The Geodesic Spring on the Euclidean Sphere with Parallel-Transport-Based Damping

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In this article differential geometric methods are applied to the design of a tracking algorithm which compels a particle that is constrained to the unit sphere $S^2$ into a moving target that is also constrained to the sphere. By tracking we mean that both the position and velocity of the chasing particle are made to match that of the target particle. A mechanical system analog of this mathematical scenario is the pointing of the end effector of a robotic arm whose range of motion is restricted to the surface of a sphere (see Figure 1). At the heart of this article is the 1D mass-spring-damper (MSD) system on the real line, which is reviewed and summarized in Sidebar 1 (see page 11), so as to be easily extensible to a unit sphere $S^2$-generalization. This spherical generalization, depicted by $(\nabla \dot{\nabla})$ in Figure 1, is the so-called geodesic spring tracking algorithm. Essentially, the spring $(\nabla)$ pulls the robotic arm close to its goal configuration, and the damper $(\dot{\nabla})$ dissipates the kinetic energy of the arm so that it eventually reaches and stops at the goal configuration.

There are some new mathematical avenues explored in this article; however, with the belief that a good example can often be as useful as a good theorem, this article’s main purpose is to instantiate the more research-oriented articles on which it is founded. Specifically, while the geodesic spring (with damping) design is theoretically formulated on a general Riemannian manifold in [1], in this article we actually implement this general result on a specific manifold, namely $S^2$. In so doing, one is confronted with the geometric, coordinate-free, nonlinear, and almost global nature of the control design that, at least to these authors, provides an additional layer of understanding of the general theorem.

The authors’ broader hope for this article is that, by providing an example which is completely expressed in closed form, some insight may be gained by the reader into the control and...
stabilization of systems on manifolds. While this article is narrowly focused on the manifold \( S^2 \), the potential usefulness of its results are limited only by the imagination of the reader to formulate a problem in terms of a specific Riemannian manifold of interest. For example, as exceptionally relayed and detailed by [2], in his investigations of surface curvature, Gauss invented a mapping of normal vectors on a surface \( S^2 \). This map, now called the Gauss map, and some of the techniques of this article on \( S^2 \) are used by [3] to devise a geometric-based guidance law for an interceptor problem. That is, unlike this article, where the goal is to match both the position and velocity of a reference trajectory \( r(t) \in S^2 \), the goal in an interceptor problem is to match only position \( r(t_f) \in \mathbb{R}^3 \) for some intercept time \( t_f \); the velocity at which the interception is made is immaterial. An additional avenue for exploration of the techniques of this article is to a Riemannian manifold called the Poincaré upper-half plane, denoted \( \mathbb{H}^2 \), where, as in \( S^2 \), the geodesic and parallel transport equations both have closed-form realizations. Perhaps then the control design of this article on \( \mathbb{H}^2 \) might find some application to electrical impedance tomography [4] and to microwave technology [5], where the geometry of \( \mathbb{H}^2 \) is shown to play a role.

A broad outline of this article is as follows. In Sidebar 1 (page 11) we review the 1D MSD. Those readers already familiar with the material from this section and who want to learn about its generalization to the geodesic spring system on a Riemannian manifold could proceed to “History of the Geodesic Spring”, where a general theorem is outlined. In “MSD System on \( S^2 \)” we instantiate this general theorem on the sphere \( S^2 \subset \mathbb{R}^3 \), and in “Implementation of MSD System on \( S^2 \)” we present several examples and simulations.

**History of the Geodesic Spring**

The evolution of the 1D MSD system to its general form on a Riemannian manifold can be traced to the evolution of a configuration error function \( \phi \) and the transport map \( \mathcal{T} \) pairing. For the purposes of this article, we think of a Riemannian manifold \( Q \) simply as a surface \( S \subset \mathbb{R}^3 \) that is equipped with an inner product \( \langle \cdot, \cdot \rangle \), which effectively induces a definition of distance between points \( q, r \in Q \). A more formal definition of Riemannian manifold, though not necessary for this article, can be found in most books on differential geometry (for example, [6], [7]). In general, the value \( \phi(q, r) \) is a measure of the difference (not necessarily distance) between \( q \) and \( r \) and satisfies \( \phi(q, r) \to 0 \) as \( q \to r \). The transport map \( \mathcal{T} \) is discussed shortly. The geometric setting of [8] is the Euclidean sphere \( S^2 \subset \mathbb{R}^3 \). One of the control problems solved in [8, Problem 2.2, p. 6] is to steer asymptotically a point particle \( q \) constrained to \( S^2 \) to a fixed reference \( r \in S^2 \). The authors use the function

\[
\frac{1}{2} \left( \text{dist}_{S^2}(q, r) \right)^2 = \frac{1}{2} \left( \arccos((r, q)) \right)^2 = \frac{1}{2} \delta^2
\]

to obtain a geodesic spring force \( F_{gr} \) on \( S^2 \) (illustrated in Figure 2) that stabilizes a curve \( q(t) \) to fixed \( r \). Since \( q, r \in S^2 \) and \( \arccos((r, q)) \to 0 \) as \( q \to r \), then (1) is an example of a configuration error function \( \phi(q, r) \) with sample plots shown in Figure 3. A comparison of this plot with Figure B1 along with the discussion in Sidebar 1, shows that (1) plays the role of a quadratic potential function on \( S^2 \) centered at \( r \in S^2 \). Extending the quadratic potential \( V(q) \) on \( \mathbb{R} \) to the quadratic potential \( \phi(q, r) \) on \( S^2 \) is the first step in generalizing the 1D MSD to \( S^2 \).

