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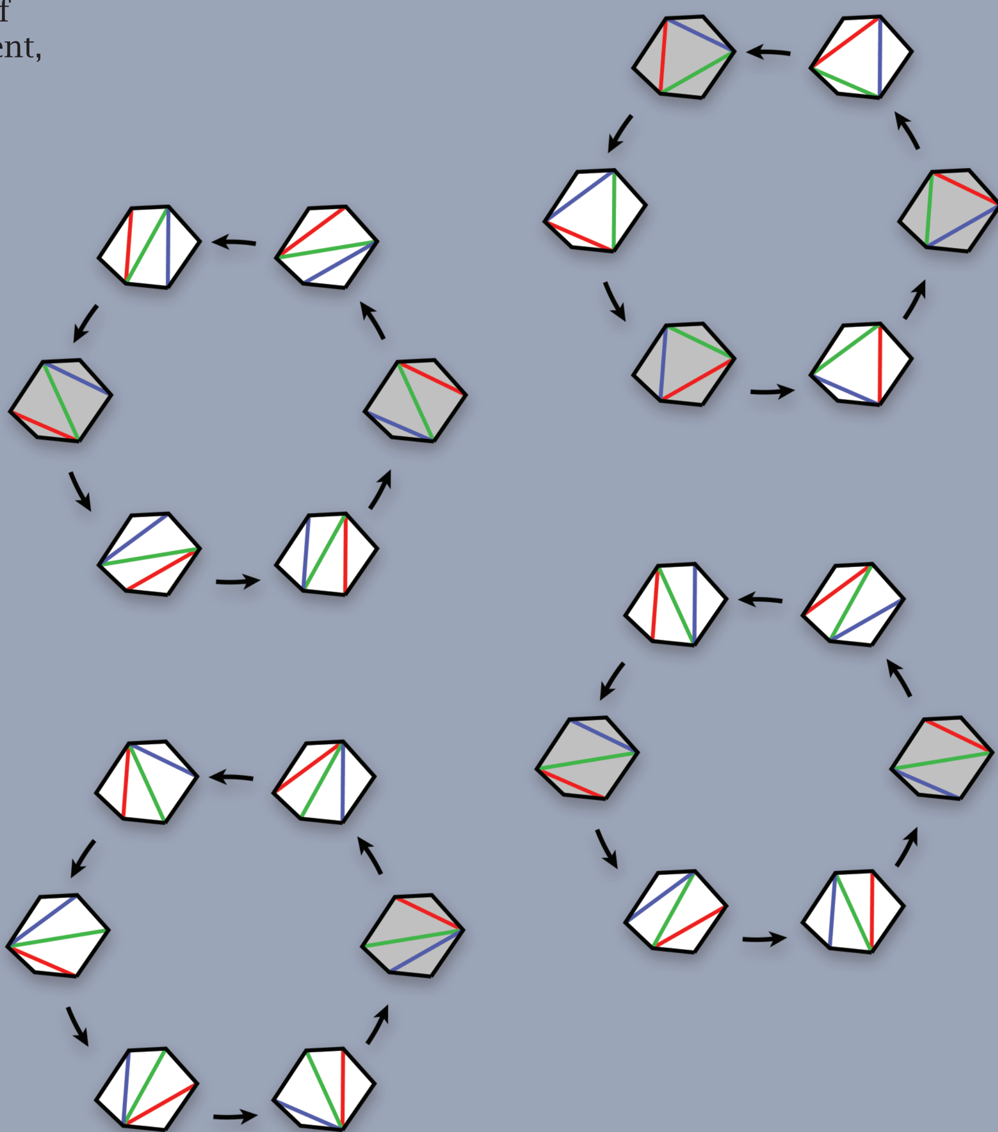

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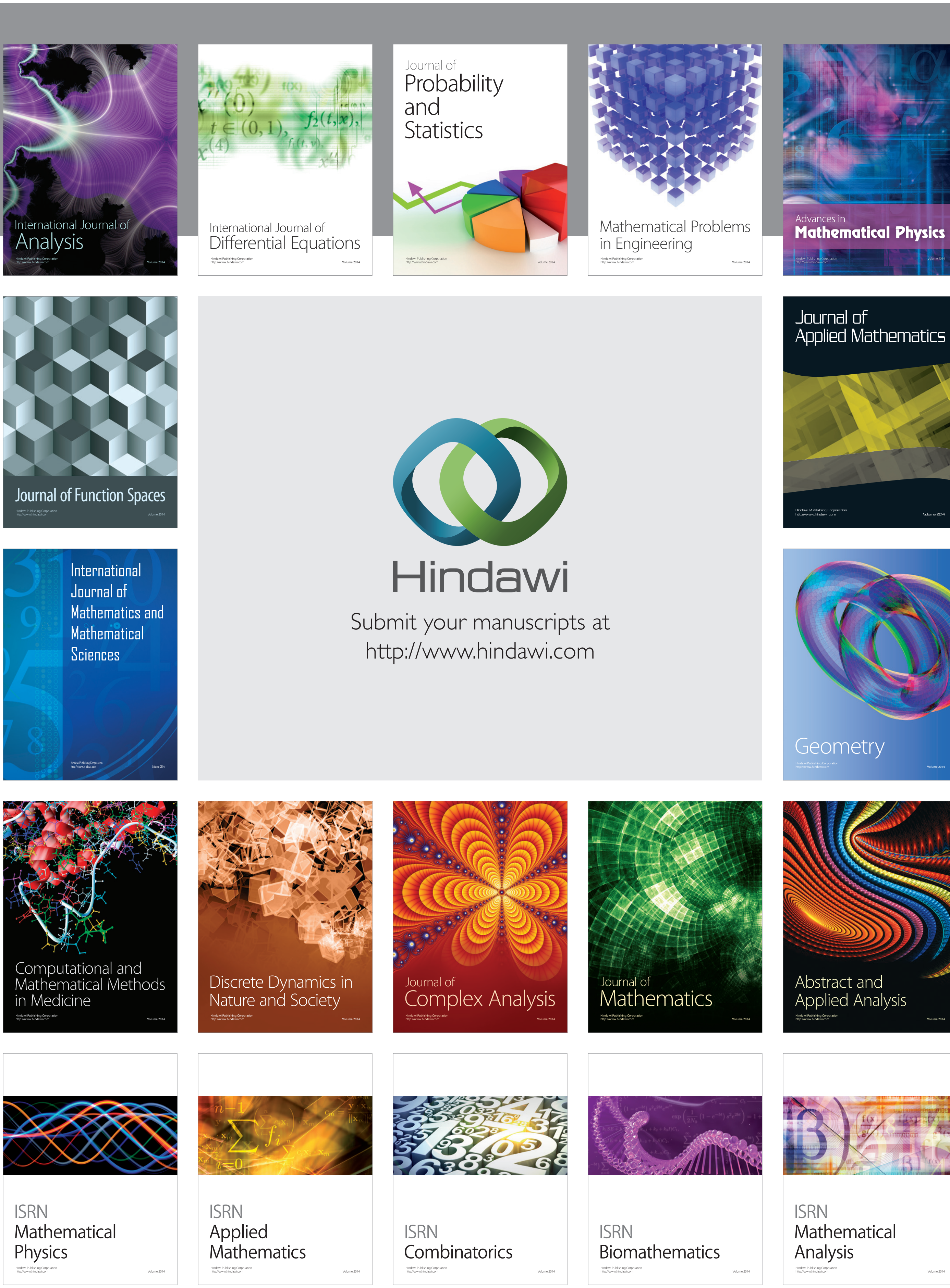
About the cover: Remarkable combinatorics in the presence of cyclic symmetry *(see page 205)*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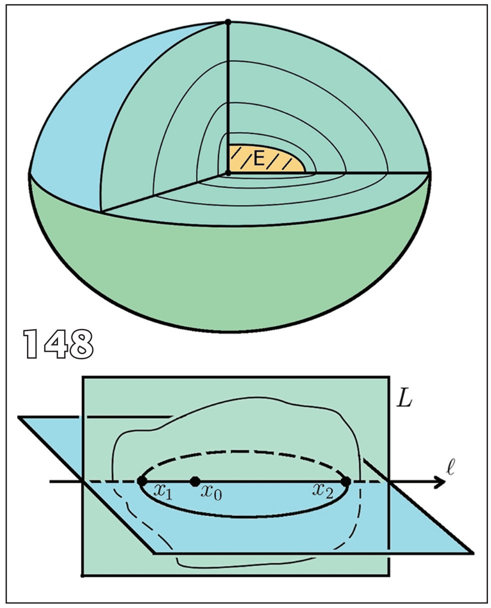
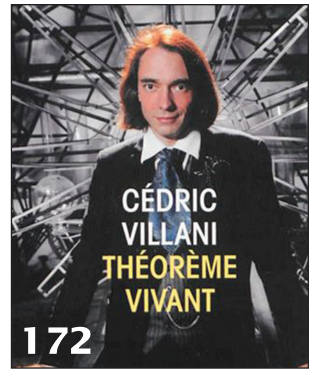
Notices
of the American Mathematical Society

February 2014

Communications

Commentary





The February issue showcases a mixture of mathematics and mathematical culture. We have a feature article on the potential theory of the ellipsoid and another on two-person fair division of indivisible items. On the other hand, we have an article on the relationship of the National Security Agency and the National Institute of Standards and Technology. There is also a piece on the JUMP Math program for getting kids interested in mathematics at an early age. Finally, there are the experiences of an American mathematician who spent a year in Moscow in the 1970s.

—*Steven G. Krantz, Editor*

Features

Notices
of the American Mathematical Society

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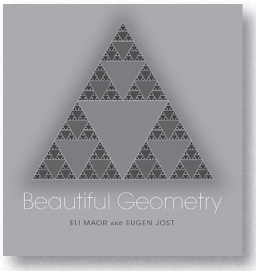

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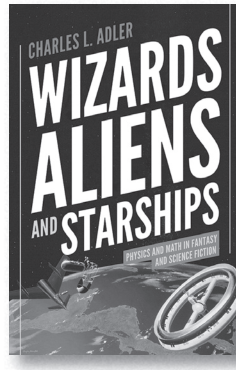

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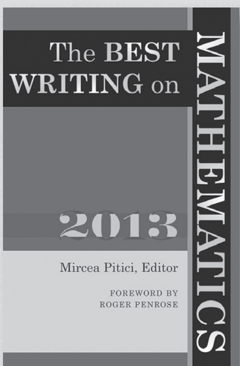

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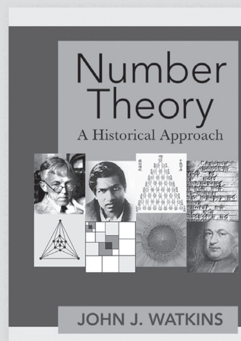

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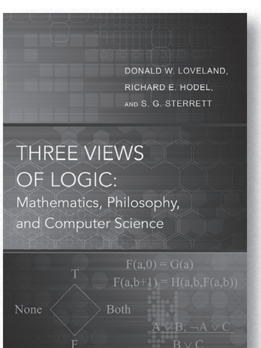

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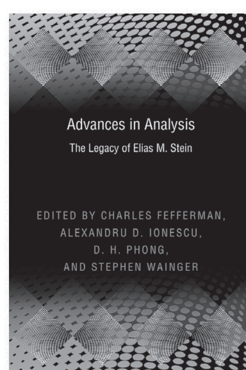

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Letters to the Editor

Is There a Better Format for the Presentation of Mathematical Subjects?

The ongoing concern with mathematics education that was expressed by two articles in the November 2013 issue of the *Notices*, makes me wonder if part of the problem might not be the format in which mathematics is presented—a format that goes back to Euclid, some 300 years b.c.e. For years, in my own studies, I have relied on a different format that has proven to be far more efficient for learning and problem solving. The format is based on several ideas from computer science: object-oriented programming (task orientation), separating the What from the How, and structured programming. In brief: each subject is conceived as involving a set of "entities." For example, in high school algebra, one of these entities is "equation." Associated with each entity is a template (the same for all entities), which is a list consisting of: definition of entity, ways of representing entity, common tasks performed on the entity, types of the entity, theorems pertaining to the entity, closely related entities. Each item in the list is then followed by a reference in the student's notes and/or in the textbook, to details on the item.

Thus in the case of the entity "equation," the list of common tasks includes: convert an equation into polynomial form, determine the type of an equation (linear, quadratic, etc.), solve an equation, add a term to both sides of an equation, multiply both sides of an equation by a term, divide both sides of an equation by a term.

Another characteristic of this format is the writing down of procedures to perform the more difficult tasks. The goal here is to have something that can be looked up and rapidly re-used days, weeks, months, years after the procedure was first learned. (It is not enough to more or less know how to do most integrals in a calculus course: the goal is to write down a procedure (it is known that no algorithm exists).)

Another characteristic is that all, or most, proofs are written in structured proof format (analogous to structured program format), which makes the devising of proofs, and the understanding of existing proofs, much more rapid.

The format can be applied to all subjects from primary school up to at least all the advanced mathematics subjects that I am familiar with. It makes all mathematical subjects look "the same." Primary school students can be introduced to it by being asked to consider all the tasks associated with, for example, a bicycle, or an iPad, or a TV set.

The format is not an alternative to the traditional textbook and classroom format, but in my experience it is a great enhancement to it.

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(Received October 31, 2013)

Ludwig van Beethoven and the Metronome

I read with great interest the article about the metronome ["Was something wrong with Beethoven's metronome?", by Sture Forsén, Harry B. Gray, L. K. Olof Lindgren, and Shirley B. Gray, *Notices*, October 2013], which presents an analysis of what happens if the weights are not in proper position and discusses the big question mark left behind by Beethoven's metronome markings. The article is very enjoyable and informative.

Nevertheless, there is one significant omission that devalues the article.

While reading the article, I was curious to see what the final conclusion would be. After all, Beethoven gave a metronome number not only for the first movement of the *Hammerklaviersonate*, but for *all* movements. And *all* the markings—including the one for the slow movement, that is, the third movement—are too fast. (At least, every musician would agree on this. There is a benchmark recording of the Beethoven sonatas by Friedrich Gulda; in the accompanying text he says, concerning the *Hammerklaviersonate*, that he intended to follow Beethoven's markings as much as possible; but even he plays the first movement a bit slower than Beethoven's indication—which is still incredibly fast!—and the same is true for the other movements, except perhaps for the second.)

The analysis in the article shows that—depending on whether the bottom weight is too low or too high—fast tempos are made faster (respectively, slower) and slow tempos are made slower (respectively, faster). I therefore wondered how this relates to the above-mentioned fact, namely, that *all* of Beethoven's metronome indications in the *Hammerklaviersonate* are too fast. The indication for the third movement (Adagio sostenuto) is: eighth = 92. Is this a slow tempo, a medium tempo? This should have been addressed, as should have the indications in the other movements. Otherwise, the theory as presented stands on weak grounds.

Finally, I can offer the following anecdote that my piano professor at the Vienna University of Music and Performing Arts (when I was a pianist in a previous life) loved to tell: it concerns Igor Stravinsky, who on some occasion was asked by a journalist how fast he would would want a certain piece of his to be played. Stravinsky thought for a moment, and then indicated a tempo. My teacher found it very amusing that the tempo was completely different from the metronome indication that Stravinsky had given for the piece.

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(Received November 15, 2013)

Pseudo-Education Marches On

In the October 2013 *Notices* article "Teaching mathematics with women in mind," Professors Deshler and Burroughs wrote the following under the heading "What Are We Teaching Our Students": "In recent years a focus on conceptual understanding has led to a curriculum reform movement in mathematics across all school levels, a movement that is focused on conceptual understanding rather than procedural understanding of mathematics."

Unfortunately, they failed to expose that the assorted "reforms" of the past twenty-four years, which I have followed closely, have simply enhanced the continuing mathematics pseudo-education of American students. One example of this grim reality is the brief essay written by one of my students in 1995, which is posted at: http://mathforum.org/kb/message.jspa?messageID=1461554.

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(Received October 4, 2013)

Adjuncts and Teaching

Catching up on old *Notices*, I read Prof. Reys's article ["Getting evidence-based teaching practices into mathematics departments: Blueprint or fantasy?," by Robert Reys, *Notices*, August 2013] with some interest, noticing that the article does not address a major issue, and that is the increasing burden of teaching being born by badly paid and overworked adjuncts, many of whom are quite talented, but who operate outside the main life of the department and who often do not have the time nor access to resources that would allow them to seriously rethink their teaching methodology.

Any serious efforts to change the culture will have to include them, one hopes with serious improvements in their remuneration and working conditions.

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(Received November 26, 2013)

AMS in Arabic Means "Yesterday"

In response to "[Contemporary pure] math is far less than the sum of its [too numerous] parts," by Doron Zeilberger, *Notices*, December 2013:

Opposing "rigorous" mathematical proof to "field" (experimental) mathematics as ethically/socially top down vs. bottom up is a cultural practice with little taxonomic value. Pure math can be as usefully defined as mathematics having conceptual distance or having no immediate application, except for other mathematicians. The proof as purity paradigm is gold standard/virginity testing stuff. It is a cultural practice, an old meme, not a paradigm that exactly gives flight to the imagination. Mathematical proof is "pure" when it extends proof/analysis/logic, and is "field" mathematics when it connects mathematical disciplines, and is "applied" when it merely verifies. Verification proof is mathematics accounting.

Field mathematics—the pure mathematics casually located adjunct to proofy math and frequently accompanied by and ambiguated with applied mathematics—doesn't need much defense, and using QFT [quantum field theory] to do so seems to invite criticism. (If I had to make an ignorant over-arching comment on QFT based on conceptual distance of the title, I would dismiss it as magnitudinal incrementalism that was probably visible twenty or thirty years ago. Pure mathematics might have a go at this.)

Partially outsourcing mathematics to machines is a done deal but it is annoying because of known limitations (100 years of quantum physics without quantum computers) and the implied administrative/power relations overhead. If they weren't dumb, metered, and politicized and more of them had names like "scratchpad" instead of "megalyth-o-tron" they would probably be better received. How many filters does a person need to pass through before qualifying to use one? Similarly, "John Henry" would probably not be a good name for a highly politically, commercially, or bureaucratically leveraged research computer, which pretty much includes all of them. A concern that their use is in equal measure "insipid" as "intrepid" reflects a certain amount of self-understanding. We are sometimes so, why not somewhat they? Fair allocation minimally requires excess flop/hrs. (or quantum equivalent). What are the benefits to managing an unused supercomputer (probably similar to pet ownership)?

What I liked about Doron's letter was that it was all over the place and unabashedly wrong. That is called self-expression. Thank you.

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Outdoing the Soviets

Amusingly, and be it intentionally or not, the December 2013 issue of the *Notices* has on page 1431 an item by Doron Zeilberger on how mathematics should allegedly be, while on pages 1448-58, another item by Christopher Hollings on the vagaries of past Soviet ideology in mathematics. During their about seven decades of ideological rampages, the Soviets got to the conclusion, see (4) on top of page 1455, that: "Nevertheless, the growth of practical applications should not hinder work in abstract areas of mathematics." As for Zeilberger, he is—more than two decades after the pitiful collapse of the Soviets, who managed to go down the drain without one single bullet being fired—trying to delight us with some "radical" views of how mathematics should be, views which are, to put it mildly, incomparably more raw, primitive, and one-sided than those of the so shamefully and utterly failed and fallen Soviets. Such a strange contrast is, of course, one of the assumed individual privileges in democracy. And as those familiar with Systems Theory may know, the more complex an entity, the more its various instances may spread across a wider spectrum. And we humans are, beyond our bodies, by far the most complex entities known to us on Planet Earth. Well, Zeilberger either knows this or not, or likes it or not, but he does manage to prove how wide the spectrum is across which we do indeed happen to spread.

By the way, the Soviets, among others, fell because of considerably less raw and primitive views than those of a Zeilberger.

Democracies do not seem to fall because of types like Zeilberger.

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(Received December 9, 2013)

The Purpose of Rigor

Regarding Doron Zeilberger's opinion piece in the December 2013 issue of the *Notices*: The purpose of mathematical rigor is not so that mathematicians can feel superior to physicists, although this may be a fringe benefit. Rather, the purpose of rigor is to know what is actually true. After Cauchy "proved" in 1821 that the limit of a sequence of continuous functions is necessarily continuous, Abel showed that "this theorem admits exceptions." Many theorems in the physics literature likewise admit exceptions. These results are not meaningless or worthless, but they do challenge mathematicians to find the version that holds without exception. Rigorous mathematics is not the only kind, but it does have a valuable place.

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(Received December 10, 2013)

Reply from Zeilberger

Nothing is absolutely certain in this world, and a traditional rigorous proof gives you only the illusion of absolute certainty, since, until now, with a few exceptions (most notably the Four Color Theorem and Kepler's conjecture), such proofs were done entirely by humans, who are notoriously unreliable.

Most of the "crises" of mathematics were illusionary, assuming the fictional infinity and the "continuous" "real" numbers. They are akin to crises in religion where good guys suffer and God is not behaving as he (or she or it) should, and to millions of pages of scholastic drivel.

Traditional "rigorous proof" is yet another religious dogma, which did some good for a long time (as did the belief in God). Of course, it is not surprising that people can get deeply offended when someone denies the existence of their "God".

But the God of (alleged!) rigorous proof is dead (well, not yet, but it should be!), and we should allow diversity. Rigorous proofs should still be tolerated, but they should lose their dominance, and the A*nnals of Mathematics* should mostly accept articles with mathematics that has only semi-rigorous or non-rigorous proofs (of course, aided by our much more powerful and superior silicon brethren), because this way the horizon of mathematical knowledge (and mathematical insight!), broadly defined, would grow exponentially wider.

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(Received December 11, 2013)

Further Remarks on a Quartic Algorithm

As noted in Dan Jurca's October 2013 *Notices* letter, when approximating $\sqrt{S}$ starting from some initial guess x_0, the default form of the Bakhshali algorithm (*Notices* August 2013, page 845) is indeed less computationally efficient than simply performing two steps of the (Newton-) Heron algorithm

$$x_{n+1}=(x_n+S/x_n)/2.$$

However, a little algebra shows that a single step of the Bakhshali method can be written as

$$x_{n+1} = \frac{2(x_n^2+S)^2-(x_n^2-S)^2}{4x_n(x_n^2+S)},$$

which is still quartically convergent, but now involves only one division. Since division is the major bottleneck—taking between five and twelve times longer than multiplication on modern computer processors [1] [2]—this single-step Bakhshali method remains computationally competitive with two steps of Heron's method.

Incidentally, the Bakhshali algorithm can be derived by applying the multiplicity-corrected Newton's method to the function

$$f(x) = \frac{(x^2-S)^3}{3x^2+S}$$

or moreover to $\sqrt[3]{f(x)}$ or to $f(x)^n$ for integers $n>0$. However, just as with Heron's method, the algorithm itself came a millennium or so before Newton's method was around to provide such post-hoc "derivations" of these inspirational historical formulae.

[1] http://www.intel.com/content/www/us/en/architecture-and-technology/64-ia-32-architectures-optimization-manual.html

[2] http://developer.amd.com/resources/documentation-articles/developer-guides-manuals/

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Two-Person Fair Division of Indivisible Items: An Efficient, Envy-Free Algorithm

Steven J. Brams, D. Marc Kilgour, and Christian Klamler

The problem of fairly dividing a divisible good, such as cake or land, between two people probably goes back to the dawn of civilization. The first mention we know of in Western literature of the well-known procedure, "I cut, you choose," occurs in the Hebrew Bible, wherein Abraham and Lot divide the land that lies before them, with Abraham obtaining Canaan and Lot obtaining Jordan (Genesis 13: 5-13).

Since then, a plethora of procedures have been suggested for dividing a cake among two or more players [8], [14]. Although not all the desirable properties one might hope for can be achieved with a finite number of cuts [3], this problem pales in comparison to that of fairly allocating indivisible items.

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We thank the editor and referees for their valuable comments, which helped us to clarify and strengthen the analysis. We also thank Haris Aziz, Richard D. Potthoff, and Ariel Procaccia for helpful comments on an earlier draft. The second author was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The third author was supported by the Austrian Science Fund (FWF): P23724-G11, "Fairness and Voting in Discrete Optimation".

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In this paper we present two algorithms for the fair division of indivisible items between two players. Both assume that the players can *strictly* rank the items from best to worst, and both use *only* these rankings to make allocations. Unlike more demanding fair-division algorithms, which ask players to give more detailed information (e.g., specify their cardinal utilities for each item) or make more difficult comparisons (evaluate different bundles of items), our algorithms are easy to apply and, therefore, eminently practicable.

The first algorithm asks the two players to make simultaneous or, equivalently, independent choices in sequence, starting with their most preferred item and progressively descending to less preferred items that have not already been allocated. The second algorithm requires that the players submit their complete preference rankings in advance to a referee (or computer).

The first algorithm was proposed by Brams and Taylor [8] as a "query step" for allocating indivisible items fairly between two players, *A* and *B*. We call it BT, and it works as follows: At any point in the allocation process, if *A* and *B* name different items, BT allocates them immediately; if *A* and *B* name the same item, it goes into a "contested pile", whose items are not allocated.

The second algorithm, which we describe in the section "The BT and AL Algorithms" and call

AL, also allocates items sequentially, one to each player, based on the players' rankings. Like BT, it does not necessarily allocate all the items—some may go into a contested pile.

However, under AL the contested pile is never larger, and may be smaller, than under BT. Furthermore, if the contested piles under AL and BT contain the same number of items, each player will never strictly prefer the items it receives under BT (we henceforth use the gender-neutral "it" rather than "he" or "she" for a player).

BT and AL share the property that, when they assign an item to one player, they simultaneously assign another item to the other player. Thus, A and B are allocated equal numbers of items.

The allocations given by both BT and AL are *envy-free* (EF): A's items can be matched pairwise to B's items such that A prefers each of its items to the corresponding item of B; there is a similar pairwise matching of B's items to A's. But only AL gives EF allocations that are efficient or *Pareto-optimal* (PO): There is no other EF allocation that is at least as good for A and B and better for one or both players, based on their rankings. (If there were such an allocation, the AL allocation would be *Pareto-dominated*.)[1] Also, AL allocations are *maximal*: There is no EF allocation that allocates more items to the players.

