WHAT IS...

a Resplendent Structure?

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Resplendent structures were introduced by Jon Barwise and John Schlipf in [1]. The reader may want to consult a dictionary for the meaning of the word "resplendent", but checking a thesaurus is even more informative. The long list of synonyms starts with "splendid, brilliant, dazzling, ..." and ends with "majestic". The last section of [1] is devoted to historical remarks, but the authors do not mention why this particular name was chosen. Before we can explain what makes resplendent structures resplendent, we need a short introduction to model theory.

Graphs, groups, fields, Lie algebras, and many other mathematical structures consist of a set (the universe of the structure) with a set of functions and relations on it. Such objects are called *first-order structures*. In the discussion below we will just call them structures. Since every function can be interpreted as a relation by considering its graph instead, to simplify the discussion we will consider only relational structures, i.e., structures with no functions. In general there are no restrictions on the number of relations on a structure, but to avoid some technicalities, in this article we will assume that structures have only finitely many relations.

The real number field $(\mathbb{R}, +, \times)$ consists of a universe $\mathbb{R}$ and two ternary relations $+$ and $\times$. A *reduct* of a structure is obtained by forgetting some of its relations. For example, the group $(\mathbb{R}, +)$ is a reduct of $(\mathbb{R}, +, \times)$. An *expansion* is obtained by adding relations. For example, $(\mathbb{R}, +, \times, <)$ and $(\mathbb{R}, +, \times, \exp)$ are both expansions of $(\mathbb{R}, +, \times)$. There is an important difference between these two expansions. The relation $x < y$ is defined in $(\mathbb{R}, +, \times)$ by the formula $(x \neq y) \wedge \exists z (y = x + z \times z)$; hence any property of the real numbers expressed in terms of $+$, $\times$, and $<$ can be expressed in terms of $+$ and $\times$ only. The status of exponentiation is different. It follows from the Tarski-Seidenberg elimination of quantifiers for real closed fields that the relation $y = \exp(x)$ cannot be defined in terms of $+$ and $\times$, so the latter expansion is an essential one. Here by definability we mean definability in first-order logic. Each structure determines its language, which consists of names of all of its relations and all properly formed formulas that can be written using those names, variables, parentheses, the equality symbol $=$, Boolean connectives, and quantifiers. Such formulas are called *first order*. If $\mathcal{M}$ is a structure with a universe M, then an n-ary relation $R \subseteq M^n$ is *definable* if there is a formula $\varphi(\bar{x}, \bar{y})$ of the language of $\mathcal{M}$ and some m-tuple $\bar{b}$ from M such that R is the set of those n-tuples $\bar{a}$ for which $\varphi(\bar{a}, \bar{b})$ is true in $\mathcal{M}$.

A *theory* is a set of sentences in a given first-order language. If T is a theory and all sentences of T are true in a structure $\mathcal{M}$, then we say that $\mathcal{M}$ is a *model* of T.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are structures such that the universe of $\mathcal{M}$ is a subset of the universe of $\mathcal{N}$ and the relations of $\mathcal{M}$ are restrictions to the universe of $\mathcal{M}$ of the relations of $\mathcal{N}$. Then we say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$ if for each first-order sentence $\varphi(\bar{a})$ involving a finite string of parameters $\bar{a}$ from $\mathcal{M}$, $\varphi(\bar{a})$ is true in $\mathcal{M}$ if and only if it is true in $\mathcal{N}$. If $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, then we also say that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$. One can show that $(\mathbb{R}, <)$ is an elementary extension of $(\mathbb{Q}, <)$, but $(\mathbb{R}, \times)$ is not an elementary extension of $(\mathbb{Q}, \times)$,

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since, for example, the sentence $\exists x\ [x^2 = 2]$ is true in the former structure but not in the latter.

Let $\mathcal{M}$ be a structure, and let R be a relation symbol not in the language of $\mathcal{M}$. Let $\varphi(R)$ be a statement in the language of $\mathcal{M}$ with this new symbol. Is there a relation R_M on the universe of $\mathcal{M}$ such that $\varphi(R)$ is true in the expanded structure $(\mathcal{M}, R_M)$ when R is interpreted as R_M? For example, if $(G, +)$ is a group, is there a ternary relation $\times_G$ on G such that $(G, +, \times_G)$ is a field? In other words, if $\varphi(\times)$ is the conjunction of the field axioms in the language with $+$ and $\times$ as ternary relations, is there an expansion of $(G, +)$ satisfying $\varphi(\times)$? There may be some obvious obstacles. Suppose $(G, +)$ is a six-element group. There are no six-element fields, so G cannot be a universe of a field. The fact that a universe of a structure has exactly six elements can be expressed by a first-order sentence ψ. The sentence ψ is true in $(G, +)$, and it is *inconsistent* with $\varphi(\times)$ in the sense that there is no structure in which both $\varphi(\times)$ and ψ would be true. For an infinite example, let us consider $(\mathbb{Z}, +)$. The sentence $[1 + 1 = 0 \vee \exists x(x + x = 1)]$ is true in every field but is false in $(\mathbb{Z}, +)$; hence no expansion of $(\mathbb{Z}, +)$ is a field. But what happens if there are no such inconsistencies?