In an effort to generalize the damping force, [9, Chapter 11], [10], and [11, Chapter 4] work within the context of Riemannian manifolds, of which the surface \( S^2 \subset \mathbb{R}^3 \) is but one example. Within this broader geometric setting, Bullo et al. introduce the general definitions of a pair of mappings called the configuration error and the transport map, which are denoted by \( \phi \) and \( \mathcal{T} \), respectively. Recall from Sidebar 1 that, when the tracker \( q(t) \) is confined to the real line \( \mathbb{R} \) and when the target \( r \in \mathbb{R} \) is fixed, the difference \( e(t) = q(t) - r \) (the position error) makes sense as a difference in \( \mathbb{R} \) and further that the time derivative of the error (the velocity error) is simply \( \dot{e} = d e(t) / dt \). In the transition to a Riemannian manifold \( Q \) (think of \( Q \) as a surface \( S \subset \mathbb{R}^3 \)) with \( q(t) \in S \) and a moving reference \( r(t) \in S \), some care needs to be taken when differentiating the velocity vectors \( \dot{q} \) and \( \dot{r} \), which lie in two separate tangent planes: one at \( q(t) \) and the other at \( r(t) \), respectively. This differencing is achieved by Bullo et al. through the introduction of a transport map \( \mathcal{T} \), which transports \( \dot{r} \) into the tangent plane at \( q \), denoted \( \mathcal{T} \dot{r} \), where the differencing with \( \dot{q} \) makes sense using the vector space properties of the tangent plane \( T_q S \). That is, as a result of the transport
map \( \mathcal{T} \), a new definition of the velocity error on a Riemannian manifold is given by

\[
\dot{e} \triangleq \dot{q} - \mathcal{T}\dot{r} \in T_qS.
\]

With these general mappings \( \phi \) and \( \mathcal{T} \), Bullo et al. generalize the 1D MSD error system from Sidebar 1 as follows. Newton’s equation for a free particle constrained to a Riemannian manifold is the geodesic equation

\[
\frac{Dq}{dt} = 0,
\]

where \( \frac{D}{dt} \) denotes covariant differentiation along the curve \( q(t) \). The general idea of covariant differentiation on a Riemannian manifold is addressed in most differential geometry books and can be illustrated in terms of, and indeed historically developed from, differentiation on a surface \( S \subset \mathbb{R}^3 \). Specifically, the covariant derivative at \( q \in S \) is the standard derivative followed by a projection onto the tangent plane to \( S \) at \( q \),

\[
\frac{D}{dt} = \text{proj}_q \circ \frac{d}{dt}.
\]

Defining a proportional (spring) force \( F_p \), a derivative (damping) force \( F_D \), and an additional so-called feed-forward force \( F_{\mathcal{T}\mathcal{T}} \) (which compensates for the nonzero accelerations of \( r(t) \))

\[
\begin{align*}
F_p & \triangleq -k_p \text{proj}_q (d_q\phi) \triangleq -k_p e \in T_qS, \\
F_D & \triangleq -k_D (\dot{q} - \mathcal{T}\dot{r}) \triangleq -k_D \dot{e} \in T_qS, \\
F_{\mathcal{T}\mathcal{T}} & \triangleq \frac{D}{dt} \left[ \mathcal{T}\dot{r} \right] \in T_qS,
\end{align*}
\]

we find that Newton’s equation with forcing \( \Sigma F \) is the closed-loop system.

\[
\frac{Dq}{dt} = \Sigma F = F_p + F_D + F_{\mathcal{T}\mathcal{T}},
\]

This simplifies to the error system

\[
\frac{De}{dt} + k_D \dot{e} + k_P e = 0.
\]

A word or two on notation is in order at this point. The vector \( \nu_r = d_r\phi \) (the derivative of \( \phi \) with respect to \( r \)) is in the tangent plane \( T_rS \), but the vector \( \text{proj}_{\nu_r}(\nu_r) \) is in the tangent plane (see Sidebar 2, page 12). From now on we denote the vectors \( \text{proj}_{\nu_r}(d_r\phi) \) and \( \text{proj}_{\nu_q}(d_q\phi) \) by \( d_r\phi|_r \) and \( d_q\phi|_q \), respectively. Assuming that \( \phi \) and \( \mathcal{T} \) satisfy a compatibility condition

\[
(4) \quad \langle d_r\phi|_r, \dot{r} \rangle = -\langle d_q\phi|_q, \mathcal{T}\dot{r} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the dot (inner) product on \( \mathbb{R}^3 \) applied to tangent vectors to \( S \), the time derivative of the total energy \( L = \frac{k_p}{2} \langle \dot{e}, \dot{e} \rangle + k_D \phi(q, r) \) along the solution to the forced dynamics reduces to

\[
(5) \quad \frac{dL}{dt} = k_p \left[ \langle d_q\phi|_q, \dot{q} \rangle + \langle d_r\phi|_r, \dot{r} \rangle \right] + k_D \left[ \frac{D\dot{e}}{dt}, \dot{e} \right]
\]

\[
= -k_D \langle \dot{e}, \dot{e} \rangle \leq 0 \quad [k_D > 0].
\]

Equation (5) indicates that the energy function is decreasing and guarantees that \( q = q(t) \) Lyapunov stabilizes to \( r = r(t) \). Assuming technical conditions on the uniformity of \( \phi \) [9, p. 536] and boundedness assumptions on, for example, \( k_P \) and \( k_D \) [9, pp. 540–541], it is proven in [9, Chapter 11] that \( q = q(t) \) exponentially stabilizes to \( r = r(t) \). This theorem (hereafter referred to as Bullo’s Theorem) can be summarized as

**Bullo’s Theorem.** The solution \( e \) to the error system

\[
\frac{de}{dt} + k_D \ddot{e} + k_P e = 0
\]

satisfies \( e \to 0 \) as \( t \to \infty \) so long as the mappings \( (\phi, \mathcal{T}) \) that define \( e \triangleq d_q\phi|_q \) and \( \dot{e} \triangleq \dot{q} - \mathcal{T}\dot{r} \) are compatible in the sense of

\[
\langle d_r\phi|_r, \dot{r} \rangle = -\langle d_q\phi|_q, \mathcal{T}\dot{r} \rangle.
\]

The objective now is to actually construct such a compatible pair of functions. In an attempt to implement Bullo’s Theorem, [1] showed that the geodesic distance and parallel transport on a Riemannian manifold lead to such a compatible pair of mappings.
A difficulty faced with the use of this pair is a reliance on numerical solutions to the geodesic equation (as a nonlinear, two-point boundary value problem) and to the parallel-transport equations (as a linear, possibly time-dependent, initial value problem). In [12], the geodesic distance and parallel transport pairing was utilized on a Riemannian manifold called the double gimbal torus, which encapsulates the physics of the double gimbal mechanism (DGM) shown in Figure 4. For a special DGM (called the flat DGM), the parallel-transport and geodesic equations can be expressed in closed form, and the closed-loop equations (3) reduce to a system of uncoupled 1D MSD error systems. However, for a general DGM (that is, nonflat), the formulation of the closed-loop dynamics still requires numerical solutions.