Both BT and AL are *manipulable*: It is possible for a player to improve its allocation by ranking items insincerely (i.e., not according to its preferences). Practically speaking, however, successful manipulation of either algorithm would require that a player have essentially complete information about the preference ranking of its opponent, which is highly unlikely in most real-life situations.

In many disputes, including divorce and estate division, only an allocation in which the disputants receive about the same number of items will be perceived as fair. BT and AL work well for that purpose, especially when they allocate most, if not all, the items. While BT is a paragon of simplicity, AL is not much harder to apply, as we show in "The BT and AL Algorithms", which should facilitate its acceptance as a practicable procedure.

The paper proceeds as follows. In the next section we define envy-freeness formally, illustrate it with examples, provide a necessary and sufficient condition for an allocation to be EF, and give a condition on the players' preferences that is necessary for the existence of an EF allocation.

[1] *A BT allocation, like an AL allocation, is what we later call locally Pareto-optimal: There is no other allocation of the items that each algorithm allocates that is at least as good for A and B and better for one or both players. Because an AL allocation can allocate more or better items to one or both players, however, it may globally Pareto-dominate a BT allocation.*

In "The BT and AL Algorithms" we define and illustrate BT and AL, showing that AL generally allocates more or better items to the players than BT. Then we use AL to prove that the necessary condition of "Envy-Free Allocations" is also sufficient for the existence of an EF allocation.

In "Other Properties of EF Allocations" we prove that an AL allocation is PO and maximal, but it, like a BT allocation, may be manipulable, as we illustrate with an example.

In "The Probability of Envy-Free Allocations", we calculate the probability that an EF allocation of all the items exists when all possible rankings are equiprobable. As the number of items approaches infinity, this probability approaches 1. In the last section, "Summary and Conclusions", we summarize our results, comparing AL and BT to other fair-division algorithms, and draw several conclusions.

Envy-Free Allocations

Consider the task of dividing a set of indivisible items between two players, A and B, so that each player receives an equal number of items. For example, if the items are numbered 1 to 6, the allocation might be $\{1, 3, 5\}$ to A and $\{2, 4, 6\}$ to B. We assume that each player can strictly rank all the items from most to least preferred. Roughly speaking, an allocation is EF if each player prefers the subset of items it receives to the subset of items received by its opponent and so is not envious.

The precise definition of envy-freeness uses only the players' rankings to assess whether each player prefers its own subset of items to its opponent's subset. Denote the sets of items received by A and B by S_A and S_B, respectively. Recall that $|S_A| = |S_B|$. An allocation (S_A, S_B) is EF iff there exist an injection $f_A : S_A \to S_B$ and an injection $f_B : S_B \to S_A$ such that for each item x received by A, A prefers x to $f_A(x)$, and for each item y received by B, B prefers y to $f_B(y)$. Thus, a player pairwise prefers the items it receives in an EF allocation to the items received by its opponent.

Suppose the players' preferences for items, going from left to right, are as indicated below:

Example 1.

$$A: \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6$$
$$B: \underline{2}\ \underline{4}\ \underline{6}\ 1\ 3\ 5$$

The underscored allocation $\{1, 3, 5\}$ to A and $\{2, 4, 6\}$ to B is EF, as demonstrated by 1-1 mappings from A's items to B's, and B's items to A's, such that each player prefers each of its own items to the item of its opponent to which it is mapped. These mappings are: for A, $f_A(1) = 2$, $f_A(3) = 4$, $f_A(5) = 6$; and for B, $f_B(2) = 1$, $f_B(4) = 3$, and $f_B(6) = 5$. To simplify notation, we write these mappings as $f_A(1,3,5) = (2,4,6)$ and $f_B(2,4,6) = (1,3,5)$.

We emphasize that each player pairwise prefers its own item to the corresponding item of its opponent. For example, A receives item 3 and prefers 3 to $f_A(3) = 4$, which B receives, but A does not prefer item 3 to item 2, another item received by B. However, A prefers item 1, another item it receives, to item 2. In this example, the functions f_A and f_B are inverses, but this property is not essential and, indeed, cannot be achieved in some examples, as we will show later.

By comparison, the allocation $\{1,2,3\}$ to A and $\{4,5,6\}$ to B is not EF. This can be proven by checking exhaustively all possible injections from $\{1,2,3\}$ to $\{4,5,6\}$ and from $\{4,5,6\}$ to $\{1,2,3\}$, showing that no pair of them has the required property. But there is an easier proof, based on the following characterization:[2]

Lemma 1. *An allocation is EF iff, for each item x received by a player (say, A), the number of items received by B that A prefers to x is not greater than the number of items received by A that A prefers to x.*

Proof. To show that the given property is necessary for an allocation to be EF, suppose that A receives x in an EF allocation, and it prefers r of its own items to x, and s of B's items to x. We show that $r \geq s$. Consider the mapping f_A defined above. Suppose that, for some item y received by A, $f_A(y)$ is preferred to x. Because A prefers y to $f_A(y)$, A must also prefer y to x. It follows that each of B's s items that A prefers to x must be the image under f_A of an item received by A that A also prefers to x. There are r such items, which implies that $r \geq s$. A similar argument, beginning with an item received by B, completes the proof of necessity.

To show sufficiency, suppose that an allocation satisfies the given property. We construct a 1-1 mapping, f_A, of A's items to B's items such that A always prefers the item it receives to the corresponding item that B receives. To see that A must receive its most preferred item, x_1, assume otherwise. Then B receives at least one item that A prefers to the most preferred item it receives, whereas A receives no such items, contradicting the required property. Therefore, x_1 must have been assigned to A. Let x_k denote A's kth most preferred of the items it receives, and define $f_A(x_k)$ to be A's kth most preferred of the items B receives. Because the number of B's items that A prefers to x_k cannot exceed $k-1$, it follows that A prefers x_k to $f_A(x_k)$. The mapping f_A thus defined and the mapping f_B constructed analogously show that the allocation is EF. □

[2] *As pointed out by a referee, our characterization is related to Hall's marriage theorem* [10]. *Hall's marriage condition is stated in terms of set cardinalities, whereas ours incorporates preferences (relations on sets).*

An alternative way to state Lemma 1 is as follows: If an allocation is EF, then whenever a player receives an item x, it must also receive at least half of all the allocated items that it strictly prefers to x. Assuming that all items are allocated, if a player receives an item x that it ranked kth in its original ranking, then it must also receive at least $(k-1)/2$ items that it strictly prefers to x.

It is clear that the allocation $\{1,2,3\}$ to A and $\{4,5,6\}$ to B in Example 1 is EF for A: Because A receives its top three items, it cannot prefer any items that B receives.

But the story is different for B, as can be shown using Lemma 1. It receives item 5, which is 6th in its ranking, and prefers only two of the items it receives, 4 and 6, to item 5. The allocation cannot be EF for B, because it prefers more items in A's subset (three: 1, 2, and 3) than in its own subset (two: 4 and 6). (In general, in an EF allocation a player must receive its most preferred item, and it cannot receive its least preferred item.) Another proof can be based on the fact that B receives item 4, which it ranks 2nd, but it receives no item that it prefers to item 4 (A receives item 2, which B ranks 1st).

We can now characterize all pairs of preference rankings for which EF allocations exist. Specifically, we present Condition D below, which we will show is necessary and sufficient for the existence of an EF allocation. The proof of necessity is given below; the proof of sufficiency will be given after we describe AL, which we show in the next section always produces an EF allocation if Condition D is satisfied.

Assume there are n items that A and B rank. For an EF allocation to be possible, the number of items allocated to each player must be the same, so the total number of items allocated is even.

Before stating Condition D, we begin with a sequence of simpler conditions. We say that A's and B's rankings satisfy Condition $C(k)$ iff

Condition C*(k)*. *The set consisting of A's k most preferred items is equal to the set consisting of B's k most preferred items.*

Note that Condition $C(k)$ refers to an equality of sets: A's *ranking* of its first k items may or may not be the same as B's. What is required is that the same k items be most preferred by A and by B.

It turns out to be important whether $C(k)$ is true when k is odd. In Example 1, where $n = 6$, $C(k)$ is false for every odd k:

$k = 1$: $\{1\}$ for A is different from $\{2\}$ for B.

$k = 3$: $\{1,2,3\}$ for A is different from $\{2,4,6\}$ for B.

$k = 5$: $\{1,2,3,4,5\}$ for A is different from $\{2,4,6,1,3\}$ for B.

We can now state Condition D in terms of Condition C(k).

Condition D. *Condition C(k) fails for all odd values of k, $1 \le k \le n$.*

In other words, Condition D states that, for all odd k, at least one of A's top k items is not a top k item for B, which, as we showed above, is true in Example 1.

Note that Condition D cannot be true if n is odd, because C(k) must hold for $k = n$ (as the set of all items is the same for both players). Thus, Condition D can be true only if the number of items to be allocated is even.

An EF allocation is *complete* iff it allocates all n items. Then our first theorem gives the following characterization.

Theorem 1. *Let n be even. A pair of strict preference rankings of n items admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Necessity). To show that Condition D must hold in order for an EF allocation of all n items to exist, we show that if Condition D fails, then there can be no EF allocation. Now Condition D fails iff there is some odd value of k such that C(k) holds. Assume such a value of k, and let S be the subset consisting of A's (or B's) top k items.

Suppose that an EF allocation exists. Because S contains an odd number k of items, it follows that one of A and B, say A, must receive fewer than half of the items in S. Suppose that A receives $r < k/2$ items from S. Moreover, because each player must receive the same number of items in an EF allocation, A must receive at least one item that does not lie in S; that is, it is not among A's k most preferred items.

Let y be the item most preferred by A among the items that A receives that are not in S. If y is hth ranked by A in A's original ranking, we must have $h \ge k + 1$. Moreover, A receives exactly r items that it prefers to y. According to Lemma 1, we must have $r \ge (h - 1)/2 \ge k/2$. But, as noted above, $r < k/2$. This contradiction shows that no EF allocation can exist, establishing Condition D as necessary. □

We postpone the proof of sufficiency, which depends on the performance of AL. We describe this algorithm and BT next.

The BT and AL Algorithms

In this section we formally state the rules of BT and AL. Both algorithms allocate a set of indivisible items in a series of stages. In the case of BT, the players can be thought of as simultaneously or independently choosing the most preferred unallocated item at each stage, so the players need not give a complete ranking of items at the outset. By contrast, in the case of AL the players submit their complete rankings to a referee (or a computer), which makes choices solely on the basis of the rankings.

BT Rules

1) Players A and B name their most preferred item of those that have not yet been allocated.

2) If A and B name different items, each player receives the item it names. If they name the same item, it goes into the *contested pile* (CP).

3) If all items have been allocated to the players or put in CP, stop. Otherwise, go to step 1.

AL Rules

We begin with an informal description of AL, which also works by descending the preference rankings of the players. If the players have not yet been assigned any items, then if there is an item at the top of both players' rankings, it is put into CP, and this step is repeated until each player most prefers a different unallocated item. When this happens, AL assigns each player its preferred item.

After the first assignment of items to the players is made, new assignments are made

(i) when the players prefer different items or
(ii) when they prefer the same item, provided a new assignment—of the preferred item to one player and a less preferred item to the other—does not cause envy and so is *feasible.*

When there is a commonly preferred item, the feasibility of assigning it to either player is assessed, one player at a time. Only if there is no such assignment is the commonly preferred item put in CP.

Formally, we start AL at stage 0, which may be repeated. In each stage t ($t = 0, 1, 2, \dots$), exactly t items have already been assigned to each player. AL proceeds until there are no unallocated items.

Stage 0

Compare the most preferred unallocated items of A and B. If they are identical, place the commonly preferred item in CP and repeat stage 0. If they are different, assign each player its most preferred item. Then go to stage $t = 1$.

Stage t

1) If one unallocated item remains, place it in CP and stop. If no unallocated items remain, stop. Otherwise, compare A's and B's most preferred unallocated items. If they are the same, go to step 2. If they are different, assign each player its most preferred item and go to stage $t + 1$.

2) Determine whether the unallocated item that A and B both most prefer, say i, which we call the *tied item*, can be assigned to either A or B as follows: Let $j_{A1}, j_{A2}, \ldots$ represent, in order of A's preference, the unallocated items that A finds less preferable than i. Let $j_{B1}, j_{B2}, \ldots$ represent, in order of B's preference, the unallocated items that B finds less preferable than i.

3) Consider all possible assignments of i to B and j_{A1} first, then j_{A2}, etc., to A. Such an assignment is *feasible* as long as the number of items assigned to B or unassigned, including i, that A prefers to the *compensation item* it receives, j_{A1} or j_{A2} or ..., is at most t.[3] Stage $t + 1$ must be implemented for each feasible assignment of i to B. If the number of items assigned to B, or unassigned, that A prefers to j_{A1}, including i, is greater than t, then no assignment of i to B is feasible.

4) Consider all possible assignments of i to A and j_{B1} first, then j_{B2}, etc., to B. Such an assignment is feasible as long as the number of items assigned to A or unassigned, including i, that B prefers to its compensation item, j_{B1} or j_{B2} or ..., is at most t. Stage $t + 1$ must be implemented for each feasible assignment of i to A. If the number of items assigned to A, or unassigned, that B prefers to j_{B1}, including i, is greater than t, then no assignment of i to A is feasible.

5) If the assignment of i to A is infeasible, and the assignment of i to B is infeasible, then put i in CP. Then repeat stage t for the remaining unallocated items.

Whereas BT gives only one EF allocation, AL may give many, because for $t > 1$ there may be multiple ways to implement AL, as we will illustrate later. Although AL is more complex than BT, it is *not* so for the players, who only need to submit their rankings of items.

The chief difference between BT and AL is in how CP is defined, as we next illustrate with two examples. In each example we assume that the players are *sincere*, ranking each item according to their true preferences. Later we assume that the players may not be sincere; in particular, they may seek to manipulate BT or AL to their advantage.

Example 2.

$$A:\ 1\ 2\ 3\ 4$$
$$B:\ 2\ 3\ 4\ 1$$

When BT is applied to Example 2, A indicates that its first choice is item 1, and B that its first choice is item 2; by BT rule 2, the players receive their preferred items because they are different. At stage 2 both A and B indicate that item 3 is their preferred item of those remaining, so it goes into CP, as does item 4 at stage 3, by BT rule 2. In summary, A receives item 1, B receives item 2, and CP = $\{3, 4\}$.

Under AL the players' top-ranked items—1 for A and 2 for B—are different, so item 1 goes to A and item 2 goes to B in stage 0. Now proceed to stage $t = 1$. Of the unallocated items, both players most prefer $i = 3$, making it the tied item. For A, one unallocated item, $j_{A1} = 4$, is less preferred than 3. We consider assigning 3 to B and $j_{A1} = 4$ to A, but then B will be assigned two items, namely, 2 and 3, that A prefers to $j_{A1} = 4$, which exceeds $t = 1$, so we cannot assign 3 to B.

For B, too, the only unallocated item less preferred than $i = 3$ is $j_{B1} = 4$. We consider assigning 3 to A and $j_{B1} = 4$ to B. This assignment is feasible, because B prefers only one item to $j_{B1} = 4$ that is allocated to A (item 1). Thus, AL produces the allocation $S_A = \{1, 3\}$, $S_B = \{2, 4\}$, in which CP = $\emptyset$. Example 2 shows that AL may sometimes produce a complete allocation when BT does not.

Example 2 also shows that, under AL, the 1-1 mappings f_A and f_B—of A's items into B's and B's into A's—need not be inverse functions. In particular, the allocation given by AL is EF for A because $f_A(1, 3) = (2, 4)$, and it is EF for B because $f_B(2, 4) = (3, 1)$.[4]

Our next example shows that AL, as well as BT, may produce only partial allocations, and these allocations may differ.

Example 3.

$$A:\ 1\ 2\ 3\ 4\ 5\ 6$$
$$B:\ 2\ 3\ 5\ 4\ 1\ 6$$

When BT is applied to Example 3, A and B initially receive their most preferred items, 1 and 2, respectively. Next, because both players name item 3, it goes into CP. Then A and B receive the items they name, 4 and 5, respectively. Finally, both players name item 6, so it goes into CP. Altogether, A receives $\{1, 4\}$, B receives $\{2, 5\}$, and CP = $\{3, 6\}$. This allocation is EF, where $f_A(1, 4) = (2, 5)$ and $f_B(2, 5) = (1, 4)$ or $(4, 1)$.

Under AL, because the players' top-ranked items are different, item 1 goes to A and item 2 goes to B in stage 0. In stage 1 both players prefer $i = 3$. For A the most preferred unallocated item less preferred than $i = 3$ is $j_{A1} = 4$. But we cannot assign $i = 3$ to B and $j_{A1} = 4$ to A, because B would

[3] *The "compensation" is in lieu of not receiving the tied item i, which A prefers.*

[4] *The mappings f_A and f_B are inverses iff $f_B(f_A(x)) = x$ for all x in A's subset. When an EF allocation exists despite a common preference (e.g., for item 3 at stage $t = 1$ in Example 2), it can be shown that the mappings f_A and f_B cannot be inverses. Thus, in Example 2, $f_A(1) = 2$, so $f_B(f_A(1)) = 3 \neq 1$.*

be assigned more than one item (namely, items 2 and 3) that A prefers to $j_{A1} = 4$.

For B the first unallocated item less preferred than $i = 3$ is $j_{B1} = 5$. We can assign $i = 3$ to A and $j_{B1} = 5$ to B, because only one item assigned to A (item 3) is preferred by B to $j_{B1} = 5$. But we cannot proceed further, because after $j_{B1} = 5$, the next unallocated item in B's preference ranking is $j_{B2} = 4$. However, assigning $i = 3$ to A and $j_{B2} = 4$ to B is infeasible, because more than one item—in fact, two items, namely, 3 and 5—that B prefers to $j_{B2} = 4$ would be unallocated or assigned to A. Therefore, there is only one way to proceed to stage 2, namely by assigning items 1 and 3 to A and items 2 and 5 to B.

In stage 2, A and B both prefer item 4 and the next most preferred item 6. As already noted, in an EF allocation neither player can be assigned item 6, the common last choice. Consequently, both 4 and 6 are put in CP. In summary, AL produces exactly one allocation in Example 3: $S_A = \{1, 3\}$, $S_B = \{2, 5\}$, and CP $= \{4, 6\}$.

Example 3 illustrates another difference between BT and AL. Neither algorithm may produce a complete allocation. Each yields a CP that contains two items, one of which is item 6. In the case of BT, the other item is 3, whereas under AL it is 4. Necessarily, the AL and BT allocations also differ, with $S_A = \{1, 4\}$ under BT and $S_A = \{1, 3\}$ under AL. Note that $S_B = \{2, 5\}$ under both BT and AL.

Consider two allocations, (S_A, S_B) and (S'_A, S'_B), where all four subsets are of equal cardinality but do not necessarily contain the same items. We say that (S_A, S_B) *Pareto-dominates* (S'_A, S'_B) iff there are injections $g_A : S_A \to S'_A$ and $g_B : S_B \to S'_B$ such that A finds x at least as preferable as $g_A(x)$ for all $x \in S_A$, B finds y at least as preferable as $g_B(y)$ for all $y \in S_B$, and for at least one of x or y this preference is strict. In words, one allocation Pareto-dominates another if it is at least as good for both players and better for at least one of them, based on pairwise comparisons.

Note that the Pareto-comparison of (S_A, S_B) and (S'_A, S'_B) depends only on the assumptions that the four subsets have equal cardinality, that S_A does not overlap S_B, and that S'_A does not overlap S'_B. In particular, the sets of items allocated, $S_A \cup S_B$ and $S'_A \cup S'_B$, need not be identical, making it possible to Pareto-compare two allocations when unallocated items remain or when the CPs are different.

In Example 3, A prefers its AL allocation, $\{1, 3\}$, to its BT allocation, $\{1, 4\}$, because while both allocations contain item 1, A prefers item 3 to item 4. Here B is indifferent between its BT and AL allocations, which are both $\{2, 5\}$.

Thus the AL allocation Pareto-dominates the BT allocation in Example 3. Note also that both players agree that CP $= \{3, 6\}$, given by BT, is preferable to CP $= \{4, 6\}$, given by AL, reflecting the fact that one player (A) prefers its AL allocation to its BT allocation, while the other player (B) is indifferent.

Examples 2 and 3 illustrate the following proposition:

Theorem 2. *The number of items allocated to the players under AL is never less, and may be more, than under BT. If the number of items allocated to the players is the same under BT and AL but some items are different, then the AL allocation Pareto-dominates the BT allocation.*

Proof. A commonly preferred item i, which we called a tied item, may be assigned to a player under AL but is never assigned under BT. Thus, one or more tied items may go into CP under BT that would not under AL, so the number of items allocated under AL may be greater and will never be less than the number allocated under BT.

When a tied item is allocated under AL, the consequence may be the creation of later tied items, which would not have occurred if the tied item had been put in CP, as it would have under BT. Thus, the total number of items in CP may be the same as under BT, but ties that occur later involve less preferred items, so an AL allocation—even if it does not reduce the cardinality of CP—Pareto-dominates the corresponding BT allocation if they differ. □

Theorem 3. *An AL allocation is a maximal EF allocation: There is no other EF allocation that allocates more items to the players.*

Proof. AL continues until all items are either assigned to one player or put in CP. Hence, any EF allocation that *contains* an AL allocation must transfer some items from CP to the players. But AL puts an item, i, in CP only if it is tied and the assignment of i to either player and any less preferred item to its opponent cannot preclude the opponent from being envious. Thus, items cannot be transferred from the CP to the AL allocation. □

This is not to say that AL finds all maximal EF allocations. In Example 3, we found two maximal EF allocations—of two items to each player: one by AL and a different one by BT—but the AL allocation Pareto-dominates the BT allocation. Indeed, Theorem 2 shows that such dominance must be the case when these two allocations are the same size but not identical.

So far we have shown that, for any pair of strict preference rankings of n items:

1. an AL allocation may give each player more items than the BT allocation;

2. an AL allocation may give each player the same number of items as the BT allocation, but the sets may not be the same, in which case the AL allocation Pareto-dominates the BT allocation;
3. the AL and BT allocations may be exactly the same.

Possibility 3 occurs in Example 1, wherein both algorithms give $\{1, 3, 5\}$ to A and $\{2, 4, 6\}$ to B. It also occurs in two extreme cases: (i) when the players rank all items exactly the same (in which case all items go into CP) and (ii) when their rankings are diametrically opposed and n is even (in which case each player will obtain its more preferred half of the items and CP will be empty).

It is apparent that BT always gives an EF allocation, because it allocates items to players only when they prefer different ones at the same time. This implies that f_A and f_B are inverses. But recall that Example 2 showed that it is possible that the mappings of an AL allocation are not inverses. Examples 2 and 3 also showed that AL may give larger or more preferred EF allocations than BT.

Earlier we proved the necessity part of Theorem 1: that Condition D—for every odd k, $1 \le k \le n$, at least one of A's top k items is not a top k item of B—is necessary for the existence of an EF allocation of all n items, i.e., a complete EF allocation. We next show that Condition D is also sufficient by adding the proof of sufficiency to Theorem 1, which we repeat below.

Theorem 1 (continued). *Let n be even. A pair of strict preference rankings of n items admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Sufficiency). We show that Condition D is sufficient for the existence of a complete EF allocation by proving that AL produces a complete EF allocation unless Condition D fails. Specifically, we show that, if AL puts any item in CP, then for some odd k, the subset comprising A's k most preferred items must equal the subset comprising B's k most preferred items.

Suppose that we are applying AL to find an EF allocation. At stage 0, if A's and B's top-ranked items are the same, AL will put this item in CP. Thus, if AL puts an item in CP at stage 0, then Condition $C(k)$ must be satisfied for $k = 1$; i.e., A's and B's most preferred items are identical.

Next suppose that A's and B's top-ranked items are different and that AL has reached stage $t > 0$, so that both players have received t items without violating envy-freeness. For an item to be added to CP, it must be the case that (i) both players prefer it to all other unallocated items (i.e., it is a tied item) and (ii) the allocation of the tied item to either player will cause its opponent to be envious.

Assume the tied item is i. If it is possible to assign i to B and j_{A1}—A's most preferred unallocated item after i—to A while preserving envy-freeness, the number of items assigned to B, including i, that A prefers to j_{A1} must be at most t. If it is not possible to assign i to B and j_{A1} to A, then the number of items assigned to B that A prefers to j_{A1} must exceed t. Because only t items were assigned to each player prior to i, then the number of items assigned to B, including i, that A prefers to j_{A1} must equal exactly $t + 1$. In particular, i itself plus the items previously assigned to A or to B must be the first $2t + 1$ items in A's preference ranking.

An analogous argument can be made for B. If it is not possible to assign i to A and j_{B1} to B, then it must be the case that the subset consisting of i, the items previously assigned to A, and the items previously assigned to B must be the first $2t + 1$ items in B's preference ranking.

When the players have the same $2t + 1$ items in their preference rankings—no matter which player receives tied item i—Condition $C(k)$ holds for $k = 2t + 1$, so Condition D fails. To conclude, AL puts an item into CP when Condition D fails, which means that Condition $C(k)$ must hold for some odd k. On the other hand, when Condition D holds, AL never puts an item in CP, so a complete EF allocation must exist. □

Although Condition D is both necessary and sufficient for the existence of an EF allocation, it does not say what the EF allocation(s) are.[5] For that purpose we need AL.

As noted previously, both AL and BT always allocate to each player the same number of items, although AL may allocate more items in toto (Example 2). Therefore, the number of items allocated to CP, if it is not empty, will be even or odd depending on whether the total number of items to be allocated is even or odd. In particular, if n is odd, then CP must contain at least one item.

We showed earlier (Theorem 2) that, if AL and BT give different EF allocations to the players, then AL's allocation must include more, or more preferred, items; furthermore, it gives a maximal EF allocation (Theorem 3). We next assess how well AL and BT do according to other properties.

Other Properties of EF Allocations

We begin with an example that illustrates how AL may produce more than one complete EF allocation.

