Yang-Mills and Beyond

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As recently as 2010, three groups received the J. J. Sakurai Prize for showing how spontaneous symmetry breaking explains the mechanism for generating vector boson masses, thus enabling the solution of one of the puzzles of the standard model of elementary particles. Since the Yang-Mills field strength incorporating a non-abelian feature is not only one of the cornerstones of the standard model, which is the basis of the ongoing search for the Higgs boson, but also since Yang-Mills theory influenced the development of algebraic and differential geometry (e.g., see Donaldson [1] and [2], and Dragomir et al. [3]), perhaps it is not out of order to explore the development of the formula for the field strength, F. In modern notation it is given in the form $F = dB + B \wedge B$, where B is the gauge field. Although Yang-Mills (YM) gauge theory can be done using fiber-bundle theory—see, for instance, the review articles of Daniel and Viallet [4] and Marateck [5]—it is historically useful to analyze the way it developed using classical gauge theory.

In their seminal paper [6], Yang and Mills invented the non-abelian field strength[1] to satisfy certain criteria but did not explain how it could be derived. In the penultimate section here we show how the Yang-Mills field strength derives from Yang's gauge transformation. The preceding sections place Yang-Mills theory in historical perspective and cover material relating to the field strength. We begin in the next section by reminding the reader what gauge invariance is, starting with its use in electromagnetism, then showing phase-invariance and gauge-invariance use in quantum electrodynamics, thus laying the groundwork for Yang's derivation of the YM field transformation. We then review the history of gauge theory in the beginning of the quantum mechanics era.

In their 1954 paper [6], when Yang and Mills discuss the phase factor-gauge transformation relationship, they cite Pauli's review paper [7]. It is interesting that, although Pauli in that paper presents the electromagnetic field strength in terms of a commutator, for whatever reason Yang and Mills did not use the commutator approach to obtain the YM non-abelian field strength; they obtained it by generalizing the electromagnetic field strength and presumably found the result by trial and error. In the section on Yang-Mills field strength we present the derivation of this field strength using the commutator approach. In the following section Yang's field transformation is derived in a slightly different way than Yang did, and it is indicated that this transformation reduces to the gauge transformation for electromagnetism when the theory is considered abelian. In the penultimate section we show that the YM field strength derives from the electromagnetic field strength and that the commutation part is dictated by the field transformation. The final section indicates how Yang-Mills theory was combined with spontaneous symmetry breaking, the Goldstone theorem, and subsequent work to contribute to the standard model of particle physics.

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[1] *Their intent was to describe the strong interactions.*

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Gauge Theory

In order to appreciate gauge invariance in quantum mechanics, let's back up and investigate gauge invariance in electromagnetism. There a gauge transformation is a transformation of $\mathbf{A}_\mu$, the four-vector potential, that leaves the **B** and **E** fields unchanged and hence Maxwell's equations invariant. The nonhomogeneous Maxwell equations[2] can be expressed as

$$d * F = J, \tag{1}$$

where J is the current and $*$ is the Hodge star operator; and the homogeneous ones[3] can be expressed as

$$dF = 0, \tag{2}$$

where[4] $F = dA$. Here A, a one-form, is a connection on a fiber bundle and the field strength F, a two-form, is its curvature. Since $d^2 = 0$, then $dJ = 0$. This is the continuity equation.[5] It is evident that F and thus Maxwell's equations are invariant under the following transformation:

$$A \to A + d\alpha(x). \tag{3}$$

Because α is a function of x, (3) is an example of a local gauge transformation. The invariance of Maxwell's equations under (3) is called *gauge invariance* or *gauge symmetry*. In electromagnetism, gauge invariance facilitates problem solving. In quantum electrodynamics, it's much more useful, as will be seen when abelian gauge invariance is discussed below.

The Lagrangian density that produces the Dirac equation is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{4}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Here **c**, the speed of light, and $\hbar$, Plank's constant divided by 2π, are set to one. A global phase transformation, e.g., $\psi' = e^{-iq\alpha}\psi$, is one in which the phase α is constant. It corresponds to a rotation in the complex plane. Since, in quantum mechanics, a phase transformation does not alter the physical reality, it is an example of a symmetry and can be used to reveal an underlying physical principle. One can show—although we will not—using the Euler-Lagrange equations for a Lagrangian with a global phase transformation, the existence of a conserved current. This is an example of Noether's theorem that a conserved current is associated with a continuous symmetry of the Lagrangian.

