A representative of a string.

Theorem 4. *Let S be a string which lives in the strip R. Then S can hide itself in the phase plot of the Riemann zeta function on R if it has chromatic number zero.*

In view of the extreme richness of Jordan curves and colorings, this result is a real miracle. Figure 20 shows phase plots of zeta in the critical strip. The regions with saturated colors belong to R. The rightmost figure depicts the domain considered on p. 342 of Conrey's paper [8].

What about the converse of Theorem 4? If there exist strings with *nonzero chromatic number* that can hide themselves in the strip R, their potential hiding places must be Jordan curves with nonvanishing chromatic number in the phase plot. By the argument principle, this would imply that zeta has zeros in R. If we assume this, for a moment, then such strings indeed exist: they are perfectly hidden and wind themselves once around such a zero. So the converse of Theorem 4 holds if and only if R contains no zeros of zeta, which is known to be equivalent to the *Riemann hypothesis* (see Conrey [8], Edwards [12]).

Phase Flow and Diagrams

Mathematical creativity is based on the interplay of problem posing and problem solving, and it is our belief that the former is even more important than the latter: often the key to solving a problem lies in asking the right questions.

Illustrations have a high density of information and stimulate imagination. Looking at pictures helps in getting an intuitive understanding of mathematical objects and finding interesting questions, which then can be investigated using rigorous mathematical techniques.

This section intends to demonstrate how phase plots can produce novel ideas. The material presented here is the protocol of a self-experiment that has been carried out by the first author in order to check the creative potential of phase plots.

Let us start by looking at Figure 1 again. It depicts the phase of a *finite Blaschke product*, which

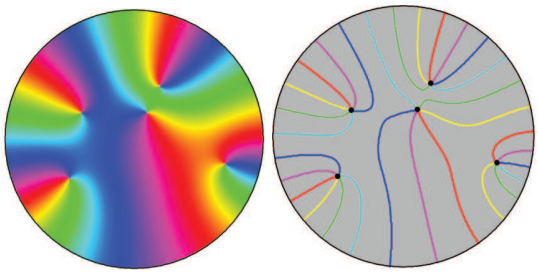

Figure 21. The phase plot of a Blaschke product with five zeros.

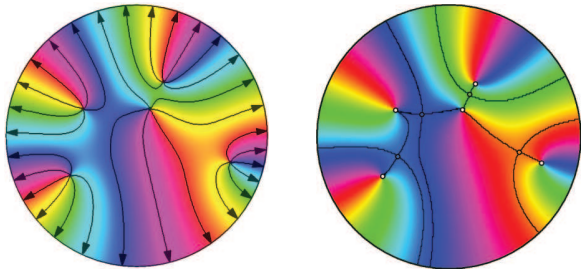

Figure 22. Phase transport to the boundary, zeros of f and f' with invariant manifolds.

is a function of the form

$$f(z) = c \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \quad z \in \mathbb{D},$$

with $|z_k| < 1$ and $|c| = 1$. Blaschke products are fundamental building blocks of analytic functions in the unit disc and have the property $|f(z)| = 1$ for all $z \in \mathbb{T}$. The function shown in Figure 1 has eighty-one zeros z_k in the unit disk.

Looking at Figure 1 for a while leaves the impression of a cyclic ordering of the zeros. Let us test this with another example having only five zeros (Figure 21, left). The picture seems to confirm the expectation: if we focus attention to the yellow color, any of these lines connects a zero with a certain point on the boundary, thus inducing a cyclic ordering.

However, looking only at one specific color is misleading. Choosing another one, for instance blue, can result in a different ordering. So what is going on here? More precisely: What is the *global structure* of the phase plot of a Blaschke product? This could be a good question.

An appropriate mathematical framework in which to develop this idea is the theory of *dynamical systems*. We here sketch only the basic facts; for details see [39].

With any meromorphic function f in a domain D we associate the dynamical system

$$\dot{z} = g(z) := \frac{f(z)\overline{f'(z)}}{|f(z)|^2 + |f'(z)|^2}. \tag{2}$$

The function g on the right-hand side of (2) extends from D_0 to a smooth function on D. This system induces a *flow* Φ on D, which we designate as the *phase flow* of f.

The fixed points of (2) are the zeros of f (repelling), the poles of f (attracting), and the zeros of f' which are not zeros of f (saddles). The remaining orbits are the components of the isochromatic lines of the phase plot of f when the fixed points are removed. Thus the orbits of the phase flow endow the phase plot with an additional structure and convert it into a *phase diagram* (for details see [39]).