Let $\bar{a}$ be a tuple of elements from the universe of a structure $\mathcal{M}$. By $\mathrm{Th}(\mathcal{M}, \bar{a})$ we denote the set of all sentences $\varphi(\bar{a})$ in the language of $\mathcal{M}$ that are true in $\mathcal{M}$. Suppose that $\varphi(R, \bar{a})$ is a sentence in the language of $\mathcal{M}$ with new relation symbol R and with parameters $\bar{a}$ such that the set of sentences $\mathrm{Th}(\mathcal{M}, \bar{a}) \cup \{\varphi(R, \bar{a})\}$ is consistent, meaning that there is a structure in which all those sentences are true. Then—and this is an exercise in model theory—$\mathcal{M}$ has an elementary extension $\mathcal{N}$ such that for some relation R_N on the universe of $\mathcal{N}$ the sentence $\varphi(R, \bar{a})$ is true in the expanded structure $(\mathcal{N}, R_N)$. In other words, a relation satisfying a consistent first-order property $\varphi(R, \bar{a})$ always exists on the universe of some elementary extension of $\mathcal{M}$. We are interested in the situation in which a relation like that can be already found on $\mathcal{M}$. This leads to the following definition.

Let $\mathcal{M}$ be a first-order structure. We say that $\mathcal{M}$ is *resplendent* if for any first-order sentence $\varphi(R, \bar{a})$ with a new relation symbol and a tuple $\bar{a}$ of parameters from $\mathcal{M}$ such that the set of sentences $\mathrm{Th}(\mathcal{M}, \bar{a}) \cup \{\varphi(R, \bar{a})\}$ is consistent, $\varphi(R, \bar{a})$ is true in $(\mathcal{M}, R_M)$ for some relation R_M on the universe of $\mathcal{M}$. In other words, if there is a relation satisfying some first-order requirement in an elementary extension of $\mathcal{M}$, then there is also such a relation on $\mathcal{M}$.

Technically, any finite structure is resplendent. This is an interesting exercise in model theory but is otherwise an uninteresting fact. More interesting examples are not hard to find, but one has to use some model theory. The crucial fact is that every structure has a resplendent elementary extension of the same cardinality. The result itself is not difficult to prove, but it has many important consequences. Let us see how it implies that the structure $(\mathbb{Q}, <)$ is resplendent. Let $(D, <)$ be a resplendent countable elementary extension of $(\mathbb{Q}, <)$. By elementarity of the extension, $(D, <)$ is a dense linear ordering without endpoints. Since, up to isomorphism, there is only one countable dense linear ordering without endpoints, $(D, <) \cong (\mathbb{Q}, <)$. Hence $(\mathbb{Q}, <)$ is resplendent. This argument works in a more general setting. Let κ be a cardinal number. A theory T in a first-order language is *κ-categorical* if, up to isomorphism, there is exactly one model of T of cardinality κ. Arguing as above, one can show that if T is κ-categorical and $\mathcal{M}$ is a model of T of cardinality κ, then $\mathcal{M}$ is resplendent. It follows that, among others, the countable random graph, any uncountable algebraically closed field, and any infinite-dimensional vector space over a finite field are all resplendent.

For an example of a structure that is not resplendent let us consider the ring $(\mathbb{Z}, +, \times)$ and the sentence $\varphi(I)$ in the language of rings with additional unary relation symbol I:

$$\exists x\, I(x) \wedge \exists x \neg I(x)$$
$$\wedge \forall x, y[x + 1 = y \to (I(x) \leftrightarrow I(y))].$$

The sentence is saying that I is a proper nonempty subset of the universe that is closed under successors and predecessors. Clearly, there are no such subsets of $\mathbb{Z}$. If $(K, +, \times)$ is a proper elementary extension, then $\mathbb{Z}$ is a proper convex subset of K; hence $\varphi(I)$ is true in $(K, +, \times, \mathbb{Z})$. It follows that $(\mathbb{Z}, +, \times)$ is not resplendent. A similar argument shows that the field $(\mathbb{R}, +, \times)$ is not resplendent (but recall that the field of complex numbers is).