[2] *In covariant form it is* $\partial^\mu F_{\mu\nu} = j_\nu$.
[3] *In covariant form it is* $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0$, *where* $\epsilon^{\alpha\beta\gamma\delta}$ *is the Levi-Civita symbol.*
[4] *In covariant form it is* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
[5] *In covariant form it is* $\partial^\nu j_\nu = 0$ *or* $\nabla \cdot \vec{\mathbf{J}} + \partial\rho/\partial t = 0$.

Let's now see the effect that a local gauge transformation has on the Lagrangian density in (4). Substituting the local phase transformation

$$\psi \to \psi' = e^{-iq\alpha(x)}\psi \tag{5}$$

into (4) spoils the invariance of the Lagrangian and has it depend on the choice of phase, that is,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ + q\partial\!\!\!/\alpha(x) - m)\psi. \tag{6}$$

In order to regain the invariance, substitute the covariant form $A \to A + \partial_\mu\alpha(x)$ of the gauge transformation in (3) and $\partial_\mu \to \partial_\mu + iqA_\mu$ (operating on ψ) into equation (6), getting $\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\psi$. Letting $D_\mu = \partial_\mu + iqA_\mu$ and adding to the Lagrangian the electromagnetic energy density[6] $-(\frac{1}{2})F \wedge *F$ produce the gauge invariant form $\mathcal{L} = \bar{\psi}(iD\!\!\!\!/ - m)\psi - (\frac{1}{2})F \wedge *F$, where $D\!\!\!\!/$ is $\gamma^\mu(\partial_\mu + iqA_\mu)$. Moreover, if J^μ is set equal to $i\bar{\psi}q\gamma^\mu\psi$, the Lagrangian density can be written as $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - J \wedge A - (\frac{1}{2})F \wedge *F$. The second and third terms yield the nonhomogeneous Maxwell equations; the homogeneous ones are satisfied trivially, because the definition of F satisfies them automatically [8]. Note that F, and therefore $F \wedge *F$, is invariant under the gauge transformation in (3). If there were a term in the Lagrangian for the mass μ of the photon, because of the Klein-Gordon equation it would take the form $\frac{1}{2}\mu^2 A_\nu A^\nu$, but this term is not gauge invariant. So gauge invariance dictates that the mass of the photon must be zero. Okun [9] comments[7] that, although a tiny photon mass would destroy the gauge invariance of quantum electrodynamics (QED), it would not affect the renormalizability of QED and the resulting excellent agreement of it with experiment; the really important principle in QED is the conservation of charge. The article [9] cites Okun's previous work on this. Gauge invariance in which the phase, here α, is a function of x is called *gauge invariance of the second kind* and the Lagrangian is said to exhibit local symmetry. Global gauge invariance is called *gauge invariance of the first kind*.

When $\mathbf{D}_\mu$ operates on ψ it takes the same form in the primed system that it does in the unprimed system, i.e.,

$$D'_\mu\psi' = e^{-iq\alpha(x)}D_\mu\psi. \tag{7}$$

For that reason it is called the *covariant derivative*. As we will see in a later section, Yang

[6] *In covariant form it is* $-(\frac{1}{4})F_{\mu\nu}F^{\mu\nu}$.
[7] *Okun wrote the paper in 1998 for the celebration of the 100th anniversary of Fock's birth. It appeared in English (see* http://www.itep.ru/theor/persons/lab180/okun/fock.pdf*) in the UNESCO-sponsored* Quantum Theory in Honour of Vladimir A. Fock *(1998). Among other things, it relates the beginning of the collaboration between Jackson and Okun* [16]. *Okun revised it in 2010* [9] *and added an appendix.*

used this idea to derive the field transformation for the non-abelian case. Note that $\partial_\mu \psi' = (-iq\partial_\mu \alpha \psi + \partial_\mu \psi)e^{-iq\alpha(x)}$ and D'_μ, which equals $\partial_\mu + iq(A_\mu + \partial_\mu \alpha)$, operates on $e^{-iq\alpha(x)}\psi$, so (7) is obvious. Gauge invariance determines the type of interaction; here, this is manifested by the inclusion of the vector potential in the covariant derivative. This is called the *gauge principle*, and $\mathbf{A}_\mu$ is called the *gauge field* or *gauge potential*. The phase factors $e^{-iq\alpha(x)}$ form the gauge group $U(1)$ of unitary 1×1 matrices. Since the elements of the group commute, it is an abelian group. Associated with this gauge group is the gauge field $\mathbf{A}_\mu$. The quantum of the gauge field is the gauge boson, which in the case of the electromagnetic field is the photon. A boson has integral spin and therefore obeys Bose-Einstein statistics.