Example 4.

$$A: \ 1\,2\,3\,4\,5\,6\,7\,8$$
$$B: \ 3\,4\,5\,6\,7\,8\,1\,2$$

[5] *We postpone until the next section examples showing that AL may produce multiple EF allocations.*

In stage 0, AL assigns item 1 to A and item 3 to B. In stage 1, AL assigns item 2 to A and item 4 to B. Then, in stage 2, there is a tie on item 5. The tie cannot be resolved by assigning $i = 5$ to B, because $j_{A1} = 6$, and the assignment of items 3, 4, and 5 to B would mean that of the items that A prefers to item 6, fewer than half (i.e., only items 1 and 2) are assigned to A.

But the tie can be resolved by assigning $i = 5$ to A, in which case B can receive either $j_{B1} = 6$ or $j_{B2} = 7$. Thus stage 3 can begin with A assigned $\{1, 2, 5\}$ and B assigned $\{3, 4, 6\}$, or with A assigned $\{1, 2, 5\}$ and B assigned $\{3, 4, 7\}$. In the first case, A is assigned item 7 and B item 8 in stage 3; in the second case, A is assigned item 6 and B item 8 in stage 3. The two resulting EF allocations are underscored below:

$$\text{(i) } A: \ \underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ 6\ \underline{7}\ 8 \qquad \text{(ii) } A: \ \underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ \underline{6}\ 7\ 8$$
$$B: \ \underline{3}\ \underline{4}\ 5\ \underline{6}\ 7\ \underline{8}\ 1\ 2 \qquad B: \ \underline{3}\ \underline{4}\ 5\ 6\ \underline{7}\ \underline{8}\ 1\ 2$$

In (i) a player's minimal ranking for an item it receives is 7th (item 7 for A), whereas in (ii) this minimal ranking is 6th (item 6 for A and item 8 for B). We call (ii) the *maximin allocation*: It maximizes the minimum rank of the players, which may be desirable in certain situations.

A complete allocation is called *locally Pareto-optimal* (LPO) if there is no other allocation of the same items that Pareto-dominates it; i.e., the items cannot be redistributed between the players so that each player is at least as well off, and some player is better off, where comparisons are always pairwise. For example, if there are $n = 2$ items and A prefers item 1 to item 2 and B prefers item 2 to item 1, then the allocation of 2 to A and 1 to B is not LPO, because both players would be better off if 1 were assigned to A and 2 to B. Recall from the "The BT and AL Algorithms" section that we defined the Pareto-optimality of allocations that were not constrained by the "same items" condition.

Call an allocation *sequential* if it assigns each player its most preferred item when it is that player's turn to choose according to some sequence (e.g., *ABAB* or *AABB*). Note that the players need not alternate in a sequence, though each player must have the same number of turns to choose. The resulting allocation of items, called a *sincere sequence of choices*, clearly depends on the sequence.[6]

Theorem 4 (Brams and King, 2005). *An allocation of a fixed set of items is LPO iff it is the product of a sincere sequence of choices.*

[6] *Choices may be strategic, not sincere, if the players know each other's preferences. Backward induction can then be used to determine subgame perfect Nash equilibria using algorithms discussed in* [12], [7], [9, *chs.* 2 *and* 3], [2, *ch.* 9], *and* [13].

Strictly speaking, BT and AL are not sequential algorithms, because items are assigned to the players simultaneously. But if at some stage the players' first choices are different, then the players can be considered to receive items in either order, *AB* or *BA*, because the items received by A and B would be the same. Therefore, when A and B most prefer (and receive) different items, the assignment can be considered as part of a sincere sequence of choices.

Now suppose that, at some stage, the players' top choices are the same. Under BT, this item always goes into CP and hence will not be part of the allocation to A and B. Under AL, by comparison, this item will go into CP iff the item cannot be assigned to either player so as to maintain envy-freeness. Nonetheless, the resulting allocation will be LPO under both algorithms in the sense that no reallocation of items can Pareto-dominate what each algorithm yields, as we next prove.

Theorem 5. *Both BT and AL produce LPO allocations.*

Proof. We have already noted that the BT allocation of items that do not end up in CP is a sincere sequence of choices. To show that the same is true of an AL allocation, we need only check that it is true at any point when both players prefer the same item. Suppose that the tied item, i, is assigned to B, while some compensation item, j_{A1} or j_{A2} or ..., is assigned to A. Recall that $j_{A1}, j_{A2}, \ldots$ represent, in order of A's preference, the unallocated items that A finds less preferable than i. Clearly, an allocation in which B receives i and A receives j_{A1} is the result of a sincere choice sequence, in the order *BA*. If B receives i and A receives, say, j_{Ah} where $h > 1$, then the allocation is the result of a sincere choice sequence, $B \ldots A$, where the missing entries are determined by the eventual allocation of the unallocated items, including $j_{A1}, j_{A2}, \ldots, j_{Ah-1}$. (This may be considered an "out-of-order" assignment in that it does not make assignments strictly according to the players' preferences.)

In an AL allocation, every item that is assigned to a player who would prefer a different item among all unallocated items receives a *deferred-compensation item*, such as j_{Ah}. When all items that precede j_{Ah} in A's order have been allocated, it will be possible to identify a sincere choice sequence containing equally many A's and B's that corresponds to an AL allocation. By Theorem 4, this allocation will be LPO. □

We have shown that complete allocations under BT and AL are both EF and LPO, but partial allocations will satisfy both properties only for the items that are allocated to the players (i.e., that do not go into CP). Moreover, as Theorem 2 shows, when a BT allocation produces the same number of items as an AL allocation but the items are different, then the AL allocation will Pareto-dominate the BT allocation.

The reason that the AL allocation in Example 3 Pareto-dominates the BT allocation is that, while each algorithm allocates four of the six items to A and B, AL assigns a preferred item (3) to A and BT does not, which puts this item in CP before assigning item 4 to A. This enables A to do better under AL than it does under BT without changing the allocation to B (but changing the contents of CP).

We note that LPO allocations need not be EF. In Example 2, for instance, the allocation of $\{1, 2\}$ to A and $\{3, 4\}$ to B is LPO in that any other allocation of the four items is less preferred by A. But B might envy A (for receiving the two items that bracket its two middle items), so we call such an allocation *envy-possible* (it does not ensure envy). In contrast, allocating $\{2, 4\}$ to A and $\{1, 3\}$ to B is *envy-ensuring* [6], because it ensures that each player envies the other.

In Example 4 both EF allocations are LPO, because they can be produced by sincere sequences. A sincere sequence that produces (i) is $ABABABAB$, whereas a sincere sequence that produces (ii) is $ABABAABB$ (there are several other sincere sequences that give each allocation).

In all examples so far in which there is a complete EF allocation (Examples 1, 2, and 4), A and B rank all the items differently (they also do so in Example 3 for the four items that do not go into CP). By contrast, if they ranked all items the same, there would be no EF allocation, because all items would go into CP.

It seems plausible, therefore, that different rankings of the items by the players might be a sufficient condition for there to be a complete EF allocation. However, this conjecture fails for

Example 5.

$$A:\ 1\ 2\ 3\ 4\ 5\ 6$$
$$B:\ 2\ 3\ 1\ 5\ 6\ 4$$

Because the top $k = 3$ items $\{1, 2, 3\}$ are the same for both A and B, Condition C(3) holds. Therefore, Condition D fails, so by Theorem 1 there can be no complete EF allocation.

The fact that Condition D fails in Example 5 does not tell us what *partial* EF allocation is possible. For this purpose, we need to apply AL.

In stage 0, AL assigns item 1 to A and item 2 to B. In stage 1, there is a tie on item 3. It cannot be resolved by assigning $i = 3$ to either A or B. Therefore, we must put item 3 into CP, after which AL allocates item 4 to A and item 5 to B. The remaining unallocated item, 6, must then go into CP. In summary, $A = \{1, 4\}$, $B = \{2, 5\}$, and $\text{CP} = \{3, 6\}$. Coincidentally, BT produces the same EF allocation.

Our next example shows that AL can give exponentially many EF allocations, all of which are complete and maximin (unlike Example 4).

Example 6.

$$A:\ 1\ 2\ 3\ 4$$
$$B:\ 4\ 2\ 3\ 1$$

It is easy to see that AL produces two allocations: $A = \{1, 2\}$ and $B = \{3, 4\}$, and $A = \{1, 3\}$ and $B = \{2, 4\}$. Now add four more items for which the players' preferences copy those of Example 6.

Example 6′.

$$A:\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$
$$B:\ 4\ 2\ 3\ 1\ 8\ 6\ 7\ 5$$

AL allocates the first four items in two ways, as before, and then allocates the second four items in two ways. Thus, there are $2 \times 2 = 4$ different EF allocations in Example 6′: $S_A = \{1, 2, 5, 6\}$, $S_B = \{3, 4, 7, 8\}$; $S_A = \{1, 2, 5, 7\}$, $S_B = \{3, 4, 6, 8\}$; $S_A = \{1, 3, 5, 6\}$, $S_B = \{2, 4, 7, 8\}$; and $S_A = \{1, 3, 5, 7\}$, $S_B = \{2, 4, 6, 8\}$.

Adding an additional four items in a similar way produces eight different EF allocations, and this doubling pattern continues. Examples of this family contain n items to be allocated; AL produces $2^{n/4}$ distinct EF allocations, all of which are complete and maximin. It follows that the number of EF allocations can grow exponentially in n, so no polynomial-time algorithm will find all EF allocations in this family.[7]

But finding just one EF allocation can be done in polynomial time by checking at every stage in which there is a tied item at most two possible

[7] *The rate of growth of the number of complete EF allocations in the family based on Example 6 is not maximal. For example, it is not difficult to show that there are six distinct complete and maximin EF allocations for the following 8-item example:*

$$A:\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$
$$B:\ 7\ 8\ 3\ 4\ 5\ 6\ 1\ 2$$

which exceeds the four distinct EF allocations of the 8-item example in the text. Copying preferences in the manner discussed in the text yields an exponent of approximately 0.323n in this example, compared to 0.25n in the example in the text. We recently discovered that Bouveret, Endriss, and Lang [1]*, using a different methodology (SCI-nets), analyze algorithms for finding EF and LPO allocations and describe their computational complexity. Some of our findings echo theirs (e.g., on "necessary envy-freeness"), but others (e.g., our Condition D and our results on maximality and manipulability) do not.*

assignments: (i) the tied item is assigned to A, with B getting its next-best item; and (ii) the tied item is assigned to B, with A getting its next-best item. If both (i) and (ii) fail, the tied item goes into CP.

We need to check only the next-best item of the player not getting the tied item because, if (i) and (ii) fail, then no lower-ranked item will give an EF allocation. Thus failure can be confirmed by testing two allocations at every stage.

To conclude, AL is an exponential-time algorithm if one wishes to generate all EF allocations. But if one EF allocation suffices, with the algorithm terminating at a stage as soon as one assignment (either to one of the players or to CP) has been found, it is polynomial time, making it applicable to the division of large numbers of items.

Up to now we have assumed that the players rank items sincerely.[8] Call an algorithm *manipulable* if a player, by submitting an insincere preference ranking, can obtain a preferred allocation.

Theorem 6. *AL and BT are manipulable.*

Proof. We begin with AL, for which there are two EF allocations in Example 7:

Example 7.

$$\text{(i) } \begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ \underline{4}\ 5\ 6 \\ B: & \underline{2}\ \underline{6}\ 4\ \underline{5}\ 3\ 1 \end{array} \qquad \text{(ii) } \begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6 \\ B: & \underline{2}\ \underline{6}\ \underline{4}\ 5\ 3\ 1 \end{array}$$

Allocation (i) is maximin (the lowest rank of a player is 4th), whereas allocation (ii) is not (the lowest rank of a player is 5th). BT gives only a partial EF allocation—$A = \{1, 3\}$, $B = \{2, 6\}$, and $\text{CP} = \{4, 5\}$—which presumably will be unsatisfactory for the players compared to one of the two complete AL allocations.

Now assume that, instead of reporting its sincere preferences in Example 7, B reports its preferences to be B'—interchanging items 4 and 6—whereas A continues to be sincere. This yields the following unique AL allocation:

Example 7′ (manipulated by B).

$$\begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6 \\ B': & \underline{2}\ \underline{4}\ \underline{6}\ 5\ 3\ 1 \end{array}$$

Thereby B obtains its top three items, whereas without manipulation B's allocation of these items was only one of two possibilities—and not the maximin one (had this property been used to choose between the two AL allocations without manipulation). BT gives exactly the same result, so B's misrepresentation helps it under BT, compared with obtaining only its top two items when it is sincere. □

[8] *The implications of insincere behavior are studied in* [15]. *Variations on the rules for making fair allocations, such as accepting or rejecting one or more items in a round, are analyzed in* [16].

We conclude that both AL and BT are manipulable if one player (B in Example 7) knows its adversary's (A's) sincere ranking and exploits its knowledge. But such manipulation seems improbable, short of A's having complete information about B's ranking of items, and A's being in the dark about the possibility of B's misrepresentation. Furthermore, the determination of an optimal misrepresentation strategy, especially when the number of items is large, is far from trivial, particularly in the case of AL because of its greater complexity. It is further complicated if there is a random selection from multiple EF allocations.

In the face of these difficulties, we think that A and B, especially when using AL, are likely to be sincere in submitting preference rankings to a referee. This presumption is reinforced by the fact that, if the players are sincere, they can ensure themselves of an EF, LPO, and maximal allocation, though it may not be complete.

The Probability of Envy-Free Allocations

There are many pairs of preference rankings for which there is no complete EF allocation. This is certainly true if both players rank all items the same, but it is also true if both players agree only on their top-ranked item, because whoever does not obtain that item will envy the other player. Similarly, no complete EF allocation is possible if the two players rank only their last-choice item the same, because whoever obtains it may envy the other.

On the other hand, if a complete EF allocation exists, it need not be unique, as we showed with several examples. To calculate the probability of a complete EF allocation, fix A's preference ranking as 1 2 3 . . . and assume all preference rankings of B are equiprobable. If $n = 2$ items and A's ranking is 1 2, then B's ranking can be 1 2 or 2 1. In the former case, there will be envy if A receives item 1 and B receives item 2, whereas in the latter there will not be envy, so the probability that an EF allocation exists is $\frac{1}{2}$.

If $n = 4$, then B can have any of $4! = 24$ preference rankings. To calculate the probability of a complete EF allocation, we note that Condition D requires that (i) the first choices of A and B be different and (ii) the first three choices of A and B be different.

Let us instead count the number of ways that Condition D can fail. For (i) to fail, B's first choice must be 1, for which there are $3! = 6$ orderings. For (ii) to fail, B's fourth choice must be 4, of which there are $3! = 6$ orderings. But Condition D fails if either (i) fails or (ii) fails, and both may fail simultaneously. However, we have double-counted the cases in which both (i) and (ii) fail, which requires that B's first choice be 1 and B's fourth

Even Number of Items n	2	4	6	8	10	12
Probability of Complete EF Allocation	0.500	0.583	0.678	0.750	0.800	0.834

choice be 4, for which there are 2 orderings. We conclude that Condition D can fail in $6+6-2=10$ ways. Thus, there are $24-10=14$ preference rankings for B for which Condition D holds.

We have shown that, when there are $n=4$ items, $14/24 \approx 0.583$ of the possible allocations admit a complete EF allocation, which can be extended to other values of n (see table above).[9] For even values of n, the probability that an EF allocation exists is on the order of $(n-2)/n$, so it tends to 1 as n approaches infinity. To see this, note that the probability that $C(k)$ holds—that a randomly chosen permutation of $\{1,\ldots,n\}$ fixes the subset $\{1,\ldots,k\}$—is $k!(n-k)!/n!$. Condition D fails iff $C(k)$ holds for at least one odd k. Therefore, the probability that $C(k)$ fails cannot exceed the sum of these probabilities over odd k from 1 to $n-1$. The terms $k=1$ and $k=n-1$ are each $1/n$, and the other terms are $0(1/n^3)$, so this sum is $2/n - 0(1/n^2)$.[10]

Summary and Conclusions

Given that two players can rank a set of indivisible items from best to worst, the main algorithm we have analyzed (AL) finds an allocation giving the players the same number of items that is EF, PO, and maximal—and complete if such an allocation exists. A simpler algorithm (BT), which is also EF and LPO, may allocate fewer preferred items to the players and so may not be maximal or, if it is maximal, will be Pareto-dominated by an AL allocation if the BT allocation is different.

A possible advantage of BT, besides its simplicity, is that the players can make sequential decisions: they can decide, based on the items they have already acquired, which of the remaining items to try to obtain next. By contrast, AL requires that the players rank all items in advance, so if the players' valuations are interdependent (i.e., the acquisition of one item affects the value of others), they cannot take advantage of possible synergies among the items. This suggests the importance of packaging individual items into subsets whose elements are complementary (e.g., matching sofas instead of two individual sofas) so that the packages are as independent as possible.

Because AL and BT are manipulable, players can sometimes do better by misrepresenting their preferences. But without complete information about an opponent's preferences, BT, and especially AL (because of its greater complexity), would be difficult to exploit. Indeed, trying but failing to do so could result in an allocation that is neither EF nor PO. Thus players would seem to have good reason to be sincere in using these algorithms.

At least one, but not necessarily all, allocations produced by AL will be maximin. This seems to be an important property to ensure balanced allocations—one player does not suffer because it receives an especially low-ranked item.

There may be many complete maximin EF allocations. If they are all known, one could be selected at random. But, to avoid algorithms that require exponential time, it might be preferable to stop AL at the first EF allocation (if any) that it finds to ensure that it can be implemented in polynomial time.

If all possible preference rankings of players are equiprobable, then the probability that a complete EF allocation exists increases rapidly with the number of items and approaches 1 as this number approaches infinity. But equiprobability is not a realistic assumption in many real-life situations, wherein the players' preferences are correlated. How the degree of correlation affects the proportion of items that are allocated to the players—versus those that go into CP—remains to be investigated.

In order to allocate the items in CP for which the players have identical rankings, Brams, Kilgour, and Klamler [5] developed an algorithm called the *undercut procedure*, whereby a player proposes a "minimal bundle" of items to keep for itself. Its opponent can either accept the complementary subset or undercut the minimal bundle by one item, which becomes the division that is implemented.[11] The allocations it produces are EF. Combined with AL, however, it can be used to allocate *all* the items, including those that AL puts into CP.

Alternative two-person procedures—including *adjusted winner* [8], [9], in which players assign points to items, and a *swapping procedure* in which players can make trades after an initial allocation [4]—produce fair divisions that satisfy other desiderata.[12] However, both procedures

[9] *We thank Richard D. Potthoff for assistance with this calculation.*

[10] *We thank a referee for this proof of convergence to a limit probability of 1.*

[11] *The extent to which this procedure is vulnerable to strategic manipulation is analyzed in* [15], [16].

[12] *One desideratum is equitability, in which players perceive that they receive the same fraction of the total value. Procedures for finding equitable as well as envy-free allocations of indivisible items are analyzed in* [11]. *Unlike BT and AL, they require that players specify preferred bundles of items, which makes them more akin to the undercut procedure for allocating the items in CP.*

require that the players provide more information than a simple ranking of items and, in the case of adjusted winner, that one item, which is not identifiable in advance, be divisible.

The fact that only AL requires that the players indicate their preference rankings is clearly an advantage, but in some applications it may be desirable to elicit and use information about the intensity of the players' preferences. But when obtaining such information is difficult, AL offers a compelling alternative—for example, in allocating the marital property in a divorce or the items in an estate, especially when the players have different tastes (e.g., for memorabilia or artworks).

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JUMP Math: Multiplying Potential

John Mighton

When people complain about problems in American education, they often speak as if those problems would be solved if students in the U.S. were able to perform as well on international tests of reading and mathematics as students from countries that achieve the highest scores. Nations like Finland and Singapore are singled out in the media as having superior educational systems because their students do better on tests like PISA and TIMMS.

It's worth looking at the results of these tests closely, but more for what they reveal about our beliefs about children and their potential than for what the tests prove about education. From the way people talk about the tests, you can see clearly what they expect the average child to achieve at school.

In 2006 only 10 percent of American students scored above level 5 in mathematics on the PISA tests (this is the level of proficiency required to take courses involving math at university), compared to 30 percent in top-performing countries such as Taiwan, Hong Kong, South Korea, and Finland. However, in each of the top performing countries, roughly 40 percent of students scored at level 3 or below. Students at level 3 would have trouble holding a job that required fairly basic mathematics.

Many people have suggested that American educators should find out how math is taught in the top-performing countries so it can be taught in the same way in the U.S. I expect this is a good idea, but we might also want to find out how countries that produce such strong students still manage to teach so little to almost half their populations. Answering this question might do as much to help the U.S. improve the teaching of mathematics as any efforts to emulate the educational practices of other countries.

Wide differences in mathematical achievement among students appear to be natural: in every school in every country only a minority of students are ever expected to excel at or love learning mathematics. In the many schools I have visited on several continents, I've always seen a significant number of students who are two or three grade levels behind by grade five. In my home province of Ontario, where children do rather well on international tests, only 58 percent of grade-six students met grade-level standards on the provincial exams last year.

Fourteen years ago I started a charity called JUMP Math in my apartment because I wanted to help students who struggle in math. The first JUMP students were referred by local schools and were matched with volunteer tutors. Most of these students had serious learning disabilities and were years behind in math, so I believed that the best way to help them was to provide them with one-on-one instruction. But JUMP soon outgrew my apartment, and teachers in schools where it was offered began to ask me to teach some lessons in their classrooms. In my first lessons I was surprised to see that the weakest students often became more engaged in the classroom than they did in tutorials—they loved putting up their hands and coming up to the board when the lesson was taught in a way that they could understand.

In designing lessons that would work for the whole class, I had to learn to break explanations and challenges into small steps so students who were initially weaker could experience success, to provide adequate review and practice for those who needed it, and to raise the level of difficulty incrementally so children would get more excited and their brains would work efficiently. I soon began to design special "bonus" questions that didn't introduce any new skills or vocabulary so faster students could independently explore small variations on the concepts they had learned while I spent time with students who needed extra help. As weaker students became more confident and attentive, they began to work much more quickly so they could get their bonus questions too. Their excitement at succeeding in front of their peers seemed to greatly increase their rate of learning.

It was clear that teachers didn't have time to develop lessons of this type, so JUMP hired a team of mathematicians and educators to help me write online teachers' guides that cover the full curriculum from grades one to eight in great detail.

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JUMP is now used by about 100,000 students in Canada and the U.S. as their main resource for mathematics for grades one to eight. In the U.S. many school boards are piloting versions of our materials that are aligned to the Common Core State Standards.

In a randomized controlled study presented at the Society for Research in Child Development in 2011, cognitive scientists Tracy Solomon and Rosemary Tannock from the Hospital for Sick Children and the University of Toronto found that students from eighteen regular classrooms using JUMP showed twice the rate of progress on a number of standardized tests of math ability as students receiving standard instruction in eleven other classrooms. As randomized controlled studies rarely show such striking differences between students in different math programs, the U.S. Department of Education has funded a much larger multiyear study by the same team.

Based on my observations of thousands of students and on data gathered in studies of JUMP (see `jumpmath.org` for a summary of these studies), I am convinced that the vast majority of students have far more potential to learn and enjoy learning math than they exhibit at school. To fully appreciate the extent of this hidden potential and of the losses that we incur as a society when we fail to nurture this potential, it helps to consider a case study.

In the fall of 2007 fifth-grade teacher Mary Jane Moreau of Mabin School in Toronto gave her students a standardized math assessment called the Test of Mathematical Abilities (TOMA). The class average was in the 54th percentile, with a wide range of scores, including one student who ranked as high as the 75th percentile and another at just the 9th percentile. A fifth of the pupils in the class were identified as learning disabled. After testing her students, Mary Jane abandoned her usual teaching approach (which meant pulling together lessons with the best materials she could find) and followed the JUMP lesson plans with fidelity. After a year of JUMP, the average score of her students on the grade-six TOMA rose to the 98th percentile, with the lowest mark in the 95th percentile. At the end of grade six, Mary Jane's entire class signed up for the Pythagoras Math competition, a prestigious contest for sixth-graders. One of the most able students was absent on the day of the exam, but of the seventeen who participated, fourteen received awards of distinction (with the other three close behind). Students who write the Pythagoras competition are almost all in the top five percentile in achievement, but the average score for students in this (initially unremarkable) class was higher than the average for students writing the Pythagoras.

The most challenged ten-year-old student in Mary Jane's class improved her score on the TOMA from the 9th percentile to the 95th percentile after only one year of JUMP. But ten-year-old brains are more developed and less plastic than four-year-old brains, so grade five is not the ideal grade for an intervention. It seems reasonable to assume that Mary Jane's student could have achieved much more in grade five if she had been enrolled in a math program as good as or better than JUMP from an early age. Indeed, if every child were taught according to their true potential from the first day of school, then I would predict that by grade five the vast majority of students (over 95 percent) could learn and love learning as much as the top one or two percent do now.

I should point out that this is not a prediction about JUMP, as it requires that children be taught "according to their true potential." JUMP has produced some extremely strong results in pilots and studies, but the program may not, in its present form, produce the results I think are possible. JUMP has partnered with many distinguished cognitive scientists and educational researchers to try to determine what works in our approach and what needs to be improved. Better programs than JUMP will certainly be developed, and JUMP itself will continue to evolve. I hope that readers will not allow any doubts they have about JUMP in its present form to distract them from considering what may be possible for children in the future.

In the randomized controlled study, teachers used JUMP with varying degrees of fidelity but still managed to double the average rate of progress of their students. I expect the results of the study would have been stronger if every teacher had followed the program with fidelity. But even if I am wrong about how effective JUMP can be when it is implemented properly, my beliefs about what children can achieve are likely to be true, as they are well supported by independent evidence from cognitive science. One day this evidence will be more widely known, and educators will be inspired to set higher expectations for students and schools, whether or not they use particular programs such as JUMP.