The works cited here are by no means a complete list and represent only those most directly related to this article. For a broader range of references addressing control design on spheres and the use of geodesics in optimal control, see, for example, [13], [14], and the references and introductions of works cited in this section.

MSD System on $S^2$

The goal of this section is to first give the details of the closed-form constructions of both the compatible configuration error and transport map pairing and the closed-loop dynamics (3). The configuration error function $\phi$ is recognizable as the quadratic potential well centered on $r(t) \in S^2$ from (1) shown in Figure 3. As illustrated in Figure B1, this potential is sufficient to attract the curve $q(t) \in S^2$ into the vicinity of $r(t)$. The main focus of "A Compatible Pair of Mappings on $S^2" below is the geometric construction of a transport map, shown to be parallel transport along the geodesics of $S^2$. As illustrated in Figure B1, this transport map is used to define a source of friction on the potential surface that eventually slows the curve $q(t)$ to $r(t)$ located at the bottom of the quadratic potential well.

A Compatible Pair of Mappings on $S^2$

Define a configuration error function $\phi$ and transport map $T$ pairing on $S^2$ by

$$
(\phi, T_{r-q}) = \frac{1}{2} (\arccos(\beta))^2, \beta I_3 + \left(1 - \beta \right) \omega \otimes \omega + \omega \otimes \omega,
$$

where $\beta \triangleq \langle q, r \rangle = \cos(\theta)$, $\omega \triangleq r \times q$, and $\omega \otimes \omega$ is defined by its action on a vector $v$ as $(\omega \otimes \omega)(v) \triangleq \langle \omega, v \rangle \omega$. Furthermore, the mapping $\tilde{\cdot}$ is defined by

$$
\tilde{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}
$$

and has the property $\tilde{v} \ast w = v \times w$, where $\ast$ is matrix-vector multiplication. The geometric interpretation of the transport map $T_{r-q}$ is revealed as parallel transport by considering the Frenet-Serret frame along the segment of the great circle between $r$ and $q$. The vectors $r$ and $q$ lead to the orthonormal set $\{r, \text{vers}(q^{-r})\}$, which can then be used to define the curve

$$
\sigma(\tau) \triangleq \cos(\tau) r + \sin(\tau) \text{vers}(q^{-r}).
$$

Geometrically, $\sigma(\tau), \tau \in [0, 2\pi)$, is the great circle in the plane defined by the orthonormal vectors $\{r, \text{vers}(q^{-r})\}$ with $\sigma(0) = r$ and $\sigma(\theta) = \beta r + \sin(\theta) \text{vers}(q^{-r}) = q$. In general, the legs of the Frenet-Serret frame $F(\tau)$ along an arbitrary curve $\sigma(\tau)$ are defined by the vectors

$$
T(\tau) = \text{vers} \left( \frac{d\sigma(\tau)}{d\tau} \right)
$$

and

$$
N(\tau) = \text{vers} \left( \frac{d}{d\tau} T(\tau) \right),
$$

and $B(\tau) = T(\tau) \times N(\tau)$. Specifically along the great circle $\sigma(\tau)$ defined in (8), the Frenet-Serret frames at $r = \sigma(0)$ and $q = \sigma(\theta)$ are computed to be

$$
F_r = \{T_r, N_r, B_r\} = \{\text{vers}(q^{-r}), -r, \text{vers}(r \times q)\},
$$

$$
F_q = \{\sin(\theta) N_r + \cos(\theta) T_r, \cos(\theta) N_r - \sin(\theta) T_r, B_r\}.
$$

Comparing (9) and (10), the two frames $F_r$ and $F_q$ are related by the equation $F_q = T \ast F_r$ where $T$ is a rotation matrix given by

$$
T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Consequently, the equation $F_q = T \ast F_r$ is interpreted geometrically to mean that $F_q$ (the frame at $q$) is obtained by an anticlockwise rotation of $F_r$ (the frame at $r$) around the $r \times q$ axis through the angle subtended by the great circle between $r$ and $q$. With the means now to transport the Frenet-Serret frame field from $r$ to $q$, the transport of $v_r$ (a vector at $r$) to $v_q$ (a vector at $q$) is a two-step process:

**Step 1.** Compute the components of $v_r$ relative to the frame $F_r$ by finding the coefficients $c_i$ of $v_r = \sum_{i=1}^3 c_i F^i_r$. These components are readily
complied to be $c_i = \langle v_r, F^i_r \rangle$, since $F_r$ is an orthonormal frame field.

**Step 2.** Define the vector $v_q$ with the components from Step 1 and the frame $F_q$ to obtain $v_q = \sum_i c^i_q F^i_q$.

Takentogether these two steps define a transport mapping called *parallel transport* along the great circle between $r$ and $q$.

Figure 5 illustrates both the parallel transport of the frame field $F_r$ to $F_q$ and a specific vector $v_r$ to $v_q$. While this definition of parallel transport is geometrically intuitive, it is a coordinate-based free expression. In a coordinate system given the ordered, orthonormal basis $B = \{T_r, N_r, B_r\}$, the projection of the derivative of the vector $v_r$ on $S^2$ is an anticlockwise rotation around the $r \times q$ axis through the angle subtended by the great circle between $r$ and $q$.

