Srinivasa Ramanujan: Going Strong at 125, Part I

Krishnaswami Alladi, Editor

The 125th anniversary of the birth of the Indian mathematical genius Srinivasa Ramanujan falls on December 22, 2012. To mark this occasion, the Notices *offers the following feature article, which appears in two installments. The article contains an introductory piece describing various major developments in the world of Ramanujan since his centennial in 1987, followed by seven pieces describing important research advances in several areas of mathematics influenced by him. The first installment in the present issue of the* Notices *contains the introductory piece by Krishnaswami Alladi, plus pieces by George Andrews, Bruce Berndt, and Jonathan Borwein. Pieces by Ken Ono, K. Soundararajan, R. C. Vaughan, and S. Ole Warnaar will appear in the second installment in the January 2013 issue of the* Notices.

Krishnaswami Alladi

Ramanujan's Thriving Legacy

Srinivasa Ramanujan is one of the greatest mathematicians in history and one of the most romantic figures in the mathematical world as well. Born on December 22, 1887, to a poor Hindu brahmin family in Erode in the state of Tamil Nadu in South India, Ramanujan was a self-taught genius who discovered a variety of bewildering identities starting from his school days. There is a legend that the Hindu goddess Namagiri in the neighboring town of Namakkal used to come in his dreams and give him these formulae. Unable to find individuals in India to understand and evaluate his findings, Ramanujan wrote two letters in 1913 to G. H. Hardy of Cambridge University, England, listing several of his most appealing formulae. The depth and startling beauty of these identities convinced Hardy that Ramanujan was on a par with Euler and Jacobi in sheer manipulative ability. At Hardy's invitation, Ramanujan went to Cambridge in 1914. The rest is history. In the five years he was in England, Ramanujan published several important papers, some of which stemmed from his discoveries in India; he collaborated with G. H. Hardy and wrote two very influential papers with him. For his pathbreaking contributions, Ramanujan was honored by being elected Fellow of Trinity College, Cambridge, and Fellow of the Royal Society (FRS) even though he did not have a college degree! The rigors of life in England during the First World War, combined with his own peculiar habits, led to a rapid decline in his health. Ramanujan returned to Madras, India, in 1919 a very sick man and died shortly after on April 20, 1920. Even during his last days, his mathematical creativity remained undiminished. He wrote one last letter to Hardy in January 1920 outlining his discovery of the mock theta functions, considered to be among his deepest contributions. In the decades after Ramanujan's death, we have come to realize the depth, breadth, and significance of his many discoveries. Ramanujan's work has had a major influence on several branches of mathematics, most notably in number theory and classical analysis, and even in some areas of physics.

During the Ramanujan Centennial in 1987, eminent mathematicians from around the world gathered in India to pay homage to this singular genius. It was an appropriate time to reflect on his legacy and consider the directions in which his work would have influence in the future. In this quarter century since the centennial, many significant developments have taken place in the world of Ramanujan: (i) there have been major research advances in recent years that have widened the arena of impact of Ramanujan's work; (ii) books on Ramanujan's work, of interest to students

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and researchers alike, and books on Ramanujan's life with appeal to the general public have been published; (iii) journals bearing Ramanujan's name have been established, one of which is devoted to all areas of research influenced by him; (iv) annual conferences devoted to various aspects of Ramanujan's work are being held in his hometown in India; (v) international prizes bearing his name and encouraging young mathematicians have been created; and (vi) movies and plays about Ramanujan inspired by his remarkable life have also appeared or are in production.

This feature in the *Notices of the AMS* is devoted to describing some of the significant developments in the world of Ramanujan since the centennial. My opening article will broadly address the developments listed under (i)–(vi). This will be followed by articles by leading mathematicians on various aspects of research related to Ramanujan's work.

Photo courtesy of Krishnaswami Alladi.

The bust of Ramanujan now stands in the central hall of his home in Kumbakonam. The bust was added after SASTRA University purchased Ramanujan's home. In the background is a replica of his famous passport photograph.

Some Major Research Advances

I focus here on certain aspects of the theory of partitions and q-hypergeometric series with which I am most familiar and provide just a sample of some very important recent results. Research relating to Ramanujan's work in many other areas has been very significant, and some of these ideas will be described by others in the following articles.

Ramanujan's mock theta functions are mysterious, and the identities he communicated to Hardy about them are deep. Since George Andrews unearthed the *Lost Notebook of Ramanujan* at Trinity College Library in Cambridge University in 1976, he has been analyzing the identities contained therein, especially those on the mock theta functions. We owe much to Andrews for explaining the mock theta function identities in the context of the theory partitions (see for instance [12]). In Andrews-Garvan [21] the mock theta conjectures are formulated. These relate the q-hypergeometric mock theta function identities to partition identities; the mock theta conjectures were proved by Hickerson [40] around the time of the centennial.

In spite of their resemblance to the classical theta functions, the exact relationship between the mock theta functions and theta functions remained unclear. In the last decade, dramatic progress on this matter has been achieved by Ken Ono and his coworkers, most notably Kathrin Bringmann, in establishing the links between mock theta functions and Maass forms, thereby providing the key to unlock this mystery [32], [33], [34].

Ramanujan's congruences for the partition function modulo the primes 5, 7, and 11 and their powers are among his most significant discoveries. More specifically, Ramanujan discovered that $p(5n+4)$ is a multiple of 5, $p(7n+5)$ is a multiple of 7, and $p(11n+6)$ is a multiple of 11, where $p(n)$ is the number of partitions of n. In the ensuing decades, this led to a fruitful study of congruences satisfied by coefficients of modular forms. Yet it was unclear whether the partition function itself satisfied congruences with respect to other primes. Oliver Atkin [22] was the first to discover congruences for the partition function of the type $p(An+b) \equiv C(mod\, p)$ with a prime modulus $p > 11$, but unlike the Ramanujan congruences, we have A not equal to p. In recent years Scott Ahlgren [1] and Ken Ono [44] have discovered partition congruences to larger prime moduli and have shown that there are indeed infinitely many such congruences.

Ramanujan (see [46]) wrote down two spectacular identities for the generating functions of $p(5n+4)$ and $p(7n+5)$, from which his congruences modulo 5 and 7 follow. Proofs of Ramanujan's partition congruences were given by Atkin using the theory of modular forms. In 1944, Freeman Dyson [35] gave a combinatorial explanation of Ramanujan's partition congruences mod 5 and 7 using the now famous statistic he called the *rank*. He then conjectured the existence of a statistic he dubbed the *crank* that would explain the mod 11 congruence. A combinatorial explanation of Ramanujan's partition congruence modulo 11 was first given in the Ph.D. thesis of Frank Garvan [36] using vector partitions and a vector crank. This was then converted to ordinary partitions by Andrews and Garvan [20] when they got together at the Ramanujan Centennial Conference at the University of Illinois, Urbana, in the summer of 1987, and so finally the crank conjectured by Dyson was found. Perhaps the Goddess of Namakkal made sure that the solution to this problem should actually happen during the Ramanujan Centennial! Since 1987 there has been an enormous amount of work on the rank and on cranks for various partition statistics by several mathematicians: Garvan, Kim, and Stanton [37] found cranks for t-cores; Mahlburg [43] found cranks for the partition congruences to moduli larger than 11; and Andrews [16] has noticed new connections with Durfee symbols of partitions. In the last decade, Richard Stanley [49] introduced a refinement of Dyson's rank, and this has led to important refinements and extensions of Ramanujan's partition congruences (see for instance Andrews [15] and Berkovich-Garvan [24]).

In the entire theory of partitions and q-hypergeometric series, the Rogers-Ramanujan identities are unmatched in simplicity of form, elegance, and depth. This pair of identities connects partitions into parts that differ by at least two with partitions into parts congruent to $\pm i \pmod 5$, for $i = 1, 2$. Nowadays, by a Rogers-Ramanujan (R-R) type identity, we mean an identity which in its analytic form equates a q-hypergeometric series to a product, where the series represents the generating function of partitions whose parts satisfy gap conditions and the product is the generating function of partitions whose parts satisfy congruence conditions. This subject has blossomed into an active area of research thanks to systematic analysis of R-R type identities, primarily by Andrews ([11], Chapter 7). In the 1980s the Australian physicist Rodney Baxter showed how certain R-R type identities arise as solutions of some models in statistical mechanics (see Andrews [13], Chapter 8). Soon after, Andrews, Baxter, and Forrester [18] determined the class of R-R type identities that arise in such models in statistical mechanics. For this work Baxter was recognized with the Boltzmann Medal of the American Statistical Society.