The methods on which JUMP is based are ones that cognitive scientists are now promoting for the development of expertise in general. In "The expert mind", an article that appeared in *Scientific American* in July 2006, Philip Ross examines the implications of a century of research on how experts develop abilities in chess and other fields and how the expert mind processes and receives information. His conclusions lend strong support to the notion that abilities can be nurtured in students through rigorous instruction and practice:

> The preponderance of psychological evidence indicates that experts are made, not

born. What is more, the demonstrated ability to turn a child quickly into an expert—in chess, music and a host of other subjects—sets a clear challenge before the schools. Can educators find ways to encourage students to engage in the effortful study that will improve their reading and math skills? Instead of perpetually pondering the question "Why can't Johnny read?" perhaps educators should ask: "Is there anything in the world he can't learn to do?"

H. Wu has warned against drawing false dichotomies in math education (for instance, between concepts and deep understanding versus procedures and algorithms). One dichotomy is particularly damaging to students: the false opposition between "explicit" or "direct instruction" versus "discovery" or "student-centered" instruction. Current research in cognitive science suggests that effective lessons should combine elements of both approaches. In 2011 A. Alfieri et al. conducted a meta-analysis of 164 studies of discovery-based learning and concluded that "unassisted discovery does not benefit learners," whereas discovery combined with "feedback, worked examples, scaffolding and elicited explanations do[es]." An effective lesson can be student-centered but still led by the teacher.

Research in cognitive science suggests that, while it is important to teach to the strengths of the brain (by allowing students to explore and discover concepts on their own), it is also important to take account of the weaknesses of the brain. Our brains are easily overwhelmed by too much new information, we have limited working memories, we need practice to consolidate skills and concepts, and we learn bigger concepts by first mastering smaller component concepts and skills.

Teachers are often criticized for low test scores and failing schools, but I believe that they are not primarily to blame for these problems. For decades teachers have been required to use textbooks and teaching materials that have not been evaluated in rigorous studies. As well, they have been encouraged to follow many practices that cognitive scientists have now shown are counterproductive. For example, teachers will often select textbooks that are dense with illustrations or concrete materials that have appealing features because they think these materials will make math more relevant or interesting to students. But psychologists such as Jennifer Kaminski have shown that the extraneous information and details in these teaching tools can actually impede learning.

To improve their practice, teachers must be made aware of the growing body of research in cognitive science that shows that higher-level abilities are grounded in practice and the acquisition of basic skills and knowledge and that overly complex lessons can overwhelm the brain. They must be allowed to innovate and test methods that are supported by solid research, and they must never be compelled to adopt programs that have not been rigorously evaluated.

The JUMP method is called "guided discovery". In a JUMP lesson students develop and explore ideas on their own, but the lesson is a carefully scaffolded series of questions and challenges in which one idea naturally leads to the next. Students are provided with many supports of the kind that research has identified as effective, such as immediate feedback and worked examples. They are also given many opportunities to practice and consolidate concepts and are assessed frequently so they can get excited about their success and so the teacher can be sure no one is falling behind.

Some lessons in JUMP allow for more open-ended exploration, but here is an example of a structured lesson on long division. I have found that this approach enables kids to both discover the steps of the algorithm and understand the underlying concepts while learning to perform the algorithm proficiently.

I tell students that the notation $3\overline{)72}$ can be interpreted to mean: 3 friends wish to share 7 dimes and 2 pennies (72 cents) as equally as possible. I then ask students to draw a picture to show how they would divide the dimes among the friends. If students use a circle for each friend and an X for each dime, the diagram would look like this:

(X X) (X X) (X X) X

I ask students to tell me the meaning of their diagram: Each friend gets two dimes and there is one dime left over. I then tell students that if they happened to see someone carrying out the first few steps of the long division algorithm, this is what they would see:

$$\begin{array}{r} 2 \\ 3\overline{)72} \\ \underline{-6} \\ 1 \end{array}$$

I challenge students to figure out what the steps in the algorithm mean by identifying where they see each number in their diagram. Students readily make the following connections between their diagram and the algorithm:

$$\begin{array}{rl} 2 & \Leftarrow \text{ each friend got two dimes} \\ 3\overline{)72} & \\ \underline{-6} & \Leftarrow \text{ 6 dimes were given away altogether} \\ 1 & \Leftarrow \text{ there was 1 dime left over} \end{array}$$

I ask students to complete their diagram to show me how much money still has to be divided among the friends. If students use a circle to represent a penny, their diagram looks like this:

(XX) (XX) (XX) XOO $\Leftarrow$ 1 dime and 2 pennies haven't been given out yet

I invite three students to come to the front of the class so I can demonstrate how I would divide the remaining coins among the three friends. I give two students a penny each and one student a dime. The students always protest that my way of dividing up the coins isn't fair: they tell me they would exchange the dime for ten pennies and divide the twelve pennies among the friends. I inform students that this process of "regrouping" the tens (dimes) as ones (pennies) is actually a step in the long division algorithm. Most adults call this the "bring down" step, but very few understand it:

$$\begin{array}{r} 2 \\ 3\overline{)72} \\ -6 \\ \hline 12 \end{array} \quad \Leftarrow \text{ when you "bring down"}$$

…the number in the ones (pennies) column, you implicitly change the number in the tens (dimes) column into the smaller unit (pennies). Then you combine all of your smaller units (to give twelve pennies altogether).

I then ask students to show me in their diagrams how they would divide the (twelve) remaining pennies among the friends. I also ask them to connect the numbers in their diagram with the remaining steps of the algorithm:

(XXOOOO) (XXOOOO) (XXOOOO)

$$\begin{array}{rl} 24 & \Leftarrow \text{ each friend received four pennies (24 cents altogether)} \\ 3\overline{)72} & \\ -6 & \\ \hline 12 & \\ -12 & \Leftarrow \text{ twelve pennies were given out altogether} \\ \hline 0 & \Leftarrow \text{ no pennies were left over} \end{array}$$

At each step in this process I give students several practice questions so I can verify that they understood the step.

Mary Jane loved teaching math and was recognized as an excellent teacher before she started using JUMP. But after reading the JUMP Teachers' Guides, she said she realized that many of the concepts she had previously taught in one step actually involved two or three steps or required skills or knowledge that she didn't normally assess or teach. She found that the more closely she followed the guides the better her students did.

Research has shown that many elementary teachers (unlike Mary Jane) are mathphobic or have very rudimentary knowledge of math. The JUMP writers and I wrote the guides, in part, because we saw that schools could not afford to provide enough professional development for teachers to make up for these deficits. In following the online lesson plans, teachers learn the math as they teach. Many have become excited about their new understanding of the subject and have formed volunteer networks to support and mentor other teachers. Two mathphobic teachers in a Vancouver network recently completed master's degrees in math education after they were inspired by their success with the program.

The principles on which JUMP lessons are built (adequate review and practice, rigorous scaffolding, continuous assessment, incrementally harder challenges, and differentiated instruction) are not new or even controversial in education, although we have tried to apply these principles with a great deal of rigor. If there is anything different about JUMP, it may lie in the belief that extreme hierarchies of ability are caused, at least in part, by the presumption that these hierarchies are natural.

Children are unlikely to fulfill their potential in math until math programs are designed to take into account the way academic hierarchies can inhibit learning. As early as grade one, children begin to compare themselves to their peers and identify themselves as "smart" or "dumb" in subjects such as math. When children decide they aren't talented in math, their brains work less efficiently: they stop paying attention, taking risks, and persevering in the face of difficulty, and they often develop anxieties or behavioral problems. By making all of her students feel capable from the first day of school, Mary Jane was able to produce a class of students who were, to a surprising degree, equally capable.

No method of teaching is likely to produce a school full of students who all have exactly the same capacity for success, but the results of teachers like Mary Jane suggest that students have far more potential in math than they exhibit at school. To bring about significant change in education, we must insist that every child has a right to fulfill their intellectual potential, just as they have a right to develop healthy bodies. We don't have to wait until we have recruited an army of superhuman teachers or invented some miraculous new technology to guarantee this right. We already have the teachers we need to transform our schools. We simply need to give them the means to teach children using effective methods that are backed by rigorous evidence.

A Tale of Ellipsoids in Potential Theory

Dmitry Khavinson and Erik Lundberg

Dirichlet's Problem

Let us start our story with the Dirichlet problem. This problem of finding a harmonic function in a, say, smoothly bounded domain $\Omega \subset \mathbb{R}^n$ matching a given continuous function f on $\partial\Omega$ gained huge attention in the second half of the nineteenth century due to its central role in Riemann's proof of the existence of a conformal map of any simply connected domain onto the disk. Later on, Riemann's proof was criticized by Weierstrass, and, after considerable turmoil, it was corrected and completed by Hilbert and Fredholm; cf. [27] for a very nice historical account. Here we want to focus on algebraic properties of solutions to the Dirichlet problem when Ω is an ellipsoid and the data f possess nice algebraic properties. Thus, we first present the following proposition.

Proposition 1. *Consider the ellipsoid*

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1 \leq 0 \right\},$$

where $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. The solution u to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = p, \end{cases} \tag{1}$$

where p is a polynomial of n variables, is a harmonic polynomial. Moreover,

$$\deg u \leq \deg p. \tag{2}$$

Remark 1. Proposition 1 was widely known in the nineteenth century for $n = 2, 3$ (perhaps due to Lamé) and was proved with the use of ellipsoidal harmonics. It is still widely known nowadays for balls but often disbelieved for ellipsoids. The first author has won a substantial number of bottles of cheap wine betting on its truthfulness at various math events and then producing the following proof that was related to him by Harold S. Shapiro. The idea of the proof goes back at least to Fischer [11]; we do not know who thought of it first, but we hope the reader will agree that this proof deserves to be called, following P. Erdős, the "proof from the book."

Proof. Denote by $P_{n,m} = P_m$ the finite-dimensional space of polynomials of degree less than or equal to m in n variables. Let $q(x) = \sum \frac{x_j^2}{a_j^2} - 1$ be the defining quadratic for $\partial\Omega$. Consider the linear operator $T : P_m \to P_m$ defined by

$$T(r) := \Delta(qr).$$

The maximum principle yields at once that $\ker T = 0$, so T is injective. Since $\dim P_m < \infty$, this implies that T is surjective.

Hence, given $P \in P_m$ with $m \geq 2$, we can find a polynomial $r \in P_{m-2}$ such that $Tr = \Delta P$. The function

$$u = P - qr$$

is then the solution of (1). □

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Proposition 1 was extended [20] to the case of entire data. Namely, entire data f (i.e., an entire function of variables $x_1, x_2, \ldots, x_n$) yields an entire solution to the Dirichlet problem in ellipsoids. This result was sharpened by Armitage in [1], who showed that the solution's order and type are dominated by that of the data.

One might get bold at this point and ask if Proposition 1 extends to, say, rational or algebraic data; i.e., does a smooth data function in (1) that is a rational (algebraic) function of $x_1, x_2, \ldots, x_n$ imply rational (algebraic) solution u? The answer is a resounding "no", but the proofs become technically more involved; see [3].

The Dirichlet Problem, Ellipsoids, and Bergman Orthogonal Polynomials

It was conjectured in [20] that Proposition 1 (without the degree condition (2)) characterizes ellipsoids. Recently, using "real Fischer spaces", H. Render confirmed this conjecture for many algebraic surfaces [28]. In two dimensions, the conjecture was confirmed under a degree-related condition on the solution in terms of the data [21]. This utilized a surprising equivalence, established by M. Putinar and N. Stylianopoulos [26], of the conjecture to the existence of finite-term recurrence relations for Bergman orthogonal polynomials. In order to state the degree conditions and the associated recurrence conditions, assume that Ω is a domain in $\mathbb{R}^2$ with C^2-smooth boundary. Let $\{p_m(z)\}$ be the Bergman orthogonal polynomials (orthogonal with respect to area measure over Ω). These are analytic polynomials of the complex variable z. Consider the following properties for Ω.

(a) There exists C such that for a polynomial data of degree m there always exists a polynomial solution of the Dirichlet problem posed on Ω of degree less than or equal to $m + C$.

(b) There exists N such that for all k, m, the solution of the Dirichlet problem with data $\bar{z}^k z^m$ is a harmonic polynomial of degree $\le (N-1)k + m$ in z and of degree less than or equal to $(N-1)m + k$ in $\bar{z}$.

(c) There exists N such that $\{p_m\}$ satisfy a (finite) $(N+1)$-recurrence relation; i.e., there are constants $a_{m-j,m}$ such that

$$zp_m = a_{m+1,m}p_{m+1} + a_{m,m}p_m + \cdots + a_{m-N+1,m}p_{m-N+1}.$$

(d) The Bergman orthogonal polynomials of Ω satisfy a finite-term recurrence relation; i.e., for every fixed $\ell > 0$, there exists an $N(\ell) > 0$, such that $\langle zp_m, p_\ell \rangle = 0$, $m \ge N(\ell)$.

(e) For any polynomial data there exists a polynomial solution of the Dirichlet problem posed on Ω.

Properties (d) and (e) are essentially equivalent [26], and (a) $\Rightarrow$ (b), (b) $\Leftrightarrow$ (c), and (c) $\Rightarrow$ (d). In [21] the authors used ratio asymptotics of orthogonal polynomials to show that (b) and equivalently (c) each characterize ellipses. The weaker statement that (a) characterizes ellipsoids was proved in arbitrary dimensions [22]. For more about the Khavinson-Shapiro conjecture stated in [20], we refer the reader to [21], [17], [22], [26], [28], and the references therein.

The Mean Value Property for Harmonic Functions

The mean value property for harmonic functions can be rephrased as saying that *the average of any harmonic function over concentric balls is a constant*. As we formulate precisely below, there is a mean value property for ellipsoids which says *the average of any harmonic function over confocal ellipsoids is a constant*.

Consider a heterogeneous ellipsoid

$$\Gamma := \left\{ x \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{a_j^2} - 1 = 0 \right\},$$

where $a_1 > a_2 > \cdots > a_N > 0$, and let Ω be its interior.

Definition. A family of ellipsoids $\{\Gamma_\lambda\}$,

$$\Gamma_\lambda = \left\{ x \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{a_j^2 + \lambda} - 1 = 0 \right\},$$

where $-a_N^2 < \lambda < +\infty$, is called a confocal family (for $N = 2$ these are ellipses with the same foci).

Note that the shapes of confocal ellipsoids differ; as $\lambda \to \infty$, Γ_λ looks like a sphere, and when $\lambda \to -a_N^2$,

$$\Gamma_\lambda \to \left\{ x \in \mathbb{R}^N : x_N = 0, \sum_{j=1}^{N-1} \frac{x_j^2}{a_j^2 - a_N^2} - 1 \le 0 \right\} =: E.$$

E is called the *focal ellipsoid*.

The following classical theorem goes back to Maclaurin, who considered prolate spheroids in $\mathbb{R}^3$ ($a_1 > a_2 = a_3$). General ellipsoids were treated later by Laplace [23, Chapter 2].

Theorem 1. *Let u be an entire harmonic function. Then*

$$\frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} u(x)dx = const. \tag{3}$$

for all $\lambda : \lambda > -a_N^2$.

From now on, for the sake of brevity, we shall only consider the case $N \ge 3$.

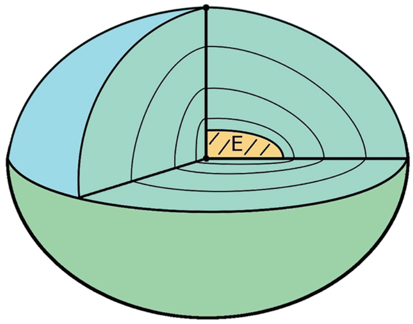

Figure 1. The mean value over confocal ellipsoids is constant.

Remark 2. Maclaurin's theorem is a corollary (via a simple change of variables; see [5, Chapter VI, Section 16] or [17, Chapter 13]) of the following result of Ásgeirsson: *Suppose* $u = u(x, y)$, *where* $x \in \mathbb{R}^{m_1}$, $y \in \mathbb{R}^{m_2}$ *satisfy the ultrahyperbolic equation*

$$\Delta_x u = \Delta_y u.$$

Then, if $\mu_i(x, y, r)$, $i = 1, 2$, *denote, respectively, the mean values of* u *over* m_i*-dimensional balls of radius* r *centered at* (x, y), *we have* $\mu_1(x, y, r) = \mu_2(x, y, r)$.

Here we offer a purely algebraic approach to Maclaurin's theorem [17, Chapter 13]. The following notions are due to E. Fischer [11] (see also [31, Chapter IV]). Let H_k be the space of homogeneous polynomials of degree k. If $f \in H_k$, then (using the standard multi-index notation of L. Schwartz [1], [31])

$$f(z) = \sum_{|\alpha|=k} f_\alpha z^\alpha.$$

Introduce an inner product on H_k (called the Fischer inner product), by letting

$$\left\langle z^\alpha, z^\beta \right\rangle = \begin{cases} 0, & \alpha \neq \beta, \\ \alpha!, & \alpha = \beta. \end{cases} \tag{4}$$

If $f = \sum_{|\alpha|=k} f_\alpha z^\alpha$, $g = \sum_{|\alpha|=k} g_\alpha z^\alpha$, then $\langle f, g \rangle = \sum_{|\alpha|=k} a! f_\alpha \overline{g_\alpha}$. With respect to the Fischer inner product, the operators $\left(\frac{\partial}{\partial z}\right)^\alpha$ and multiplication by z^α are adjoint. Also, it follows from the definition (4) that $\frac{1}{m!}\left(z \cdot \bar{\xi}\right)^m$ is a reproducing kernel for H_m; i.e., for all $f \in H_m$,

$$\frac{1}{m!}\left\langle f, \left(z \cdot \bar{\xi}\right)^m \right\rangle = f(\xi). \tag{5}$$

Indeed, it is enough to check this for monomials, and for all multi-indices α, $|\alpha| = m$, we see that

$$\frac{1}{m!}\left\langle z^\alpha, \left(z \cdot \bar{\xi}\right)^m \right\rangle = \frac{1}{m!}\left\langle z^\alpha, \sum_{|\beta|=m} \frac{m!}{\beta!} z^\beta \bar{\xi}^\beta \right\rangle$$
$$= \frac{1}{\alpha!}\left\langle z^\alpha, z^\alpha \bar{\xi}^\alpha \right\rangle = \xi^\alpha.$$

Let $\mathcal{H}_m \subset H_m$ denote the space of harmonic polynomials of degree m.

Lemma 1. *The linear span of harmonic polynomials* $\left(z \cdot \bar{\xi}\right)^m$ *for all* $\xi \in \Gamma_0 = \left\{\xi \in \mathbb{C}^N : \sum_{j=1}^N \xi_j^2 = 0\right\}$ (*the isotropic cone*) *equals* $\mathcal{H}_m$.

Proof of Lemma 1. Let us assume for the sake of contradiction that there is a nonzero polynomial $u \in \mathcal{H}_m$ satisfying

$$\left\langle u, \left(z \cdot \bar{\xi}\right)^m \right\rangle = 0, \quad \forall \xi \in \Gamma_0.$$

Using the reproducing kernel condition (5), we have $u(\xi) = 0$ for all $\xi \in \Gamma_0$. By Hilbert's Nullstellensatz

$$u(\xi) = \left(\sum_{j=1}^N \xi_j^2\right) q(\xi), \quad \text{for some } q \in H_{m-2}.$$

But then, since u is harmonic, we have

$$0 = \langle \Delta u, q \rangle = \left\langle u, \left(\sum_{j=1}^N \xi_j^2\right) q \right\rangle = \langle u, u \rangle,$$

where we have used the fact that multiplication and differentiation are adjoint. Hence, $u \equiv 0$. ☐

Proof of Maclaurin's theorem. It suffices to check (3) for harmonic homogeneous polynomials, and in view of Lemma 1, we just have to check it for polynomials

$$\left(z \cdot \bar{\xi}\right)^m, \quad \xi \in \Gamma_0.$$

Fix λ. Let $b_i = \left(a_i^2 + \lambda\right)^{1/2}$ be the semiaxes of Ω_λ. We have to show that

$$\frac{1}{|\Omega|}\int_\Omega \left(x \cdot \bar{\xi}\right)^m dx = \frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \left(x \cdot \bar{\xi}\right)^m dx, \quad \forall \xi \in \Gamma_0.$$

Changing variables in both integrals $x_k = a_k y_k$, $x_k = b_k y_k$, we see that it suffices to show the following:

$$\int_B \left(\sum_{k=1}^N a_k y_k \overline{\xi_k}\right)^m dy = \int_B \left(\sum_{k=1}^N b_k y_k \overline{\xi_k}\right)^m dy, \tag{6}$$

where B is the unit ball in $\mathbb{R}^N$. Since $\xi \in \Gamma_0$ implies that

$$\sum_{k=1}^N \left((a_k \bar{\xi}_k)^2 - (b_k \bar{\xi}_k)^2\right) = -\lambda^2 \overline{\sum_{k=1}^N \xi_k^2} = 0,$$

verifying (6) reduces to checking the following assertion.

Assertion. The polynomial

$$P(t) := \int_B \left(\sum_{k=1}^N x_k t_k\right)^m dx$$

depends only on $\sum_{k=1}^{N} t_k^2$, for $t \in \mathbb{C}^N$.

The assertion follows from the rotation invariance of P [17, Chapter 13]. □

The following application is noteworthy. Let Ω be an ellipsoid with semiaxes $a_1 > a_2 > \cdots > a_N > 0$, and let

$$u_\Omega(x) := C_N \int_\Omega \frac{dy}{|x-y|^{N-2}}, \quad x \in \mathbb{R}^N \setminus \Omega$$

be the exterior potential of Ω.

As above, E denotes the focal ellipsoid. The following corollary of Maclaurin's theorem describes a so-called *mother body* [14], i.e., a measure supported inside the ellipsoid which generates the same gravitational potential (outside the ellipsoid) as the uniform density but is *minimally supported* in some sense (see the discussion in [14]). In this case the mother body is supported on E, a set of codimension one with connected complement.

Corollary 1. *For* $x \in \mathbb{R}^N \setminus \bar{\Omega}$

$$u_\Omega(x) = C_N \int_E \frac{d\mu(y)}{|x-y|^{N-2}},$$

where

$$d\mu(y) = 2\left(\prod_{j=1}^{N} a_j\right)\left(\prod_{j=1}^{N-1}\left(a_j^2 - a_N^2\right)\right)^{-1/2} \times \left(1 - \sum_{j=1}^{N-1} \frac{y_j^2}{a_j^2 - a_N^2}\right)^{1/2} dy' \Bigg|_E$$

(dy' *is Lebesgue measure on* $\{y_N = 0\}$).

Sketch of proof. Since the integrand is harmonic, we have by MacLaurin's theorem

$$u_\Omega(x) = \frac{\prod_{j=1}^{N} a_j}{\prod_{j=1}^{N}\left(a_j^2 + \lambda\right)^{1/2}} \int_{\Omega_\lambda} \frac{C_N}{|x-y|^{N-2}}\, dy.$$

After simplifying this integral using Fubini's theorem, the corollary is established by applying the Lebesgue dominated convergence theorem as $\lambda \to -a_N^2$ [17, Chapter 13]. □

We note in passing that finding relevant mother bodies for oblate and prolate spheroids (supported on a disk and segment, respectively) could be a satisfying exercise.

Since the density of the distribution $d\mu$ is real analytic in the interior of E (viewed as a set in $\mathbb{R}^{N-1}$), we note the following corollary:

Corollary 2. *The potential* $u_\Omega(x)$ *extends as a (multivalued) harmonic function into* $\mathbb{R}^N \setminus \partial E$.

An extension of this fact and a "high ground" view of the mother body, based on holomorphic PDE in $\mathbb{C}^n$, is discussed in the section "The Cauchy Problem: A View from $\mathbb{C}^n$".

The Equilibrium Potential of an Ellipsoid. Ivory's Theorem

Considering that force is the gradient of potential, the following theorem, due to Newton, can be paraphrased in a rather catchy way: "there is no gravity in the cavity".

Theorem 2 (Newton's theorem)**.** *Let* $t > 1$*, and consider the ellipsoidal shell* $S := t\Omega \setminus \Omega$ *between two homothetic ellipsoids. The potential* U_S *of uniform density on* S *is constant inside the cavity* Ω*.*

In fact, ellipsoids are characterized by this property; i.e., Newton's theorem has a converse [7], [8], [24], [17]. A modern approach to Newton's theorem and far-reaching generalizations due to V. I. Arnold and A. Givental are sketched in the epilogue.

A consequence of Newton's theorem is that the gravitational potential U_Ω of Ω is a quadratic polynomial inside Ω. Namely,

$$U_\Omega(x) = B - \sum_{i=1}^{N} A_j x_j^2, \quad \text{for } x \in \Omega,$$

with $B = C_N \int_\Omega \frac{dV(y)}{|y|^{N-2}} = U_\Omega(0)$, where $C_N = \frac{1}{\mathrm{Vol}(S^{N-1})}$. Indeed, denoting by $\Omega_t = t\Omega$ (for $t > 1$) the dilated ellipsoid, one computes that its gravitational potential is $u_t(x) = t^2 u(x/t)$. Since Newton's theorem implies that (where u is the potential of the original ellipsoid) $u_t - u = const$ inside Ω, the smaller ellipsoid, then taking partial derivatives ∂^α, with respect to x, $|\alpha| = 2$, yields that $\partial^\alpha u_t(x) = \partial^\alpha u(x/t) = \partial^\alpha u(x)$. Thus all these partial derivatives are homogeneous of degree zero inside Ω. They are also obviously continuous and, hence, are constants, thus yielding U_Ω to be a quadratic as claimed.

Denoting $\Gamma := \partial\Omega$, consider the single layer potential

$$V(x) = C_N \int_\Gamma \frac{\rho(y)}{|x-y|^{N-2}} dA(y),$$

where $\rho(y)$ is a mass density and $dA(y)$ on Γ is the surface area measure. Also, $V(x)$ is called an *equilibrium potential* if $V(x) \equiv 1$ on Γ and hence inside Ω. We again focus on the case $N \geq 3$. The quantity

$$\sigma := \lim_{|x|\to\infty} |x|^{N-2} V(x) = C_N \int_\Gamma \rho(y) dA(y)$$

is called capacity.

On the way to proving Ivory's theorem, we note an explicit formula for the equilibrium potential.