The mappings $\phi$ and $\mathcal{T}$ from (6) are now shown to satisfy the compatibility condition (4) where $d_q \phi|_q = -\hat{q}^2 (d_q \phi)$ and $d_r \phi|_r = -\hat{r}^2 (d_r \phi)$ are the vectors on $S^2$ obtained by projecting the vectors $d_q \phi$ and $d_r \phi$ in $\mathbb{R}^3$ onto the tangent planes $T_q S^2$ and $T_r S^2$, respectively (cf. Sidebar 2). A straightforward computation determines the derivative of $\phi$ with respect to $q$ and its projection to

$$d_q \phi = - (\theta/ \sin(\theta)) r, d_q \phi|_q = -\theta \text{vers}(r^{1\rightarrow q}).$$

Similarly, the projection of the derivative of $\phi$ with respect to $r$ is $d_r \phi|_r = -\theta \text{vers}(q^{1\rightarrow r})$. Therefore the right-hand side of (4) is computed to be

$$\begin{align*}
\theta \beta (v_r) &= \theta (v_r) - \theta \beta (v_r) \\
&= \theta \beta (v_r) - \theta (v_r) = \theta \text{vers}(r^{1\rightarrow q} - \theta \beta \text{vers}(r^{1\rightarrow q})).
\end{align*}$$

Since the equality in (12) holds for an arbitrary vector $v_r \in T_q S^2$, $\phi$ and $\mathcal{T}_{r\rightarrow q}$ form a compatible configuration error and transport map pair, and, consequently, they can be used to implement Bullo’s Theorem on $S^2$.

Recall from Sidebar 1 that the potential function $V(q) = \frac{1}{2}(\text{dist}_R(q, r))^2$ generated the spring force $F_P = -k_F \nabla V = \pm k_F \text{dist}_R(q, r)$, where the direction of the force $\pm$ is chosen so that $F_P$ points from position $q$ to fixed reference $r$. Unlike the 1D case, where there are only two directions $\pm$ (left, right) to select from, on the sphere there are an infinite number of directions to choose from, namely, one for every vector $v_q \in T_q S^2$. Strategically selecting

\[\text{vers}(r^{1\rightarrow q})\] the direction vector for $F_P$ on $S^2$ is at the heart of the geodesic-spring design.
of the force $F_P$ with strength given by the geodesic distance $\theta$ between $q$ and $r$ (see Figure 6).

**Control Forces on $\mathbb{S}^2$**

Having shown that $\phi$ and $T$ from (6) are a compatible pair on $\mathbb{S}^2$, we can now theoretically apply Bullo’s Theorem on $\mathbb{S}^2$. To actually implement this theorem, we must compute the resulting proportional derivative and feedforward forces $(F_{PD} = F_P + F_D$ and $F_{TF}$, respectively) which stabilize the configuration curve $q$ to the reference curve $r$ in the closed loop. Unlike the general Riemannian manifold case in which the geodesic equations (as a nonlinear, two-point boundary value problem) and the parallel transport equations (as a linear, possibly time-dependent, initial value problem) have at best only numerical solutions, closed form expressions of these forces on $\mathbb{S}^2 \subset \mathbb{R}^3$ lead to a result that generalizes the MSD error system to $\mathbb{S}^2$.

**Result 2.1. (MSD System on $\mathbb{S}^2$)** Let $q = q(t) \in \mathbb{S}^2$ be the solution to the nonlinear dynamic equations for a free particle

$$\frac{Dq}{dt} = \text{proj}_q \left( \frac{dq}{dt} \right) = -\hat{\Delta}^2 \left( \frac{dq}{dt} \right) = \hat{q} - \langle \hat{q}, q \rangle q = 0$$

and let $r = r(t) \in \mathbb{S}^2$ be a known reference curve. The compatible pair of functions $(\phi, T)$ defined by (6) generates closed-form expressions of the external forces $\Sigma F = F_{PD} + F_{TF}$ [$k_P > 0, k_D > 0$] given by

$$F_{PD} = F_P + F_D = -k_P \hat{\alpha}_q \phi|_q - k_D \left( \hat{q} - T_{r-q} \hat{r} \right),$$

where $\hat{\alpha} \triangleq \langle 1 - \beta \rangle \langle \omega, \hat{r} \rangle$ and

$$F_{TF} = \frac{D}{dt} \left[ T_{r-q} \hat{r} \right] = q \times (\hat{r} \times r) + \langle q, \hat{r} \times r \rangle (\hat{q} \times q) + \hat{\alpha}_q \omega + \hat{\alpha} \langle \hat{r}, q \rangle + \hat{\alpha}_r \langle \omega, \hat{r} \rangle - \hat{\alpha} \langle \omega, \hat{r} \times q \rangle,$$

with

$$\hat{\alpha}_q \triangleq \langle 1 - \beta \rangle \frac{\langle \hat{r} \times q, \hat{r} \rangle + \langle q, \hat{r} \rangle}{\langle \omega, \hat{r} \rangle - 2 \langle \omega, \hat{r} \rangle \langle r, q \rangle \langle \omega, \hat{r} \rangle \langle r, q \rangle \langle \omega, \hat{r} \rangle},$$

$$\hat{\alpha}_r \triangleq \langle 1 - \beta \rangle \frac{\langle r, \hat{r} \times q \rangle - 2 \langle \omega, \hat{r} \rangle \langle r, q \rangle}{\langle \omega, \hat{r} \rangle - 2 \langle \omega, \hat{r} \rangle \langle r, q \rangle \langle \omega, \hat{r} \rangle \langle r, q \rangle \langle \omega, \hat{r} \rangle}.$$
and bounding assumptions [9, pp. 536, 540–541] are met and provided that \( q \) and \( r \) never become antipodally positioned.

