WHAT IS...

a Biholomorphic Mapping?

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Let $\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}$ denote complex Euclidean space, and let $D_1, D_2 \subset \mathbf{C}^n$ be domains. A mapping $f(z_1, \ldots, z_n) = (f_1, \ldots, f_n) : D_1 \to D_2$ is *holomorphic* if each of the coordinate functions f_j is holomorphic. If f is one-to-one and onto, then there is an inverse function $f^{-1} : D_2 \to D_1$, and this may be shown to be holomorphic. In this case, we say that f is *biholomorphic*. Biholomorphic maps preserve many of the basic properties of analytic function theory and serve as the isomorphisms in the category of analytic objects.

Let us first consider the case $n = 1$. The derivative of f at a point z_0 is a complex number $f'(z_0)$. For a biholomorphic mapping, the derivative is nonzero, and if we write $f'(z_0) = re^{i\theta}$, $r > 0$, then, infinitesimally at z_0, f looks like the translation $z_0 \mapsto f(z_0)$, followed by the rotation by angle θ about $f(z_0)$ and a dilation by the factor r centered at $f(z_0)$. Such a map is called *conformal* since it preserves angles at the infinitesimal level. In other words, if γ_1 and γ_2 are two smooth curves both passing through z_0 and crossing with an angle of α with respect to each other, then the image curves $f(\gamma_1)$ and $f(\gamma_2)$ will both pass through $f(z_0)$ and cross each other with the same angle α. In dimension 1, biholomorphic maps are conformal.

A domain D in the plane $\mathbf{C}$ is *simply connected* if its complement in the Riemann sphere is connected. A topological fact is that a simply connected domain in the plane $\mathbf{C}$ is topologically equivalent (homeomorphic) to the disk $\Delta = \{|z| < 1\}$, but this is less than obvious, since simply connected domains can have very complicated boundaries.

Riemann Mapping Theorem (RMT). *Suppose that $D \subset \mathbf{C}$ is simply connected and $D \neq \mathbf{C}$. Then there is a biholomorphic map $\varphi : D \to \Delta$.*

There are many consequences to the RMT. In particular, there are a lot of conformal maps: for every proper, simply connected domain, there is a conformal map to the disk. On the other hand, $\mathbf{C}$ is not biholomorphically equivalent to Δ because a holomorphic map $\psi : \mathbf{C} \to \Delta$ is necessarily bounded, so by the Liouville Theorem ψ is constant. Thus, by the RMT, there are only two classes of simply connected domains in $\mathbf{C}$ which are inequivalent up to conformal equivalence, namely the class of Δ and the class of $\mathbf{C}$. In fact, if $f : \mathbf{C} \to D$ is biholomorphic, then $D = \mathbf{C}$ and f has the form $z \mapsto \alpha z + \beta$ with $\alpha, \beta \in \mathbf{C}$.

The case $n > 1$ is different. Many important one-dimensional results do not hold for $n > 1$, and we find new, and sometimes unexpected, phenomena. First, biholomorphic maps are not necessarily conformal (consider $(z_1, z_2) \mapsto (z_1, 2z_2)$). Further, if $n > 1$, we find "rigidity" results, and distinct domains are "typically" inequivalent. On the other hand, for unbounded domains, we find larger spaces of holomorphic mappings.

Biholomorphic self-maps of a domain D are called *automorphisms*, and the automorphism group $\mathrm{Aut}(D)$ is a biholomorphic invariant. However, for a "typical" domain $D \subset \mathbf{C}^n$, $n > 1$, the automorphism group $\mathrm{Aut}(D)$ consists only of the identity transformation. There is no single domain in $\mathbf{C}^n$, $n > 1$, which plays the special role that Δ plays in $\mathbf{C}$. Leading competitors are

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the polydisk $\Delta^n := \{\max(|z_1|, \dots, |z_n|) < 1\}$ and the ball $\mathbf{B}^n := \{|z_1|^2 + \cdots + |z_n|^2 < 1\}$. An exercise in several complex variables is to determine the automorphism groups of Δ^n and $\mathbf{B}^n$. These groups are not isomorphic, so the bidisk is not biholomorphically equivalent to the ball. Thus there is no RMT in dimension > 1.

One obstruction for the existence of holomorphic mappings for $n > 1$ is given by rigidity phenomena. For instance, suppose that U is a domain with $U \cap \partial\mathbf{B}^n \neq \emptyset$. If $f : U \to f(U)$ is biholomorphic and if $f(U \cap \partial\mathbf{B}^n) \subset \partial\mathbf{B}^n$, then f coincides with an automorphism of $\mathbf{B}^n$. This was shown by H. Poincaré for $n = 2$ and by H. Alexander for $n > 2$. More generally, if $D_1, D_2 \subset \mathbf{C}^n$ are simply connected domains, bounded, with real analytic, strictly convex boundaries, and if $f : U \to f(U)$ is a biholomorphism taking $U \cap \partial D_1 \neq \emptyset$ to $f(U) \cap \partial D_2$, then f extends to a biholomorphic map $\hat{f} : D_1 \to D_2$. S. Webster showed that if D_1 and D_2 are both defined by polynomials, then f must be algebraic. Such rigidity phenomena give another obstruction to formulating an RMT for $n > 1$.