The introduction of gauge invariance has promoted the four-vector potential $\mathbf{A}_\mu$ from being a mathematical construct in electromagnetism—as J. D. Jackson related to Okun [9], the three-vector version of (3) was known to Maxwell—to causing the shift in the interference pattern in the Aharonov-Bohm solenoid effect [10]. Moreover, gauge invariance determines interactions and thus the observable interaction forces. We have already discussed its use in electromagnetic theory: as the generalized four-vector $\mathbf{A}_\mu$ in the covariant derivative, $\mathbf{A}_\mu$ becomes the gauge field that mediates the electromagnetic interaction, i.e., the photon. In the theory of the weak interactions, $\mathbf{A}$ represents the intermediate vector bosons $W^\pm$ and Z^0 fields, and in the strong interaction, $\mathbf{A}$ represents the colored gluon fields. This application is not limited to physics. In mathematics $\mathbf{A}_\mu$ is understood as the connection on fiber-bundles in differential geometry.

Now that we have seen the mathematical basis of gauge theory, we can appreciate its historical development. As we discussed in [5], modern gauge theory has its roots in Weyl's 1918 paper [11], in which he tried to combine electromagnetism and gravity by requiring the theory to be invariant under a local scale change of the metric $g_{\mu\nu} \to g_{\mu\nu}e^{\alpha(x)}$, where x is a 4-vector. This was criticized by Einstein, because lengths and time intervals for a parallel transported vector would be path dependent in contradiction to the sharpness of spectral lines. It is in this paper, however, that Weyl introduced the German term for gauge invariance, *Eichinvarianz*. The first appearance of "gauge invariance" in English was in Weyl's 1929 English version [12] of his famous 1929 paper [13]. At the 1985 International Symposium on Particle Physics in the 1950s: Pions to Quarks [14], Yang noted that if in Einstein's objection to Weyl's 1918 paper you replaced scale with phase, substituted electrons for clocks, and had a magnetic flux within the circuit, you would reproduce the Aharonov-Bohm effect. In 1926, Fock [15] showed that for a quantum theory of charged particles interacting with the electromagnetic field, invariance under a gauge transformation of the potentials (3) required multiplication of the wave function by the now well-known phase factor,[8] which in present-day notation is given by (5). Fock, however, at that time did not give a name to this principle.

By 1929 Maxwell's equations had been combined with quantum mechanics to produce the start of quantum electrodynamics. Weyl in [13] turned from trying to unify electromagnetism and gravity to following (without attribution) Fock's suggestion [15] and introduced as the phase factor an exponential in which the phase $\lambda(x)$ is preceded by the imaginary unit i, i.e., $e^{i\lambda(x)}$. This multiplies the wave function in the wave equations. In [12] he states it quite succinctly:

> …the laws are invariant under the simultaneous substitution of $e^{i\lambda}\psi$ for ψ and $\varphi_p - \frac{\partial\lambda}{\partial x_p}$ for φ_p, where λ is an arbitrary function of position in space and time.

Here φ_p are the four potentials. So the change of scale in Weyl's 1918 paper becomes a change of phase. It was here that Weyl suggested that gauge invariance be used as a symmetry principle to derive the electromagnetic interactions. Of course, by that time, electromagnetic theory was already well established. The importance of gauge invariance, however, was in its application to quantum field theory—it was instrumental in charting the way field theory would develop. As Jackson and Okun [16] so eloquently put it:

> Historically, of course, Weyl's 1929 papers were a watershed. They enshrined as fundamental the modern principle of gauge invariance in which the existence of the 4-vector potentials (and field strengths) follow from the requirement that the matter equation be invariant under gauge transformation such as [our (5)] of the matter fields. This principle is the touchstone of the theory of gauge fields, so dominant in theoretical physics in the second half of the 20th century.