Subsequently there has been significant work on q-hypergeometric identities and integrable models in conformal field theory in physics. The leader in this effort is Barry McCoy, who in the 1990s, with his collaborators Alexander Berkovich, Anne Schilling, and Ole Warnaar [25], [26], [27], [28], discovered new R-R type identities from the study of integrable models in conformal field theory. For this work McCoy was recognized with the Heineman Prize in Mathematical Physics and an invited talk at the ICM in Berlin in 1998.

My own work in the theory of partitions and q-hypergeometric series has been in the study of R-R type identities and their refinements [4], especially on a partition theorem of Göllnitz, one of the deepest R-R type identities. In collaboration with Basil Gordon, George Andrews, and Alexander Berkovich, I was able to obtain generalizations and refinements of Göllnitz's partition theorem in 1995 [9] and later, in 2003, a much deeper four-parameter extension of the Göllnitz theorem [10].

Books on Ramanujan's Life and Work

Hardy's twelve lectures on Ramanujan [39] are a classic in exposition. For several decades after Ramanujan's death, this book and the *Collected Papers* [46] of Ramanujan were the primary sources for students and young researchers aspiring to enter the Ramanujan mathematical garden. In 1987, *The Lost Notebook of Ramanujan* was published [48]. Students today are very fortunate, because Bruce Berndt has edited the original three notebooks of Ramanujan and published this material in five volumes [29]. In these volumes one can find proofs of Ramanujan's theorems and how they are related to past and contemporary research. For this monumental contribution, Berndt was recognized with the AMS Steele Prize. In the past five years, Bruce Berndt and George Andrews have joined forces to edit Ramanujan's *Lost Notebook*. This is expected to also require five volumes, of which three have already appeared [19].

Ramanujan is now making an impact beyond mathematics into society in general around the world. Throughout India, Ramanujan's life story is well known, and Ramanujan is a hero to every Indian student. But with the publication of Robert Kanigel's book *The Man Who Knew Infinity* [41], Ramanujan's life story has reached the world over. The impact of this book cannot be underestimated (see [2]). Subsequently, Bruce Berndt and Robert Rankin have published two wonderful books. The first one, called *Ramanujan—Letters and Commentary* [30], collects various letters written to, from, and about Ramanujan and makes detailed commentaries on each letter. For instance, if a letter contains a mathematical statement, there is an explanation of the mathematics with appropriate references. If there is a statement about Ramanujan being elected Fellow of the Royal Society, there is a description about the procedures and practices for such an election (see review [3]). The second book, *Ramanujan—Essays and Surveys* [31], is a collection of excellent articles by various experts on Ramanujan's life and work and about various individuals who played a major role in Ramanujan's life (see review [5]). This book also contains an article by the great mathematician Atle Selberg, who describes how he was inspired by Ramanujan's mathematical discoveries. Most recently, David Leavitt has published a book [42] entitled *The Indian Clerk* which focuses on the close relationship between Ramanujan and Hardy (see review [38]). All of these books will not only appeal to mathematicians but to students and lay persons as well.

Since the Ramanujan Centennial, I have written articles annually comparing Ramanujan's life and mathematics with those of several great mathematicians in history, such as Euler, Jacobi, Galois, Abel, and others. These articles, which were written to appeal to the general public, appeared in *The Hindu*, India's national newspaper, each year around Ramanujan's birthday in December. The collection of all these articles, along with my book reviews and other Ramanujan-related articles, has just appeared as a book [8] for Ramanujan's 125th anniversary.

There are many other books on Ramanujan that have been published, and my list above is certainly not exhaustive. However, I must add to this list the comprehensive CD on Ramanujan that Srinivasa Rao brought out in India a few years ago. In this CD we have scanned images of Ramanujan's notebooks, his papers, several articles about Ramanujan, many important letters, etc.

Journals Bearing Ramanujan's Name

Inspired by what I saw and heard during the Ramanujan Centennial, I wanted to create something that would be a permanent and continuing memorial to Ramanujan, namely, *The Ramanujan Journal*, devoted to all areas of mathematics influenced by Ramanujan. This idea received enthusiastic support from mathematicians worldwide, many of whom have served, or are serving, on the editorial board. The journal was launched in 1997 by Kluwer Academic Publishers, which published four issues of one hundred pages per year in one volume. Now the journal is published by Springer, which publishes nine issues per year of one hundred fifty pages each in three volumes. The rapid growth of this journal is a further testimony to the fact that Ramanujan's work is continuing to have major influence on various branches of mathematics.

There is another journal bearing Ramanujan's name: *The Journal of the Ramanujan Mathematical Society*, which is distributed by the AMS and is devoted to all areas of mathematics. It bears Ramanujan's name since he is India's most illustrious mathematician.

Photo courtesy of Krishnaswami Alladi.

Left to right: Krishnaswami Alladi, George Andrews (Pennsylvania State University), and SASTRA Vice-chancellor R. Sethuraman in front of a bust of Srinivasa Ramanujan at SASTRA University, Kumbakonam—December 22, 2003.

Prior to these two journals, in the 1970s Professor K. Ramachandra (now deceased) founded *The Hardy-Ramanujan Journal*, devoted to elementary and classical analytic number theory, and published it privately in India.

Annual Conferences in Ramanujan's Hometown

A recent major event in the world of Ramanujan is the acquisition in 2003 of Ramanujan's home in Kumbakonam, South India, by SASTRA University. This private university, founded about twenty years ago, has grown by leaps and bounds. Since the university has purchased Ramanujan's home, we have the active involvement of administrators, academicians, and students in the preservation of Ramanujan's legacy. To mark the occasion of the purchase of Ramanujan's home, SASTRA University conducted an international conference at its Kumbakonam campus December 20–22, 2003, and had it inaugurated by Dr. Abdul Kalam, president of India. I was invited to bring a team of mathematicians to this conference. George Andrews gave the opening lecture of this conference as well as the concluding Ramanujan Commemoration Lecture on December 22, Ramanujan's birthday. At the conclusion of the conference, the participants suggested that SASTRA should conduct conferences annually in Kumbakonam on various areas of mathematics influenced by Ramanujan. I have helped SASTRA organize these annual conferences. Each year several leading researchers visit SASTRA in December to speak at these conferences, and by participating in them, I have had the pleasure of spending Ramanujan's birthday, December 22, every year in Ramanujan's hometown (see [6]).

Prizes Bearing Ramanujan's Name

At the inauguration of the second Ramanujan conference at SASTRA University in December 2004, Vice Chancellor Professor R. Sethuraman graciously offered to set apart US$10,000 annually to recognize pathbreaking research by young mathematicians.

In response to his offer of such support, I suggested that an annual prize be launched the next year to be given to mathematicians not exceeding the age of thirty-two for outstanding contributions to areas influenced by Ramanujan. The age limit was set at thirty-two because Ramanujan achieved so much in his brief life of thirty-two years. Thus the SASTRA Ramanujan Prize was born. In 2005, the year the prize was launched, the prize committee felt that two brilliant mathematicians, Manjul Bhargava (Princeton) and Kannan Soundararajan (then at Michigan, now at Stanford), both deserved the prize for spectacular contributions to algebraic and analytic number theory respectively. The vice chancellor generously informed the prize committee that there would be two full prizes that year, and it would not be split. The prestige of any prize is determined by the caliber of the winners. The SASTRA Ramanujan Prize could not have had a better start (see [7], [45]). The prize is now one of the most prestigious and coveted. The subsequent winners are: Terence Tao (UCLA) in 2006, Ben Green (Cambridge University) in 2007, Akshay Venkatesh (Stanford) in 2008, Kathrin Bringmann (Cologne) in 2009, Wei Zhang (Harvard) in 2010, Roman Holowinsky (Ohio State) in 2011, and Zhiwei Yun (MIT and Stanford) in 2012. This prize has been given in December each year in Ramanujan's hometown, Kumbakonam, during the SASTRA Ramanujan Conference.