Intuitively, the phase flow Φ transports a colored substance ("phase") from the zeros to the poles and to the boundary of the domain (see Figure 22). The left part of Figure 23 illustrates how "phase" of measure 2π emerging from the zero in the highlighted domain is transported along the orbits of Φ until it is finally deposited along (parts of) the boundary.

In the general case, where $f : D \to \hat{\mathbb{C}}$ is meromorphic on a domain D and $G \subset D$ is a Jordan domain with boundary J in D_0, the phase transport from any zero (pole) of f in G generates a (signed) *measure* on J. The result is a *quantitative version of the argument principle* that tells us in which way the phase of the zeros (poles) is distributed along J (see Figure 23, right).

The question about the structure of phase plots of Blaschke products can now be rephrased in the setting of dynamical systems: What are the *basins of attraction* of the zeros of f with respect to the (reversed) phase flow?

The key to solving this problem is given by the *invariant manifolds* of the saddle points, i.e., the points $a_j \in \mathbb{D}$, where $f'(a_j) = 0$ and $f(a_j) \neq 0$. Removing all *unstable manifolds* of the points a_j from $\mathbb{D}$ results in an open set B, which is the union of connected components B_j. Any component B_j contains exactly one zero b_j of f, where multiple zeros are counted only once.

The intersection of every set $\overline{B_j}$ with $\mathbb{T}$ is not empty and consists of a finite number of arcs A_{ji}. The complete set of these arcs covers the unit circle, and two arcs are either disjoint or their intersection is a singleton. These *separating points* are the endpoints of unstable manifolds that originate from saddle points. For later use we renumber the arcs A_{ji} as $A_1, A_2, \ldots, A_s$ in a counterclockwise direction.

It is obvious that the number s of separating points cannot be less than the number of distinct zeros of f. In order to get an upper bound of s, f has m distinct zeros with multiplicities $\beta_1, \ldots, \beta_m$ and k saddle points where f' has zeros of multiplicities $\alpha_1, \ldots, \alpha_k$, respectively. Then we have

$$\alpha_1 + \cdots + \alpha_k = n - 1, \quad \beta_1 + \cdots + \beta_m = n.$$

The first equation follows from the well-known fact that the derivative of a Blaschke product of order

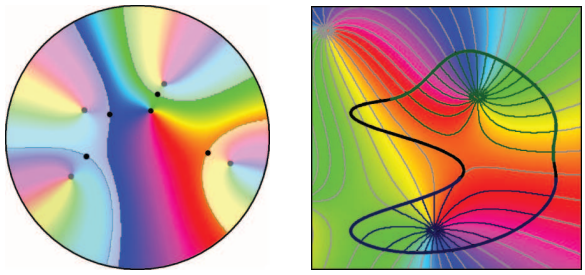

Figure 23. The phase flow and the argument principle.

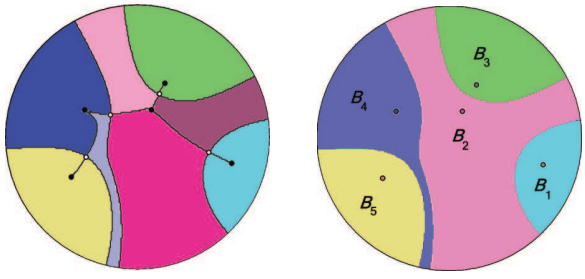

Figure 24. Invariant manifolds of the saddles and basins of attraction of the zeros.

n has exactly $n-1$ zeros in $\mathbb{D}$, this time counting multiplicity. From any saddle point a_j exactly $\alpha_j + 1$ rays emerge that belong to the unstable manifold of a_j. Since any separating point must be the endpoint of one such line, the total number s of separating points cannot be greater than $k + \alpha_1 + \cdots + \alpha_k = k + n - 1$. Thus we finally get

$$m \le s \le n + k - 1.$$

Examples show that both estimates are sharp.

It turns out ([34]) that the global topological structure of the phase plot is completely characterized by the sequence S of integers, which associates with any of the arcs $A_1, \ldots, A_s$ (in consecutive order) the number of the corresponding zero. This sequence depends on the specific numbering of the zeros and the arcs, but an appropriate normalization makes it unique. For example, the Blaschke product depicted in Figure 24 is represented by the sequence $S = (1,2,3,2,4,5,4,2)$.