It is interesting that in 1938 Oskar Klein gave a paper [17] at a conference[9] in Warsaw, Poland, in which he presented an expression for the

[8] *Fock used $e^{ip/\hbar}$, where $p = p_1 - e/cf$ and f is an arbitrary function of space-time coordinates.*

[9] *The participants included some of the great physicists of the era. Among the attendees were N. Bohr, L. Brillouin, L. de Broglie, C. Darwin, A. Eddington, R. Fowler, G. Gamow, S. Goudsmit, O. Klein, H. Kramers, L. de Kronig, P. Langevin, C. Moeller, J. von Neumann, F. Perrin, L. Rosenfeld, and E. Wigner.*

electromagnetic field strength coming from the contribution of the charged vector boson that looks similar to the non-abelian Yang-Mills field strength but actually is not [18]. When questioned by Moeller at the end of his talk, Klein in effect generalized his SU(2) theory to one that was like an $\mathrm{SU}(2) \times U(1)$ gauge theory, thus anticipating the electroweak part of the standard model. Klein never referred to this work in any of his subsequent publications, and unfortunately it was forgotten. Pauli, in letters written to Abraham Pais in 1953, using dimensional reduction derives what he calls the "analogue of the field-strengths", which is the same form as the non-abelian Yang-Mills field strength. Pauli did not pursue this further. These letters, Klein's paper, and Weyl's 1929 article, along with his 1918 article and Fock's, and other key articles appear in translation in a work by O'Raifeartaigh [19] with his comments. Yang, on page 19 of his selected papers [20], cites Weyl's gauge theory results as reported by Pauli [7] as a source for YM gauge theory, although Yang didn't find out until much later that these were Weyl's results. Moreover, Pauli's article did not explicitly mention Weyl's geometric interpretation. It was only long after Yang and Mills published their article that Yang realized the connection between their work and geometry. In fact, on page 74 of his selected papers, Yang says:

> What Mills and I were doing in 1954 was generalizing Maxwell's theory. We knew of no geometrical meaning of Maxwell's theory, and we were not looking in that direction.

Independently of Yang and Mills and slightly after them, Shaw [21] and Utiyama [22] separately developed non-abelian gauge theories similar to the YM one. Utiyama's theory also included a gauge theory for gravity and electromagnetism. Part of Shaw's thesis and Utiyama's paper are also included in O'Raifeartaigh's book [19].

Yang-Mills Field Strength

Pauli, in equation 22a of Part I of his 1941 review article [7], gives the electromagnetic field strength in terms of a commutator.[10] In present-day usage it is

(8) $$[D_\mu, D_\nu] = i\epsilon F_{\mu\nu},$$

where $\mathbf{D}_\mu$ is the covariant derivative $\partial_\mu - i\epsilon A_\mu$. Mathematically the $F_{\mu\nu}$ in (8) corresponds to the curvature or field strength. In their 1954 paper [4] Yang and Mills do not mention this relation, although they do cite Pauli's 1941 article [7]. They use

(9) $$\psi = S\psi',$$

where ψ is a wave function and S is a local isotopic spin rotation represented by an SU(2) matrix, to obtain the gauge transformation in equation (3) of their paper:

(10) $$B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon,$$

where B_μ is the gauge field. They[11] then define the field strength as

(11) $$F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu).$$

The first part corresponds to a curl; the second part is a commutator. They didn't know at that time that this corresponds to Cartan's second structural equation, which in differential geometry notation is $\Omega = \mathbf{dA} + [\mathbf{A}, \mathbf{A}]$, where A is a connection on a principal fiber bundle.

On page 19 of his selected papers [20], Yang states:

> Starting from [$\psi = S\psi'$ and $S(\partial_\mu - i\epsilon B'_\mu)\psi' = (\partial_\mu - i\epsilon B_\mu)\psi$] it was easy to get [our (10)]. Then I tried to define the field strength $F_{\mu\nu}$ by $F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu$ which was a "natural" generalization of electromagnetism.

Yang returned to this work when he collaborated with Mills when they shared an office at Brookhaven. They published their results in their 1954 paper [6]. There they introduce (11) (their equation (4)) by saying:

> In analogy to the procedure of obtaining gauge invariant field strengths in the electromagnetic case, we define now
>
> (4) $$F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu).$$
>
> One easily shows from [$B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon$] that
>
> (5) $$F'_{\mu\nu} = S^{-1}F_{\mu\nu}S$$
>
> under an isotopic gauge transformation. Other simple functions of B than (4) do not lead to such a simple transformation property.

Later we will demonstrate that substituting (10) into the electromagnetic field strength $F'_{\mu\nu} = \partial_\nu B'_\mu - \partial_\mu B'_\nu$ dictates adding a nonelectromagnetic term, i.e., the non-abelian term, so that their (5), the covariant transformation, is satisfied.