The proof of this result is purely computational and requires only repeated use of the straightforward fact that differentiation of both the inner product and the cross product satisfy the “product” rules

\[
\frac{d}{dt} \left[ \langle v(t) \times w(t) \rangle \right] = \langle \dot{v}(t) \times w(t) + v(t) \times \dot{w}(t) \rangle,
\]

\[
\frac{d}{dt} (\langle v(t), w(t) \rangle) = \langle \dot{v}(t), w(t) \rangle + \langle v(t), \dot{w}(t) \rangle.
\]

However, the derivative of the transport map presents some subtleties. Since the configuration \( q(t) \) is time-varying and the reference \( r(t) \) can be, in general, time-varying, the derivative of \( T_r \) is a total derivative

\[
\frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right] = \frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right] + \frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right],
\]

where

\[
T_{r(t) - q(t)} \dot{r}(t) = q(t) \times (\dot{r}(t) \times r(t)) + \left[ 1 - \langle q(t), r(t) \rangle \right] \text{proj}_{r(t) \times q(t)} \dot{r}(t)
\]

and where

\[
\frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right] = q \times (\dot{r}(t) \times r(t)) + \alpha_q (r(t) \times q) + \alpha_q (\dot{r}(t) \times q),
\]

\[
\frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right] = q \times (\dot{r} \times r) + \alpha_r (r \times q(t)) + \alpha_r (\dot{r} \times q(t)),
\]

with \( \alpha, \alpha_q, \alpha_r \) defined as in the statement of the result. Notice that a distinction is being made between the time-dependent and the fixed quantities. For example, \( \frac{d}{dt} \left[ T_{r(t) - q(t)} \dot{r}(t) \right] \) indicates that the regular time derivative is taken of \( T_{r(t) - q(t)} \dot{r}(t) \) with \( q \) fixed and \( r(t) \) time varying. Now, recall that while \( v_q = T_{r(t) - q(t)} \in T_q S^2 \), the derivative (or acceleration) \( \frac{d}{dt} (v_q) \) does not generally lie in \( T_q S^2 \).

To ensure that the derivative of \( v_q \) lies in \( T_q S^2 \), one must follow the derivative by a projection. That is, covariant differentiation must be used rather than regular differentiation. Computing the projections \( -\dot{q}(\cdot) \) and \( -\dot{q}(t)(\cdot) \) of (13) and (14), respectively, and adding the results complete the computation of the feedforward force \( F_{FF} \).

**Implementation of MSD System on \( S^2 \)**

With the external forces \( SF = F_{PPD} + F_{FF} \) now defined in terms of the intrinsic geometry of \( S^2 \), Result 2.1 is coordinate free and can be implemented using any set of coordinates on \( S^2 \) (see Sidebar 3). This distinction is important to note. It is only after we have formulated the

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Figure 9. Tracking on the Euclidean Sphere (3D view of Figure 10 (a, b, c), respectively). In all three cases, red is the reference curve \( r(t) \), black the solution to \( \dot{D}q/\dot{dt} = 0 \), and blue the configuration curve \( q(t) \) solution to \( \dot{D}q/\dot{dt} = SF = F_{PPD} + F_{FF} \). Notice that when the forces \( SF \) are applied, the blue solution turns away from the black curve and tracks the red curve. Therefore, the chasing objective is achieved. The control algorithm is coordinate free, meaning that it can be easily adapted to any coordinate system. The algorithm allows for near-global, large-angle maneuvers of the chasing particle.
control law that coordinates are introduced. This is in contrast to first introducing coordinates, say spherical coordinates, to formulate the free dynamics $\frac{Dq}{dt} = 0$ and then devising the control forces $\Sigma F = F_{FPD} + F_{TF}$ to achieve the tracking objective. This method would be a coordinate-based approach and works for the chosen coordinate system and no other.

In this section we give two examples that, in spherical coordinates, have closed-form solutions due to the fact that the reference curve $r(t)$ and the configuration curve $q(t)$ are restricted to the equator (a geodesic). We also present simulations (shown in both stereographic and projection coordinates in Figure 10) where the reference curve is not the equator (i.e., not a geodesic). Exact solutions to this case are unknown to this author and so numerical solutions are considered. The goal of each example is to demonstrate that no matter the coordinate system, the geodesic spring design with parallel-transport-based damping exhibits the behavior of a mass-spring-damper error system in the geodesic distance.

**Examples**

Two examples of Result 2.1 with closed-form solutions are presented. These two examples are informed by a general theorem [1, p. 5] which proves that for the configuration and transport

**Figure 10. Simulations of Result 2.1 in Spherical and Projection Coordinates.** In the projection coordinates simulations, the reference curve (red) is the coordinate form of a “short” circle. Since this reference curve is not a geodesic, the closed-loop dynamics do not present a closed-form solution. Instead, numerical solutions must be considered. The geodesic distance is computed by $\arccos(r(t), q(t))$ where, in projection coordinates, for example, $r(t) = \Phi_{sp}([u_r(t), v_r(t)])$ and $q(t) = \Phi_{sp}([u(t), v(t)])$. See Figure 9 for the trajectories on the sphere.
map pairing \((\phi, T)\) given by (6), the geodesic distance exhibits the behavior seen from the 1D MSD in Figure B2. The first example is the simplest example of the closed-loop error system from Result 2.1, namely, the fixed reference case.

**Example 1** \((\dot{r} = 0)\). In spherical coordinates, let \(r\) be a fixed reference with coordinate description \(R = \{\theta, \phi_r\} = \{2, \pi/2\}\), and assume the initial conditions \(\{\theta(0) = 3, \phi(0) = \pi/2, \dot{\theta}(0) = 0, \dot{\phi}(0) = 0\}\) for the configuration curve \(q(t)\) with coordinate description \(Q(t) = \{\theta(t), \phi(t)\}\). In other words, both \(q(t)\) and \(r\) start on the equator and have zero initial velocity. Because the reference is fixed, the transport map of \(r\) and its derivatives is 0, and the closed-loop error system of Result 2.1 reduces to \(\frac{D}{dt} \Phi_F = F_{PD}\) with

\[
F_{PD} = F_P + F_D = -k_P d_{\phi}\dot{\phi}|_{q} - k_D \dot{q} \{k_P > 0, k_D > 0\}
\]

\[
= k_P \frac{-\arccos(\beta)}{\sqrt{1 - \beta^2}} (r - \beta \dot{q}) - k_D \dot{q} \{\beta \in (\beta, \dot{q})\},
\]

The first and third equations of the closed-loop error system are

\[
\dot{\theta} + 2ct \phi \dot{\phi} + k_d \dot{\theta} + k_p \theta_1 = 0,
\]

\[
\dot{\phi} - c_\phi s_\phi \dot{\theta}^2 + k_d \dot{\phi} + k_p \theta_2 = 0,
\]

where

\[
\frac{\theta_1}{2} = \arccos\left(\frac{c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta}{s_\phi \sqrt{1 - (c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta)^2}}\right)
\]

\[
\frac{\theta_2}{2} = \arccos\left(\frac{c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta}{s_\phi \sqrt{1 - (c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta)^2}}\right)
\]

and where, for example, \(c_\theta - \phi = \cos(\theta - \phi)\), \(s_\phi = \sin(\phi)\), and \(c\theta - \phi = \cot(\phi)\).