There is another very prestigious prize with Ramanujan's name attached, namely, the ICTP Ramanujan Prize. The International Centre for Theoretical Physics (ICTP) in Trieste, Italy, was created in the sixties by the great physicist Abdus Salam with a particular commitment to encourage scientists from developing countries. Keeping this vision of Salam, the ICTP Ramanujan Prize launched in 2003 is a US$10,000 annual prize for mathematicians under the age of forty-five from developing countries for outstanding research in any branch of mathematics. Whereas the SASTRA Ramanujan Prize is open to mathematicians the world over but focuses on areas influenced by Ramanujan, the ICTP Ramanujan Prize is open to all areas of mathematics but focuses on candidates from developing countries. The ICTP Prize has Ramanujan's name because Ramanujan is the greatest mathematician to emerge from a developing country. The ICTP Ramanujan Prize is given in cooperation with the Abel Foundation.

Finally, there is the Srinivasa Ramanujan Medal given by the Indian National Science Academy for outstanding contributions to mathematical sciences. There is no age limit for this medal. Since its launch in 1962, the medal has been given fourteen times.

Movies and Plays about Ramanujan

Ramanujan's life story is so awe inspiring that movies and plays about him have been and are being produced. The first was a superb documentary about Ramanujan in the famous *Nova* series of the Public Broadcasting System (PBS) on television, which described some of his most appealing mathematical contributions in lay terms and some of the most startling aspects of his life, such as the episode of the taxi cab number 1729. This documentary was produced before the Ramanujan Centennial but after Andrews's discovery of the *Lost Notebook*.

Since the Ramanujan Centennial, there have been many stage productions, including the *Opera Ramanujan*, which was performed in Munich in 1998. In 2005 I had the pleasure of seeing the play *Partition* at the Aurora Theater in Berkeley along with George Andrews. In this and other stage productions, there is usually a "playwright's fancy", namely, a modification of the true life story of Ramanujan for a special effect. The producer of *Partition*, Ira Hauptman, focuses on Ramanujan's monumental work with Hardy on the asymptotic formula for the partition function, but for a special effect describes Ramanujan's attempts to solve Fermat's Last Theorem and Ramanujan's conversation with the Goddess of Namakkal regarding this matter. As far as we know, Ramanujan never worked on Fermat's Last Theorem, but this digression for a special effect is acceptable artist's fancy (see [8], article 23).

In 2006 Andrews and I served on a panel to critique a script entitled *A First Class Man* which had received support from the Sloan Foundation. The reading of the script was done at the Tribeca Film Institute in New York when the distinguished writer David Freeman was present. Andrews and I helped in clearing up certain mathematical misconceptions in the script and also noted that the script had changed the Ramanujan life story in certain ways that were not appropriate. Regardless of how the life of Ramanujan is portrayed in these stage productions, they have had the positive effect of drawing the attention of the public around the world to the life of this unique figure in history.

In 2007 a play entitled *A Disappearing Number* was conceived and directed by the English playwright Simon McBurney for the Theatre Complicite Company. It first played at the Theater Royal in Plymouth, England, and won three very prestigious awards in England in 2007. This play was performed at the International Congress of Mathematicians in Hyderabad, India, in August 2010.

The latest theatrical production is a movie that is now being produced in India based on Kanigel's book *The Man Who Knew Infinity* and featuring one of the most popular actors in the Tamil cinema world as Ramanujan.

Events in 2011–12

Since December 2012 is the 125th anniversary of Ramanujan's birth, the entire year starting from December 22, 2011, was declared as the Year of Mathematics by the prime minister of India at a public function in Madras on December 26, 2011. On that day, a stamp of Ramanujan was released and Robert Kanigel was given a special award for his biography of Ramanujan. Also, the Tata Institute released a reprinting of the *Notebooks* of Ramanujan in the form of a collector's edition [47]. The mathematics year has featured several conferences and public lectures on Ramanujan throughout India.

The Ramanujan Journal is bringing out a special volume for the 125th birth anniversary in December 2012. The 125th anniversary celebrations will conclude in 2012 with an International Conference on the Works of Ramanujan at the University of Mysore, India, December 12–13; an International Conference on the Legacy of Ramanujan at SASTRA University in Kumbakonam during December 14–16; and an International Conference on Ramanujan's Legacy in India's capital New Delhi during December 17–22. In 2012 only, the SASTRA Ramanujan Prize will be given outside of Kumbakonam, namely, at the New Delhi conference.

Just as there was a Ramanujan Centennial Conference at the University of Illinois, Urbana, in the summer of 1987 before the centennial celebrations in India that December, there was a Ramanujan 125 Conference at the University of Florida, Gainesville, in November 2012, one month before the 125th anniversary celebrations in India. Finally, at the Joint Annual Meetings of the AMS and MAA in January 2013 in San Diego, there will be a Special Session on Ramanujan's mathematics to mark his 125th birth anniversary.

We have much to be proud of regarding what has happened in the quarter century after the Ramanujan Centennial. The articles in this *Notices* feature convince us that the world of Ramanujan's mathematics is continuing to grow, and so we have much to look forward to in the next quarter century.

References

[1] S. Ahlgren and K. Ono, Congruence properties for the partition function, *Proc. Nat. Acad. Sci. USA* **98** (2001), 12882–12884.

[2] K. Alladi, The discovery and rediscovery of mathematical genius, Book Review of *The Man Who Knew Infinity* (by Robert Kanigel), *The American Scientist* **80** (1992), 388–389.

[3] ——, Book Review of *Ramanujan: Letters and Commentary* (by Bruce C. Berndt and Robert A. Rankin), *The American Math. Monthly* **103** (1996), 708–713.

[4] ——, *Refinements of Rogers-Ramanujan type identities*, in Proc. Fields Inst. Conf. on Special Functions, q-Series and Related Topics, Fields Inst. Communications, 14, Amer. Math. Soc., Providence, RI, 1997, pp. 1–35.

[5] ——, Book Review of *Ramanujan: Essays and Surveys* (by Bruce C. Berndt and Robert A. Rankin), *American Math. Monthly* **110** (2003), 861–865.

[6] ——, A pilgrimage to Ramanujan's hometown, *MAA Focus* **26** (2006), 4–6.

[7] ——, The first SASTRA Ramanujan prizes, *MAA Focus* **26** (2006), 7.

[8] ——, *Ramanujan's Place in the World of Mathematics*, Springer, New Delhi, 2012, 180 pp.

[9] K. Alladi, G. E. Andrews, and B. Gordon, Generalizations and refinements of a partition theorem of Göllnitz, *J. Reine Angew. Math.* **460** (1995), 165–188.

[10] K. Alladi, G. E. Andrews, and A. Berkovich, A new four parameter q-series identity and its partition implications, *Invent. Math.* **153** (2003), 231–260.

[11] G. E. Andrews, *The theory of partitions*, in Encyl. Math., 2, Addison-Wesley, Reading, MA 1976.

[12] ——, An introduction to Ramanujan's lost notebook, *Amer. Math. Monthly* **86** (1979), 89–108.

[13] ——, *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conf. Ser. in Math., 66, Amer. Math. Soc., Providence, RI, 1986.

[14] ——, The hard-hexagon model and the Rogers-Ramanujan identities, *Proc. Nat. Acad. Sci. USA* **78** (1981), 5290–5292.

[15] ——, On a partition function of Richard Stanley, Stanley Festschrift, *Elec. J. Comb.* **11** (2004/06).

[16] ——, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, *Invent. Math.* **169** (2007), 37–73.

[17] ——, The meaning of Ramanujan, now and for the future, *Ramanujan J.* **20** (2009), 257–273.

[18] G. E. Andrews, R. J. Baxter, and P. J. Forrester, Eight-vertex SOS model and generalized Rogers-Ramanujan type identities, *J. Stat. Phys.* **35** (1984), 193–266.

[19] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Parts I–V*, Springer, New York, 2005, 2009, 2012, to appear.

[20] G. E. Andrews and F. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc. (N.S.)* **18** (1988), 167–171.

[21] ——, Ramanujan's "lost" notebook, VI: The mock theta conjectures, *Adv. in Math.* **73** (1989), 242–255.

[22] A. O. L. Atkin, Multiplicative congruence properties and density problems for $p(n)$, *Proc. London Math. Soc.* **18** (1968), 563–576.