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[10] *It is presented in Pauli's equation (22a):* $D_i D_k - D_k D_i = -i\epsilon f_{ik}$*, where* $D_k = (\partial/\partial x_k) - i\epsilon\phi_k$*,* ϕ_k *is the electromagnetic potential,* ϵ *is the charge, and* $f_{ik} = (\partial\phi_k/\partial x_i) - (\partial\phi_i/\partial x_k)$ *is the field strength.*

[11] *Yang had earlier started studying this problem as a graduate student at the University of Chicago and derived (10). When he returned to this problem as a visitor at Brookhaven, he, in collaboration with Mills, obtained (as we will explain) the field strength. See page 17 in Yang's selected papers* [20].

Using the YM covariant derivative $(\partial_\mu - i\epsilon B_\mu)$, let's see how the YM field strength is obtained from the commutator

$$[D_\mu, D_\nu] = (\partial_\mu - i\epsilon B_\mu)(\partial_\nu - i\epsilon B_\nu) - (\partial_\nu - i\epsilon B_\nu)(\partial_\mu - i\epsilon B_\mu) \tag{12}$$

operating on the wave function ψ. Note that $-\partial_\mu(B_\nu\psi) = -(\partial_\mu B_\nu)\psi - B_\nu\partial_\mu\psi$ and $\partial_\nu(B_\mu\psi) = (\partial_\nu B_\mu)\psi + B_\mu\partial_\nu\psi$. So we get a needed $-B_\nu\partial_\mu$ and a $B_\mu\partial_\nu$ term to cancel $B_\nu\partial_\mu$ and $-B_\mu\partial_\nu$, respectively. Therefore the right-hand side of (12) becomes

$$i\epsilon(\partial_\nu B_\mu - \partial_\mu B_\nu) - \epsilon^2[B_\mu, B_\nu]. \tag{13}$$

So $[D_\mu, D_\nu] = i\epsilon F_{\mu\nu}$, which is (8). The failure of the covariant derivatives to commute is caused by a nonvanishing $F_{\mu\nu}$ and demonstrates the presence of curvature. We will show that (8) is related to $F'_{\mu\nu} = S^{-1}F_{\mu\nu}S$ when we discuss finding the field strength below.

The Field Transformation

We present a detailed derivation of the gauge transformation by using the transformation

$$\psi' = S\psi \tag{14}$$

instead of the traditional $\psi = S\psi'$, i.e., the one Yang and Mills used. In order to obtain the gauge transformation of the Yang and Mills paper,

$$B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon \tag{15}$$

requires you to use[12] $\partial_\mu S^{-1} = -S^{-1}(\partial_\mu S)S^{-1}$. The approach indicated by (14) is marginally more straightforward, since it doesn't require differentiating the inverse of a matrix.

The covariant derivative, $D_\mu = \partial_\mu - i\epsilon B_\mu$, transforms the same way as ψ does:

$$D'\psi' = SD\psi, \tag{16}$$

as we showed in (7): $D'_\mu\psi' = e^{-iq\alpha(x)}D_\mu\psi$ in the electromagnetic case. The left-hand side of (16) becomes

$$(\partial_\mu - i\epsilon B'_\mu)S\psi = (\partial_\mu S)\psi + S\partial_\mu\psi - i\epsilon B'_\mu S\psi. \tag{17}$$

But (17) equals $S\partial_\mu\psi - i\epsilon SB_\mu\psi$. Cancelling $S\partial_\mu\psi$ on both sides, we eventually get

$$B'_\mu = SB_\mu S^{-1} - i(\partial_\mu S)S^{-1}/\epsilon. \tag{18}$$

In differential geometry terms, this is expressed as $B \to SBS^{-1} - i/\epsilon(dS)S^{-1}$. Equation (18) reduces to (3) for the abelian case (where B and S commute) and where $S = e^{+i\epsilon\alpha(x)}$.