Why the first and the third equations? Note that computing the free particle dynamics on \(S^2\) by \(\frac{D}{dt} \Phi_F = \dot{q} - \{\dot{q}, q\}\) \(q = 0\), where \(q(t) = \Phi_F\{\theta(t), \phi(t)\}\) yields three equations:

\[
\dot{\phi} - c_\phi s_\phi \dot{\theta}^2 = 0,
\]

\[
\frac{\theta_1}{2} = \frac{\arccos\left(\frac{c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta}{s_\phi \sqrt{1 - (c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta)^2}}\right)}{2}
\]

\[
\frac{\theta_2}{2} = \frac{\arccos\left(\frac{c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta}{s_\phi \sqrt{1 - (c_\phi \phi_r + c_\phi \phi_\theta + c_\phi \phi_\theta)^2}}\right)}{2}
\]

Solving (17) for \(\dot{\phi}\) and then substituting the result into (18) and (19) yields the same result:

\[
\dot{\theta} + 2ct \phi \dot{\phi} = 0.
\]

Consequently, only the first and third (or second) of the free particle equations are distinct. The equations

\[
\dot{\phi} - c_\phi s_\phi \dot{\theta}^2 = 0,
\]

\[
\dot{\theta} + 2ct \phi \dot{\phi} = 0
\]

are the typical form (in spherical coordinates) of the geodesic equations on \(S^2 \subset \mathbb{R}^3\). With the addition of the external forces \(\Sigma F\), the closed-loop dynamics \(\frac{D}{dt} \Phi_F = \Sigma F\) also have redundancies. As in the free particle case, the first and third equations are distinct.

Given the geodesic spring motivation of the closed-loop error system, (15) and (16) have a closed-form solution. For the chosen initial conditions, the solution must stay on the equator \(\phi = \phi_r = \pi/2\) (since \(\dot{\phi}(0) = 0\)) and behave like a 1D MSD measured from \(\theta_r\). The claim is \(Q(t) = [\theta(t), \phi(t)] = [\theta_r + \dot{\theta}(t), \pi/2]\), with \(\dot{\theta}(t)\) the solution to the 1D MSD initial value problem \(\dot{\theta} + k_D \dot{\theta} + k_P \dot{\theta} = 0\) with \(\dot{\theta}(0) = \theta_r - \dot{\theta}(0) = 1\), and \(\theta(0) = \theta(0) = 0\) is the exact solution to (15) and (16). Since \(\dot{\phi} = \dot{\phi} = 0\), \(c_\phi = 0\), and \(s_\phi = 1\), (16) is trivially satisfied. Under the same assumptions and since \(\dot{\theta} = \dot{\theta}, \dot{\theta}_r - \dot{\theta} = \dot{\theta}_r\), (15) simplifies to an equation in \(\theta\) as

\[
\dot{\theta} + k_D \dot{\theta} + k_P (\dot{\theta}_r - \dot{\theta}) \rightarrow \dot{\theta} + k_D \dot{\theta} + k_P \dot{\theta} = 0.
\]

For example, when \(k_D = 2\) and \(k_P = 1\), \(\theta(t) = e^{-t}(1 + t)\) is the critically damped solution to \(\dot{\theta} + k_D \dot{\theta} + k_P \dot{\theta} = 0\) with \(\theta(t) = 0\) as \(t \rightarrow \infty\). It follows that \(\theta(t) = \theta_r + \dot{\theta}(t)\) is the critically damped solution to (15) where \(\dot{\theta} - \dot{\theta}_r\). Figure 8 shows the solution (blue curve) \(q(t) = \Phi_F\{\phi_r + \theta(t) - \theta(t), \pi/2\}\) on \(S^2\). Figures 7(a) and 7(c) corroborate the undamped and critically damped solutions.

**Example 2** \((\dot{r} \neq 0)\). This second example extends the first example to consider a moving reference \(r(t)\) along the equator. In spherical coordinates, let \(r(t)\) be the reference with coordinate description \(R(t) = [\theta(t), \phi_r(t)] = [2 - \frac{t}{2}, \pi/2]\), and assume the initial conditions \(\{\theta(0) = 3, \phi(0) = \pi/2, \dot{\theta}(0) = 0, \dot{\phi}(0) = 0\}\) for the configuration curve \(q(t)\) with coordinate description \(Q(t) = [\theta(t), \phi(t)]\). Since the reference is no longer fixed and yet lies on the geodesic (equator), it follows that \((\omega, r) = 0\); therefore, the middle term of the transport map is 0. Consequently, the closed-loop error system of Result 2.1 reduces to \(\frac{D}{dt} \Phi_F = F_{PD} + F_{TF}\) with

\[
F_{PD} = F_P + F_D = -k_P d_{\phi}\dot{\phi}|_{q} - k_D \dot{q} \{\dot{q} - T_{r-t} \dot{r}\}
\]

\[
= k_P \frac{-\arccos(\beta)}{\sqrt{1 - \beta^2}} (q - \beta \dot{q}) - k_D \dot{q} \{\dot{q} - \dot{q} \times (\dot{r} \times r)\},
\]

\[
F_{TF} = \frac{D}{dt} \{T_{r-t} \dot{r}\} = \dot{q} \times (\dot{r} \times r).
\]

The third equation of the closed-loop system is trivially satisfied along the equator \(\phi = \phi_r = \pi/2\). In terms of the new variable \(\Theta = \theta - \theta, \phi\) defines \(\dot{\phi} = \dot{\theta}_r\), which defines \(\dot{\theta} = \dot{\theta}, \dot{\theta}_r = \dot{\theta}_r - \frac{t}{2}\), the first equation simplifies to

\[
\frac{k_D}{2} + k_P (\dot{\theta}_r - \dot{\theta}) + k_D \dot{\theta} + \dot{\theta} = 0 \rightarrow k_P \dot{\theta} + k_D \dot{\theta} + \dot{\theta} = 0.
\]

Therefore, as in the first example, the solution to the first and third equations of the closed-loop system is given by \(Q(t) = [\theta(t), \phi(t)] = [\theta_r(t) + \phi_r(t), \pi/2]\), where \(\theta(t)\) is a solution to
\[ \ddot{q} + k_D \dot{q} + k_P q = 0, \quad q(0) = 1, \dot{q}(0) = 1/5. \] Figures 7(b) and 7(d) corroborate the underdamped and critically damped solutions in this example.