[23] A. O. L. ATKIN and F. GARVAN, Relations between the ranks and cranks of partitions, *Ramanujan J.* **7** (2003), 343–366.
[24] A. BERKOVICH and F. GARVAN, On the Andrews-Stanley refinement of Ramanujan's partition congruence modulo 5 and generalizations, *Trans. Amer. Math. Soc.* **358** (2006), 703–726.
[25] A. BERKOVICH and B. MCCOY, Continued fractions and Fermionic representations for characters of $M(p,p')$ minimal models, *Lett. Math. Phys.* **37** (1996), 49–66.
[26] ______, *Rogers-Ramanujan identities: A century of progress from mathematics to physics*, Doc. Math. 1998 Extra Vol. III ICM, 163–172.
[27] A. BERKOVICH, B. MCCOY, A. SCHILLING, and O. WARNAAR, Bailey flows and Bose-Fermi identities for conformal coset models $(A_1^{(1)})_N \times (A_1^{(1)})_{N'}/(A_1^{(1)})_{N+N'}$, *Nucl. Phys. B* **499** (1997), 621–649.
[28] A. BERKOVICH, B. MCCOY, and A. SCHILLING, Rogers-Schur-Ramanujan type identities for $M(p,p')$ models of conformal field theory, *Comm. Math. Phys.* **191** (1998), 325–395.
[29] BRUCE C. BERNDT, *Ramanujan's Notebooks, Parts I–V*, Springer, New York, 1985, 1989, 1991, 1994, 1998.
[30] BRUCE C. BERNDT and ROBERT A. RANKIN, *Ramanujan: Letters and Commentary*, Amer. Math. Soc., Providence, RI, and LMS, London, 1995.
[31] ______, *Ramanujan: Essays and Surveys*, Amer. Math. Soc., Providence, RI, and LMS, London, 2001.
[32] K. BRINGMANN and K. ONO, The $f(q)$ mock theta function conjecture and partition ranks, *Invent. Math.* **165** (2006), 243–266.
[33] ______, Dyson's rank and Maass forms, *Annals of Math.* **171** (2010), 419–449.
[34] J. K. BRUINIER and K. ONO, Heegner divisors, *L*-functions, and harmonic weak Maass forms, *Annals of Math.* **172** (2010), 2135–2181.
[35] F. J. DYSON, Some guesses in the theory of partitions, *Eureka* (Cambridge) **8** (1944), 10–15.
[36] FRANCIS GERARD GARVAN, *Generalizations of Dyson's rank*, Ph.D. Thesis, The Pennsylvania State University (1986), 136 pp.
[37] FRANK GARVAN, DONGSU KIM, and DENNIS STANTON, Cranks and *t*-cores, *Invent. Math.* **101** (1990), 1–17.
[38] H. HALBERSTAM, Review of *The Indian Clerk* (by David Leavitt), *Notices Amer. Math. Soc.* **55** (2008), 952–956.
[39] G. H. HARDY, *Ramanujan—Twelve Lectures on his Life and Work*, Cambridge University Press, Cambridge, 1940.
[40] D. HICKERSON, A proof of the mock theta conjectures, *Invent. Math.* **94** (1988), 639–660.
[41] ROBERT KANIGEL, *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, Charles Scribner's, New York, 1991.
[42] DAVID LEAVITT, *The Indian Clerk*, Bloomsbury, New York, 2007.
[43] KARL MAHLBURG, Partition congruences and the Andrews-Garvan-Dyson crank, *Proc. Nat. Acad. Sci. USA* **102** (2005), 15373–15376.
[44] K. ONO, Distribution of the partition function modulo *m*, *Annals of Math.* **151** (2000), 293–307.
[45] ______, Honoring a gift from Kumbakonam, *Notices Amer. Math. Soc.* **53** (2006), 640–651.
[46] S. RAMANUJAN, *Collected Papers*, Cambridge University Press, Cambridge, 1927.
[47] ______, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957 (reprinted as Collector's Edition in 2011).
[48] ______, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1987.
[49] R. STANLEY, Problem 10969, *Amer. Math. Monthly* **109** (2000), 760.

George E. Andrews

Partitions and the Rogers-Ramanujan Identities

In his first letter to G. H. Hardy [19, p. xxvii], Ramanujan asserts:

$$\text{“The coefficient of } x^n \text{ in } \frac{1}{1-2x+2x^4-2x^9+2x^{16}-\cdots}$$
$$(1)\qquad = \text{the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}, \text{”}$$

and, in a footnote, Hardy states, “This is quite untrue. But the formula is extremely interesting for a variety of reasons.” Perhaps one of the most important justifications for calling it “extremely interesting” is that it eventually led to the Hardy-Ramanujan formula for $p(n)$.

Each positive integer can be partitioned into unordered sums of positive integers. For example, there are seven partitions of the number 5: 1+1+1+1+1, 1+1+1+2, 1+2+2, 1+1+3, 2+3, 1+4, 5. We denote by $p(n)$ the number of partitions on n, so $p(5) = 7$.

Inspired by (1), Hardy and Ramanujan eventually proved [13] the following:

Theorem 1. *Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\frac{d}{dn}\left(e^{C\lambda_n/q}/\lambda_n\right),$$

where C and λ_n are defined by $C = \frac{2\pi}{\sqrt{6}}$ and $\lambda_n = \sqrt{n-\frac{1}{24}}$, and for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula

$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\frac{1}{4}(2-pq-p)+\frac{1}{12}\left(q-\frac{1}{q}\right)\right.\right.$$
$$\left.\left.(2p-p'+p^2p')\right\}\pi i\right]$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\frac{1}{4}(q-1)+\frac{1}{12}\left(q-\frac{1}{q}\right)\right.\right.$$
$$\left.\left.(2p-p'+p^2p')\right\}\pi i\right],$$

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Photo courtesy of Krishnaswami Alladi.

Krishnaswami Alladi and George Andrews in front of Srinivasa Ramanujan's home, Sarangapani Sannidhi Street, Kumbakonam, December 20, 2003.

where (a/b) is the symbol of Legendre and Jacobi and p' is any positive integer such that $1 + pp'$ is divisible by q; that

$$A_q(n) = \sum_{(p)} \omega_{p,q} e^{-2np\pi i/q};$$

and that α is any positive constant; and ν, the integer part of $\alpha\sqrt{n}$.

Then

$$p(n) = \sum_1^{\nu} A_q \phi_q + O(n^{-\frac{1}{4}}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_1^{\nu} A_q \phi_q.$$

In 1937 Rademacher was preparing lecture notes on this amazing theorem. To simplify things a bit, he replaced $\frac{1}{2} e^{C\lambda_n/q}$ by $\sinh(C\lambda_n/q)$ in the definition of $\phi_q(n)$ [17, p. 699]. To his amazement, he was now able to prove that, in fact,

$$p(n) = \sum_{q=1}^{\infty} A_q \phi_q.$$

In the celebrations surrounding the one hundredth anniversary of Ramanujan's birth, Atle Selberg [22] notes a further major surprise:

> If one looks at Ramanujan's first letter to Hardy, there is a statement there [ed: i.e., (1) above] which has some relation to his later work on the partition function, namely about the coefficient of the reciprocal of a certain theta series (a power series with square exponents and alternating signs as coefficients). It gives the leading term in what he claims as an approximate expression for the coefficient. If one looks at that expression, one sees that this is the exact analogue of the leading term in the Rademacher formula for $p(n)$ which shows that Ramanujan, in whatever way he had obtained this, had been led to the correct term of that expression.
>
> In the work on the partition function, studying the paper, it seems clear to me that it must have been, in a way, Hardy who did not fully trust Ramanujan's insight and intuition, when he chose the other form of the terms in their expression, for a purely technical reason, which one analyses as not very relevant. I think that if Hardy had trusted Ramanujan more, they should have inevitably ended with the Rademacher series. There is little doubt about that.

The subsequent impact of the "circle method" introduced by Hardy and Ramanujan (cf. [25]) is discussed by Vaughan. Further amazing formulas for $p(n)$ have been found by Folsom, Kent, and Ono [10], along with spectacular results on the divisibility of $p(n)$; these aspects will be discussed by Ono.