Let us now return to Figure 1 again. Picturing once more that "phase" is a substance emerging from sources at the zeros that can exit the domain only at its boundary, is it not then quite natural that phase plots of Blaschke products *must* look like they do?

And if you are asking yourself what "natural" means, then this is already another question.

Concluding Remarks

Phase plots result from splitting the information about the function f into two parts (phase and modulus), and one may ask why we do not separate f into its real and imaginary parts. One reason is that often zeros are of special interest; their presence can easily be detected and characterized using the phase, but there is no way to find these from the real or imaginary part alone.

And what is the advantage of using $f/|f|$ instead of $\log|f|$? Of course, zeros and poles can be seen in the analytic landscape, but they are much better represented in the phase plot. In fact there is a subtle asymmetry between modulus and argument (respectively, phase). For example, Theorem 3 has no counterpart for the modulus of a function.[5]

Since phase plots and standard domain coloring produce similar pictures, it is worth mentioning that they are based on different concepts and have a distinct mathematical background.

Recall that standard domain coloring methods use the complete values of an analytic function, whereas phase plots depict only its phase. Taking into account that phase can be considered as a periodization of the argument, which is (locally) a harmonic function, reveals the philosophy behind phase plots: Analytic functions are considered as harmonic functions, endowed with a set of singularities having a special structure. Algebraically, phase plots forget about the linear structure of analytic functions, while their multiplicative structure is preserved.

This approach has at least two advantages. The first one is almost trivial: phase has a small range, the unit circle, which allows visualizing all functions with one and the same color scheme. Moreover, a one-dimensional color space admits a better resolution of singularities. Mathematically more important is the existence of a simple parameterization of *analytic* and *meromorphic* phase plots by their boundary values and their singularities (Theorem 3). There is no such result for domain colorings of analytic functions.

The following potential fields of applications demonstrate that phase plots may be a useful tool for anyone working with complex-valued functions.

1. A trivial but useful application is *visual inspection* of functions. If, for example, it is not known which branch of a function is used in a certain software, a glance at the phase plot may help. In particular, if several functions are composed, software implementations with different branch cuts can lead to completely different results. You may try this with the MATHEMATICA functions `Log(Gamma)` and `LogGamma`. Another useful exercise in teaching is to compare the phase plots of $\exp(\log z)$ and $\log(\exp z)$.

2. A promising field of application is visual analysis and synthesis of *transfer functions* in *systems*

[5] *There is such a result for* outer functions, *but it is impossible to see if a function is outer using only the boundary values of its modulus.*

theory and *filter design*. Since here the modulus (gain) is often more important than phase, it is recommended to use the left color scheme of Figure 7.

3. Further potential applications lie in the area of *Laplace* and complex *Fourier transforms*, in particular to the method of steepest descent (or stationary phase).

4. Phase plots also allow one to guess the *asymptotic behavior* of functions (compare, for example, the phase plots of $\exp z$ and $\sin z$) and to find *functional relations*. A truly challenging task is to rediscover the functional equation of the Riemann zeta function from phase plots of ζ and Γ (see [40], pp. 44–46).

5. Complex *dynamical systems*, in the sense of iterated functions, have been investigated by Felix Huang [18] and Martin Pergler [30] using domain coloring methods. The problem of scaling the modulus disappears when using phase plots; see the pictures of François Labelle [24] and Donald Marshall [26].

6. Geir Arne Hjelle [17] developed an interactive Java applet that visualizes interpolation problems for Blaschke products using phase portraits.

7. The utility of phase plots is not restricted to analytic functions. Figure 25 visualizes the function

$$h(z) := \operatorname{Im}\left(e^{-\frac{i\pi}{4}} z^n\right) + i \operatorname{Im}\left(e^{\frac{i\pi}{4}} (z-1)^n\right),$$

with $n = 4$. This is Wilmshurst's example [42] of a harmonic polynomial of degree n having the maximal possible number of n^2 zeros. For background information we recommend the paper on gravitational lenses by Dmitry Khavinson and Genevra Neumann [23]. To understand the construction of the depicted function it is important to keep track of the zeros of its real and imaginary parts. In Figure 25 these (straight) lines are visualized using a modified color scheme that has jumps at the points $1, i, -1$ and $-i$ on the unit circle.