**Simulations**

Simulations of Result 2.1 are performed in spherical and stereographic coordinates using Mathematica’s NDsolve command.

**Sidebar 1. 1D Mass-Spring-Damper (MSD) System**

Consider a particle of mass \( m \) constrained to move on a frictionless rod (analogous to the real line \( \mathbb{R} \)). Assume that the particle, initially at position \( q_0 \), is set into motion by an initial force that imparts an initial velocity \( v_0 \) to the particle but after which is subject to no other outside forces. That is, the particle is free with external forces \( \Sigma F = 0 \). As a differential equation, the free dynamics for the particle are given by Newton’s equation \( \Sigma F = ma \), specifically,

\[
\frac{d^2 q}{dt^2} = 0
\]

with solution \( q(t) = v_0 t + q_0 \). Depending on the direction (left, –) or (right, +) of the initial velocity, the particle’s position \( q(t) \) tends to \( \pm \infty \) as \( t \to \infty \). Forcing the particle to approach a fixed reference position \( r \) (rather than \( \pm \infty \)) can be achieved by connecting the particle to the rod with a spring-damper where the linear spring has a stiffness modeled by the constant \( k_P > 0 \) and the damper provides a resistance modeled by the constant \( k_D > 0 \). Define the proportional (spring) force \( F_P \) and the derivative (damping) force \( F_D \) by

\[
F_P \triangleq -k_P e(t) \quad \text{and} \quad F_D \triangleq -k_D \frac{d}{dt} \left[ e(t) \right],
\]

where \( e(t) \) is the error between the particle’s position \( q(t) \) and the fixed reference position \( r \). It follows that Newton’s equation \( m \ddot{q} = \Sigma F \) with external forces \( \Sigma F = F_P + F_D \) simplifies to the 1D mass-spring-damper (MSD) error system

\[
\ddot{e} + 2 \delta \dot{e} + \omega^2 e = 0,
\]

where \( \delta = k_D/2m \), \( \omega = \sqrt{k_P/m} \), and \( \dot{e}, \ddot{e} \) denote in this case the first and second time derivatives of \( e(t) \). The solution to this error system is a standard result in introductory ordinary differential equations. For any initial conditions \( q_0, v_0 \), the error \( e(t) \to 0 \) as \( t \to \infty \). In other words, the position of the forced particle \( q(t) \) approaches the reference position \( r \) as \( t \to \infty \). Consequently, depending on the selections for \( \delta \) and \( \omega \), the control objective \( q(t) \to r \) is achieved in an underdamped, overdamped, or critically damped manner (see Figure B2).

While the control objective has been achieved, the result is rather unsatisfactory in the sense that we are not left with much understanding about how this result was achieved. What is the big picture, and how might knowing it aid in extending this spring design to the sphere? To begin answering this question, consider more closely the forces \( F_D \) and \( F_P \). The control objective is partially achieved by the addition of the spring force \( F_P \), which can be interpreted in terms of the quadratic potential function

\[
V(q) = \frac{1}{2} \text{dist}_2(q,r)^2 \triangleq \frac{1}{2} |q-r|^2.
\]

This potential can be viewed as a quadratic well centered at the fixed reference \( r \), as shown in Figure B1, that generates a (restoring) force \( F_P = -k_P \nabla V \) on the particle directed toward the reference \( r \) with a strength proportional to the distance between the particle’s position \( q \) and reference \( r \). In other words,

\[
F_P = \pm k_P \text{dist}_2(q,r) = -k_P(q-r) = -k_P e,
\]

where the direction +(right) or –(left) of the force is given by the expression \(-(q-r)/|q-r|\). Due to energy considerations, the spring force \( F_P \) alone is not enough to achieve the control objective. Specifically, for zero damping \( \delta = 0 \Rightarrow F_D = 0 \), the total energy \( L \) (kinetic energy \( \frac{1}{2} \dot{e}^2 \) plus potential energy \( \frac{1}{2} \omega^2 (q-r)^2 \)) of the particle along the solution to \( m \ddot{e} = F_P \) is a constant function of the initial position and velocity of the particle, and therefore the spring force \( F_P \) alone cannot dissipate the energy. Introducing the external damping force \( F_D \), the total derivative of the total energy of the particle along the solutions to \( m \ddot{e} = F_D + F_P \) (equivalently, \( \dot{e} = -2 \delta \dot{e} - \omega^2 e \)) is

\[
\frac{dL}{dt} = \dot{q}^2 + \omega^2 (q-r) \dot{q} = \dot{e}^2 + \omega^2 \dot{e}^2 = \dot{e}(-2 \delta \dot{e} - \omega^2 e) + \omega^2 \dot{e}^2
\]

\[
= -2 \delta \dot{e}^2 \leq 0.
\]

For the cases \( \delta^2 > \omega^2 \) and \( \delta^2 = \omega^2 \), Figure B2 (bottom, middle, right) illustrates that \( e \neq 0 \), in which case \( \frac{dL}{dt} \) is strictly negative, and therefore the total energy of the particle must dissipate to 0. Even when \( e = 0 \) (\( \delta^2 < \omega^2 \)), Figure B2 (bottom, left) illustrates that the total energy of the particle eventually dissipates to 0. In all three cases, the control objective \( q(t) \to r \) is achieved.