The Rogers-Ramanujan identities also form an intriguing and surprising component of Ramanujan's career. In 1916 P. A. MacMahon began Chapter III, Section VII, of his monumental *Combinatory Analysis* [16, p. 33] as follows:

> RAMANUJAN IDENTITIES
>
> Mr Ramanujan of Trinity College, Cambridge, has suggested a large number of formulae which have applications to the partition of numbers. Two of the most interesting of these concern partitions whose parts have a definite relation to the modulus five. Theorem I gives the relation
>
> $$\begin{aligned} 1 + \frac{x}{1-x} &+ \frac{x^4}{(1-x)(1-x^2)} \\ &+ \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \cdots \\ &+ \frac{x^{i^2}}{(1-x)(1-x^2)\ldots(1-x^i)} + \cdots \\ &= \frac{1}{(1-x)(1-x^6)(1-x^{11})\ldots(1-x^{5m+1})\ldots} \\ &\times \frac{1}{(1-x^4)(1-x^9)(1-x^{14})\ldots(1-x^{5m+4})\ldots}, \end{aligned}$$
>
> where on the right-hand side the exponents of x are the numbers given by the congruences $\equiv 1 \mod 5, \equiv 4 \mod 5$.
>
> This most remarkable theorem has been verified as far as the coefficient of x^{89} by actual expansion so that there is practically no reason to doubt its truth; but it has not yet been established.

MacMahon then notes that this identity implies the following assertion about partitions:

The theorem asserts that the partitions, whose parts are limited to be of the forms $5m+1, 5m+4$, are equi-numerous with those which involve neither repetitions nor sequences.

Subsequently, MacMahon notes that, in the second identity,

$$1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\cdots$$
$$+\frac{x^{i^2+i}}{(1-x)(1-x^2)\dots(1-x^i)}+\cdots$$
$$=\frac{1}{(1-x^2)(1-x^7)(1-x^{12})\dots(1-x^{5m+2})}$$
$$\times\frac{1}{(1-x^3)(1-x^8)(1-x^{13})\dots(1-x^{5m+3})\dots},$$

the exponents of x on the right-hand side are of the form $5m+2$ and $5m+3$.

This relation has also been verified by actual expansion to a high power of x, but it has not been established. As with the first identity, MacMahon notes the implications for partitions:

> states that of any given number such partitions are equi-numerous with those whose parts are all of the forms $5m+2, 5m+3$.

So here in 1916 was an immensely appealing unsolved problem. As Hardy remarks [14, p. 91]:

> Ramanujan rediscovered the formulae sometime before 1913. He had then no proof (and knew that he had none), and none of the mathematicians to whom I communicated the formulae could find one. They are therefore stated without proof in the second volume of MacMahon's Combinatory Analysis.
>
> The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the *Proceedings of the London Mathematical Society,* came accidentally across Rogers' paper. I can remember very well his surprise, and the admiration which he expressed for Rogers' work. A correspondence followed in the course of which Rogers was led to a considerable simplification of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again.

In his account of Ramanujan's notebooks, Berndt [9, pp. 77–79] gives a detailed history of subsequent work on the Rogers-Ramanujan identities.

Surprisingly, this aspect of the theory of partitions languished for the next forty years. W. N. Bailey [8], [7] and his student Lucy Slater [24], [23] did generalize the series-product identities, but D. H. Lehmer [15] and H. L. Alder [1] published papers suggesting that there were no generalizations of an obvious sort. Even Hans Rademacher [18, p. 73] asserted that "It can be shown there can be no corresponding identities for moduli higher than 5."

It turned out that Lehmer and Alder had shown only that the wrong generalization is the wrong generalization. The subject truly opened up with B. Gordon's magnificent partition-theoretic generalization [12] in 1961.

Theorem 2. *Let $B_{k,i}(n)$ denote the number of partitions of n of the form $b_1+b_2+\cdots+b_s$, where $b_j \ge b_{j+1}, b_j - b_{j+k-1} \ge 2$ and at most $i-1$ of the $b_j = 1$. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then $A_{k,i}(n) = B_{k,i}(n)$, for $1 \le i \le k$.*

The cases $k=2$, $i=1,2$ are the partition theoretic consequences of the original identities as stated by MacMahon.

The question of analogous identities remained open until 1974, when the following result appeared [15].

Theorem 3. *For $1 \le i \le k$,*

$$\sum_{n_1,\dots,n_{k-1}\ge 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty}\frac{1}{1-q^n},$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Since the 1970s there has been a cornucopia of Rogers–Ramanujan-type results pouring forth. As mentioned earlier, Berndt [9, p. 77] gives many references to recent work.

It should be added that Schur (mentioned in Hardy's brief history given above) had a strong influence on the combinatorial/arithmetical aspects of Rogers-Ramanujan theory. Both his independent discovery of the Rogers-Ramanujan identities [20] and his modulus 6 theorem of 1926 [21] greatly impacted subsequent developments. These include the Ph.D. thesis of H. Göllnitz [11] and the "weighted words" method of Alladi and Gordon [2], the latter culminating in a generalization of Göllnitz's deep modulus 12 theorem by Alladi, Andrews, and Berkovich [3].

The analytic side of the study was greatly assisted by the discovery of "Bailey chains" in 1984 [6]. Further accounts of this part of the Rogers-Ramanujan identities are provided in S. O. Warnaar's survey [26] "50 Years of Bailey's lemma" and in Chapter 3 of the *Selected Works of George E. Andrews* [4].

The Rogers-Ramanujan identities also lead directly to a study of a related continued fraction, a topic dealt with at lenghth in [9].

References

[1] Henry L. Alder, The nonexistence of certain identities in the theory of partitions and compositions, *Bull. Amer. Math. Soc.*, 54:712–722, 1948.

[2] Krishnaswami Alladi, The method of weighted words and applications to partitions. In *Number Theory (Paris, 1992–1993)*, volume 215 of London Math. Soc. Lecture Note Ser., pages 1–36, Cambridge Univ. Press, Cambridge, 1995.

[3] Krishnaswami Alladi, George E. Andrews, and Alexander Berkovich, A new four parameter q-series identity and its partition implications, *Invent. Math.*, 153(2):231–260, 2003.

[4] G. E. Andrews and A. V. Sills, *Selected Works of George E. Andrews: With Commentary*, Imperial College Press, 2012.

[5] George E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli, *Proc. Nat. Acad. Sci. U.S.A.*, 71:4082–4085, 1974.

[6] ______, Multiple series Rogers-Ramanujan type identities, *Pacific J. Math.*, 114(2):267–283, 1984.

[7] W. N. Bailey, Some identities in combinatory analysis, *Proc. London Math. Soc. (2)*, 49:421–425, 1947.

[8] ______, Identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)*, 50:1–10, 1948.

[9] Bruce C. Berndt, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991.

[10] Amanda Folsom, Zachary A. Kent, and Ken Ono, *l*-adic properties of the partition function, *Advances in Mathematics*, 229(3):1586–1609, 2012.

[11] H. Göllnitz, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.*, 225:154–190, 1967.

[12] Basil Gordon, A combinatorial generalization of the Rogers-Ramanujan identities. *Amer. J. Math.*, 83:393–399, 1961.

[13] G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115], in *Collected Papers of Srinivasa Ramanujan*, pages 276–309, AMS Chelsea Publ., Amer. Math. Soc., Providence, RI, 2000.

[14] G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, AMS Chelsea Publishing Series, Amer. Math. Soc., Providence, RI, 1999.

[15] D. H. Lehmer, Two nonexistence theorems on partitions, *Bull. Amer. Math. Soc.*, 52:538–544, 1946.

[16] Percy A. MacMahon, *Combinatory Analysis. Vols. I, II (bound in one volume)*, Dover Phoenix Editions, Dover Publications Inc., Mineola, NY, 2004. Reprint of *An Introduction to Combinatory Analysis* (1920) and *Combinatory Analysis. Vols. I, II* (1915, 1916).

[17] H. Rademacher, On the partition function $p(n)$, *Proc. London Math. Soc.*, 43(2):241–254, 1937.

[18] ______, *Lectures on Elementary Number Theory*, Tata Institute of Fundamental Research, Bombay. Lectures on mathematics and physics. Mathematics. Tata Institute of Fundamental Research, 1963.

[19] Srinivasa Ramanujan, *Collected Papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Amer. Math. Soc., Providence, RI, 2000, edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, third printing of the 1927 original, with a new preface and commentary by Bruce C. Berndt.

[20] I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.*, pages 302–321, 1917. Reprinted in *Gesammelte Abhandlungen*. Springer-Verlag, 1973.

[21] ______, Zur additiven Zahlentheorie, *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.*, pages 488–495, 1926. Reprinted in *Gesammelte Abhandlungen*, Springer-Verlag, 1973.

[22] A. Selberg, *Collected Papers, Vol. I*, Springer-Verlag, 1989.

[23] L. J. Slater, A new proof of Rogers's transformations of infinite series, *Proc. London Math. Soc. (2)*, 53:460–475, 1951.