So how resplendent are resplendent structures? Each infinite resplendent structure has nontrivial automorphisms, and each countable resplendent structure has continuum many. Moreover, each infinite resplendent structure is isomorphic to a proper elementary substructure of itself. There is no room here for a full discussion, but let us just make one comment about proofs of such results. In the whole spectrum of models of a given theory T, resplendent models typically form a smaller class, often with a well-defined complete set of isomorphism invariants. Often, to show that a resplendent model $\mathcal{M}$ has some property P, one constructs a resplendent elementary extension $\mathcal{N}$ of $\mathcal{M}$ that has property P and has the same isomorphism invariants as $\mathcal{M}$. It follows that $\mathcal{N}$ is isomorphic to $\mathcal{M}$; hence $\mathcal{M}$ has property P.

Countable resplendent structures are even more than resplendent; they are *chronically resplendent*. The definition is the same as that of resplendency, except that we also require that the expansion $(\mathcal{M}, R_M)$ is also resplendent. This feature is particularly useful in applications.

Let $\mathcal{M}$ be a countable structure and suppose that A is a relation on the universe of $\mathcal{M}$ such that the expanded structure $(\mathcal{M}, A)$ is resplendent and A is not definable in $\mathcal{M}$. Then A has continuum many automorphic images in $\mathcal{M}$. Here is a natural example. Let $(K, +_K, \times_K)$ be a proper countable elementary extension of $(\mathbb{Z}, +, \times)$. It can be shown that for all such extensions $(K, +_K)$ is resplendent. We know that $\times$ (as a relation on $\mathbb{Z}$) is not definable in $(\mathbb{Z}, +)$.[1] Since the extension is elementary, $\times_K$ is not definable from $+_K$ in K. We conclude that $\times_K$ has continuum many automorphic images in $(K, +_K)$; hence there are continuum many different multiplications $\circ$ on K such that $(K, +_K, \circ)$ has all the first-order properties of $(\mathbb{Z}, +, \times)$.

Resplendent models have many applications in model theory, but here is an application of a more algebraic nature. Let $\mathcal{R} = (R, +, \times)$ be an ordered field. A set $I \subseteq R$ is an *integer part* of $\mathcal{R}$ if I is a discretely ordered subring such that 1 is the least positive element and for each $x \in R$ there is some $i \in I$ such that $i \le x < i + 1$. By a result of Mourgues and Ressayre [3], every real closed field has an integer part. Of course, the real field $(\mathbb{R}, +, \times)$ has the unique integer part $(\mathbb{Z}, +, \times)$, but other fields can have many. By the already-mentioned Tarski-Seidenberg elimination of quantifiers, every definable subset of a real closed field is a finite union of intervals. Hence, no integer part of a real closed field can be first-order definable. It follows that if $\mathcal{R}$ is a resplendent countable real closed field, then $\mathcal{R}$ has continuum many integer parts. In a recent paper [2], D'Aquino, Knight, and Starchenko proved that a countable real closed field has an integer part that is a nonstandard model of Peano Axioms (PA) if and only if the field is resplendent. One direction is straightforward once we note that in the definition of resplendency the first-order property $\varphi(R, \bar{a})$ can be replaced by any effectively presented (computable) list of such properties (still with a fixed tuple of parameters $\bar{a}$). An example would be a list of sentences $\Phi(I) = \{\varphi_n(I) : n < \omega\}$ expressing that I is an integer part that satisfies the Peano Axioms (which can be effectively listed). Clearly, all sentences in $\Phi(I)$ are true in the structure $(\mathbb{R}, +, \times, \mathbb{Z})$. The axioms of real closed fields are complete, which implies that for each real closed field $\mathcal{R}$, $\mathrm{Th}(\mathcal{R}) = \mathrm{Th}(\mathbb{R}, +, \times)$. It follows that Φ is consistent with $\mathrm{Th}(\mathcal{R})$; hence every resplendent real closed field has an expansion satisfying $\Phi(I)$, and, by the results mentioned before, any countable resplendent real closed field has continuum many such expansions. It is interesting that a similar argument shows that every resplendent real closed field has integer parts satisfying countably many different theories (all contradicting the Peano Axioms). This follows from a theorem of Shepherdson [4] according to which any discretely ordered ring satisfying the axioms of Peano Arithmetic restricted to formulas without quantifiers is an integer part of a real closed field.

[1] *It follows, for example, from the fact that $(\mathbb{Z}, +)$ is a decidable structure and $(\mathbb{Z}, +, \times)$ is not.*

References

[1] Jon Barwise and John Schlipf, An introduction to recursively saturated and resplendent models, *J. Symbolic Logic* **41**, no. 2 (1976), 531–536.

[2] P. D'Aquino, J. F. Knight, and S. Starchenko, Real closed fields and models of Peano arithmetic, *J. Symbolic Logic* **75**, no. 1 (2010), 1–11.

[3] M.-H. Mourgues and J. P. Ressayre, Every real closed field has an integer part, *J. Symbolic Logic* **58**, no. 2 (1993), 641–647.

[4] J. C. Shepherdson, A non-standard model for a free variable fragment of number theory, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **12** (1964), 79–86.