WHAT IS...
the p-adic Mandelbrot Set?

Joseph H. Silverman

Before attempting to answer the title question, we must first answer two preliminary questions: "What are the p-adic numbers?" and "What is the (classical complex) Mandelbrot set?" We start with the former.

A standard characterization of the real numbers $\mathbb{R}$ is as the smallest field containing $\mathbb{Q}$ that is complete with respect to the "usual" absolute value $|r|_\infty = \max\{r, -r\}$ on $\mathbb{Q}$, where we recall that a field is complete if every Cauchy sequence converges. But there are other absolute values on $\mathbb{Q}$. Ostrowski showed that, up to a natural equivalence, there is one absolute value for each prime number p. Writing a rational number r as $r = p^k \frac{a}{b}$ with p not dividing ab, the p-adic absolute value of r is defined by $|r|_p = p^{-k}$. Intuitively, two rational numbers r and s are p-adically close if the numerator of their difference is divisible by a large power of p. Then the field of p-adic numbers $\mathbb{Q}_p$ is the smallest field containing $\mathbb{Q}$ that is complete with respect to the p-adic absolute value $|\cdot|_p$. The p-adic numbers were invented by Hensel in the nineteenth century. They are analogous in many ways to $\mathbb{R}$. For example, the field $\mathbb{Q}_p$ is locally compact for the topology induced by $|\cdot|_p$, and one can do p-adic analysis with p-adic power series. A significant difference between $\mathbb{R}$ and $\mathbb{Q}_p$ is that the p-adic absolute value satisfies the "ultrametric" triangle inequality $|r+s|_p \le \max\{|r|_p, |s|_p\}$. This implies that the unit disk $\{x \in \mathbb{Q}_p : |x|_p \le 1\}$ is a compact sub*ring* of $\mathbb{Q}_p$, which is nice, but it also implies that $\mathbb{Q}_p$ is totally disconnected, which is not so nice. Also, although the fields $\mathbb{R}$ and $\mathbb{Q}_p$ are complete, they are not algebraically closed; so just as it is often better to work with the complete algebraically closed field of complex numbers $\mathbb{C}$, it is also often better to work with the field $\mathbb{C}_p$, the smallest complete algebraically closed field containing $\mathbb{Q}_p$. But we note that $\mathbb{C}_p$ is a monster of a field; it is not even locally compact!

The classical (degree 2 complex) Mandelbrot set $\mathcal{M}_\infty$ arises in studying the dynamics of the simplest family of nonlinear functions, which is the set of quadratic polynomials $f_c(x) = x^2 + c$. Dynamicists study the effect of repeatedly applying the map f_c to an initial point $a \in \mathbb{C}$; i.e., they study how the points in the orbit $\mathcal{O}_{f_c}(a) = (a, f_c(a), f_c^2(a), \ldots)$ move around $\mathbb{C}$, where $f_c^n = f_c \circ f_c \circ \cdots \circ f_c$ denotes the nth iterate of f_c. Of particular interest is the *Julia set* $\mathcal{J}(f_c)$ *of* f_c, which is the set of initial points where the iterates of f_c behave chaotically. The Julia set may also be described as the boundary of the set of initial points $a \in \mathbb{C}$ whose orbit $\mathcal{O}_{f_c}(a)$ is bounded. Surprisingly, the geometry of $\mathcal{J}(f_c)$ is heavily influenced by the orbit of the single point 0. For example, a famous theorem of Fatou and Julia (discovered independently) says that if $\mathcal{O}_{f_c}(0)$ is bounded, then $\mathcal{J}(f_c)$ is connected (although generally quite fractal-like), and otherwise $\mathcal{J}(f_c)$ is totally disconnected. This dichotomy divides the parameter space of c values into two pieces. The *Mandelbrot set* $\mathcal{M}_\infty$ is the set of parameters $c \in \mathbb{C}$ such that the orbit $\mathcal{O}_{f_c}(0)$ is bounded, or equivalently, such that $\mathcal{J}(f_c)$ is connected. You have undoubtedly seen pictures of the incredibly complicated and beautiful Mandelbrot set. It has become one of the most ubiquitous images in all of mathematics, and

Joseph H. Silverman is professor of mathematics at Brown University. His email address is `jhs@math.brown.edu`.

DOI: http://dx.doi.org/10.1090/noti1038

the study of its geometry has occupied generations of mathematicians.