Besides these and other concrete applications one important feature of phase plots is their potential to bring up interesting questions and produce novel ideas. If you would like to try out phase plots on your own problems, you may start with the following self-explaining MATLAB code:[6]

```
xmin=-0.5; xmax=0.5; ymin=-0.5; ymax=0.5;
xres = 400; yres = 400;
x = linspace(xmin,xmax,xres);
y = linspace(ymin,ymax,yres);
[x,y] = meshgrid(x,y); z = x+i*y;
f = exp(1./z);
p = surf(real(z),imag(z),0*f,angle(-f));
set(p,'EdgeColor','none');
caxis([-pi,pi]), colormap hsv(600)
view(0,90), axis equal, axis off
```

[6] *This is a contribution to Nick Trefethen's project of communicating ideas by exchanging ten-line computer codes.*

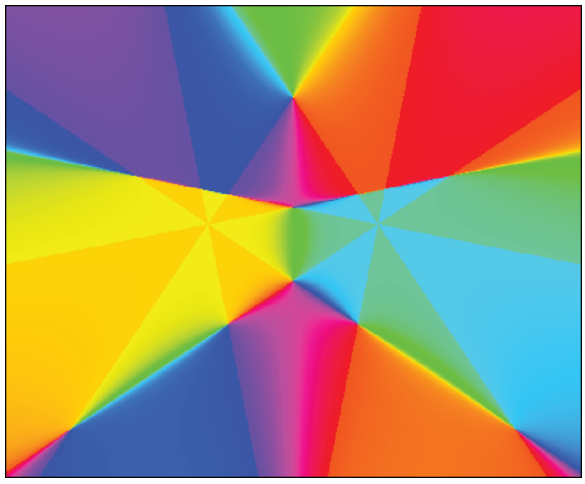

Figure 25. A modified phase plot of Wilmshurst's example for $n = 4$.

Though the phase of a function is at least of the same importance as its modulus, it has not yet been studied to the same extent as the latter. It is our conviction that *phase plots are problem factories*, which have the potential to change this situation. The forthcoming book [37] aims to give a comprehensive introduction to complex functions using phase portraits. The reactions to a mathematical calendar [33] featuring phase portraits of analytic functions have shown that the beauty of these images may also stimulate the interest of a more general public.

Technical Remark

All images in this article were created using MATHEMATICA and MATLAB.

Acknowledgment

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The rewriting of the paper would not have been possible without the valuable comments and critical remarks of several referees.

References

[1] T. ARENS, F. HETTLICH, CH. KARPFINGER, U. KOCKELKORN, K. LICHTENEGGER, and H. STACHEL, *Mathematik* (German), Spektrum Akademischer Verlag, 2008.