We will use[13] $S = e^{i\alpha(x)\cdot\sigma}$ to simplify (18), where σ are the Pauli spin matrices. For α infinitesimal, $S = 1 + i\alpha\cdot\sigma$ produces

$$B'_\mu = (1 + i\alpha\cdot\sigma)B_\mu(1 - i\alpha\cdot\sigma) - i(1/\epsilon)\partial_\mu(1 + i\alpha\cdot\sigma)(1 - i\alpha\cdot\sigma). \tag{19}$$

[12] *The following can be obtained by differentiating $S^{-1}S = I$.*

[13] *In this section our S is the inverse of the Yang-Mills S.*

Remembering that $(a\cdot\sigma)(b\cdot\sigma) = a\cdot b + i\sigma\cdot(a\times b)$, setting $B_\mu = \sigma\cdot b_\mu$, and since α is infinitesimal, dropping terms of order α^2, results in

$$b'_\mu = b_\mu + 2(b_\mu \times \alpha) + (1/\epsilon)\partial_\mu\alpha, \tag{20}$$

which is equation (10) in the YM paper [6].

Finding the Field Strength

As opposed to the electromagnetic case where the transformation of $F_{\mu\nu}$ is gauge invariant, we will show at the end of this section that, in the nonabelian case, the field strength $\mathbf{F}_{\mu\nu}$ transformation is gauge covariant:

$$F'_{\mu\nu} = S^{-1}F_{\mu\nu}S. \tag{21}$$

But first let us find an expression for the field strength. Let's start our quest by examining (21) in the primed system,

$$F'_{\mu\nu} = \partial_\nu B'_\mu - \partial_\mu B'_\nu, \tag{22}$$

and express it in terms of the nonprimed system fields. To do this we calculate $\partial_\nu B'_\mu$ from the expression $B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon$, (10), obtaining

(23)
$$\partial_\nu B'_\mu = -S^{-1}(\partial_\nu S)S^{-1}B_\mu S + S^{-1}(\partial_\nu B_\mu)S + S^{-1}B_\mu\partial_\nu S + i/\epsilon[-S^{-1}(\partial_\nu S)S^{-1}\partial_\mu S + S^{-1}\partial_\nu\partial_\mu S].$$

So

(24)
$$\begin{aligned}\partial_\nu B'_\mu - \partial_\mu B'_\nu &= -S^{-1}[(\partial_\nu S)S^{-1}B_\mu - (\partial_\mu S)S^{-1}B_\nu]S \\ &\quad + S^{-1}[\partial_\nu B_\mu - \partial_\mu B_\nu]S + S^{-1}[B_\mu\partial_\nu - B_\nu\partial_\mu]S \\ &\quad + i/\epsilon[-S^{-1}(\partial_\nu S)S^{-1}\partial_\mu S + S^{-1}(\partial_\mu S)S^{-1}\partial_\nu S].\end{aligned}$$

We see that the $+S^{-1}[\partial_\nu B_\mu - \partial_\mu B_\nu]S$ term satisfies (21) if the field strength had only the electromagnetic-like contribution. The other terms must either represent the transformed nonelectromagnetic-like part of $\mathbf{F}_{\mu\nu}$ or be cancelled by adding the nonelectromagnetic terms to (22). Since S is used only for the transformation, it should not appear in the expression for $\mathbf{F}_{\mu\nu}$.

The i/ϵ term in (24) dictates that a term multiplied by $i\epsilon$ be added to (22). Since $S^{-1}(\partial_\mu S)$ and $S^{-1}\partial_\nu S$ appear in the expressions for $\mathbf{B}'_\mu$ and $\mathbf{B}'_\nu$, respectively, the product of $S^{-1}(\partial_\mu S)$ and $S^{-1}\partial_\nu S$ that appears in the last term of (24) suggests that we should start our quest to eliminate extra terms here by adding $i\epsilon B'_\mu B'_\nu$ to the equation. This product gives

$$i\epsilon S^{-1}B_\mu B_\nu S - i/\epsilon S^{-1}(\partial_\mu S)S^{-1}\partial_\nu S - S^{-1}B_\mu\partial_\nu S - S^{-1}(\partial_\mu S)S^{-1}B_\nu S. \tag{25}$$

All but the first term (which represents the transformation of $i\epsilon B_\mu B_\nu$) cancel components of the extraneous terms in (24). And the full expression,

$i\epsilon(B'_\mu B'_\nu - B'_\nu B'_\mu)$, cancels all of the extraneous terms except the transformation of $i\epsilon(B_\mu B_\nu - B_\nu B_\mu)$. After performing the cancellation, we get

(26)
$$\partial_\nu B'_\mu - \partial_\mu B'_\nu + i\epsilon(B'_\mu B'_\nu - B'_\nu B'_\mu) = S^{-1}[\partial_\nu B_\mu - \partial_\mu B_\nu + i\epsilon(B_\mu B_\nu - B_\nu B_\mu)]S,$$

which satisfies (21).