There are also natural questions about the boundary behavior of a biholomorphism. For instance, if $D_1, D_2 \subset \mathbf{C}^n$ are bounded with real analytic boundaries, does a biholomorphism $f : D_1 \to D_2$ extend continuously (or holomorphically) to $\overline{D_1}$? While the affirmative answer has been proved for $n = 1$ and 2, the question remains open for $n > 2$.

It is natural to ask: What are the domains D for which $\mathrm{Aut}(D)$ is "large" in some sense? H. Cartan showed that, if D is bounded, then $\mathrm{Aut}(D)$ is a Lie group. If $p \in D$, then we define $\mathrm{Aut}_p(D) = \{f \in \mathrm{Aut}(D) : f(p) = p\}$. If D is bounded, then by the Cauchy estimates, there is a bound (independent of f) on $\partial^\alpha f(p)$ for any derivative $\partial^\alpha = \partial^{(\alpha_1, \dots, \alpha_n)} = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_n)^{\alpha_n}$. Applying this bound simply to the differential $df^n(p) = (df(p))^n$ for all $n \in \mathbf{Z}$, we conclude that $df(p) \in U(n)$ is (essentially) an element of the unitary group. Thus $f \mapsto \rho(f) := df(p)$ gives a representation $\rho : \mathrm{Aut}_p(D) \to U(n)$. Next suppose that $Df(p) = I$ is the identity. Setting $p = 0$, we may assume that $f(z) = z + h(z) + O(|z|^{m+1})$, where $h(z)$ is homogeneous of degree $m \geq 2$. Iterating this, we see that $f^j(z) = z + jh(z) + O(|z|^{m+1})$. If $|\alpha| = m$, then we have $|\partial^\alpha f^j(p)| = |j\partial^\alpha h(p)|$. Since this must hold for all j, we may apply the Cauchy estimates to conclude that $h = 0$. Thus the representation ρ is injective.

It follows from the injectivity of ρ that $\mathrm{Aut}_p(D)$ is compact if D is bounded. Thus $\mathrm{Aut}(D)$ can be noncompact (a natural sense of being "large") only if the orbit $O(p) = \{f(p) : f \in \mathrm{Aut}(D)\}$ is noncompact. Let us give some smooth, bounded domains with noncompact automorphism groups. Start with coordinates $(z, w) = (z_1, \dots, z_n, w)$ on $\mathbf{C}^n \times \mathbf{C}$ and write $w = u + iv$. Any domain of the form $D = \{(z, w) : v + \psi(z, \bar{z}) < 0\}$ is invariant under real translations $(z, w) \mapsto (z, w + s)$ for all $s \in \mathbf{R}$, so $\mathrm{Aut}(D)$ is not compact. Given weights $\delta_1, \dots, \delta_n > 0$, we define the (weighted) Cayley transform $\Phi : (z, w) \mapsto (z^*, w^*)$ by $w = (1 - iw^*/4)(1 + iw^*/4)^{-1}$, $z_j = z_j^*(1 + iw^*/4)^{-2\delta_j}$. This defines a biholomorphic map of any domain $D \subset \{v < 0\}$ to its image $\Phi(D) \subset \mathbf{C}^{n+1}$. The weights of monomials are given by $\mathrm{wt}(z^J) = \mathrm{wt}(z_1^{j_1} \cdots z_n^{j_n}) = \sum_\ell \delta_\ell j_\ell$, $\mathrm{wt}(\bar{z}^J) = \mathrm{wt}(z^J)$, and $\mathrm{wt}(z^J \bar{z}^K) = \mathrm{wt}(z^J) + \mathrm{wt}(\bar{z}^K)$. Now let us suppose that $\psi = \sum_{J,K} a_{J,K} z^J \bar{z}^K$ is a nonnegative polynomial which is homogeneous and "balanced" with respect to this selection of weights in the following sense: $\mathrm{wt}(J) = \mathrm{wt}(K) = 1/2$ for all (J, K) such that $a_{J,K} \neq 0$. It follows that the Cayley transform takes $D = \{\psi(z, \bar{z}) + v < 0\}$ to the smoothly bounded domain $\Phi(D) = \{\sum_{J,K} a_{J,K} z^J \bar{z}^K + |w|^2 < 1\}$. Since there is great freedom in choosing weights and coefficients, this gives a large number of smoothly bounded domains with noncompact automorphism groups. As natural as these domains seem, it is not known whether all smoothly bounded domains with noncompact automorphism groups are biholomorphically equivalent to some $\Phi(D)$.