In this article, the forces \( F_D \) and \( F_P \) are extended to the sphere \( S^2 \subset \mathbb{R}^3 \) in order to stabilize a particle \( q \in S^2 \) to a time-dependent reference curve \( r(t) \in S^2 \). The overarching control methodology is to create a potential well, this time on the sphere, centered on the moving reference \( r(t) \). The resulting potential force along with a geometrically defined damping force achieves the control objective \( q(t) \to r(t) \).
The potential function $V(q)$ can be viewed as a quadratic potential well centered at $r \in \mathbb{R}$ in which the particle $q$ is constrained to roll about. As $q$ is released from its position of maximum potential $V_0$, its energy is converted to purely kinetic energy at the bottom of the well and again to maximum potential at $V_1 = V_0$. The particle $q$ will continue to roll around the point $r$ but never settle on it. (b) With the inclusion of a damping force, thought of as a source friction on the potential surface, the potential energy of particle $q$ dissipates from $V_1$ to $V_3$, and eventually $V = 0$ where the reference position $r$ is located. Consequently, in the presence of damping, $q$ settles on $r$. The overarching control methodology in this article is to create a quadratic potential surface (well), this time centered on a moving reference $r(t) \in S^2$ (see Figure 2). The resulting potential force, along with a geometrically defined damping force, which defines a source of friction on the potential surface will achieve the control objective $q(t) \to r(t)$ on the sphere.

Figure B1. Illustration of the Potential Function $V(q) = \frac{1}{2} \left( \text{dist}_{\mathbb{R}}(q, r) \right)^2 = \frac{1}{2} (q - r)^2$ and Damping.

Sidebar 2. Visualization of Notation

The following equations illustrate the interaction between the inner product, cross-product, and projection and are a collection of the computations needed to prove the results of this article:

- $v \times (w \times z) = (v, z)w - (v, w)z$,  
- $(y, (w \times z) \times v) = -(v, (w \times z) \times y)$,  
- $-\dot{r}^2 q = -(q \times r) \times r = q - \beta r \overset{\Delta}{=} q^{\perp r}$,  
- $||q^{\perp r}|| = ||r^{\perp q}|| = \sin(\theta)$,  
- $(r \times q) \times r^{\perp q} = q^{\perp r} + \beta r^{\perp q} = \sin^2(\theta) q$,  
- $(r \times q) \times \text{vers}(q^{\perp r}) = -\sin(\theta) r$. 

Figure B2. 1D Mass-Spring-Damper (MSD) Summary.
where \( \beta \overset{\Delta}{=} \langle q, r \rangle = \cos(\theta) \) and where vers(\( \cdot \)) is the length-normalization mapping on vectors \( v \) defined by vers(\( v \)) = \( v / \| v \| \). The mapping \( \gamma \) is defined in (7). Equation (B7), the standard triple cross-product formula, is the main equation from which the other equations are derived. In general, the vectors \( q \) and \( v \) are not in the tangent plane to \( S^2 \) at \( r \) (denoted \( T_r S^2 \)). According to (B9) and as shown in Figure B3, \( q^r = -\hat{r} q \in T_r S^2 \) and \(-\hat{r} v \in T_q S^2 \). That is, the operators \(-\hat{r} (\cdot)\) and \(-\hat{r} (\cdot)\) act as projections onto the tangent planes to \( S^2 \) at \( r \) and \( q \), respectively. Therefore the covariant derivative operator at \( q \in S^2 \) from (2) can be explicitly given by

\[
\frac{d}{dt} |_{q} = -\hat{\gamma}^2 \circ \frac{d}{dt}.
\]

Note that \( r^1 q \overset{\Delta}{=} -\gamma^2(q) \) is the vector at \( q \) pointing tangent to the geodesic segment (black solid curve) connecting \( q \) to \( r \). In this article, we construct a force \( F_p \propto r^1 q \) that compels a particle \( q(t) \in S^2 \) towards a moving target \( r(t) \in S^2 \). It is important that the applied forces, like \( F_p \) for example, be tangent to the sphere. Otherwise, the curve \( q(t) \) will leave the sphere, and, using the robotic arm analogy, the tool tip of the robotic arm will no longer be constrained to the sphere.

**Sidebar 3. Spherical and Projection Coordinates**

Spherical and stereographic projection coordinates are the two coordinate systems employed in this article. The general concept of coordinates on \( S^2 \) can be illustrated using the example of stereographic projection. Since we are viewing the unit sphere \( S^2 \) as a subset of \( \mathbb{R}^3 \), one description of the sphere is the algebraic relation \( x^2 + y^2 + z^2 = 1 \) between the three variables \((x, y, z)\). Imagine now that a light source and planar screen are placed at the north and south poles of a translucent sphere, respectively. Any region on the sphere will then cast a shadow onto the screen to create a two-dimensional representation of that region (see Figure B4). The two variables which describe the position of the shadow are the (stereographic) projection coordinates of \( S^2 \).

The spherical coordinate \((\theta, \phi)\) and stereographic (south pole) projection coordinate \((u, v)\) representations of the sphere \( \Phi_{sc} : D_{sc} \subset \mathbb{R}^2 \to S^2_{sc} \subset S^2 \) and \( \Phi_{sp} : D_{sp} \subset \mathbb{R}^2 \to S^2_{sp} \subset S^2 \) are given by

\[
\Phi_{sc}(\theta, \phi) = [\cos(\phi) \sin(\theta), \sin(\phi) \sin(\theta), \cos(\phi)],
\]

\[
\Phi_{sp}(u, v) = \left[ \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right],
\]

where \( D_{sc} = (0, 2\pi) \times (0, \pi) \) and \( D_{sp} = (-\infty, \infty) \times (\pi, \infty) \) are the domains chosen so that \( \Phi_{sc} \) and \( \Phi_{sp} \) are one-to-one mappings. For completeness, the mapping which assigns to each point \((x, y, z) \in \mathbb{R}^3\) its projection coordinates \((u, v)\) is

\[
\Phi_{sp}^{-1}(x, y, z) = \left[ \frac{2x}{1 + z}, \frac{2y}{1 + z} \right].
\]

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