The YM Lagrangian density consists of

$$-(\frac{1}{4})Tr(F^{\mu\nu}F_{\mu\nu}) \tag{27}$$

(similar to the kinetic part of the electromagnetic Lagrangian) plus a Dirac Lagrangian for a fermion doublet, giving

(28)
$$\mathcal{L} = -(\frac{1}{4})Tr(F^{\mu\nu}F_{\mu\nu}) - \bar{\psi}\gamma^\mu(\partial_\mu - i\epsilon B_\mu)\psi - m\bar{\psi}\psi.$$

The Euler-Lagrange equations used to obtain the field equations are

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{29}$$

for each independent field ϕ. Setting ϕ to $\bar{\psi}$ we get

$$\gamma^\mu(\partial_\mu - i\epsilon B_\mu)\psi + m\psi = 0, \tag{30}$$

and setting ϕ to $\mathbf{B}$, we get

$$\partial^\mu \mathbf{F}_{\mu\nu} - i\epsilon[\mathbf{B}^\mu, \mathbf{F}_{\mu\nu}] = -i\epsilon\bar{\psi}\gamma_\nu\psi = -\mathbf{J}_\nu. \tag{31}$$

As we have seen, in differential geometry terms, the field strength takes the form

$$F = dB + B \wedge B. \tag{32}$$

The Bs do not commute. Therefore, as opposed to the electromagnetic case, here the field strength has a $B \wedge B$ contribution. This represents the interaction of the quanta of the **B** field, which is due to the isospin they carry, and for the nonneutral ones, their charge. The Euler-Lagrange equations obtained from the kinetic part of the Lagrangian are

$$d_B F = 0, \tag{33}$$

the Bianchi identity, and

$$d_B * F = 0, \tag{34}$$

where d_B is the covariant exterior derivative.

We conclude this section by showing how the covariant transformation in (21) is related to the commutator in (8). We shall start with the transformation

$$\psi' = S^{-1}\psi. \tag{35}$$

This means that the covariant derivative transforms as

$$D'_\mu\psi' = S^{-1}D_\mu\psi, \tag{36}$$

and repeated operations of D_μ will transform in the same way. So we can write

$$[D'_\mu, D'_\nu]\psi' = S^{-1}[D_\mu, D_\nu]\psi \tag{37}$$

and obtain

$$F'_{\mu\nu}S^{-1} = S^{-1}F_{\mu\nu}. \tag{38}$$

Operating on the right by S gives us the required result,

$$F'_{\mu\nu} = S^{-1}F_{\mu\nu}S. \tag{39}$$

Concluding Remarks

It was proven in the 1960s that if the non-abelian Lagrangian would have an explicitly added mass term that was not gauge invariant, the theory would not be renormalizable and therefore unable to produce finite results. See, for instance, [23] and [24]. Therefore the gauge particles predicted by the original YM theory must have zero mass and infinite range. This contradicts the fact that the weak and strong interactions are short range. To solve this problem, physicists took over the spontaneous symmetry breaking (SSB) technique from superconductivity. In this approach, symmetry breaking is not imposed explicitly but arises from the theory itself. SSB occurs when the Lagrangian has a symmetry and the ground state, known in quantum field language as the *vacuum*, does not. If the vacuum is unique, however, it must share the symmetry of the Langrangian. Therefore the vacuum must be degenerate for SSB to occur. The continuous symmetry seen in the Lagrangian for the Mexican hat potential diagram is exemplified by the infinite number of degenerate minima on the orbit at the bottom of the hat. These degenerate vacua are related to each other by the symmetry operation of O(2) rotation. As soon as one vacuum state is chosen, the symmetry is broken. As early as 1937 Landau saw the possibility of SSB in second-order phase transitions [25]. (See Michel's talk on page 377 of [14].) After World War II, Ginzburg and Landau [26] devised a model of SSB to describe superconductors. Witten [27] describes this model and *inter alia* the standard model; one can see from this the Landau-Ginzburg model relationship to the standard model. In the west, Nambu [28], [29] and Anderson [30] saw that SSB of the electromagnetic field occurs in superconductors and made the analogy to particle physics. In fact, in [30], Anderson ends with "…the Goldstone zero-mass difficulty is not a serious one, because we can probably cancel it off against an equal Yang-Mills zero-mass problem."