Up to this point, we have focused our attention on bounded domains. The situation is somehow the opposite for unbounded domains: there are many automorphisms of $\mathbf{C}^n$ when $n > 1$, and the structure of the group $\mathrm{Aut}(\mathbf{C}^n)$ is not well understood. Perhaps the most striking difference is the existence of *Fatou-Bieberbach domains:* such domains are strict subsets of $\mathbf{C}^n$ which are biholomorphically equivalent to $\mathbf{C}^n$. These domains arise naturally from considerations of complex dynamics, as we will show here. For parameters a and b, we consider the automorphism $f(z_1, z_2) = (z_1^3 + az_1 - bz_2, z_1)$, which is a cubic mapping of $\mathbf{C}^2$ with a cubic inverse. These maps belong to the family of Hénon mappings. These have been studied as dynamical systems $f : \mathbf{C}^2 \to \mathbf{C}^2$ and have been found to have rich dynamics, although their dynamical behaviors are still only partially understood. The map f is odd in the sense that $f(-z) = -f(z)$. The fixed points, by definition, satisfy $f(z) = z$ and are three points: $(0, 0)$ and $\pm P$. For generic a and b, the differential $Df(\pm P)$ at $\pm P$ has distinct eigenvalues $|\lambda_1| > |\lambda_2|$. We may choose a, b such that $1 > |\lambda_1| > |\lambda_2|$. We define the basin of P as $B(P) := \{z \in \mathbf{C}^2 : \lim_{n\to\infty} f^n z = P\}$, which are all the points which approach P in forward time. Since the mapping is odd, the other basin is given by $B(-P) = -B(P)$. To examine this basin in more detail, we may uniformize it. That is, we find a biholomorphic map $h : \mathbf{C}^2 \to B(P)$.

Now we treat f as a dynamical system. We set $L(z) = (\lambda_1 z_1, \lambda_2 z_2)$. After an affine change of coordinates, we may suppose that $P = 0$ and that $f(z) = L(z) + \tilde{f}(z)$, where $\tilde{f}(z) = O(|z|^2)$ denotes terms of degree 2 and higher. We will find h as a solution to the functional equation

$$(*) \qquad L \circ h = h \circ f.$$

Thus h conjugates the dynamical system $(f, B(P))$ to the linear dynamical system $(L, \mathbf{C}^2)$. We wish to show the existence of the uniform limit $h(z) := \lim_{n\to\infty} L^{-n} \circ f^n(z) = \mathrm{id} + \tilde{h}(z)$ in a neighborhood U of the origin. Such a limit clearly must satisfy $(*)$. Technically, we may show the existence of the limit directly if we further restrict a, b so that $|\lambda_2| > |\lambda_1|^2$. Now that h is defined on U, we may extend it to $f^{-1}(U)$ using the identity $h = L^{-1} \circ h \circ f$ on U. It follows that h extends holomorphically to a map $h : \mathbf{C}^2 \to B(P) = \bigcup_{n\geq 0} f^{-n}U$. Using the property that $B(P)$ is the basin, we conclude that h is biholomorphic, there are no other fixed points in $B(P)$, and $B(P)$ is disjoint from $B(-P) = -B(P)$. Thus both $B(P)$ and $-B(P)$ are disjoint Fatou-Bieberbach domains, and if we think of $B(P)$ as "large", then we must also think of the complement $\mathbf{C}^2 - B(P)$ as "large".

Now that we have the existence of Fatou-Bieberbach domains, we may also consider their increasing limits. That is, suppose that $f_j : \mathbf{C}^2_j \to f(\mathbf{C}^2_j) \subset \mathbf{C}^2_{j+1}$, $j = 1, 2, 3, \ldots$, is a sequence of biholomorphic mappings to Fatou-Bieberbach domains in $\mathbf{C}^2$. Now consider the direct limit M of this family, which is the complex manifold given as $M := \bigsqcup_j \mathbf{C}^2_j / \sim$, the disjoint union of infinitely many copies of $\mathbf{C}^2$ with the identification $z_j \sim z_{j+1}$ if $z_j \in \mathbf{C}^2_j$, $z_{j+1} \in \mathbf{C}^2_{j+1}$, and $f_j(z_j) = z_{j+1}$. There are instances where the limit M is biholomorphically equivalent to $\mathbf{C}^2$ and examples where it is not.

Further Reading

K. Fritzsche and H. Grauert, *From Holomorphic Functions to Complex Manifolds*, Graduate Texts in Mathematics, Vol. 213, Springer-Verlag, New York, 2002.