[24] ______, Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)*, 54:147–167, 1952.

[25] R. C. Vaughan, *The Hardy-Littlewood Method*, Cambridge Tracts in Mathematics, vol. 80, Cambridge University Press, Cambridge, 1981.

[26] S. Ole Warnaar, 50 years of Bailey's lemma, in *Algebraic Combinatorics and Applications (Gößweinstein, 1999)*, pages 333–347, Springer, Berlin, 2001.

Photo courtesy of Krishnaswami Alladi.

The only cot at the humble home of Ramanujan in Kumbakonam. As a young boy, Ramanujan used to sit on the windowsill and do his "sums", watching the passersby on the street.

Bruce C. Berndt

Ramanujan's Earlier Notebooks and His Lost Notebook

Since the centenary of Ramanujan's birth, a large amount of Ramanujan's mathematics that had been ensconced in his earlier notebooks and in his lost notebook has been brought to light. In particular, in 1987 only one book had been written on Ramanujan's notebooks [2], but now several have appeared [2], [1]. We now have a larger portion of Ramanujan's mathematics to appreciate, evaluate, analyze, and challenge. The *challenge* is to ascertain Ramanujan's thinking and motivation and to properly place his work within contemporary mathematics. Since other writers in this series of articles are focusing on Ramanujan's

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lost notebook, we give more concentration in this paper on his earlier notebooks.

For most of his life, both in Kumbakonam and later in Madras, Ramanujan worked on a slate. Paper was expensive for him, and so he recorded only his final results in notebooks. Writing down his proofs in notebooks would have taken precious time; Ramanujan was clearly anxious to get on with his next ideas. Moreover, if someone had asked him how to prove a certain claim in his notebooks, he undoubtedly was confident that he could reproduce his argument. We do not know precisely when Ramanujan began to record his mathematical discoveries in notebooks, but it was likely around the time that he entered the Government College in Kumbakonam in 1904. Except for possibly a few entries, he evidently stopped recording his theorems in notebooks after he boarded a passenger ship for England on March 17, 1914. That Ramanujan no longer concentrated on logging entries in his notebooks is evident from two letters that he wrote to friends in Madras during his first year in England [5, pp. 112–113; 123–125]. In a letter of November 13, 1914, to his friend R. Krishna Rao, Ramanujan confided, "I have changed my plan of publishing my results. I am not going to publish any of the old results in my notebooks till the war is over." And in a letter of January 7, 1915, to S. M. Subramanian, Ramanujan admitted, "I am doing my work very slowly. My notebook is sleeping in a corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously."

Prior to his journey to England in 1914, Ramanujan prepared an enlarged edition of his first notebook; this augmented edition is the second notebook. A third notebook contains only thirty-three pages. G. N. Watson, in his delightful account of the notebooks [16], relates that Sir Gilbert Walker, head of the Indian Meteorological Observatory and a strong supporter of Ramanujan, told him that in April 1912 he observed that Ramanujan possessed four or five notebooks, each with a black cover and each about an inch thick. We surmise that these notebooks were either a forerunner of the first notebook or that the first notebook is an amalgam of these earlier black-covered notebooks.

When Ramanujan returned to India on March 13, 1919, he took with him only his second and third notebooks. The first notebook was left with G. H. Hardy, who used it to write a paper [9] on a chapter in the notebook devoted to hypergeometric series. Hardy's paper [9] was read on February 5, 1923, and published later during that year. He begins by writing, "I have in my possession a manuscript note-book which Ramanujan left with me when he returned to India in 1919." It was not until March 1925 that Hardy returned the first notebook to the University of Madras through its librarian, S. R. Rangathanan [15, pp. 56, 57]. A handwritten copy was subsequently made and sent to Hardy in August 1925.

Meanwhile, Hardy had originally wanted to bring Ramanujan's published papers, his notebooks, and any other unpublished manuscripts together for publication. To that end, in a letter to Hardy dated August 30, 1923, Francis Dewsbury, registrar at the University of Madras, wrote, "I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem. I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office." The "four manuscript notebooks" were handwritten copies of the second and third notebooks, and the packet of miscellaneous papers evidently included the "lost notebook", which Hardy passed on to Watson and which was later rediscovered by George Andrews in the Trinity College Library in March 1976. Note that, in contradistinction to Dewsbury's request, Hardy never returned the miscellaneous papers.

As it transpired, only Ramanujan's published papers were collected for publication. However, published with the *Collected Papers* [12] were the first two letters that Ramanujan had written to Hardy, which contained approximately one hundred twenty mathematical claims. Upon their publication, these letters generated considerable interest, with the further publication of several papers establishing proofs of these claims. Consequently, either in 1928 or 1929, at the strong suggestion of Hardy, Watson and B. M. Wilson, one of the three editors of [12], agreed to edit the notebooks. The word "edit" should be interpreted in a broader sense than usual. If a proof of a claim can be found in the literature, then it is sufficient to simply cite sources where proofs can be found; if a proof cannot be found in the literature, then the task of the editors is to provide one. In an address [16] to the London Mathematical Society on February 5, 1931, Watson cautioned (in retrospect, far too optimistically), "We anticipate that it, together with the kindred task of investigating the work of other writers to ascertain which of his results had been discovered previously, may take us five years." Wilson died prematurely in 1935, and although Watson wrote approximately thirty papers on Ramanujan's work, his interest evidently flagged in the late 1930s, and so the editing was not completed.

The majestic gopuram (entrance tower) of the Sarangapani Temple as seen from the front of Ramanujan's home on Sarangapani Sannidhi Street.

Altogether, the notebooks contain over three thousand claims, almost all without proof. Hardy surmised that over two-thirds of these results were rediscoveries. This estimate is much too high; on the contrary, at least two-thirds of Ramanujan's claims were new at the time that he wrote them, and two-thirds more likely should be replaced by a larger fraction. Almost all of the results are correct; perhaps no more than five to ten claims are incorrect. The topics examined by Ramanujan in his notebooks fall primarily under the purview of analysis, number theory, and elliptic functions, with much of his work in analysis being associated with number theory and with some of his discoveries also having connections with enumerative combinatorics and modular forms. Chapter 16 in the second notebook represents a turning point, since in this chapter he begins to examine q-series for the first time and also to begin an enormous devotion to theta functions. Chapters 16–21 and considerable material in the unorganized pages that follow focus on Ramanujan's distinctive theory of elliptic functions, in particular, on theta functions. Especially in the years 1912–1914 and 1917–1920, Ramanujan's concentration on number theory was through q-series and elliptic functions.

Finally, in 1957, the notebooks were made available to the public when the Tata Institute of Fundamental Research in Bombay published a photocopy edition [13], but no editing was undertaken. The notebooks were set in two volumes, with the first containing the first notebook and with the second comprising the second and third notebooks. The reproduction is quite faithful, with even Ramanujan's scratch work photographed. Buttressed by vastly improved technology, a new, much clearer, edition was published in 2011 to commemorate the 125th anniversary of Ramanujan's birth.

In February 1974, while reading two papers by Emil Grosswald [7], [8] in which some formulas from the notebooks were proved, we observed that we could prove these formulas by using a transformation formula for a general class of Eisenstein series that we had proved two years earlier. We found a few more formulas in the notebooks that could be proved using our methods, but a few thousand further assertions that we could not prove. In May 1977 the author began to devote all of his attention to proving all of Ramanujan's claims in the notebooks. With the help of a copy of the notes from Watson and Wilson's earlier attempt at editing the notebooks and with the help of several other mathematicians, the task was completed in five volumes [2] in slightly over twenty years.

In this series of articles for the *Notices*, George Andrews, Ken Ono, and Ole Warnaar, in particular, write about Ramanujan's discoveries in his lost notebook, which is dominated by q-series. However, published with the lost notebook [14] are several partial manuscripts and fragments of manuscripts on other topics that provide valuable insight into Ramanujan's interests and ideas. We first mention two topics that we have examined with Sun Kim and Alexandru Zaharescu in two series of papers, with [3] and [4] being examples of the two series. First, Ramanujan had a strong interest in the *circle problem* and *Dirichlet divisor problem*, with two deep formulas in the lost notebook clearly aimed at attacking these problems. Second, [14] contains three very rough, partial manuscripts on diophantine approximation. In one of these, Ramanujan derives the best diophantine approximation to $e^{2/a}$, where a is a nonzero integer, a result not established in the literature until 1978 [6]. Also appearing in [14] are fragments of material that were originally intended for portions of published papers, for example, [10] and [11]. It is then natural to ask why Ramanujan chose not to include these results in his papers. In at least two instances, the discarded material depends on the use of complex analysis, a subject that Ramanujan occasionally employed but not always correctly. In particular, in the cases at hand, Ramanujan did not have a firm knowledge of the Mittag-Leffler Theorem for developing partial fraction decompositions. The fragments and incomplete manuscripts will be examined in the fourth book that the author is writing with Andrews on the lost notebook [1].