Corollary 3. *With* B *as above, in* $\mathbb{R}^N \setminus \overline{\Omega}$*, we have*

$$V(x) = \frac{1}{B}\left(\hat{\mu} - \frac{1}{2}\sum_{i=1}^{N} x_i \frac{\partial \hat{\mu}}{\partial x_i}\right), \tag{7}$$

where $\hat{\mu}(x) = C_N \int_E \frac{d\mu(y')}{|x-y'|^{N-2}}$, $y' = (y_1, y_2, \ldots, y_{N-1}, 0)$*, and* $d\mu(y')$ *is the MacLaurin quadrature measure supported on the focal ellipsoid* E *(cf. Corollary 1).*

Proof. Thus the right-hand side of (7) is harmonic in $\mathbb{R}^N \setminus \overline{\Omega}$ (in fact, in $\mathbb{R}^N \setminus E$) since $\hat{\mu}$ is harmonic there and $\Delta(x \cdot \nabla\hat{\mu}) = n\Delta\hat{\mu} = 0$. On Γ, by Maclaurin's theorem and Newton's theorem,

$$\hat{\mu} = U_\Omega(x) = B - \sum_{i=1}^{N} A_j x_j^2. \tag{8}$$

Moreover, since $U_\Omega(x)$ has continuous first derivatives throughout $\mathbb{R}^N$, we can differentiate (8) on Γ and thus obtain

$$\frac{1}{B}\left(\hat{\mu} - \frac{1}{2}\sum_{i=1}^{N} x_i \frac{\partial\hat{\mu}}{\partial x_i}\right) = \frac{1}{B}\left(B - \sum_{i=1}^{N} A_j x_j^2 + \frac{1}{2}\sum_{i=1}^{N} 2A_j x_j^2\right) = 1.$$

Thus, the right-hand side of (7) equals $V(x)$ on Γ. Both functions are harmonic in $\mathbb{R}^N \setminus \overline{\Omega}$ and vanish at infinity, and the statement follows. □

Corollary 4 (Ivory's theorem)**.** *The equipotential surfaces of the equilibrium potential* $V(x)$ *are confocal with* Γ*.*

For the proof, one simply notes that the right-hand side of (7) changes only by a constant factor when Ω is replaced by a confocal ellipsoid

$$\Omega_\lambda := \left\{ x : \sum^{N} \frac{x_j^2}{a_j^2 + \lambda} \le 1,\ \lambda \ge 0 \right\}.$$

Namely, $B \to B_\lambda$ while $\frac{d\mu_\lambda}{d\mu} = \frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(\Omega)}$.

For the classical proof of Ivory's theorem, see [23], [12, Lecture 30].

Ellipsoids in Fluid Dynamics

Let us pause for a moment to mention applications of these properties of ellipsoids to two problems in fluid dynamics. In the first problem, involving a slowly moving interface, viscosity plays an important role. In the second problem, viscosity is completely neglected, while vorticity plays the dominant role.

Moving Interfaces and Richardson's Theorem

Imagine a blob of incompressible viscous fluid within a porous medium surrounded by an inviscid fluid. Suppose there is a sink at position x_0 in the region Ω_t occupied by viscous fluid, so Ω_t is shrinking with time. Darcy's law governs the fluid velocity v in terms of the pressure P:

$$v = -\nabla P. \tag{9}$$

Incompressibility implies that

$$\nabla \cdot v = -\Delta P = 0$$

except at the sink x_0. The pressure of the inviscid fluid is assumed constant. Neglecting surface tension (a rather controversial assumption), the pressure matches at the interface, which gives a constant (say, zero) boundary condition for P, so P is nothing more than the harmonic Green's function with a singularity at x_0. The mathematical problem is then to track the evolution of a domain Ω_t whose boundary velocity is determined by the gradient of its own Green's function. See [32] for an engaging exposition of the two-dimensional case of this problem.

Given a harmonic function $u(x)$, Richardson's theorem [29] describes the time dependence of the integration of u over the domain occupied by the viscous fluid. In the language of integrable systems this represents "infinitely many conservation laws".

Theorem 3 (S. Richardson, 1972)**.** *Let* $u(x)$ *be a function harmonic in* Ω_t *for all* t*. Then*

$$\frac{d}{dt}\int_{\Omega_t} u(x)dV(x) = -Qu(x_0), \tag{10}$$

where x_0 *is the position of the sink with pumping rate* $Q > 0$*.*

An alternative setup places the viscous fluid in an unbounded domain with a single sink at infinity [7]; a reformulation of Richardson's theorem implies that the potential inside the cavity of the shell regions $\Omega_t \setminus \Omega_{s>t}$ is constant. Thus it is a consequence of Newton's theorem (Theorem 2) that an increasing family of homothetic ellipsoids is an exact solution. In fact, this is the only solution starting from a bounded inviscid fluid domain that exists for all time and fills the entire space [7].

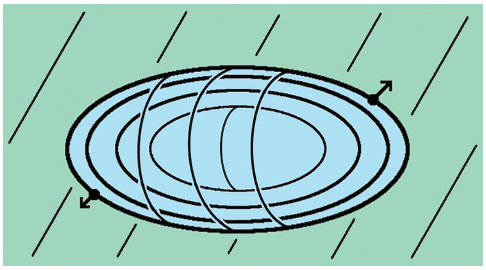

Figure 2. Viscous fluid occupies the exterior. The ellipsoid grows homothetically.

Returning to the case when the viscous fluid is bounded, suppose the initial domain Ω_0 is an ellipsoid and consider the problem of determining sinks and pumping rates such that $\{\Omega_t\}_{t=0}^{T}$ shrinks to zero volume as $t \to T$. As a consequence of the mean value property, one can solve this problem exactly, thus removing all of the fluid, that is, provided we can stretch our imaginations to allow a continuum of sinks (spread over the focal set E). Starting from the given ellipsoid Ω_0, the evolution Ω_t is a family of ellipsoids confocal to Ω_0 shrinking down to the (zero-volume) focal set E. The pumping rate is given by the time-derivative of the quadrature measure appearing in Corollary 1.

The Quasigeostrophic Ellipsoidal Vortex Model

Based on the observation that motion in the atmosphere is roughly stratified into horizontal layers, the quasigeostrophic approximation provides a simplified version of the Euler equations (governing inviscid incompressible flow). Further assumptions reduce the entire dynamics to a scalar field, the potential vorticity, which in the high Reynolds number limit forms coherent regions of uniform density. Even with these simplifications, the problem can still be quite complicated. For instance, approximating the regions of potential vorticity by clouds of point-vortices, one encounters the notoriously difficult n-body problem.

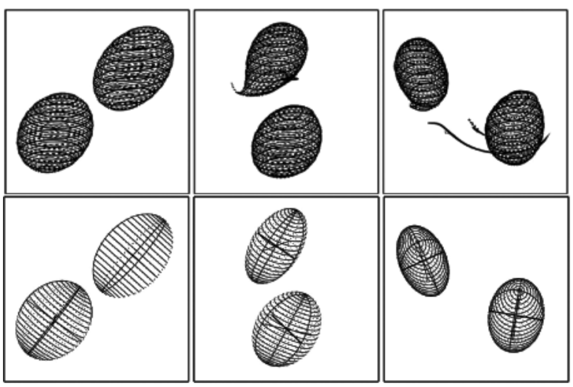

Figure 3. Top row: A vortex simulation using "contour dynamics". Bottom row: A faster, but still accurate, simulation using the ellipsoidal vortex model.

The *quasigeostrophic ellipsoidal vortex model* developed by Dritschel, Reinaud, and McKiver [9] simulates the interaction of ellipsoidal regions of vorticity (see Figure 3, included here with the kind permission of Dritschel, Reinaud, and McKiver). As these regions interact, the length and alignment of semiaxes can change, but nonellipsoidal deformations are filtered out. (Note that a single ellipsoidal vortex is stable for a certain range of axis ratios.) The effect that one ellipsoid has on another is determined by its exterior potential, and thus the mean value property can be used to replace the ellipsoid by a two-dimensional set of potential vorticity on its focal ellipse (with density determined by Corollary 1) which can be further approximated by point vortices.

Remark 3. It is interesting to single out the two-dimensional case of the moving interface problem

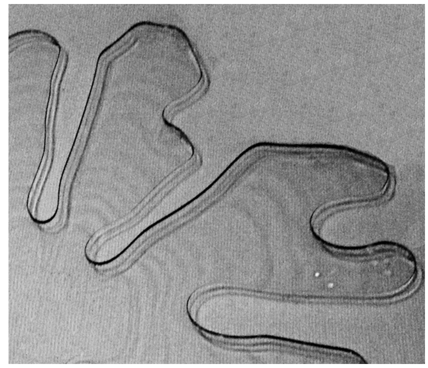

Figure 4. Viscous fingering in a Hele-Shaw cell.

which serves as a model for viscous fingering in a Hele-Shaw cell.[1] Conformal mapping techniques lead to explicit exact solutions that can even exhibit the tip-splitting depicted in Figure 4. The vortex dynamics problem also admits many sophisticated analytic solutions in the two-dimensional case [6]. For a compelling survey discussing *quadrature domains* as a common thread linking these and several other fluid dynamics problems, see [6].

The Cauchy Problem: A View from $\mathbb{C}^n$

The problem mentioned in the section "The Mean Value Property for Harmonic Functions" of analytically continuing the exterior potential U_Ω inside the region Ω occupied by mass was studied by Herglotz [15] and can be reformulated as studying the singularities of the solution to the following Cauchy problem posed on the initial surface $\Gamma := \partial\Omega$:

$$\begin{cases} \Delta M = 1, & \text{near } \Gamma, \\ M \equiv_\Gamma 0, \end{cases} \tag{11}$$

where the notation $M \equiv_\Gamma G$ indicates that M along with its gradient coincide with G and its gradient, respectively, on Γ.

The fact that M carries the same singularities in Ω as the analytic continuation u of U_Ω is a consequence of the fact that u itself is given by the piecewise function

$$u := \begin{cases} U_\Omega, & \text{outside } \Omega, \\ U_\Omega - M, & \text{inside } \Omega. \end{cases} \tag{12}$$

The reason is that u is harmonic on both sides of Γ and is C^1-smooth across Γ. (Note that, inside Ω, U_Ω denotes the physical gravitational potential which solves a Poisson equation $\Delta U_\Omega = 1$.) An extension

[1] *A Hele-Shaw cell is a lab apparatus consisting of two closely spaced sheets of glass with a small hole in the top piece; after filling the gap with viscous fluid, one may inject a bubble of less viscous fluid. This experiment is cheap and easy to perform—in fact the photograph in Figure 4 was taken in the second author's home.*

of Morera's theorem (attributed to S. Kovalevskaya) implies that u is actually harmonic across Γ, i.e., Γ is a removable singularity set for u. Thus u is the desired analytic continuation of U_Ω across Γ, and the singularities of u in Ω are carried by M.

Further reformulating the problem, note that the so-called *Schwarz potential* of Γ, $W = \frac{1}{2}|x|^2 - M$, has the same singularities as M and solves a Cauchy problem for Laplace's equation:

$$(13) \qquad \begin{cases} \Delta W = 0 \quad \text{near } \Gamma, \\ W \equiv_\Gamma \frac{1}{2}|x|^2. \end{cases}$$

This is a rather delicate (ill-posed according to Hadamard) problem, and our discussion of it will pass from $\mathbb{R}^n$ to the complex domain $\mathbb{C}^n$. Let us first consider a more intuitive Cauchy problem for a hyperbolic equation where similar behavior can be observed while staying in the real domain. Explicitly, consider

$$(14) \qquad \begin{cases} v_{xy} = 1 \quad \text{near } \gamma, \\ v \equiv_\gamma 0, \end{cases}$$

where γ is, say, a real analytic curve in $\mathbb{R}^2$.

For hyperbolic equations the mantra is "singularities propagate along characteristics." If the solution is singular at some point (x_0, y_0), then one can trace the source of this singularity back to γ by following the characteristic cone with vertex at (x_0, y_0). One expects to find a singularity in the data itself at a point where this cone intersects γ, but what if the data function has no singularities as in (14)? It is still possible for a singularity to propagate to the point (x_0, y_0) if the characteristic cone from (x_0, y_0) is tangent to γ. The point of tangency is called a *characteristic point* of γ.

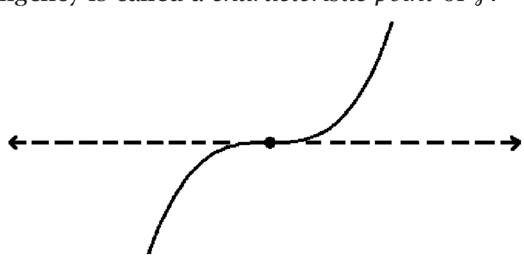

Figure 5. The solution to (14) is regular except on the tangent characteristic $\{y = 0\}$.

For example, suppose $\gamma := \{y = x^3\}$. We can solve (14) exactly:

$$v(x, y) = x \cdot y - \frac{x^4}{4} - \frac{3}{4}y^{4/3}.$$

The solution is singular on the characteristic $\{y = 0\}$ which is tangent to the initial curve γ at the point $(0, 0)$; see Figure 5.

The singularities in the solution of (13) also propagate along tangent characteristics. The important difference is that the characteristic points (the "birth places" of singularities) reside on the *complexification* of Γ, the complex hypersurface given by the same defining equation.

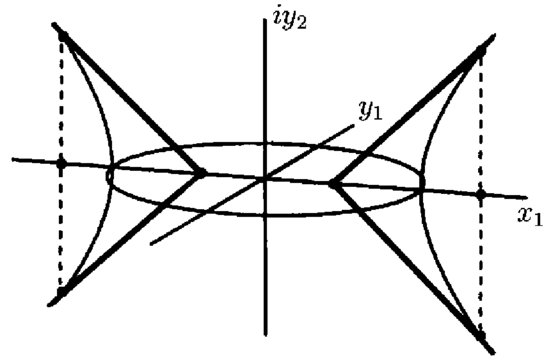

Figure 6. The characteristic lines tangent to Γ at four characteristic points intersect $\mathbb{R}^2$ precisely at the foci.

For ellipsoids, these ideas can be made precise. Namely, the following result, due to G. Johnsson [16], was proved using a globalization of *Leray's principle*, a local theory governing propagation of singularities.

Theorem 4 (G. Johnsson, [16]). *All solutions of the Cauchy problem* (13) *with entire data f on $\Gamma := \left\{ z \in \mathbb{C}^n : \sum_1^n z_1^2/a_i^2 = 1 \right\}$ extend holomorphically along all paths in $\mathbb{C}^n$ that avoid the characteristic surface Σ (consisting of all characteristic lines tangent to Γ).*

The intersection $\Sigma \cap \mathbb{R}^n = E$ is the focal ellipsoid that was discussed in previous sections. This provides, according to the properties of the Schwartz potential discussed above, a $\mathbb{C}^n$-explanation of a rather physical fact that E supports a measure solving an inverse potential problem. As Johnsson notes, there is an unexpected coincidence between potential-theoretic foci (points where singularities of W are located) and algebraic foci in the classical sense of Plücker [16]. Understanding this correspondence and extending it to higher-degree algebraic surfaces is part of a program advocated by the first author and H. S. Shapiro. The case $n = 2$ is more transparent, but for $n > 2$ it is virtually unexplored.

Epilogue

Newton's theorem can be reformulated in terms of a single layer potential obtained by shrinking a constant-density ellipsoidal shell to zero thickness (while rescaling the constant), leading to a nonconstant density $\rho(x) = 1/|\nabla q(x)|$, where $q(x)$ is the defining quadratic of the ellipsoid. This is sometimes called the *standard single layer potential* (it is different from the equilibrium potential discussed in the section "The Equilibrium Potential of an

Ellipsoid. Ivory's Theorem"). The modern approach due to V. I. Arnold and, then, A. Givental [2], [13], views the force at x_0 induced by infinitesimal charges at two points x_1, x_2 on a line ℓ through x_0 as a sum of residues for a contour integral in the complex extension L of ℓ. The vanishing of force then follows from deforming the contour to infinity. The detailed proof can be found in [17, Chapter 14].

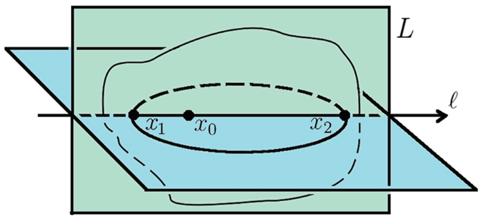

Figure 7. The force from two points is realized as a sum of residues in the complex line *L*.

The same proof can be used to extend Newton's theorem beyond ellipsoids to any *domain of hyperbolicity* of a smooth, irreducible real algebraic variety Γ of degree k. A domain Ω is called a domain of hyperbolicity for Γ if for any $x_0 \in \Omega$, each line ℓ passing through x_0 intersects Γ at precisely k points. For example, the interior of an ellipsoid is a domain of hyperbolicity, and if a hypersurface of degree $2k$ consists of an increasing family of k ovaloids, then the smallest one is the domain of hyperbolicity.

Defining the standard single layer density on Γ in exactly the same way as before, except that the sign $+$ or $-$ is assigned on each connected component of Γ depending on whether the number of obstructions for "viewing" this component from the domain of hyperbolicity of Γ is even or odd, the Arnold-Givental generalization of Newton's theorem implies, in particular, that the force due to the standard layer density vanishes inside the domain of hyperbolicity (cf. [2], [13] for more general statements and proofs).

As a final remark, returning to ellipsoids, and even taking $n = 2$, let us note an application to gravitational lensing of Corollary 1. The two-dimensional version of Maclaurin's theorem plays a key role in formulating analytic descriptions for the gravitational lensing effect for certain elliptically symmetric lensing galaxies [10], [4] (cf. [19], [25] for terminology). Here the projected mass density that is constant on confocal ellipses produces at most four lensed images [10]. The density that is constant on homothetic ellipses produces at most six images [4]. In connection to the converse to Newton's theorem, whenever the rare focusing effect in gravitational lensing produces a continuous "halo" (a.k.a. Einstein ring; cf. [19] for some striking NASA pictures) around the lensing galaxy (of any shape), the "halo" necessarily turns out to be either a circle or an ellipse [10]. But this alley leads to the beginning of another story.

Note: Due to considerations of space, the reference list has been shortened. The more complete list of references is available in [18].

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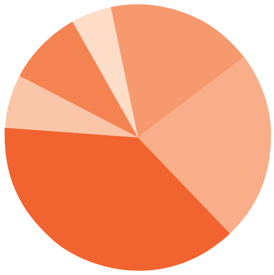


Fall 2012 Departmental Profile Report

Richard Cleary, James W. Maxwell, and Colleen Rose

This report presents a profile of mathematical sciences departments at four-year colleges and universities in the United States, as of fall 2012. The information presented includes the number of faculty in various categories, undergraduate and graduate course enrollments, number of bachelor's and master's degrees awarded during the preceding year, and the number of graduate students.

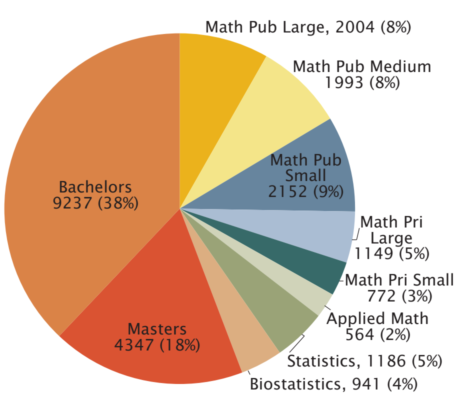
Data collected earlier from these departments on recruitment and hiring and faculty salaries were presented in the Report on 2011-2012 Academic Recrutiment and Hiring (pages 586–591 of the May 2013 issue of *Notices* of the AMS) and the 2011–2012 Faculty Salaries Report (pages 426–432 of the April 2013 issue of *Notices* of the AMS).

Detailed information, including tables which traditionally appeared in this report, is available on the AMS website at `www.ams.org/annual-survey/survey-reports.`

Faculty Size*

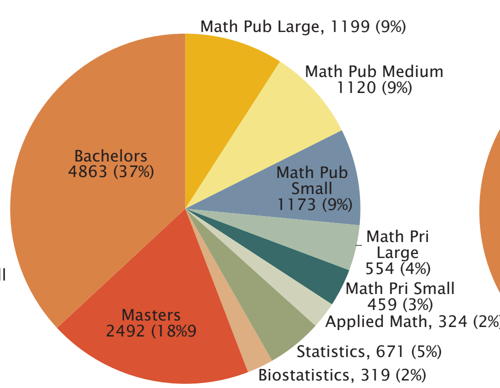
All groups reported an increase in the number of faculty for fall 2012. The estimated number of full-time faculty in all departments is 24,346 with 22,219 of these in all mathematics departments combined (Math Public, Math Private, Applied Math, Masters & Bachelors), up 1% from 22,039 last year. Full-time faculty among the doctoral mathematics departments combined (Math Public, Math Private & Applied Math) increased slightly to 8,634 from 8,528 last year. In the mathematics departments combined we estimate the number of nondoctoral full-time faculty is 3,692, down 2% from last year's estimate of 3,750. With a standard error of 85 for our 2013 estimate, this difference may be explained by sampling error. The total part-time faculty in all mathematics departments combined is estimated to be 6,907 (with a standard error of 181), up 8% from 6,419 last year.

Figure F.1: All Full-time Faculty by Department Groupings



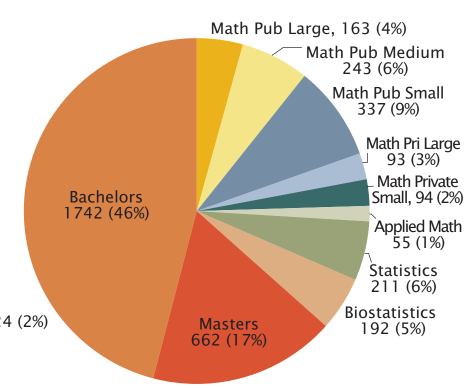
Total: 24,346

Figure F.2: Full-time Tenured Doctoral Faculty



Total: 13,173

Figure F.3: Full-time Untenured, Tenure-track Doctoral Faculty



Total: 3,791

* All 2011 figures referenced on this page were adjusted to reflect the new departmental groupings introduced for 2012 (see page 166).

Richard Cleary is a professor in the Division of Mathematics and Sciences at Babson College. James W. Maxwell is AMS associate executive director for special projects. Colleen A. Rose is AMS survey analyst.

Doctoral Faculty*

The estimated number of full-time doctoral faculty in all mathematics departments combined (Math Public, Math Private, Applied Math, Masters & Bachelors) is 18,527 (with a standard error of 174), up slightly from last year's number of 18,289. For these same groups combined, total doctoral tenured faculty remained essentially unchanged at 12,183 compared to 12,196 for fall 2011. 35% (4,863) of all doctoral tenured faculty are in Bachelors departments.

Figure D.1: Gender of Full-time Doctoral Faculty
Total: 20,551

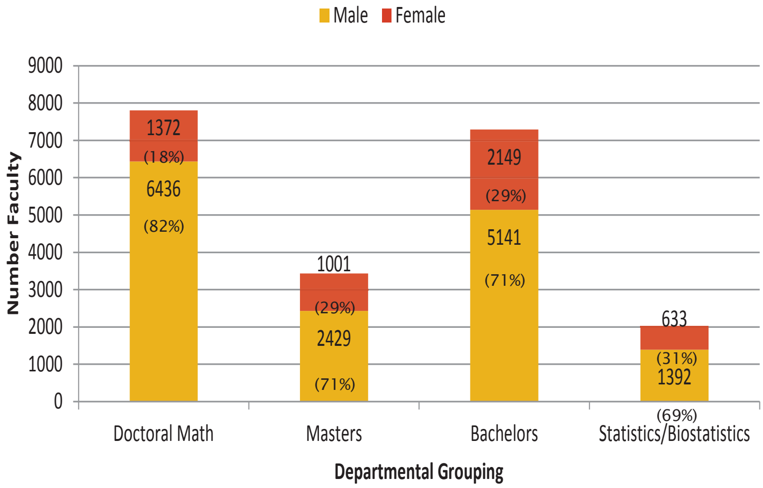
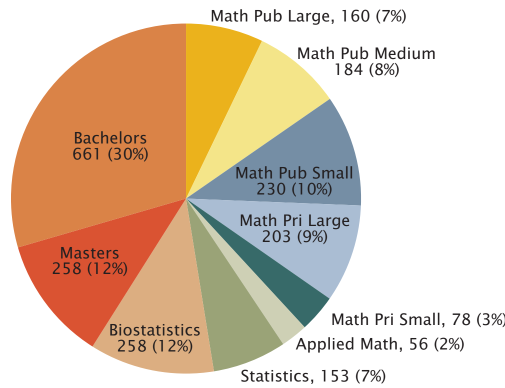


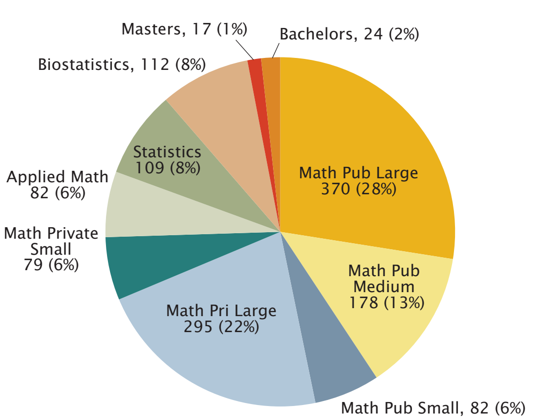
Figure D.2: Non-tenure-track Doctoral Faculty (excluding Postdocs)



Total: 2,241

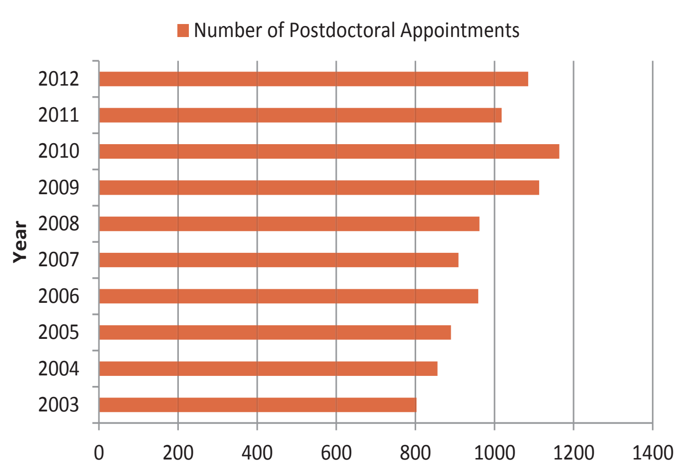
Postdoctoral appointments among the doctoral mathematics departments increased to 1,085 for fall 2012. This is a 6% increase from last year and 14% of the total full-time doctoral faculty in these departments. Females hold 19% of all postdoctoral appointments. Since 2003 total postdoctoral appointments among these departments has increased 35% and females holding postdocs increased 45% to 207 from 143.