It had been shown by Nambu [31], generalized and announced as a theorem and proven by Goldstone [32], and proven more rigorously by Goldstone, Salam, and Weinberg [33], that for every continuous symmetry that is spontaneously broken, a zero-mass, zero-spin boson appears in the theory. These are called Goldstone or Nambu-Goldstone (NG) bosons. One of the criteria for the theorem to apply is that the gauge used be manifestly Lorentz covariant. One choice is to use the Coulomb gauge as the local gauge so that, as we will see, the Goldstone bosons disappear.

In the Mexican hat potential problem, fluctuations along the degenerate vacua orbit produce a Goldstone boson. An example of Goldstone's theorem in ferromagnetism is SSB creating Bloch spin waves; in superfluids, an example is the Landau phonon. In elementary particles, the pion (its mass is[14] 135 MeV for the π^0 and 139.6 MeV for the π^+ and π^-; by comparison, the electron's mass is 0.5 MeV) is considered a Goldstone boson of spontaneous broken *approximate* symmetry; also called spontaneous broken chiral symmetry. According to Weinberg [34], for a while this view of the pion as being a form of the Goldstone boson dampened the enthusiasm to make the Goldstone bosons vanish. Even so, physicists were still faced with the problem that there is only one zero mass boson, the photon. So zero-mass particle production had to be explained away.

Now physicists were faced with two problems: zero-mass intermediate bosons (IB) and the Nambu-Goldstone zero-mass bosons that accompanied SSB. At the February 2010 American Physical Society meeting, three groups were awarded the J. J. Sakurai Prize for their work in solving this problem. The groups and their epic papers are: Englert and Brout[15] [35]; Higgs [36] (who did not attend); and Guralnick, Hagen, and Kibble [37]. Essentially, what is done is to make the gauge transformation local and have the "IB eat the NG bosons". The zero-mass IB have two polarizations. The NG bosons contribute a longitudinal polarization to the IB, making them massive. This technique is called the *Higgs mechanism*, and it is thought that it is responsible for giving particles their mass. Another massive particle (the Higgs boson) is also produced. The search for the Higgs boson is ongoing at the CERN Large Hadron Collider (LHC). The CMS (Compact Muon Solenoid) collaboration at the LHC has posted their results at [41]. They have a hint of evidence that the Higgs mass is around 124 GeV, while the ATLAS collaboration, which posts its results at [42], feels the mass is between 115 and 127 GeV. Much more data has to be taken before the two groups zero in on a value they can have confidence in.

As we have seen, the conserved current of quantum electrodynamics is associated with the U(1) gauge group, which has the photon as its gauge particle. Yang and Mills, because their theory incorporated SU(2) symmetry with $e^{-i\alpha(x)\cdot\sigma}$ as an isotopic spin rotation, had the insight to predict the existence of three gauge particles, the three vector bosons associated with SU(2). The Higgs mechanism, along with the YM non-abelian gauge theory and these mathematical groups, is incorporated into the electroweak force by the Glashow-Salam-Weinberg (GSW) $\mathrm{SU}(2)\times U(1)$ theory [38], [39], [40]. The electroweak interaction is mediated by four gauge bosons: the three massive ones, $W^\pm$ and Z^0, and the photon. The theory predicts the masses of the $W^\pm$ and Z^0: The mass of the W is approximately 80.4 GeV and that of the Z^0 is approximately 91.2 GeV. They were experimentally detected in 1983. G. 't Hooft [43], helped by techniques developed by Veltman, showed in 1971 that YM theories are renormalizable. This gave prominence to the GSW theory. In fact, the number of references to Weinberg's electroweak paper [40] surged from 1 in 1970 to 64 in 1972 [44].

In [34] Weinberg says:

> The importance of the renormalizability of the electroweak theory was not so much that the infinities could be removed by renormalization, but rather that the theory has the potentiality of describing weak and electromagnetic interactions at energies greater than 300 GeV, and perhaps all the way up to the Planck scale.[16]

[14] *An electron volt is the energy gained by an electron accelerating through one volt. A MeV is* 10^6 *ev. A GeV is* 10^3 *MeV. Masses are also measured in MeV.*

[15] *Unfortunately, Robert Brout passed away on 3 May 2011.*

[16] *The Planck scale is an energy scale around* 1.22×10^{19} *GeV.*

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