We conclude with a few remarks about Ramanujan's methods. It has been suggested that he discovered his results by "intuition" or by making deductions from numerical calculations or by inspiration from Goddess Namagiri. Indeed, like most mathematicians, Ramanujan evidently made extensive calculations that provided guidance. However, Hardy and this writer firmly believe that Ramanujan created mathematics as any other mathematician would and that his thinking can be explained like that of other mathematicians. However, because Ramanujan did not leave us any proofs for the vast number of results found in his earlier notebooks and in his lost notebook, we often do not know Ramanujan's reasoning. As Ramanujan himself was aware, some of his arguments were not rigorous by then-contemporary standards. Nonetheless, despite his lack of rigor at times, Ramanujan doubtless thought and devised proofs as would any other mathematician.

References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Parts I–V, Springer, New York, 2005, 2009, 2012, to appear.

[2] B. C. Berndt, *Ramanujan's Notebooks, Parts I–V*, Springer-Verlag, New York, 1985, 1989, 1991, 1994, 1998.

[3] B. C. Berndt, S. Kim, and A. Zaharescu, *Circle and divisor problems, and double series of Bessel functions*, *Adv. Math*, to appear.

[4] ______, *Diophantine approximation of the exponential function, and a conjecture of Sondow*, submitted for publication.

[5] B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, American Mathematical Society, Providence, RI, 1995; London Mathematical Society, London, 1995.

[6] C. S. Davis, Rational approximation to e, *J. Austral. Math. Soc.* **25** (1978), 497–502.

[7] E. Grosswald, Die Werte der Riemannschen Zetafunktion an ungeraden Argumentstellen, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1970** (1970), 9–13.

[8] ______, Comments on some formulae of Ramanujan, *Acta Arith.* **21** (1972), 25–34.

[9] G. H. Hardy, A chapter from Ramanujan's note-book, *Proc. Cambridge Philos. Soc.* **21** (1923), 492–503.

[10] S. Ramanujan, Some formulae in the analytic theory of numbers, *Mess. Math.* **45** (1916), 81–84.

[11] ______, On certain trigonometric sums and their applications in the theory of numbers, *Trans. Cambridge Philos. Soc.* **22** (1918), 259–276.

[12] ______, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[13] ______, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second edition, Tata Institute of Fundamental Research, Mumbai, 2011.

[14] ______, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[15] S. R. Ranganathan, *Ramanujan: The Man and the Mathematician*, Asia Publishing House, Bombay, 1967.

[16] G. N. Watson, Ramanujan's notebooks, *J. London Math. Soc.* **6** (1931), 137–153.

Jonathan M. Borwein

Ramanujan and Pi

Since Ramanujan's 1987 centennial, much new mathematics has been stimulated by uncanny formulas in Ramanujan's *Notebooks* (lost and found). In illustration, I mention the exposition by Moll and his colleagues [1] which illustrates various neat applications of Ramanujan's Master Theorem, which extrapolates the Taylor coefficients of a function, and relates them to methods of integration used in particle physics. I also note lovely work on the modular functions behind Apéry and Domb numbers by Chan and others [6], and finally I mention my own work with Crandall on Ramanujan's arithmetic-geometric continued fraction [12].

For reasons of space, I now discuss only work related directly to pi, and so continue a story started in [9], [11]. Truly novel series for $1/\pi$, based on elliptic integrals, were found by Ramanujan around 1910 [19], [5], [7], [21]. One is

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{1}$$

Each term of (1) adds eight correct digits. Though then unproven, Gosper used (1) for the computation of a then-record 17 million digits of π in 1985, thereby completing the first proof of (1) [7, Ch. 3]. Soon after, David and Gregory Chudnovsky found the following variant, which relies on the quadratic number field $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$, as is implicit in (1):

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{2}$$

Each term of (2) adds fourteen correct digits. (Were a larger imaginary quadratic field to exist with class number one, there would be an even more extravagant rational series for some surd divided by π [10].) The brothers used this formula several times, culminating in a 1994 calculation of π to over four billion decimal digits. Their remarkable story was told in a prize-winning *New Yorker* article [18]. Remarkably, (2) was used again in 2010 and 2011 for the current record computations of π to five and ten trillion decimal digits respectively.

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Quartic Algorithm for π

The record for computation of π has gone from 29.37 *million* decimal digits in 1986 to ten *trillion* digits in 2011. Since the algorithm below, which found its inspiration in Ramanujan's 1914 paper, was used as part of computations both then and as late as 2009, it is interesting to compare the performance in each case: Set $a_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$; then iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}},$$

(3)

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then a_k converges *quartically* to $1/\pi$; each iteration quadruples the number of correct digits. Twenty-one iterations produce an algebraic number that coincides with π to well over six trillion places.

This scheme and the 1976 Salamin–Brent scheme [7, Ch. 3] have been employed frequently over the past quarter century. Here is a highly abbreviated chronology (based on `http://en.wikipedia.org/wiki/Chronology_of_computation_of_pi`):

- 1986: David Bailey used (3) to compute 29.4 million digits of π. This required 28 hours on one CPU of the new Cray-2 at NASA Ames Research Center. Confirmation using the Salamin-Brent scheme took another 40 hours. This computation uncovered hardware and software errors on the Cray-2.
- January 2009: Takahashi used (3) to compute 1.649 trillion digits (nearly 60,000 times the 1986 computation), requiring 73.5 hours on 1,024 cores (and 6.348 Tbyte memory) of the Appro Xtreme-X3 system. Confirmation via the Salamin-Brent scheme took 64.2 hours and 6.732 Tbyte of main memory.
- April 2009: Takahashi computed 2.576 trillion digits.
- December 2009: Bellard computed nearly 2.7 trillion decimal digits (first in binary) using (2). This took 131 days, but he used only a single four-core workstation with lots of disk storage and even more human intelligence!
- August 2010: Kondo and Yee computed 5 trillion decimal digits, again using equation (2). This was done in binary, then converted to decimal. The binary digits were confirmed by computing 32 hexadecimal digits of π ending with position 4,152,410,118,610 using BBP-type formulas for π due to Bellard and Plouffe [7, Chapter 3]. Additional details are given at `http://www.numberworld.org/misc_runs/pi-5t/announce_en.html`. See also [4], in which analysis showing these digits appear to be "very normal" is made.

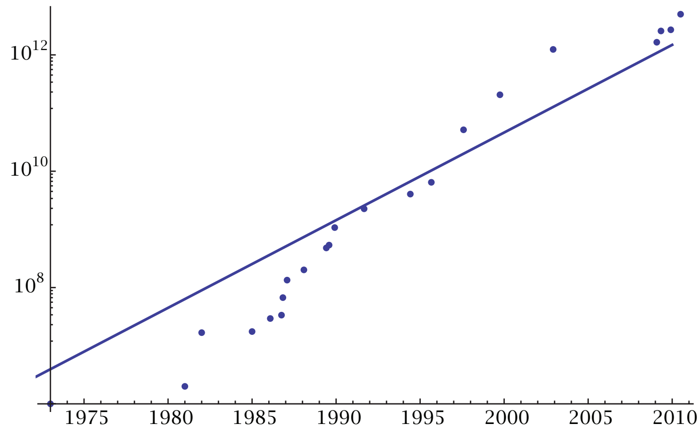

Figure 1. Plot of π calculations in digits (dots) compared with the long-term slope of Moore's Law (line).

Daniel Shanks, who in 1961 computed π to over 100,000 digits, once told Phil Davis that a billion-digit computation would be "forever impossible". But both Kanada and the Chudnovskys achieved that in 1989. Similarly, the intuitionists Brouwer and Heyting asserted the "impossibility" of ever knowing whether the sequence 0123456789 appears in the decimal expansion of π; yet it was found in 1997 by Kanada, beginning at position 17387594880. As late as 1989, Roger Penrose ventured, in the first edition of his book *The Emperor's New Mind*, that we likely will never know if a string of ten consecutive 7s occurs in the decimal expansion of π. This string was found in 1997 by Kanada, beginning at position 22869046249.