Figure D.3: Full-time Postdoctoral Faculty



Total: 1,347

Figure D.4: Postdoctoral Faculty in All Doctoral Mathematics Departments Combined by Year, Fall 2003 to Fall 2012

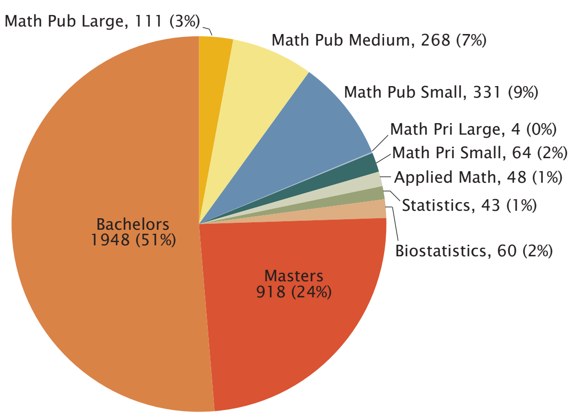


* All 2011 figures referenced on this page were adjusted to reflect the new departmental groupings introduced for 2012 (see page 166).

Nondoctoral Faculty*

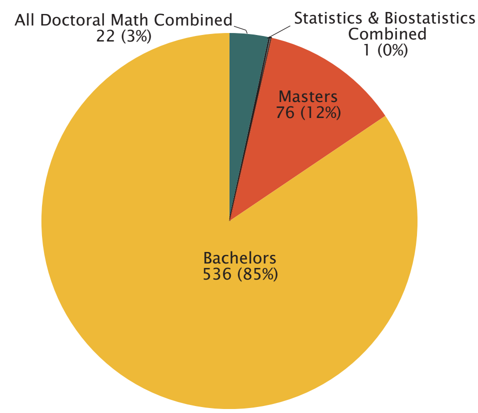
The estimated number of nondoctoral full-time faculty in all mathematics departments combined (Math Public, Math Private, Applied Math, Masters & Bachelors) is 3,692. This is down 2% from last year and is 17% of all full-time faculty (22,219) in these departments. In addition, nondoctoral tenured faculty decreased 15% from 748 to 633 this year. 195 of the nondoctoral faculty in all mathematics departments are untenured, tenure-track faculty, 4% of all untenured tenure-track faculty in these groups. Nondoctoral full-time non-tenure-track faculty increased to 2,848; this is 77% of all nondoctoral mathematics faculty.

Figure ND.1: Full-time Nondoctoral Faculty by Departmental Grouping



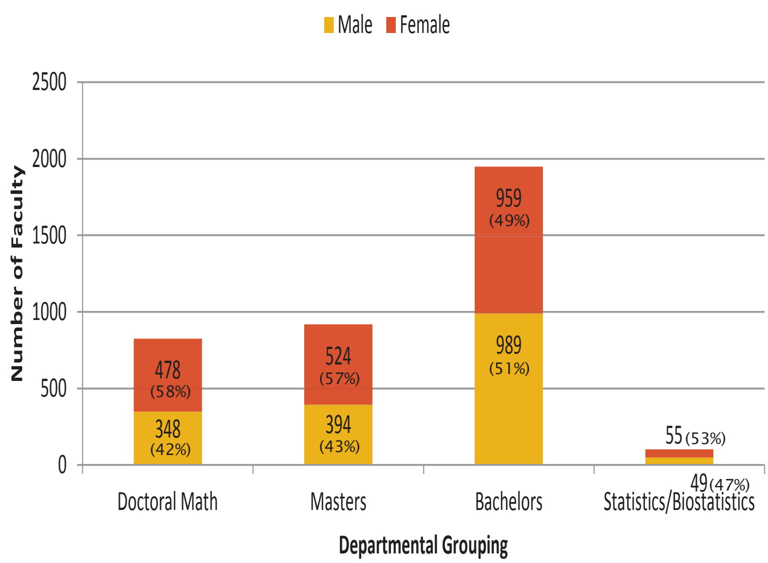
Total: 3,795

Figure ND.2: Full-time Nondoctoral Tenured Faculty



Total: 634

Figure ND.3: Gender of Full-time Nondoctoral Faculty Total: 3,795



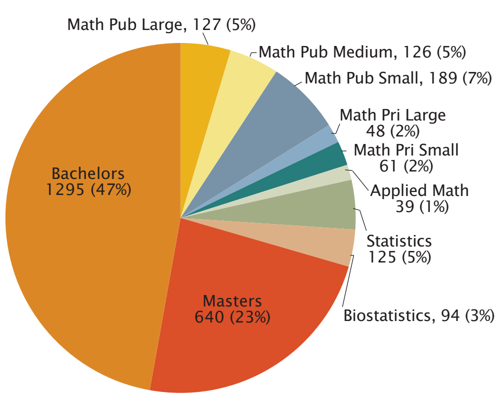
- Females account for 53% of full-time nondoctoral faculty in all mathematics groups combined (down from 54% last year), compared to females accounting for 24% of all doctoral full-time faculty and 29% of all full-time faculty.
- Total part-time nondoctoral faculty in all doctoral mathematics departments combined (Math Public, Math Private and Applied Math) is 694, 59% of all part-time faculty in these groups.

* All 2011 figures referenced on this page were adjusted to reflect the new departmental groupings introduced for 2012 (see page 166).

Female Faculty*

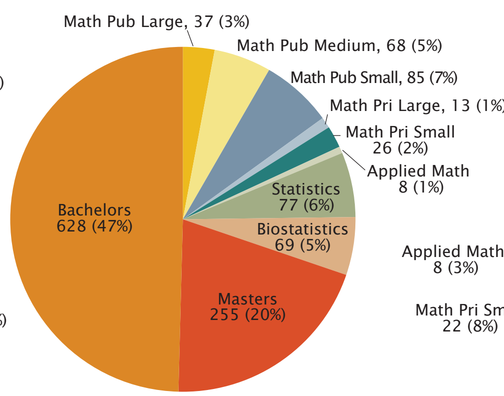
For the combined mathematics departments (Math Public, Math Private, Applied Math, Masters and Bachelors), women comprised 29% (6,482 with a standard error of 83) of the full-time faculty (22,219) in fall 2012. For the doctoral mathematics departments combined (Math Public, Math Private and Applied Math), women comprised 14% of the combined doctoral-holding tenured and tenure-track faculty and 27% of the doctoral-holding non-tenure-track (including postdocs) faculty in fall 2012. For Masters faculty these same percentages are 28 and 39, and for Bachelors faculty they are 29 and 33, respectively. Among the nondoctoral full-time faculty in all math departments combined, women comprise 53%. Females account for 41% of all part-time faculty in mathematics departments combined.

Figure FF.1: Tenured Female Doctoral Faculty



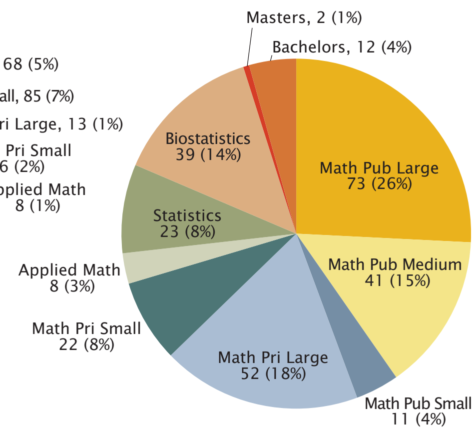
Total: 2,744

Figure FF.2: Untenured, Tenure-track Female Doctoral Faculty



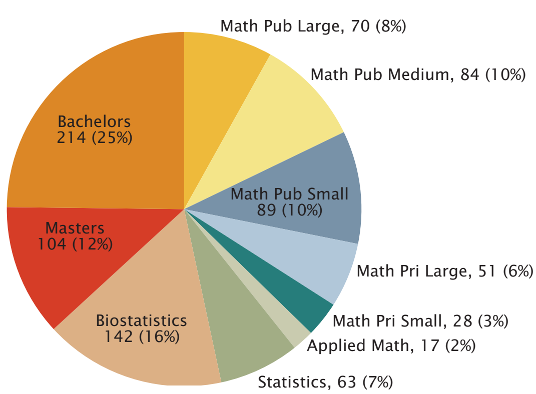
Total: 1,266

Figure FF.3: Postdoctoral Female Faculty



Total: 282

Figure FF.4: Female Doctoral Non-tenure-track Faculty (excluding Postdocs)



Total: 861

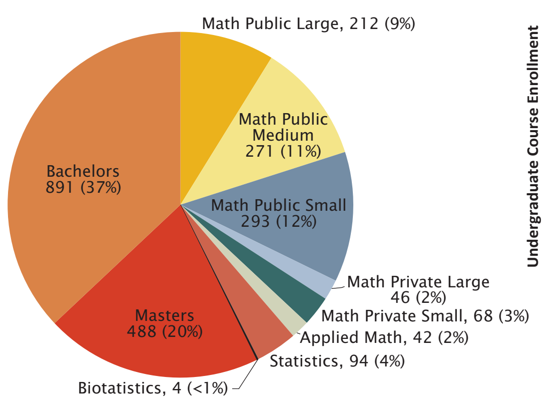
- Females hold 12% of full-time tenured and 24% of full-time untenured/tenure-track positions in all doctoral mathematics departments combined.
- 43% of all full-time female faculty (in all groups combined) are in the Bachelors Group.
- Masters departments reported the highest percentage of full-time female faculty (35%), while Math Private Large reported the lowest (14%).
- Females hold 21% of all postdoctoral appointments. 35% of all female postdocs in doctoral mathematics departments combined are found in Math Public Large departments. This group reported the highest percentage (26%) of female postdocs.
- 53% of all part-time female faculty among the mathematics departments combined are found in the Bachelors Group.

* All 2011 figures referenced on this page were adjusted to reflect the new departmental groupings introduced for 2012 (see page 166).

Undergraduate Course Enrollments

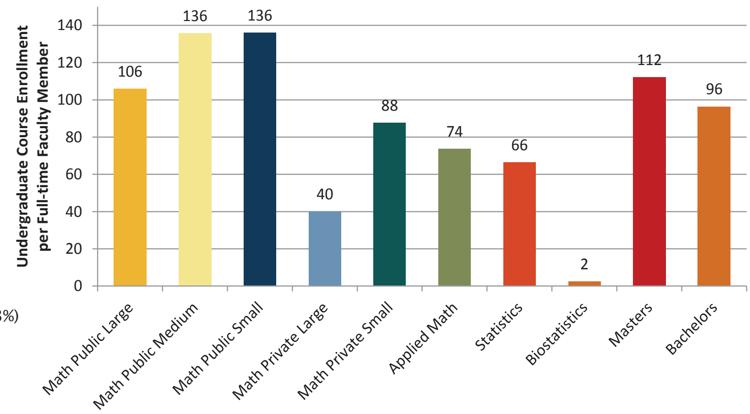
Total undergraduate enrollments for all groups combined increased by 2% (57,000) to 2,407,000 (with a standard error of 23,000). All departments combined reported an overall increase of 14% in the number of undergraduate course enrollments per full-time faculty member.

Figure UE.1: Undergraduate Course Enrollments by Department Groupings (Thousands)



Total Undergraduate Enrollments (thousands)**: 2,407**

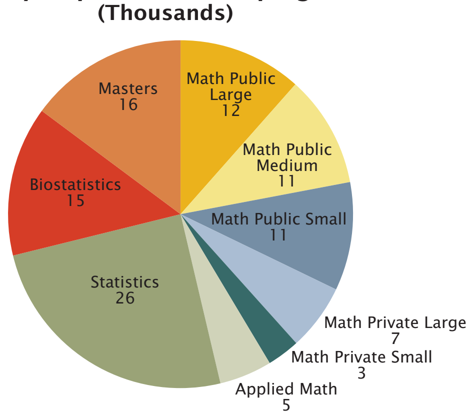
Figure UE.2: Undergraduate Course Enrollment per Full-Time Faculty Members, Fall 2012



Graduate Course Enrollments

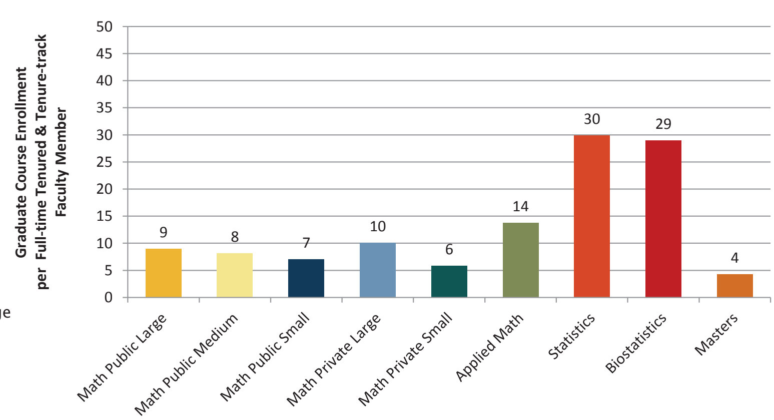
Total graduate course enrollments have increased by 3% (3,000) to 106,000 (with a standard error of 3,000). All departments combined reported an overall increase of 8% in the estimated number of graduate course enrollments per full-time tenured/tenure-track faculty member.

Figure GE.1: Graduate Course Enrollments by Department Groupings (Thousands)



Total Graduate Enrollments (thousands)**: 106**

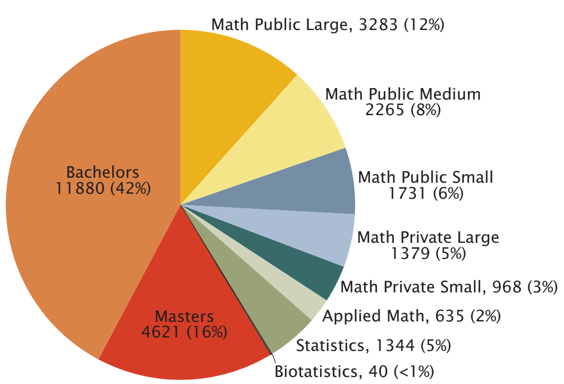
Figure GE.2: Graduate Course Enrollment per Full-Time Tenured and Tenure-track Faculty Member, Fall 2012



Undergraduate Degrees Awarded

The estimated number of undergraduate degrees awarded during 2011-2012 by all mathematics departments combined (Math Public, Math Private, Applied Math, Masters, and Bachelors) is 26,761 (with a standard error of 442), up 7% from last year's estimate of 25,054. The growth in degrees was similar for males and females. Females earned 41% (10,980) of undergraduate degreeds, almost exactly the same as last year. This year's estimated number of undergraduate degrees awarded included 477 statistics-only and 1,987 computer-science only.

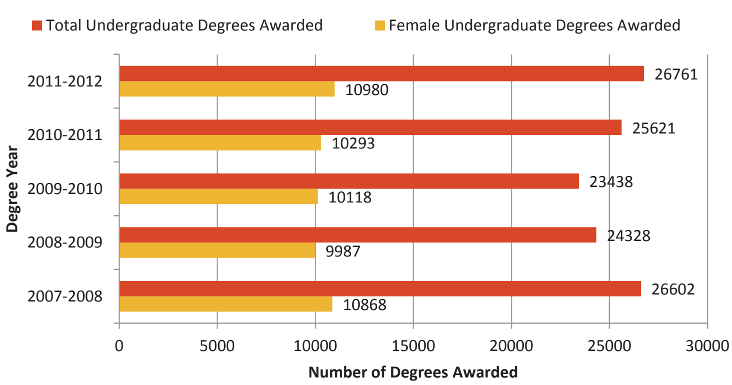
Figure UD.1: Undergraduate Degrees Awarded by Department Groupings



Total Degrees Awarded: 28,145

- Math Doctoral departments awarded 18% more degrees this year, up 1,539 from last year; 32% of all degrees awarded.
- Bachelors departments awarded 42% of all the degrees, down from 48% last year in all mathematics departments combined.
- Total statistics-only degrees increased in all mathematics departments combined by 30% to 477.
- Statistics and Biostatistics departments combined reported a 61% increase in degrees awarded, but most of the increase comes from one department that has reported tremendous growth over the past year.

Figure UD.2: Undergraduate Degrees Awarded All Mathematics Combined



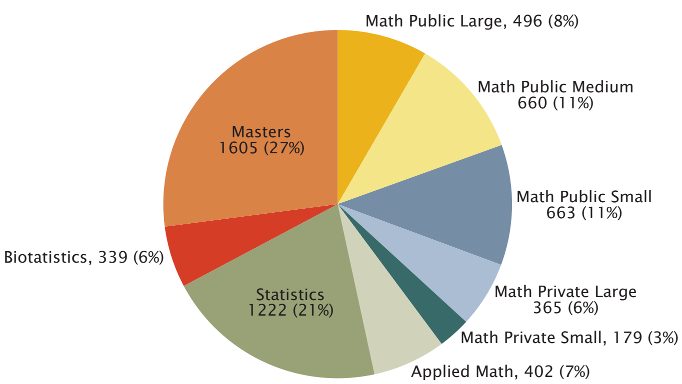
Comparing undergraduate degrees awarded this year with those awarded in 2007–2008:

- Degrees awarded have increased slightly.
- Degrees awarded to females increased by 1%.
- The percentage of total degrees awarded to females is the same, 41%.

Master's Degrees Awarded

The estimated number of master's degrees awarded during 2011-2012 in all mathematics departments combined (Math Public, Math Private, Applied Math, and Masters is 4,370, a 1% increase from last year's estimate of 4,030 (with a standard error of 131). This year's estimated graduate degrees included 1,888 statistics-only and 125 computer science-only degrees. Departments reported a slight decrease in the number of degrees awarded to females, 1,728.

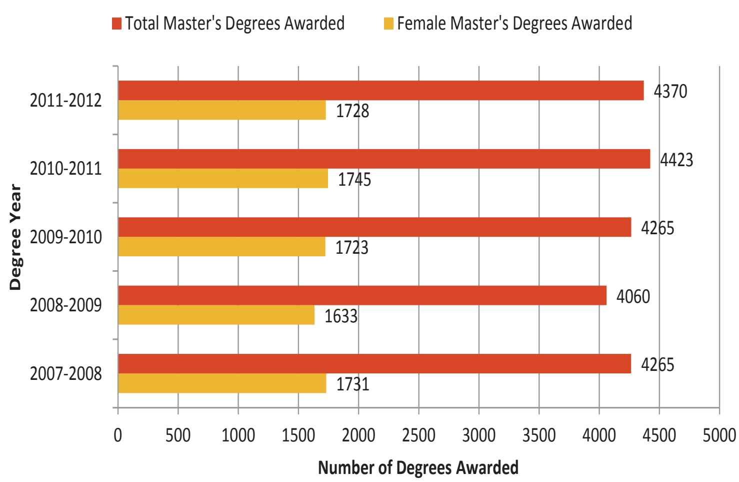
Figure MD.1: Master's Degrees Awarded by Department Groupings



Total Degrees Awarded: 5,931

- Looking at all mathematics departments combined:
  - Masters departments awarded the highest percentage of degrees (37%, down from 40% last year).
  - Math Private Small awarded the fewest degrees with 4%.
  - Females received 40% of all degrees awarded among all the mathematics departments combined; the same as last year.
  - 16% of degrees awarded to females in all mathematics departments combined were in statistics-only or computer science-only, compared to 12% for males.
- Statistics and Biostatistics combined awarded 1,561 degrees, an increase of 14% from last year; females received 50% of these degrees (up from 47% last year).

Figure MD.2: Master's Degrees Awarded All Mathematics Combined



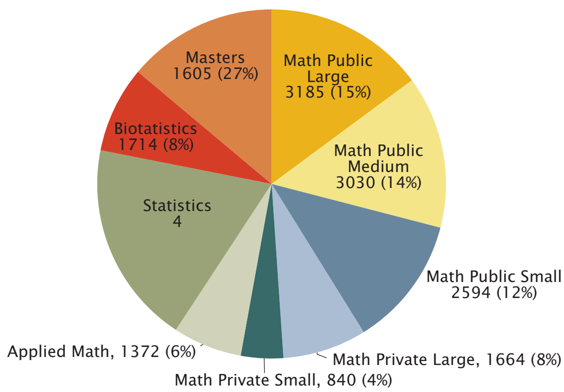
Comparing master's degrees awarded this year with those awarded in 2007–2008:

- Total degrees awarded have increased 2% overall.
- Total degrees awarded to females decreased from 41% to 40%.

Graduate Students*

The total number of full-time graduate students in all mathematics departments combined is 15,658, up from 15,122 in fall 2012. The total number of full-time graduate students in doctoral mathematics departments combined (Math Public, Math Private and Applied Math) is 12,684 (up from 12,464). The number of U.S. citizens among the doctoral mathematics departments combined dropped slightly to 6,893 and the number of U.S. citizen first-year students decreased 2% to 1,796. For Group Masters, full-time graduate students increased 8% to 2,974, the number of U.S. citizens is 2,222 (up from 2,180), and the number of first-year students is 1,302 (up from 1,244). Statistics and Biostatistics combined reported full-time graduate students as 5,749, up from 5,316.

Figure GS.1: Graduate Students by Department Groupings



Total Graduate Students: 21,407

- Full-time graduate students increased in all groups except Math Public Medium and Applied Math which decreased 2% and 3%, respectively.
- Biostatistics departments had the largest percentage increase in graduate students with 13% (up 199 from 1,515 to 1,714), while Masters departments had the largest number increase—up 326 from 2,648 to 2974.
- Females account for 36% (7,707) of the full-time graduate students; all groups reported increases except Math Public Medium, Math Private Large and Applied Math.
- First-year graduate students in Math Public Medium, Math Private Large and Biostatistics decreased by 6%, 4% and 41% respectively. All groups increased with Applied Math and Statistics increasing by 33% and 45%, respectively.
- U.S. citizen graduate students decreased slightly overall; all doctoral mathematics departments, except Math Public Small (which increased 10%) reported decreases.
- Total part-time graduate students increased slightly in all groups with Math Public Small and Masters having the largest increases at 4% and 8%, respectively.

Table GS.2: Full-Time Graduate Students in All Doctoral Math Combined by Gender and Citizenship, Fall 2006–2012

	2006	2007	2008	2009	2010	2011	2012
Total full-time graduate students	**10984**	**10937**	**10883**	**11286**	**13048**	**12464**	**12684**
Female	3279	3249	3193	3248	3839	3745	3771
% Female	30%	30%	29%	29%	29%	30%	30%
% U.S. Citizen	56%	56%	55%	56%	57%	56%	54%
% Underrepresented minorities [1]	9.0%	9.0%	9.0%	9.0%	9.0%	9.0%	9.0%
Total first-year graduate students	**2960**	**2964**	**2924**	**3040**	**3313**	**3200**	**3394**
Female	961	950	870	904	1019	1078	1036
% Female	32%	32%	30%	30%	31%	33%	31%
% U.S. Citizen	55%	56%	56%	55%	51%	50%	54%
% Underrepresented minorities	10.0%	10.0%	10.0%	10.0%	9.0%	9.0%	9.0%

[1] Underrepresented minorities includes any person having origins within the categories *American Indian or Alaska Native, Black or African American, Hispanic or Latino, and Native Hawaiian or Other Pacific Islander*.

Looking at Table GS.2 we see that although the numbers and percentages have fluctuated somewhat among the categories, the numbers of full-time and first-year graduate students have increased this year, while the percentage of U.S. citizens and female first-year graduate students has dropped. While the number of full-time and full-time first-year graduate students have both increased 15% above their level in 2006, they have dropped 3% and 2% from their seven year highs in 2010.

* All 2011 figures referenced on this page were adjusted to reflect the new 2012 groupings for comparison.

Remarks on on Statistical Procedures

The questionnaire on which this report is based, “Departmental Profile”, is sent to all doctoral and master’s departments. It is sent to a stratified random sample of bachelors departments, the stratifying variable being the undergraduate enrollment at the institution.

The response rates vary substantially across the different department groups. For most of the data collected on the Departmental Profile form, the year-to-year changes in a given department’s data are very small when compared to the variations among the departments within a given group. As a result of this, the most recent prior year’s response is used (imputed) if deemed suitable. After the inclusion of prior responses, standard adjustments for the remaining nonresponse are then made to arrive at the estimates reported for the entire groups.

Standard errors were calculated for some of the key estimates for all Doctoral Math Groups (Math Public, Math Private, and Applied Math) combined, for Groups Masters and Bachelors, and for Statistics and Biostatistics combined. Standard errors are calculated using the variability in the data and can be used to measure how close our estimate is to the true value for the population. As an example, the number of full-time faculty in Group Masters is estimated at 4,347 with a standard error of 68. This means the actual number of full-time faculty in Group Masters is most likely between 4,347 plus or minus two standard errors, or between 4,211 and 4,484. This is much more informative than simply giving the estimate of 4,347.

Estimates are also given for parameters that are totals from all groups, such as the total number of full-time faculty. For example, an estimate of the total number of full-time faculty in all groups but Statistics and Biostatistics combined is 22,219, with a standard error of 190.

The careful reader will note that a row or column total may differ slightly from the sum of the individual entries. All table entries are the rounded values of the individual projections associated with each entry, and the differences are the result of this rounding (as the sum of rounded numbers is not always the same as the rounded sum).