[2] D. N. ARNOLD, *Graphics for complex analysis*, http://www.ima.umn.edu/~arnold/complex.html.
[3] D. N. ARNOLD and J. ROGNESS, Möbius transformations revealed, *Notices of the AMS* **55** (2008), 1226–1231.
[4] B. BAGCHI, A joint universality theorem for Dirichlet *L*-functions, *Math. Z.* **181** (1982), 319–334.
[5] T. BANCHOFF, *Complex function graphs*, http://www.math.brown.edu/~banchoff/gc/script/CFGInd.html.
[6] T. BANCHOFF, *Tom Banchoff's mathematical artwork*, http://www.math.brown.edu/TFBCON2003/art/welcome.html.
[7] A. G. BENNETT, *Complex function grapher*, http://www.math.ksu.edu/~bennett/jomacg/.
[8] J. B. CONREY, The Riemann hypothesis, *Notices of the AMS* **50** (2003), 341–353.
[9] L. CRONE, *Color graphs of complex functions*, http://www1.math.american.edu/People/lcrone/ComplexPlot.html.
[10] T. A. CRONIN and C. A. KING, Spectral sensitivity of vision in the mantis shrimp, *gonodactylus oerstedii*, determined using noninvasive optical techniques, *Bio. Bull.* **176** (1989), 308–316.
[11] P. DUREN, A.-K. HERBIG, and D. KHAVINSON, Robert Jentzsch, mathematician and poet, *Mathematical Intelligencer* **30** (2008), 18–23.
[12] H. M. EDWARDS, *Riemann's Zeta Function*, Academic Press, 1974.
[13] F. A. FARRIS, Review of *Visual Complex Analysis*, by Tristan Needham, *American Mathematical Monthly* **105** (1998), 570–576.
[14] _____, *Visualizing complex-valued functions in the plane*, http://www.maa.org/pubs/amm_complements/complex.html.
[15] W. FORST and D. HOFFMANN, *Funktionentheorie erkunden mit* MAPLE, Springer, 2002.
[16] H. GRASSMANN, Zur Theorie der Farbenmischung, *Poggendorff's Annalen der Physik und Chemie* **89** (1853), 69–84.
[17] G. A. HJELLE, Blaschke Product Explorer, http://www.math.wustl.edu/~hjelle/software/blaschke/BPExplorer.php.
[18] F. K. HUANG, *An overview of complex dynamics using a computer program*, Master's thesis, Univ. of Washington, 1995, http://www.math.washington.edu/~marshall/dynamics/thesis.dvi.
[19] E. JAHNKE and F. EMDE, *Funktionentafeln mit Formeln und Kurven*, 2nd ed., B. G. Teubner, 1933.
[20] J. L. W. V. JENSEN, Recherches sur la théorie des équations, *Acta Math.* **36** (1913), 181–195.
[21] S. G. KRANTZ, *Complex Variables. A Physical Approach with Applications and* MATLAB. Chapman and Hall/CRC, 2008.
[22] A. A. KARATSUBA and S. M. VORONIN, *The Riemann Zeta-Function*, Walter de Gruyter, 1992.
[23] D. KHAVINSON and G. NEUMANN, From the fundamental theorem of algebra to astrophysics: A "harmonious" path, *Notices of the AMS* **55** (2008), 666–675.
[24] F. LABELLE, *A gallery of complex functions*, http://www.cs.berkeley.edu/~flab/complex/gallery.html.
[25] H. LUNDMARK, *Visualizing complex analytic functions using domain coloring*, http://www.mai.liu.se/~halun/complex/domain_coloring-unicode.html.
[26] D. MARSHALL, *Windows executable program for pictures of analytic functions*, http://www.math.washington.edu/~marshall/math_535/complexfcn/vcomplexfcn_pkg.exe.
[27] E. MAILLET, Sur les lignes de décroissance maxima des modules et les équations algébraiques ou transcendantes, *J. de l'Éc. Pol.* **8** (1903), 75–95.
[28] T. NEEDHAM, *Visual Complex Analysis*, Clarendon Press, Oxford University Press, 1997.
[29] _____, *Websites related to "Visual Complex Analysis"*, http://www.usfca.edu/vca/websites.
[30] M. PERGLER, *Newton's method, Julia and Mandelbrot sets, and complex coloring*, http://users.arczip.com/pergler/mp/vdocuments/ptr/.
[31] K. POELKE and K. POLTHIER, Lifted domain coloring, *Computer Graphics Forum* **28** (2009), http://geom.mi.fu-berlin.de/publications/db/lifteddomaincoloring.pdf.
[32] O. REIMERDES, *Die Niveau- und Fallinien auf Flächen insbesondere auf Modulflächen analytischer Funktionen. Inaugural-Dissertation, Christian-Albrechts-Universität zu Kiel*, Teubner, 1911.
[33] G. SEMMLER and E. WEGERT, Complex Beauties 2011, *Mathematical Calendar*, http://www.mathcalendar.net
[34] G. SEMMLER and E. WEGERT, *Topography of Blaschke products*, in preparation.
[35] J. STEUDING, *Value-Distribution of L-Functions*, Springer, 2007.
[36] E. C. TITCHMARSH, *The Theory of Functions*, second edition, Oxford University Press, 1939.
[37] E. ULLRICH, Betragflächen mit ausgezeichnetem Krümmungsverhalten, *Math. Zeitschr.* **54** (1951), 297–328.
[38] E. WEGERT, *Visual Complex Functions*, Birkhäuser, (to appear in spring 2012).
[39] _____, Phase diagrams of meromorphic functions, *Comput. Methods Funct. Theory* **10** (2010), 639–661.
[40] _____, Die Schönheit komplexer Funktionen, http://www.mathe.tu-freiberg.de/~wegert/Talks/Chemnitz2011.pdf.
[41] N. WELSCH and C. C. LIEBMANN, *Farben*, Spektrum Akademischer Verlag, 2007.
[42] A. S. WILMSHURST, The valence of harmonic polynomials, *Proc. Am. Math. Soc.* **126** (1998), 2077–2081.
[43] WOLFRAM RESEARCH, *The Wolfram Functions Site*, http://www.functions.wolfram.com.
[44] D. ZAWISCHA, *Seeing, measuring, and rendering colors*, http://www.itp.uni-hannover.de/~zawischa/ITP/effects.html.