Figure 6 shows the progress of π calculations since 1970, superimposed with a line that charts the long-term trend of Moore's Law. It is worth noting that whereas progress in computing π exceeded Moore's Law in the 1990s, it has lagged a bit in the past decade. Most of this progress is still in mathematical debt to Ramanujan.

As noted, one billion decimal digits were first computed in 1989, and the ten (actually fifty) billion digit mark was first passed in 1997. Fifteen years later one can explore, in real time, multibillion step walks on the hex digits of π at `http://carmaweb.newcastle.edu.au/piwalk.shtml`, as drawn by Fran Aragon.

Formulas for $1/\pi^2$ and More

About ten years ago Jésus Guillera found various Ramanujan-like identities for $1/\pi^N$ using integer relation methods. The three most basic—and entirely rational—identities are

(4)
$$\frac{4}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13+180n+820n^2) \left(\frac{1}{32}\right)^{2n+1},$$

(5)
$$\frac{2}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1+8n+20n^2) \left(\frac{1}{2}\right)^{2n+1},$$

(6)
$$\frac{4}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1+14n+76n^2+168n^3) \left(\frac{1}{8}\right)^{2n+1},$$

where $r(n) := (1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2)/n!$.

Guillera proved (4) and (5) in tandem by very ingeniously using the Wilf-Zeilberger algorithm [20], [17] for formally proving hypergeometric-like identities [7], [15], [21]. No other proof is known. The third, (6), is almost certainly true. Guillera ascribes (6) to Gourevich, who found it using integer relation methods in 2001.

There are other sporadic and unexplained examples based on other symbols, most impressively a 2010 discovery by Cullen:

(7)
$$\frac{2^{11}}{\pi^4} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^7 (\frac{3}{4})_n}{(1)_n^9}$$
$$\times (21+466n+4340n^2+20632n^3+43680n^4) \left(\frac{1}{2}\right)^{12n}.$$

We shall revisit this formula below.

Formulae for π^2

In 2008 Guillera [15] produced another lovely, if numerically inefficient, pair of third-millennium identities—discovered with integer relation methods and proved with creative telescoping—this time for π^2 rather than its reciprocal. They are based on:

$$(8) \quad \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} (6(n+x)+1) = 8x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2},$$

and

(9)
$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} (42(n+x)+5) = 32x \sum_{n=0}^{\infty} \frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}.$$

Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the *rising factorial*. Substituting $x = 1/2$ in (8) and (9), he obtained respectively the formulae

$$(10) \qquad \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (3n+2) = \frac{\pi^2}{4},$$
$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (21n+13) = 4\,\frac{\pi^2}{3}.$$

Calabi-Yau Equations and Supercongruences

Motivated by the theory of Calabi-Yau differential equations [2], Almkvist and Guillera have discovered many new identities. One of the most pleasing is

$$(11) \quad \frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{(532\,n^2+126\,n+9)}{10^{6n+3}}.$$

This is yet one more case where mysterious connections have been found between disparate parts of mathematics and Ramanujan's work [21], [13], [14].

As a final example, we mention the existence of *supercongruences* of the type described in [3], [16], [23]. These are based on the empirical observation that a Ramanujan series for $1/\pi^N$, if truncated after $p-1$ terms for a prime p, seems always to produce congruences to a higher power of p. The formulas below are taken from [22]:

(12)
$$\sum_{n=0}^{p-1} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^3 (\frac{3}{4})_n}{2^{4n}\,(1)_n^5} \left(3+34n+120n^2\right) \equiv 3p^2 (\mathrm{mod}\, p^5),$$

(13)
$$\sum_{n=0}^{p-1} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^7 (\frac{3}{4})_n}{2^{12n}\,(1)_n^9} \left(21+466n+4340n^2+20632n^3+43680n^4\right)$$
$$\stackrel{?}{\equiv} 21p^4 (\mathrm{mod}\, p^9).$$

We note that (13) is the supercongruence corresponding to (6), while for (12) the corresponding infinite series sums to $32/\pi^4$. We conclude by reminding the reader that all identities marked with '$\stackrel{?}{=}$' are assuredly true but remain to be proved. Ramanujan might well be pleased.

References

[1] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. Moll, and A. Straub, Ramanujan's Master Theorem, *Ramanujan J.* **29** (2012, to appear).

[2] G. Almkvist, Ramanujan-like formulas for $1/\pi^2$ à la Guillera and Zudilin and Calabi-Yau differential equations, *Computer Science Journal of Moldova* **17**, no. 1 (49) (2009), 100–120.

[3] G. Almkvist and A. Meurman, Jesus Guillera's formula for $1/\pi^2$ and supercongruences (Swedish), *Normat* **58**, no. 2 (2010), 49–62.

[4] D. H. Bailey, J. M. Borwein, C. S. Calude, M. J. Dinneen, M. Dumitrescu, and A. Yee, An empirical approach to the normality of pi, *Experimental Mathematics*. Accepted February 2012.

[5] N. D. Baruah, B. C. Berndt, and H. H. Chan, Ramanujan's series for $1/\pi$: A survey, *Amer. Math. Monthly* **116** (2009), 567–587.

[6] B. C. Berndt, H. H. Chan, and S. S. Huang, Incomplete elliptic integrals in Ramanujan's lost notebook, pp. 79–126 in *q-series from a Contemporary Perspective*, Contemp. Math., 254, Amer. Math. Soc., Providence, RI, 2000.

[7] J. M. Borwein and D. H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, 2nd ed., A K Peters, Natick, MA, 2008.

[8] J. M. Borwein, D. H. Bailey, and R. Girgensohn, *Experimentation in Mathematics: Computational Roads to Discovery*, A K Peters, 2004.

[9] J. M. Borwein and P. B. Borwein, Ramanujan and pi, *Scientific American*, February 1988, 112–117. Reprinted in *Ramanujan: Essays and Surveys,* B. C. Berndt and R. A. Rankin, eds., AMS-LMS History of Mathematics, vol. 22, 2001, pp. 187–199.

[10] ———, Class number three Ramanujan type series for $1/\pi$, *Journal of Computational and Applied Math. (Special Issue)* **46** (1993), 281–290.

[11] J. M. Borwein, P. B. Borwein, and D. A. Bailey, Ramanujan, modular equations and pi or how to compute a billion digits of pi, *MAA Monthly* **96** (1989), 201–219. Reprinted on `http://www.cecm.sfu.ca/organics`, 1996.

[12] J. M. Borwein and R. E. Crandall, On the Ramanujan AGM fraction, Part II: The complex-parameter case, *Experimental Mathematics* **13** (2004), 287–296.

[13] H. H. Chan, J. Wan, and W. Zudilin, Legendre polynomials and Ramanujan-type series for $1/\pi$, *Israel J. Math.*, in press, 2012.

[14] ———, Complex series for $1/\pi$, *Ramanujan J.*, in press, 2012.

[15] J. Guillera, Hypergeometric identities for 10 extended Ramanujan-type series, *Ramanujan J.* **15** (2008), 219–234.

[16] J. Guillera and W. Zudilin, “Divergent” Ramanujan-type supercongruences, *Proc. Amer. Math. Soc.* **140** (2012), 765–777.

[17] M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$, A K Peters, 1996.

[18] R. Preston, The Mountains of Pi, *New Yorker*, March 2, 1992, `http://www.newyorker.com/archive/content/articles/050411fr_archive01`.

[19] S. Ramanujan, Modular equations and approximations to pi, *Quart. J. Math.* **45** (1913–14), 350–372.

[20] H. S. Wilf and D. Zeilberger, Rational functions certify combinatorial identities, *Journal of the AMS* **3** (1990), 147–158.

[21] W. Zudilin, Ramanujan-type formulae for $1/\pi$: a second wind? Modular forms and string duality, in Fields Inst. Commun., 5, Amer. Math. Soc., Providence, RI, 2008, pp. 179–188.

[22] ———, Arithmetic hypergeometric series, *Russian Math. Surveys* **66**:2 (2011), 1–51.

[23] ———, Ramanujan-type supercongruences, *J. Number Theory* **129** (2009), 1848–1857.

The second installment of this article—with pieces by Ken Ono, K. Soundararajan, R. C. Vaughan, and S. Ole Warnaar—will appear in the January 2013 issue of the Notices.