Departmental Groupings

Starting with reports on the 2012 AMS-ASA-IMS-MAA-SIAM Annual Survey of the Mathematical Sciences, the Joint Data Committee has implemented a new method for grouping the doctorate-granting mathematics departments. These departments are first grouped into those at public institutions and those at private institutions. These groups are further subdivided based on the size of their doctoral program as reflected in the average annual number of Ph.D.’s awarded between 2000 and 2010, based on their reports to the Annual Survey during this period. Furthermore, doctorate-granting departments which self-classify their Ph.D. program as being in applied mathematics will join with the other applied mathematics departments previously in Group Va to form their own group. The former Group IV will be divided into two groups, one for departments in statistics and one for departments in biostatistics.

For further details on the change in the doctoral department groupings see the article in the October 2012 issue of *Notices of the AMS* at `http://www.ams.org/notices/201209/rtx120901262p.pdf`.

Math. Public Large consists of departments with the highest annual rate of production of Ph.D.’s, ranging between 7.0 and 24.2 per year.
Math. Public Medium consists of departments with an annual rate of production of Ph.D.’s, ranging between 3.9 and 6.9 per year.
Math. Public Small consists of departments with an annual rate of production of Ph.D.’s of 3.8 or less per year.
Math. Private Large consists of departments with an annual rate of production of Ph.D.’s, ranging between 3.9 and 19.8 per year.
Math. Private Small consists of departments with an annual rate of production of Ph.D.’s of 3.8 or less per year.
Applied Mathematics consists of doctoral degree granting applied mathematics departments.
Statistics consists of doctoral degree granting statistics departments.
Biostatistics consists of doctoral degree granting biostatistics departments.
Group Masters contains U.S. departments granting a master’s degree as the highest graduate degree.
Group Bachelors contains U.S. departments granting a baccalaureate degree only.

Listings of the actual departments which compose these groups are available on the AMS website at `www.ams.org/annual-survey/groups`.

Departmental Response Rates

Survey Response Rates by New Groupings

Departmental Profile
Department Response Rates

Department Group	Number	Percent	Imputed[1]
Math Public Large	22 of 26	85%	3
Math Public Medium	31 of 40	78%	9
Math Public Small	50 of 64	78%	12
Math Private Large	29 of 24	96%	1
Math Private Small	24 of 28	86%	2
Applied Math	20 of 25[2]	80%	3
Statistics	42 of 59	71%	14
Biostatistics	17 of 35	46%	12
Masters	92 of 180	51%	40
Bachelors	273 of 591[3]	46%	83

1 See paragraph two under 'Remarks on Statistical Procedures.'
2 The population for Applied Math is slightly less than for the Doctorates Granted Survey because four programs do not formally "house" faculty, teach undergraduate courses, or award undergraduate degrees.
3 This is the sampled population, the total population for Bachelors is 1,007.

About the Annual Survey

The Annual Survey series, begun in 1957 by the American Mathematical Society, is currently under the direction of the Data Committee, a joint committee of the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. The current members of this committee are Richard Cleary (chair), Charles Epstein, Amanda Gabeck, Sue Geller, Boris Hasselblatt, Loek Helminck, Ellen Kirkman, Peter March, David R. Morrison, James W. Maxwell (ex officio), illiam Velez, and Edward Waymire. The committee is assisted by AMS survey analyst Colleen A. Rose. In addition, the Annual Survey is sponsored by the Institute of Mathematical Statistics. Comments or suggestions regarding this Survey Report may be emailed to the committee at ams-survey@ams.org.

Acknowledgments

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires.

Other Sources of Data

Visit the AMS website at `www.ams.org/annual-survey/other-sources` for a listing of additional sources of data on the Mathematical Sciences.

WHAT IS... Cyclic Sieving?

Victor Reiner, Dennis Stanton, and Dennis White

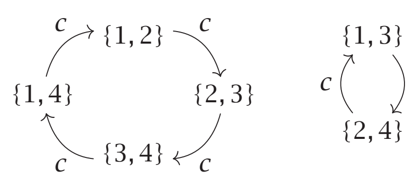
Many finite sets in combinatorics have both cyclic symmetry and a natural generating function. Surprisingly often the generating function evaluated at roots of unity counts symmetry classes. We call this the *cyclic sieving phenomenon*.

More precisely, let C be a cyclic group generated by an element c of order n acting on a finite set X. Given a polynomial $X(q)$ with integer coefficients in a variable q, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon (CSP)* if for all integers d, the number of elements fixed by c^d equals the evaluation $X(\zeta^d)$ where $\zeta = e^{\frac{2\pi i}{n}}$. In particular, $X(1)$ is the cardinality of X, so that $X(q)$ can be regarded as a *generating function* for X.

In the protoexample, X is the collection of all k-element subsets of $\{1, 2, \ldots, n\}$, and $X(q)$ is the renowned *q-binomial coefficient* or *Gaussian polynomial*

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \tag{1}$$

where $[m]!_q := [m]_q[m-1]_q \cdots [2]_q[1]_q$ and $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$. Let the generator c of C act by cycling the elements of a k-subset modulo n. One then finds [1, Theorem 1.1(b)] that this triple $(X, X(q), C)$ exhibits the CSP. For example, taking $n = 4$ and $k = 2$, one has c acting as shown here:



One can compute $X(q) = 1 + q + 2q^2 + q^3 + q^4$ from (1). Note that $X(1) = 6$, while $X((e^{\frac{2\pi i}{4}})^2) = X(-1) = 2$ counts the two subsets $\{\{1,3\},\{2,4\}\}$ fixed by c^2, and $X(e^{\frac{2\pi i}{4}}) = 0 = X((e^{\frac{2\pi i}{4}})^3)$ since no two-element subset is fixed by c or c^3.

The CSP was first defined in [1]. It has proven to be remarkably ubiquitous; see, for example, B. Sagan's excellent survey [3]. The special case of a CSP when C has order 2 was known as J. Stembridge's $q = -1$ *phenomenon* [4]. He gave interesting examples involving enumeration of *plane partitions* and *Young tableaux*.

Stembridge emphasized the value of a single q-formula $X(q)$ encompassing both the cardinality of X as $X(1)$ and a second enumeration $X(-1)$ of a symmetry class within X. A CSP triple $(X, X(q), C)$ generalizes his idea. The polynomial $X(q)$ packages as its n-th root of unity evaluations, or equivalently in its residue class modulo $q^n - 1$, all of the information about the cyclic action of C on X. In fact, given (X, C) there is always a unique (but generally *uninteresting*) choice of a polynomial $X(q)$ of degree at most $n - 1$ completing the triple, as the CSP is equivalent [1, Proposition 2.1(ii)] to the assertion that

$$X(q) \equiv \sum_{i=0}^{n-1} a_i q^i \bmod q^n - 1,$$

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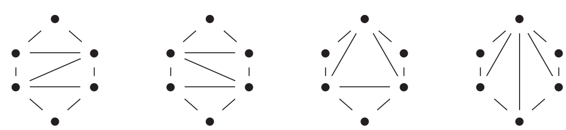
where a_i is the number of orbits of C on X in which the stabilizer cardinality divides i. Thus a CSP interprets combinatorially the coefficients of $X(q)$ when reduced mod $q^n - 1$; e.g., a_0 counts the total number of orbits on X, while a_1 counts the number of *free* orbits. Our protoexample with $n = 4$ and $k = 2$ has $X(q) \equiv 2 + q + 2q^2 + q^3 \bmod q^4 - 1$, so $a_0 = 2$ counts the two orbits in total, and $a_1 = 1$ counts the free orbit.

Here is a second example from [1]. Let X be the set of triangulations of a regular $(n+2)$-gon, with C a cyclic group of order $n+2$ rotating triangulations, and let

$$X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

a *q-Catalan number* considered by P. A. MacMahon. Then $(X, X(q), C)$ exhibits the CSP [1, Theorem 7.1]. For example, when $n = 4$, the four orbits of triangulations are represented by



while

$$\begin{aligned} X(q) &= \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q \\ &= 1 + q^2 + q^3 + 2q^4 + q^5 \\ &\quad + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12} \\ &\equiv 4 + q + 3q^2 + 2q^3 + 3q^4 + q^5 \bmod q^6 - 1, \end{aligned}$$

so that $a_0 = 4$ counts the four orbits, of which $a_1 = 1$ of them is free (the fourth orbit), while $a_2 = 3$ orbits (the first, second, fourth) have stabilizer size dividing 2, and $a_3 = 2$ orbits (the third, fourth) have stabilizer size dividing 3.

It was conjectured by the authors and verified by S.-P. Eu and T.-S. Fu that this triangulation example generalizes to a CSP triple $(X, X(q), C)$ in which X is the collection of *clusters* in a cluster algebra of finite type W à la S. Fomin and A. Zelevinsky, where C is generated by a *deformed Coxeter element* and $X(q)$ is a q-analogue of the *Catalan number* for W.

So what makes a generating function $X(q)$ "natural"? To some extent, this is in the eye of the beholder. Nevertheless, here are some conditions on $X(q)$ arising in many CSPs encountered so far:

(i) $X(q)$ is the *statistic generating function* for a map $s : X \to \{0, 1, 2, \ldots\}$; that is, $X(q) = \sum_{x \in X} q^{s(x)}$.

(ii) $X(q)$ has a simple *product formula.*

(iii) $X(q)$ at $q = p^d$ a prime power counts the points of a variety $X(\mathbb{F}_q)$ defined over the *finite field* $\mathbb{F}_q$.

(iv) $X(q^2) = \sum_i \beta_i q^i$ records the *Betti numbers* β_i of a complex variety $X(\mathbb{C})$.

(v) $X(q) = \sum_i \dim R_i q^i$ records the *Hilbert series* of some interesting graded ring $R = \bigoplus_i R_i$.

(vi) $X(q^2)$ is, up to a power of q, the *formal character* of an $SL_2(\mathbb{C})$-representation, that is, the sum $\sum_i \dim V_i q^i$ where V_i is the weight space on which a diagonal matrix with eigenvalues (q, q^{-1}) acts via the scalar q^i.

Our protoexample has each of these natural properties:

(a) After multiplying $X(q)$ by $q^{\binom{k+1}{2}}$, it is the statistic generating function for k-subsets A by their sum $s(A) = \sum_{a \in A} a$.

(b) The product formula for $X(q)$ is given in (1).

(c) $X(q)$ counts the points in the *Grassmannian of k-planes* in an n-dimensional vector space over $\mathbb{F}_q$.

(d) $X(q^2)$ records the Betti numbers for this Grassmannian over $\mathbb{C}$.

(e) When the symmetric group S_n permutes polynomials in n variables, $X(q)$ is the Hilbert series for the quotient ring[1] of the polynomials invariant under $S_k \times S_{n-k}$ after modding out the nonconstant polynomials invariant under S_n.

(f) $q^{-k(n-k)} X(q^2)$ is the formal character for the k-th exterior power of the n-dimensional $SL_2(\mathbb{C})$-irreducible.

In our triangulations example, the q-Catalan $X(q)$ has an interpretation as in (a), (b), (c) and a variation of (e). We know no interpretation like (d) or (f).

Some CSPs in the literature are proven via a *linear algebra paradigm* [1, §2]. Such proofs interpret $X(q)$ as in (d) or (e), giving a graded representation $V = \bigoplus_i V_i$ of the cyclic group C. One shows that $X(\zeta^d)$ equals the size of the c^d-fixed subset of X by computing the trace of c^d using two bases. The first basis is indexed by X and permuted by c, so that the trace of c^d is the size of the c^d-fixed subset. The second basis shows that c scales V_i by ζ^i, so that c^d has trace $X(\zeta^d)$.

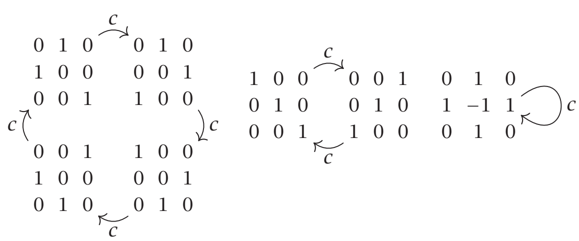
A pleasing situation where this paradigm works generalizes (e) above. It arises from the *invariant theory* of finite subgroups W of $GL_n(\mathbb{C})$ generated by *reflections*, that is, elements whose fixed space is a complex hyperplane. T. Springer developed a theory of *regular elements* in such groups, which are the elements c that have an eigenvector fixed by none of the reflections of W. Using Springer's main result, one obtains [1, Theorem 8.2] a CSP triple from the coset space $X := W/W'$ for *any* subgroup W', with C generated by a regular element left-translating coset, and $X(q)$ is the quotient of the

[1] *This graded ring is isomorphic, after doubling degrees, to the cohomology of the Grassmannian in* (d).

Hilbert series for the W'-invariant polynomials over the Hilbert series for the W-invariant polynomials.

An intriguing CSP was conjectured by D. White involving *rectangular Young tableaux* and the cyclic action of *jeu-de-taquin promotion*. It has now seen several proofs via the linear algebra paradigm, first by B. Rhoades [2] and most recently by B. Fontaine and J. Kamnitzer. Such insightful proofs are rarer than we would like. Many known instances of CSPs, such as the triangulations example, have only been verified using a product formula for $X(q)$ to evaluate $X(\zeta^d)$ and comparing with known counts of symmetry classes.

We close with a perplexing example of this nature. Let X be the set of $n \times n$ *alternating sign matrices*: the matrices with $0, \pm 1$ entries whose row and column sums are all $+1$, and nonzero entries alternate in sign reading along any row or column. Here they are for $n = 3$:



Let C be the cyclic group of order 4 whose generator c rotates matrices through 90°. Let

$$X(q) = \prod_{k=0}^{n-1} \frac{[3k+1]!_q}{[n+k]!_q}.$$

This triple $(X, X(q), C)$ exhibits the CSP, but we have no linear algebraic proof. Furthermore, $X(q)$ is only known as the generating function for *descending plane partitions* by weight and is not defined by a statistic on alternating sign matrices.

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Book Review

Théorème vivant

Reviewed by Jacques Hurtubise

Théorème vivant
Cédric Villani
Grasset et Fasquelle
(Language: French)
288 pages
ISBN-13: 978-2246798828

"But what do you actually *do*?" We have all had to answer that question. "What does a research mathematician actually do? Isn't it already known? Do you just sit down and write things out?" And so on.

It is a difficult question to answer if one wants to give some idea both of the effort and of the intellectual pleasure involved. This pleasure is quite real and keeps us all going, as arcane as it might seem to the passer-by. To quote André Weil, as does the book under review, "Tout mathématicien digne de ce nom a ressenti, ne serait-ce que quelquefois, l'état d'exaltation lucide dans lequel une pensée succède à une autre comme par miracle…. Contrairement au plaisir sexuel, ce plaisir peut durer plusieurs heures, voire plusieurs jours. (Any mathematician worthy of the name has felt, if only a few times, that state of lucid exaltation in which one thought follows another as if by miracle…. Unlike sexual pleasure, this state can go on for hours, even days.)"

Our world, also, is different. We evolve inside a mathematical culture which is to a great degree alien to the common culture. Our heroes are, by and large, unknown to the public. Though we share to a great degree the values of fellow scientists, even within a faculty of science, we are often strangers, and outliers, doing that strange stuff that is immune to experiment.

Of course, we have explained ourselves in several ways: mathematicians have written autobiographies. Cleaving to the theme of French

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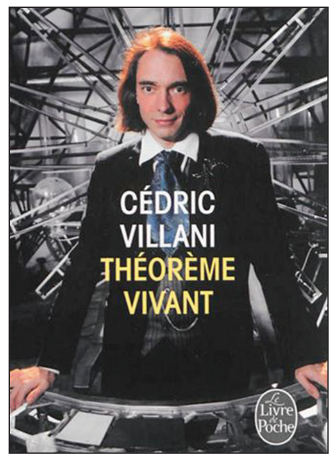

mathematics, appropriately for this review, two come to mind, which, incidentally, illustrate to a certain degree how French mathematical culture has evolved over time. One is André Weil's *Souvenirs d'apprentissage*: beautifully written, quirky, if, in the end, too preoccupied with the anecdotes of a well-filled life, having carefully avoided any intimacy. The other is *Un mathématicien aux prises avec le siècle*, by Laurent Schwartz, which follows a life of someone who, much more than most of us, was deeply involved in the political and intellectual struggles of his time: it is a beautiful book, following the life of a truly exceptional man. However, neither of these gives any real idea of what mathematical research is actually like and why it involves us so deeply. Nor do many other autobiographies.

The book *Théorème vivant*, by Cédric Villani, explores a different direction and gives a wonderfully living answer to the question. It follows, through a few years of work, a collaboration by Villani and his former student, Clément Boutot, on a difficult and important result: establishing Landau damping for the Vlasov equation, beyond linear perturbation. The Vlasov, or Vlasov-Poisson, equation governs the evolution of plasmas. It displays time reversibility and does not reflect collisions of the particles, the mechanism which would normally govern a rapid convergence to an equilibrium. Landau, in the 1940s, argued that nevertheless the electric field converged exponentially to an equilibrium and computed the rate from the linearized equation: this is Landau damping. The

Boutot-Villani theorem establishes this damping for the full equation.

The core of the book, thus, is about a theorem. It is a very major theorem, not one tossed off in a weekend, and its genesis exhibits all the rebounds that we have felt in our own work. The technique Villani uses to show this is kaleidoscopic, or, rather, by vignette; perhaps a stained-glass window would be a better analogy. One has the initial dialogue, a sort of jam session, between the two collaborators, trying to work out what they are going to do. After a while, the desired theorem crystallizes in their minds, and the hunt is on. One watches, mainly through a series of email transcripts, as their proof takes shape. We are even given the full proofs of a few lemmas, a sampling. There are ups and downs: gaps appear and are filled. The theorem and its proof evolve into something that can be presented to a specialized audience. There follows the first seminar, and in the course of preparation, doubts appear, as is often the case. The audience criticizes, and the authors go back to work. Better approaches evolve; the argument is refined; estimates treated before in a block are attacked individually, sharpened, and the theorem improves. It is submitted to a major journal, and the editors reject it: very good, but too long, not quite there, and so on. The final theorem, hardened and improved by its trial by fire, in the end is accepted. Recognition follows.

The writing is true to life: the emails have the informal style, interspersed with borrowings from English that actual French mathematicians actually use; the descriptive passages are more formal, with a very pleasant prose. The formal mathematics is, well, formal mathematics. The dialogues of the two collaborators as they are beginning to work on their theorem are perhaps a bit artificial and slick, to my mind: a transcript of my own efforts at a blackboard with a collaborator would include a lot more of "Huh? Could you repeat that again?" On the other hand, there is a stylistic challenge of summarizing a five-hour session into something that does not numb the brain. By and large, though, true to the word "living" in the title, the prose reflects life. I presume that there will be an English translation eventually, but one should read the book in French if one can.

Interspersed with the mathematics, there is indeed life, the rest of it, which again resonates with us all: we are not abstract theorem-proving machines, but people, with interests, and families, and various duties. The chapters on the development of the theorem interleave with paragraphs or sections on looking after the children, on travel, on taking a walk in the grounds of the Institute for Advanced Study, on the wonders of tea, or on the difficulties of finding good cheese in the United States. There are several beautiful discourses on music, about which Villani cares very much and which is a constant companion in his life. He is particularly moved by the songs of Catherine Ribeiro; one of them is reproduced in the book, along with a photograph of her. One has a long and eclectic list, almost like a well-known poem of Prévert, of his musical likes; unlike Prévert's poem, though, as far as I can tell, it is resolutely raccoon-free. From the animal kingdom, Blake's Tyger makes an appearance, following a reference to "tyger" phenomena for Burger's and Euler's equations.

It is quite naturally in these sections that the author's personality shines through. There are of course the personal quirks, his clothes for example. (Villani dresses like a nineteenth-century romantic poet.) Mostly, however, through these passages, the strongest sense one gets is of the relentless and sympathetic curiosity, the omnivorous cultural enthusiasm, and the boundless energy of the man, which he deploys with great generosity. The book is interspersed with personal vignettes on other mathematicians, some of them heroes from the past (Malliavin, for example), some of them his colleagues, either in his area, or outside of it, encountered in the course of the few years covered by the book. Each is accompanied by a beautiful line drawing by Claude Gondard of the person in question.

Villani has already had a major impact on the public perception of mathematics in France: he is a born communicator, and he has things to say. The book has been very popular in francophone countries; it was quite visibly displayed in the Montreal bookstore where I bought my copy. The implicit task that Villani had set for himself, of explaining what it is all about, is a difficult one, mostly due to the wide variety of audiences concerned: colleagues in the field, colleagues without, a general scientific reader, and of course members of that vast and rather undefined set—the general public. To give each a sense of what is going on, without pandering, so that each goes away with a good sense of the mathematical process, or of that mathematical process, is not easy. To my mind the book succeeds wonderfully.

A Missing Piece: Early Elementary Plane Rotations

Bob Palais

The formula for plane rotations can and perhaps should be taught at an elementary level, for its usefulness in many fields including geometry, physics, and computer animation, and because it unifies and clarifies a wide range of mathematical subjects.

A rotation of the plane that fixes the origin is determined by its effect on a single reference point. If the image of $(1, 0)$ is (X, Y), the plane rotation formula expresses the image of any point (x, y) as the multiple x of (X, Y) plus the multiple y of $(-Y, X)$, the image of $(0, 1)$:

$$(1) \quad x\langle X, Y\rangle + y\langle -Y, X\rangle = \langle xX - yY, yX + xY\rangle.$$

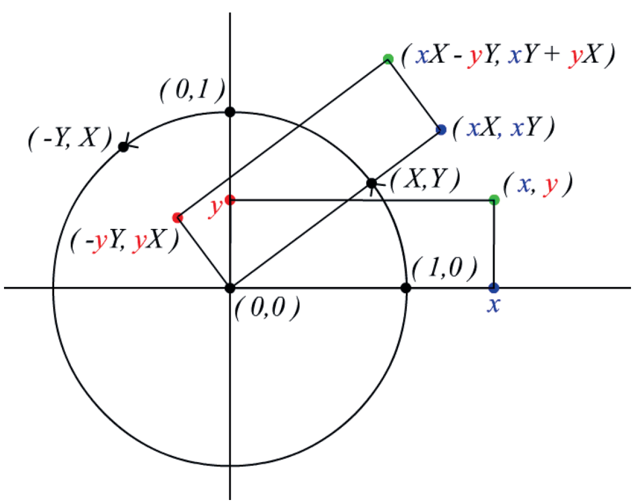

Figure 1. The plane rotation formula

The addition and scaling operations call for vector notation $\langle\ ,\ \rangle$ instead of point notation $(\ ,\ )$. The Pythagorean relation on X and Y can actually be obtained as a consequence of the formula! The derivation suggested in Figure 1 requires only basic Euclidean geometry. This makes a variety of exciting and useful mathematical topics available much earlier. Organizing the curriculum around this marriage of geometry and algebra can also provide a natural introduction to and excellent preparation for the next levels.

The key to the plane rotation formula is the observation that, when you turn your head 90° clockwise, what used to look like (X, Y) now looks like $(-Y, X)$. A rotation by a right angle is equivalent to choosing a neighboring pair of perpendicular rays from the axes of the same rectangular system as positive first and second coordinate axes. Applying this twice, the image of $\langle -Y, X\rangle$ is $\langle -X, -Y\rangle$, so a quarter-turn rotation behaves as a square root of -1. By the same reasoning, if a rotation takes $(1, 0)$ to (X, Y), then it must also take $(0, 1)$ to $(-Y, X)$. By similar triangle constructions, or simply scaling, the image of $\langle x, 0\rangle$ is $x\langle X, Y\rangle = \langle xX, xY\rangle$, and the image of $\langle 0, y\rangle$ is $y\langle -Y, X\rangle = \langle -yY, yX\rangle$. By a congruent triangle construction, or shift of origin, we arrive at (1): the plane rotation that takes $\langle 1, 0\rangle$ to $\langle X, Y\rangle$ takes $\langle x, y\rangle$ to $x\langle X, Y\rangle + y\langle -Y, X\rangle$. We will now see how this formula unifies the circular addition formulas, the Pythagorean relation, the

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geometric meanings of complex products, powers, exponentials, the dot and cross products, and more.

With $\langle x, y\rangle = \langle \cos s, \sin s\rangle$, $\langle X, Y\rangle = \langle \cos t, \sin t\rangle$, (1), the rotation formula (1) becomes

$$\langle \cos(s+t), \sin(s+t)\rangle$$
$$= \langle \cos s \cos t - \sin s \sin t, \sin s \cos t + \cos s \ \sin t\rangle.$$

This reunites the two circular addition formulas that are usually treated separately and better explains their related algebraic structure. In contrast, the kinds of derivations many students experience are like the one given in the 2010 edition of a popular precalculus text.[1] Starting from a distance formula obtained by unrelated cut and paste methods, $\sqrt{(\cos s - \cos t)^2 + (\sin s - \sin t)^2} = \sqrt{(\cos(s-t) - 1)^2 + (\sin(s-t) - 0)^2}$, two pages of algebraic manipulation lead to the cosine subtraction formula by itself. This approach provides little insight into the connected and linear structure and origins of both addition formulas. It is hardly surprising that many students fail to understand these formulas. Through no fault of their own, they may feel that math is a pointless, unmotivated, and unpleasant exercise in gymnastic memorization.

Formula (1) wants and contains the Pythagorean formula that relates the horizontal and vertical components of a reference point to which $\langle 1, 0\rangle$ can be rotated. Because if $\langle 1, 0\rangle$ is rotated to $\langle X, Y\rangle$, then $\langle X, -Y\rangle$ returns to $\langle 1, 0\rangle = \langle xX - y(-Y), x(-Y) + yX\rangle = \langle X^2 + Y^2, 0\rangle$. At the basic level, this is just reflection symmetry, or Pythagoras as a special case of the addition formula: $\cos(t-t) = \cos(0) = 1 = \cos(t)\cos(-t) - \sin(t)\sin(-t)$. When scaling is incorporated, the same analysis explains the two scaling factors of R in $X^2 + Y^2 = R^2$.