The modern theory of complex dynamics dates from the fundamental work of Fatou and Julia around 1920. The study of p-adic dynamics is more recent, where initial investigations in the 1980s by Herman, Yoccoz, and others led to an explosion of interest starting in the 1990s with groundbreaking results by Benedetto, Bézivin, Hsia, and Rivera-Letelier. The classical Mandelbrot set is defined in terms of whether $|f_c^n(0)|$ is bounded or goes to infinity as $n \to \infty$. We can use exactly the same definition to define the p-adic Mandelbrot set $\mathcal{M}_p$. Thus $c \in \mathbb{C}_p$ is in $\mathcal{M}_p$ if and only if $|f_c^n(0)|_p$ is bounded as $n \to \infty$. The only change is that we've replaced the usual absolute value on $\mathbb{C}$ with the p-adic absolute value on $\mathbb{C}_p$. However, using the ultrametric property of $|\cdot|_p$, it is very easy to see that $|f_c^n(0)|_p$ is bounded if and only if $|c|_p \le 1$, so $\mathcal{M}_p = \{c \in \mathbb{C}_p : |c|_p \le 1\}$ is the closed unit disk, which really is not very interesting.

If that were the end of the story, then we'd have wasted a lot of ink, since the title question would have a single word answer: boring. Luckily, matters become more interesting when we look at Mandelbrot sets associated with polynomials of higher degree. But first we ask why it is that the orbit of the particular point 0 for $f_c(x) = x^2 + c$ has such a profound influence on the dynamics of f_c. What makes the point 0 so important? The answer is that 0 is the (unique) critical point of f_c, i.e., the derivative $f_c'(x)$ vanishes at 0, and thus there is no neighborhood of 0 on which f_c is one-to-one. We now fix $d \ge 2$, and for each $(d-1)$-tuple $\boldsymbol{c} = (c_1, \ldots, c_{d-1}) \in \mathbb{C}^{d-1}$, we define a (normalized) degree-d polynomial

$$f_{\boldsymbol{c}}(x) = x^d + c_1 x^{d-2} + c_2 x^{d-3} + \cdots + c_{d-2}x + c_{d-1}.$$

(Every degree-d polynomial can be put into this form by conjugating by a linear polynomial. This conjugation does not materially affect the dynamics.) The polynomial $f_{\boldsymbol{c}}(x)$ has $d-1$ critical points $\gamma_1, \ldots, \gamma_{d-1}$ (counted with multiplicity) whose orbits similarly have a profound influence on the dynamics of $f_{\boldsymbol{c}}$. We say that $f_{\boldsymbol{c}}$ is *critically bounded* if the orbits $\mathcal{O}_{f_{\boldsymbol{c}}}(\gamma_i)$ of all of the critical points are bounded. Then the *degree-d Mandelbrot set* $\mathcal{M}_{\infty,d}$ is the set of $\boldsymbol{c} \in \mathbb{C}^{d-1}$ whose associated polynomials $f_{\boldsymbol{c}}$ are critically bounded. If we work over $\mathbb{C}$, then these higher degree analogues of $\mathcal{M}_{\infty,2}$ are extremely complicated, with higher-dimensional fractal-like boundaries.

It's clear how we should define the p-adic analogue of the degree-d Mandelbrot set; it is the set

$$\mathcal{M}_{p,d} = \{\boldsymbol{c} \in \mathbb{C}_p^{d-1} : f_{\boldsymbol{c}}(x) \text{ is } p\text{-adically critically bounded}\},$$

where of course we now use the p-adic absolute value to determine whether the orbits $\{f_{\boldsymbol{c}}^n(\gamma_i)\}$ of the critical points are bounded. If p is large, then $\mathcal{M}_{p,d}$ is again boring, as shown by the following result that has long been "well known to the experts," but seems to have first been written down in [1].

Theorem 1. *If $p \ge d$, then $\mathcal{M}_{p,d}$ is a polydisk,*

$$\mathcal{M}_{p,d} = \{\boldsymbol{c} \in \mathbb{C}_p^{d-1} : |c_i|_p \le 1 \text{ for all } 1 \le i \le d-1\}.$$

The fact that $\mathcal{M}_{p,2}$ is a disk for all p, combined with Theorem 1, seems to have discouraged people from studying p-adic Mandelbrot sets, but recent work by Anderson has shown that when $p < d$, the p-adic Mandelbrot set $\mathcal{M}_{p,d}$ has a complicated geometric structure that may rival the geometry of the classical complex Mandelbrot sets.

Example 2. Consider the action of the polynomial $g(x) = x^3 - \frac{3}{4}x - \frac{3}{4}$ on $\mathbb{C}_2$. The critical points of $g(x)$ are $\pm\frac{1}{2}$, and they both have finite orbits, since

$$-\frac{1}{2} \xrightarrow{g} -\frac{1}{2} \quad \text{and} \quad \frac{1}{2} \xrightarrow{g} -1 \xrightarrow{g} -1.$$

Hence $(-\frac{3}{4}, -\frac{3}{4}) \in \mathcal{M}_{2,3}$, so $\mathcal{M}_{2,3}$ is not contained in the unit polydisk. (Note that $|\frac{3}{4}|_2 = 4 > 1$.)

More generally, consider the one-parameter family of polynomials $g_t(x) = x^3 - \frac{3}{4}t^2x - \frac{1}{4}(t^3 + 2t)$, so $g(x) = g_1(x)$. The critical points of $g_t(x)$ are the fixed point $\gamma_1 = -\frac{1}{2}t$ and the point $\gamma_2 = \frac{1}{2}t$, so g_t is critically bounded if and only if the orbit of $\frac{1}{2}t$ is bounded. One can show that $g_t(x)$ is critically bounded for the sequence of parameter values $t = 1 + 2^{2k}$ converging 2-adically to 1, while it is critically unbounded for the sequence of parameter values $t = 1 + 3 \cdot 2^{2m+1}$, also converging 2-adically to 1. Thus $(-\frac{3}{4}, -\frac{3}{4})$ is on the boundary of $\mathcal{M}_{2,3}$. Computations in [1] show that the geometry of $\mathcal{M}_{2,3} \cap \mathbb{Q}_2$ in a 2-adic neighborhood of $(-\frac{3}{4}, -\frac{3}{4})$ is extremely complicated, and they further suggest that this neighborhood is geometrically equivalent (in some not yet precisely defined sense) to a 2-adic neighborhood of the critical point $\frac{1}{2}$ in the Julia set $\mathcal{J}(g_1) \cap \mathbb{Q}_2$. This is our first inkling of a potential p-adic analogue of a famous classical result over $\mathbb{C}$ that says that a Misiurewicz point c in $\mathcal{M}_{\infty,2}$ has a neighborhood that is quasi-similar to a neighborhood of the critical point 0 in the complex Julia set $\mathcal{J}(f_c)$.

If $p < d$, then the p-adic Mandelbrot set $\mathcal{M}_{p,d}$ is (generally) not contained in the unit polydisk, so it is an interesting problem to compute or estimate the p-adic radius of $\mathcal{M}_{p,d}$. For this purpose, we define the *p-adic critical radius* $R_{p,f}$ of a polynomial $f(x)$ to be the maximum of $|\gamma - \alpha|_p$ as γ ranges over the critical points of $f(x)$ (roots of $f'(x)$) and α ranges over the fixed points of $f(x)$ (roots of $f(x) - x$).

Then the *critical radius of* $\mathcal{M}_{p,d}$, denoted $R(\mathcal{M}_{p,d})$, is the maximum of $R_{p,f}$ for $f \in \mathcal{M}_{p,d}$.

Example 3. It is an easy consequence of Theorem 1 that $R(\mathcal{M}_{p,d}) = 1$ for $p \geq d$. On the other hand, the polynomial from Example 2 satisfies

$$g'(x) = 3x^2 - \frac{3}{4} = 3(x - \frac{1}{2})(x + \frac{1}{2})$$

and

$$g(x) - x = (x+1)(x - \frac{3}{2})(x + \frac{1}{2}),$$

so $R_{2,g} = 2$. Hence $R(\mathcal{M}_{2,3}) \geq 2$.

Theorem 2 (Anderson [1])**.** *Let p be a prime satisfying* $\frac{1}{2}d < p < d$. *Then* $R(\mathcal{M}_{p,d}) = p^{p/(d-1)}$. *Further,* $R(\mathcal{M}_{p,2p}) = 1$.

To prove the lower bound for $R(p,d)$, it suffices to exhibit a single polynomial in $\mathcal{M}_{p,d}$. Explicitly, one can show that the polynomial $x^{d-p}(x-\beta)^p$ has critical radius $p^{p/(d-1)}$ for the carefully chosen value $\beta = [(-p/d)^p(1-p/d)^{d-p}]^{-1/(d-1)}$. The upper bound in Theorem 2 is more difficult and requires an elaborate calculation with Newton polygons. It's possible that the argument in [1] can be extended to compute $R(\mathcal{M}_{p,d})$ for $\frac{1}{3}d < p < \frac{1}{2}d$, but it appears to be a difficult problem to evaluate $R(\mathcal{M}_{p,d})$ when, say, $p < \sqrt{d}$.

Returning now to the classical complex Mandelbrot set $\mathcal{M}_{\infty,2}$, we consider the collection $\mathcal{H}_{\infty,2}$ of *hyperbolic maps*, which is the set of parameter values $c \in \mathcal{M}_{\infty,2}$ such that the orbit of the critical point 0 of $f_c(x) = x^2 + c$ converges to an attracting cycle. In other words, $c \in \mathcal{H}_{\infty,2}$ if and only if $\lim_{n\to\infty} f_c^n(0)$ converges to a point $\alpha \in \mathbb{C}$ satisfying $f_c^m(\alpha) = \alpha$ and $|(f_c^m)'(\alpha)| < 1$ for some $m \geq 1$. It is known that $\mathcal{H}_{\infty,2}$ is an open subset of $\mathcal{M}_{\infty,2}$, and the Lebesgue measures satisfy $1.503 \leq \mu(\mathcal{H}_{\infty,2}) \leq \mu(\mathcal{M}_{\infty,2}) \leq 1.562$. The famous Hyperbolicity Conjecture asserts that $\mathcal{H}_{\infty,2}$ equals the entire interior of $\mathcal{M}_{\infty,2}$.

The p-adic analogue $\mathcal{H}_{p,2}$ is defined similarly; we simply replace $\mathbb{C}$ with $\mathbb{C}_p$ and use $|\cdot|_p$ in place of the complex absolute value. Then, although $\mathcal{H}_{p,2}$ is a subset of the (boring) closed unit disk $\mathcal{M}_{p,2}$, it turns out that $\mathcal{H}_{p,2}$ itself is quite complicated. A first reduction is to note that $\mathcal{H}_{p,2}$ is the full inverse image under the "reduction mod p map" of the set

$$\overline{\mathcal{H}}_{p,2} = \{c \in \overline{\mathbb{F}}_p : f_c^m(0) = 0 \text{ in } \overline{\mathbb{F}}_p \text{ for some } m \geq 1\}.$$

(Here $\overline{\mathbb{F}}_p$ denotes an algebraic closure of the finite field $\mathbb{F}_p$.) It is conjectured that $\overline{\mathcal{H}}_{p,2}$ is quite small, in contrast to the complex hyperbolic set $\mathcal{H}_{\infty,2}$, which has positive Lebesgue measure.

Conjecture 3. *Let* $p \geq 3$. *Then*

$$\lim_{k\to\infty} \frac{\#(\overline{\mathcal{H}}_{p,2} \cap \mathbb{F}_{p^k})}{p^k} = 0.$$

A beautiful and deep result of Jones [3] says that Conjecture 3 is true if $p \equiv 3 \pmod 4$ and that a slightly weaker statement with an alternative notion of density is true for all $p \geq 3$. Jones's proof begins by using the function field Chebotarev density theorem to reduce the problem to properties of the action of the Galois group of $\overline{\mathbb{F}_p(t)}/\mathbb{F}_p(t)$ on the iterated preimage tree of 0. He next constructs a stochastic process that encodes information about the group action and shows that this process is a martingale. Finally, additional information about the group action is combined with a martingale convergence theorem to complete the proof.

Acknowledgment

I would like to thank Jackie Anderson, Rob Benedetto, Liang-Chung Hsia, Rafe Jones, and the referees for their helpful comments.

References

[1] J. Anderson, Bounds on the radius of the p-adic Mandelbrot set, 2012, `arXiv:1207.2628`.

[2] M. Baker and R. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line*, Mathematical Surveys and Monographs, vol. 59, American Mathematical Society, Providence, RI, 2010.

[3] R. Jones, Iterated Galois towers, their associated martingales, and the p-adic Mandelbrot set, *Compos. Math.* **143** (5) (2007), 1108–1126.

[4] J. H. Silverman, *The Arithmetic of Dynamical Systems*, Chapter 5, Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007.