In 1799, Wessel saw that (1) could be interpreted to define a complex multiplication isomorphic to $(x+yi)(X+Yi) = (xX - yY) + (yX + xY)i$. It is no coincidence that, in the same paper, Wessel also first introduced the geometric interpretation of vector addition as uniting directed segments "in such a way that the second begins where the first ends. The sum is from the first to the last point of the united segments." Interpreting complex powers as iterated rotation and scaling permits a more complete connection among the cornerstones of the "College Algebra" curriculum: exponentials, polynomials and their zeros, and systems of linear equations. The Fibonacci-like dynamical systems that evolve deterministically from two initial conditions are a wonderful example that combines all this and more, and also leads naturally to the corresponding material at the calculus level. For example, the solutions of $F_{n+2} = 5F_{n+1} - 6F_n$ are $c_1 2^n + c_2 3^n$, a superposition of exponentials whose bases are solutions of the polynomial equation $x^2 = 5x - 6$, and whose coefficients are found by solving a system of linear equations to match the initial conditions. When the difference equation is replaced with $y'' = 5y' - 6y$, we just change to rates for natural exponentials: $c_1 e^{2t} + c_2 e^{3t}$. Even closer to Fibonacci, when the sum is replaced by a difference, $F_{n+2} = F_{n+1} - F_n$, the rotation formula explains the 6-step periodicity of the basic solutions $x_n = 2, 1, -1, -2, -1, 1, 2, 1, \ldots$ and $y_n = 0, 1, 1, 0, -1, -1, 0, 1, \ldots$. They also may be written $c_1 r_+^n + c_2 r_-^n$, where $r_\pm$ are solutions of $x^2 = x - 1$, each one-sixth turn from 1 on the unit circle. The points (x_n, y_n) lie on an ellipse, $x_n^2 + 3y_n^2 = 4$, corresponding to the hyperbolas $y_n^2 - 5x_n^2 = \pm 4$ that contain the standard Fibonacci-Lucas pairs $x_n = 1, 1, 2, 3, 5, \ldots$, $y_n = 1, 3, 4, 7, 11, \ldots$. The solutions of $y'' = y - 1$ are $c_1 e^{r_+ t} + c_2 e^{r_- t}$. See the footnote link for how the rotation formula can also simplify the derivation and analysis of conic sections.

Rectangular coordinates are specified using Euclid's perpendicular bisector construction by a choice of origin, a second point for direction and unit on the first axis, and an orientation for the positive direction of the second (counterclockwise in math, clockwise in computer graphics). Then any two rectangular coordinate systems may be related by a combination of shift of origin, scaling, rotation, and reflection, corresponding to complex addition, multiplication, and conjugation, respectively! It seems a shame for our students to be deprived of the one missing transformation, rotation, when providing access to it is so elementary and permits so much utility and insight.

At the calculus level, the quarter-turn rotation formula $\langle -y, x\rangle$ also expresses the physics of uniform circular motion and Hooke's spring. When motion is neither inward nor outward, velocity must be perpendicular to displacement: $\langle x, y\rangle' = \langle -y, x\rangle$ or $z' = iz$. This physically motivated and geometrically natural relationship provides a definition of the cosine and sine functions that parameterize our reference point $\langle X, Y\rangle$, and explains Euler's formula $e^{it} = \cos t + i \sin t$. All other properties follow easily from this starting point. Applied twice, $\langle x, y\rangle'' = \langle -x, -y\rangle$, or $z'' = -z$, says acceleration is opposite to displacement. From the differential equations perspective, $\langle -y, x\rangle$ is a change of variables, the linear combination $X\langle x, y\rangle + Y\langle -y, x\rangle$ is a linear superposition of solutions, and the Pythagorean symmetry reflects their invariance under time reversal. Formula (1) is also the origin of the Cauchy-Riemann equations that characterize an analytic function of a complex variable z as independent of $\overline{z} = x - iy$.

[1] *See* http://math.utah.edu/%7Epalais/AMissingPiece *for this and other comparisons and examples.*

It is well known that the negative reciprocal condition that two lines are perpendicular fails when one is vertical. The condition that two directions $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are collinear, $\mathbf{w} = c\mathbf{v}$, also fails when $\mathbf{v} = \mathbf{0}$. The vanishing of the plane cross product $\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$ is a test for collinearity that does not suffer from this exception. Directions are perpendicular if a quarter-turn rotation makes them collinear. Combining the cross product and the quarter-turn rotation formula, the vanishing of plane dot product $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ is a test for orthogonality that has no exception.

The plane rotation formula provides a great setting to introduce geometric vector algebra and matrices, where equation (1) takes the form $\left[\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. A direct calculation using plane rotation formula (1) and its consequence $X^2 + Y^2 = 1$ confirms that, if the same rotation $\mathbf{R}$ is applied to both $\mathbf{v}$ and $\mathbf{w}$, the dot product of the images is the same: $\mathbf{Rv} \cdot \mathbf{Rw} = \mathbf{v} \cdot \mathbf{w}$. We may simultaneously rotate any such vectors so that the image of the first is along the positive x_1-axis, $\langle r_1, 0 \rangle$, and that of the second is $\langle r_2 \cos t, r_2 \sin t \rangle$, where $r_1^2 = v_1^2 + v_2^2$ and $r_2^2 = w_1^2 + w_2^2$. This standard configuration exhibits the meaning of the dot product $\mathbf{v} \cdot \mathbf{w} = r_1 r_2 \cos t$, where t is the angle between $\mathbf{v}$ and $\mathbf{w}$. Any rotation of space in three or higher dimensions and the standard configuration may be constructively obtained through a sequence of coordinate plane rotations, so the dot product $\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{n} v_j w_j$ immediately inherits the same invariance and interpretation. Therefore, the generalization of the above dot product to $\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{n} v_j w_j$ immediately inherits the same invariance and interpretation. The definition, invariance, and interpretation of the three-dimensional cross product

$$\mathbf{v} \times \mathbf{w} = v_1 \langle 0, -w_3, w_2 \rangle + v_2 \langle w_3, 0, -w_1 \rangle + v_3 \langle -w_2, w_1, 0 \rangle$$

can also be easily understood in terms of coordinate plane rotations.

Collectively, these observations give rise to a useful and purely three-dimensional interpretation of quaternions and their multiplication that we will discuss elsewhere. There are many more examples that cannot be covered here, but perhaps this brief survey will stimulate more inquiry, discussion, and discovery!

Interview with Pierre Deligne

Martin Raussen and Christian Skau

Pierre Deligne is the recipient of the 2013 Abel Prize of the Norwegian Academy of Science and Letters. This interview was conducted by Martin Raussen and Christian Skau in Oslo in May 2013 in conjunction with the Abel Prize celebration. This article originally appeared in the September 2013 issue of the *Newsletter of the European Mathematical Society* and is reprinted here with permission of the EMS.

Raussen and Skau: *Dear Professor Deligne, first of all we would like to congratulate you as the eleventh recipient of the Abel Prize. It is not only a great honor to be selected as recipient of this prestigious prize, the Abel Prize also carries a cash amount of six million NOK, that is around US$1,000,000. We are curious to hear what you are planning to do with this money…*

Deligne: I feel that this money is not really mine, but it belongs to mathematics. I have a responsibility to use it wisely and not in a wasteful way. The details are not clear yet, but I plan to give part of the money to the two institutions that have been most important to me: the Institut des Hautes Études Scientifiques (IHÉS) in Paris and the Institute for Advanced Study (IAS) in Princeton.

I would also like to give some money to support mathematics in Russia. First to the Department of Mathematics of the Higher School of Economics (HSE). In my opinion, it is one of the best places in Moscow. It is much smaller than the Faculty of Mechanics and Mathematics at the [National Research] University, but has better people. The student body is small; only fifty new students are accepted each year. But they are among the best students. The HSE was created by economists. They have done their best under difficult circumstances. The department of mathematics has been created five years ago, with the help of the Independent University of Moscow. It is giving prestige to the whole HSE. There I think some money could be well used.

Another Russian institution I would like to donate some money to is the Dynasty Foundation, created by the Russian philanthropist Dmitry Zimin. For them, money is most likely not that important. It is rather a way for me to express my admiration for their work. It is one of the very few foundations in Russia that gives money to science; moreover, they do it in a very good way. They give money to mathematicians, to physicists, and to biologists; especially to young people, and this is crucial in Russia! They also publish books to popularize science. I want to express my admiration for them in a tangible way.

Photograph by Anne-Marie Astad.

From left to right: Pierre Deligne, Martin Raussen, Christian Skau.

Raussen and Skau: *The Abel Prize is certainly not the first important prize in mathematics that you have won. Let us just mention the Fields Medal that you received 35 years ago, the Swedish Crafoord Prize, the Italian Balzan Prize and the Israeli Wolf Prize. How important is it for you, as a mathematician, to win such prestigious prizes? And how important is it for the mathematical community that such prizes exist?*

Deligne: For me personally, it is nice to be told that mathematicians I respect find the work I have done interesting. The Fields Medal possibly helped me to be invited to the Institute for Advanced Study. To win prizes gives opportunities, but they have not changed my life.

I think prizes can be very useful when they can serve as a pretext for speaking about mathematics to the general public. I find it particularly nice that

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the Abel Prize is connected with other activities such as competitions directed towards children and the Holmboe Prize for high school teachers. In my experience, good high school teachers are very important for the development of mathematics. I think all these activities are marvellous.

Youth

Raussen and Skau: *You were born in 1944, at the end of the Second World War in Brussels. We are curious to hear about your first mathematical experiences: In what respect were they fostered by your own family or by school? Can you remember some of your first mathematical experiences?*

Deligne: I was lucky that my brother was seven years older than me. When I looked at the thermometer and realized that there were positive and negative numbers, he would try to explain to me that minus one times minus one is plus one. That was a big surprise. Later when he was in high school he told me about the second degree equation. When he was at the university he gave me some notes about the third degree equation, and there was a strange formula for solving it. I found it very interesting.

When I was a Boy Scout, I had a stroke of extraordinary good luck. I had a friend there whose father, Monsieur Nijs, was a high school teacher. He helped me in a number of ways; in particular, he gave me my first real mathematical book, namely *Set Theory* by Bourbaki, which is not an obvious choice to give to a young boy. I was fourteen years old at the time. I spent at least a year digesting that book. I guess I had some other lectures on the side, too.

Having the chance to learn mathematics at one's own rhythm has the benefit that one revives surprises of past centuries. I had already read elsewhere how rational numbers, then real numbers, could be defined starting from the integers. But I remember wondering how integers could be defined from set theory, looking a little ahead in Bourbaki, and admiring how one could first define what it means for two sets to have the "same number of elements", and derive from this the notion of integers. I was also given a book on complex variables by a friend of the family. To see that the story of complex variables was so different from the story of real variables was a big surprise: once differentiable, it is analytic (has a power series expansion), and so on. All those things that you might have found boring at school were giving me a tremendous joy.

Then this teacher, Monsieur Nijs, put me in contact with Professor Jacques Tits at the University of Brussels. I could follow some of his courses and seminars, though I still was in high school.

Raussen and Skau: *It is quite amazing to hear that you studied Bourbaki, which is usually considered quite difficult, already at that age. Can you tell us a bit about your formal school education? Was that interesting for you, or were you rather bored?*

Deligne: I had an excellent elementary school teacher. I think I learned a lot more in elementary school than I did in high school: how to read, how to write, arithmetic, and much more. I remember how this teacher made an experiment in mathematics that made me think about proofs, surfaces, and lengths. The problem was to compare the surface of a half-sphere with that of the disc with the same radius. To do so, he covered both surfaces with a spiralling rope. The half sphere required twice as much rope. This made me think a lot: how could one measure a surface with a length? How to be sure that the surface of the half sphere was indeed twice that of the disc?

When I was in high school, I liked problems in geometry. Proofs in geometry make sense at that age because surprising statements have not-too-difficult proofs. Once we were past the axioms, I enjoyed very much doing such exercises. I think that geometry is the only part of mathematics where proofs make sense at the high school level. Moreover, writing a proof is another excellent exercise. This does not only concern mathematics, you also have to write in correct French—in my case—in order to argue why things are true. There is a stronger connection between language and mathematics in geometry than for instance in algebra, where you have a set of equations. The logic and the power of language are not so apparent.

Raussen and Skau: *You went to the lectures of Jacques Tits when you were only sixteen years old. There is a story that one week you could not attend because you participated in a school trip…?*

Deligne: Yes. I was told this story much later. When Tits came to give his lecture he asked: Where is Deligne? When it was explained to him that I was on a school trip, the lecture was postponed to the next week.

Raussen and Skau: *He must already have recognized you as a brilliant student. Jacques Tits is also a recipient of the Abel Prize. He received it together with John Griggs Thompson five years ago for his great discoveries in group theory. He was surely an influential teacher for you?*

Deligne: Yes; especially in the early years. In teaching, the most important can be what you don't do. For instance, Tits had to explain that the center of a group is an invariant subgroup. He started a proof, then stopped and said in essence: "An invariant subgroup is a subgroup stable by all inner automorphisms. I have been able to define the center. It is hence stable by all symmetries of the data. So it is obvious that it is invariant."

For me, this was a revelation: the power of the idea of symmetry. That Tits did not need to go through a step-by-step proof, but instead could just say that symmetry makes the result obvious, has influenced me a lot. I have a very big respect for symmetry, and in almost every one of my papers there is a symmetry-based argument.

Raussen and Skau: *Can you remember how Tits discovered your mathematical talent?*

Deligne: That I cannot tell, but I think it was Monsieur Nijs who told him to take good care of me. At that time, there were three really active mathematicians in Brussels: apart from Tits himself, Professors Franz Bingen and Lucien Waelbroeck. They organized a seminar with a different subject each year. I attended these seminars, and I learned about different topics such as Banach algebras, which were Waelbroeck's speciality, and algebraic geometry.

Then, I guess, the three of them decided it was time for me to go to Paris. Tits introduced me to Grothendieck and told me to attend his lectures as well as Serre's. That was an excellent advice.

Raussen and Skau: *This can be a little surprising to an outsider. Tits being interested in you as a mathematician, one might think that he would try to capture you for his own interests. But he didn't?*

Deligne: No. He saw what was best for me and acted accordingly.

Algebraic Geometry

Raussen and Skau: *Before we proceed to your career in Paris, perhaps we should try to explain to the audience what your subject, algebraic geometry, is about.*

When Fields medalist Tim Gowers had to explain your research subjects to the audience during the Abel Prize announcement earlier this year, he began by confessing that this was a difficult job for him. It is difficult to show pictures that illustrate the subject, and it is also difficult to explain some simple applications. Could you, nevertheless, try to give us an idea what algebraic geometry is about? Perhaps you can mention some specific problems that connect algebra and geometry with each other.

Deligne: In mathematics, it is always very nice when two different frames of mind come together. Descartes wrote: "La géométrie est l'art de raisonner juste sur des figures fausses (Geometry is the art of correct reasoning on false figures)." "Figures" is plural: it is very important to have various perspectives and to know in which way each is wrong.

In algebraic geometry, you can use intuitions coming both from algebra—where you can manipulate equations—and from geometry, where you can draw pictures. If you picture a circle and consider the equation $x^2 + y^2 = 1$, different images are evoked in your mind, and you can try to play one against the other. For instance, a wheel is a circle and a wheel turns; it is interesting to see what the analogue is in algebra: an algebraic transformation of x and y maps any solution of $x^2 + y^2 = 1$ to another. This equation describing a circle is of the second degree. This implies that a circle will have no more than two intersection points with a line. This is a property you also see geometrically, but the algebra gives more. For instance, if the line has a rational equation and one of the intersection points with the circle $x^2 + y^2 = 1$ has rational coordinates, then the other intersection point will also have rational coordinates.

Algebraic geometry can have arithmetical applications. When you consider polynomial equations, you can use the same expressions in different number systems. For instance, on finite sets on which addition and multiplication are defined, these equations lead to combinatorial questions: you try to count the number of solutions. But you can continue to draw the same pictures, keeping in mind a new way in which the picture is false, and in this way you can use geometrical intuition while looking at combinatorial problems.

I have never really been working at the center of algebraic geometry. I have mostly been interested in all sorts of questions that only touch the area. But algebraic geometry touches many subjects! As soon as a polynomial appears, one can try to think about it geometrically; for example in physics with Feynman integrals, or when you consider an integral of a radical of a polynomial expression. Algebraic geometry can also contribute to the understanding of integer solutions of polynomial equations. You have the old story of elliptic functions: to understand how elliptic integrals behave, the geometrical interpretation is crucial.

Raussen and Skau: *Algebraic geometry is one of the main areas in mathematics. Would you say that to learn algebraic geometry requires much more effort than other areas in mathematics, at least for a beginner?*

Deligne: I think it's hard to enter the subject because one has to master a number of different tools. To begin with, cohomology is now indispensable. Another reason is that algebraic geometry developed in a succession of stages, each with its own language. First, the Italian school which was a little hazy, as shown by the infamous saying: "In algebraic geometry, a counterexample to a theorem is a useful addition to it." Then Zariski and Weil put things on a better footing. Later Serre and Grothendieck gave it a new language, which is very powerful. In this language of schemes one can express a lot; it covers both arithmetical applications and more geometrical aspects. But it requires time to understand the power of this language. Of course, one needs to know a number of basic theorems, but I don't think that this is the main stumbling block. The most difficult is to understand the power of the language created by Grothendieck and how it relates to our usual geometrical intuition.

Apprentice in Paris

Raussen and Skau: *When you came to Paris you came in contact with Alexander Grothendieck and Jean-Pierre Serre. Could you tell us about your first impression of these two mathematicians?*

Deligne: I was introduced to Grothendieck by Tits during the Bourbaki seminar of November 1964. I was really taken aback. He was a little strange, with his shaved head, a very tall man. We shook hands but did nothing more until I went to Paris a few months later to attend his seminar.

That was really an extraordinary experience. In his way, he was very open and kind. I remember the first lecture I attended. In it, he used the expression "cohomology object" many times. I knew what cohomology was for abelian groups, but I did not know the meaning of "cohomology object". After the lecture I asked him what he meant by this expression. I think that many other mathematicians would have thought that if you didn't know the answer, there wouldn't be any point to speak to you. This was not his reaction at all. Very patiently he told me that if you have a long exact sequence in an abelian category and you look at the kernel of one map, you divide by the image of the previous one and so on… I recognized quickly that I knew about this in a less general context. He was very open to people who were ignorant. I think that you should not ask him the same stupid question three times, but twice was all right.

I was not afraid to ask completely stupid questions, and I have kept this habit until now. When attending a lecture, I usually sit in front of the audience, and if there is something I don't understand, I ask questions even if I would be supposed to know what the answer was.

I was very lucky that Grothendieck asked me to write up talks he had given the previous year. He gave me his notes. I learned many things, both the content of the notes, and also a way of writing mathematics…. This was both in a prosaic way, namely that one should write only on one side of the paper and leave some blank space so he could make comments, but he also insisted that one was not allowed to make any false statement. This is extremely hard. Usually one takes shortcuts; for instance, not keeping track of signs. This would not pass muster with him. Things had to be correct and precise. He told me that my first version of the redaction was much too short, not enough details…. It had to be completely redone. That was very good for me.

Serre had a completely different personality. Grothendieck liked to have things in their natural generality; to have an understanding of the whole story. Serre appreciates this, but he prefers beautiful special cases. He was giving a course at Collège de France on elliptic curves. Here, many different strands come together, including automorphic forms. Serre had a much wider mathematical culture than Grothendieck. In case of need, Grothendieck redid everything for himself, while Serre could tell people to look at this or that in the literature. Grothendieck read extremely little; his contact with classical Italian geometry came basically through Serre and Dieudonné. I think Serre must have explained to him what the Weil conjectures were about and why they were interesting. Serre respected the big constructions Grothendieck worked with, but they were not in his taste. Serre preferred smaller objects with beautiful properties such as modular forms, to understand concrete questions, for instance congruences between coefficients.

Their personalities were very different, but I think that the collaboration between Serre and Grothendieck was very important and it enabled Grothendieck to do some of his work.

Raussen and Skau: *You told us that you needed to go to Serre's lectures in order to keep your feet on the ground?*

Deligne: Yes, because there was a danger in being swept away in generalities with Grothendieck. In my opinion, he never invented generalities that were fruitless, but Serre told me to look at different topics that all proved to be very important for me.

The Weil Conjectures

Raussen and Skau: *Your most famous result is the proof of the third—and the hardest—of the so-called Weil conjectures. But before talking about your achievement, can you try to explain why the Weil conjectures are so important?*

Deligne: There were some previous theorems of Weil about curves in the one-dimensional situation. There are many analogies between algebraic curves over finite fields and the rational numbers. Over the rational numbers, the central question is the Riemann hypothesis. Weil had proved the analogue of the Riemann hypothesis for curves over finite fields, and he had looked at some higher-dimensional situations as well. This was at the time where one started to understand the cohomology of simple algebraic varieties, like the Grassmannians. He saw that some point-counting for objects over finite fields reflected what happened over the complex numbers and the shape of the related space over the complex numbers.

As Weil looked at it, there are two stories hidden in the Weil conjectures. First, why should there be a relation between apparently combinatorial questions and geometric questions over the complex numbers? Second, what is the analogue of the Riemann hypothesis? Two kinds of applications came out of these analogies. The first started with Weil himself: estimates for some arithmetical functions. For me, they are not the most important. Grothendieck's construction of a formalism explaining why there should be a relation between the story over the complex numbers, where one can use topology, and the combinatorial story, is more important.

Secondly, algebraic varieties over finite fields admit a canonical endomorphism, the Frobenius. It can be viewed as a symmetry, and this symmetry makes the whole situation very rigid. Then one can transpose this information back into the geometric world over the complex numbers; it yields constraints on what will happen in classical algebraic geometry, and this is used in applications to representation theory and the theory of automorphic forms. It was not obvious at first that there would be such applications, but for me they are the reason why the Weil conjecture is important.

Raussen and Skau: *Grothendieck had a program on how to prove the last Weil conjecture, but it didn't work out. Your proof is different. Can you comment on this program? Did it have an influence on the way you proved it?*

Deligne: No. I think that the program of Grothendieck was, in a sense, an obstruction to finding the proof, because it made people think in just a certain direction. It would have been more satisfying if one had been able to do the proof following the program, because it would have explained a number of other interesting things as well. But the whole program relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.

I used a completely different idea. It is inspired by the work of Rankin and his work on automorphic forms. It still has a number of applications, but it did not realize the dream of Grothendieck.

Raussen and Skau: *We heard that Grothendieck was glad that the Weil conjecture was proved, of course, but still he was a little disappointed?*

Deligne: Yes. And with very good reason. It would have been much nicer if his program had been realized. He did not think that there would be another way to do it. When he heard I had proved it, he felt I must have done this and that, which I hadn't. I think that's the reason for the disappointment.

Raussen and Skau: *You have to tell us about the reaction of Serre when he heard about the proof.*

Deligne: I wrote him a letter when I did not have a complete proof yet, but a test case was clear. I think he got it just before he had to go to the hospital for an operation of a torn tendon. He told me later that he went into the operation theatre in a euphoric state because he knew now that the proof was roughly done.

Raussen and Skau: *Several famous mathematicians have called your proof of the last Weil conjecture a marvel. Can you describe how you got the ideas that led to the proof?*

Deligne: I was lucky that I had all the tools needed at my disposal at the same time and that I understood that those tools would do it. Parts of the proof have since been simplified by Gérard Laumon, and a number of these tools are no longer needed.

At the time, Grothendieck had ideas for putting into a purely algebraic framework the work of Solomon Lefschetz from the 1920s about families of hyperplane sections of an algebraic variety. Of particular interest was a statement of Lefschetz, later proved by William Hodge, the so-called hard Lefschetz theorem. Lefschetz's approach was topological. In contrast to what one might think, if arguments are topological there is a better chance to translate them into abstract algebraic geometry than if they are analytic, such as the proof given by Hodge. Grothendieck asked me to look at the 1924 book *L'analysis situs et la géométrie algébrique* by Lefschetz. It is a beautiful and very intuitive book, and it contained some of the tools I needed.

I was also interested in automorphic forms. I think it is Serre who told me about an estimate due to Robert Rankin. I looked carefully at it. Rankin was getting some nontrivial estimates for coefficients of modular forms by proving for some related L-functions what was needed to apply results of Landau, in which the location of the poles of an L-function gave information on the poles of the local factors. I saw that the same tool, in a much less sophisticated way, just using that a sum of squares is positive, could be used here because of the control the work of Grothendieck gave on poles. This was enough. The poles were much easier to understand than the zeros and it was possible to apply Rankin's idea.

I had all these tools at my disposal, but I cannot tell how I put them together.

Raussen and Skau: *What is a motive?*

Deligne: A surprising fact about algebraic varieties is that they give rise not to one, but to many cohomology theories, among them the l-adic theories, one for each prime l different from the characteristic, and in characteristic zero, the algebraic de Rham cohomology. These theories seem to tell the same story, over and over again, each in a different language. The philosophy of motives is that there should exist a universal cohomology theory, with values in a category of motives to be defined, from which all these theories could be derived. For the first cohomology group of a projective nonsingular variety, the Picard variety plays the role of a motivic H^1: the Picard variety is an abelian variety, and from it the H^1 in all available cohomology theories can be derived. In this way, abelian varieties (taken up to isogeny) are a prototype for motives.

A key idea of Grothendieck is that one should not try to define what a motive is. Rather, one should try to define the category of motives. It should be an abelian category with finite dimensional rational vector spaces as *Hom* groups.

Crucially, it should admit a tensor product, needed to state a Künneth theorem for the universal cohomology theory, with values in the category of motives.

If only the cohomology of projective nonsingular varieties is considered, one speaks of pure motives. Grothendieck proposed a definition of a category of pure motives and showed that, if the category defined had a number of properties, modeled on those of Hodge structures, the Weil conjectures would follow.

For the proposed definition to be viable, one needs the existence of "enough" algebraic cycles. On this question almost no progress has been made.

A Little Bit about Subsequent Work

Raussen and Skau: *What about your other results? Which of those that you worked on after the proof of the Weil conjecture are you particularly fond of?*

Deligne: I like my construction of a so-called mixed Hodge structure on the cohomology of complex algebraic varieties. In its genesis, the philosophy of motives has played a crucial role, even if motives don't appear in the end result. The philosophy suggests that, whenever something can be done in one cohomology theory, it is worthwhile to look for a counterpart in other theories. For projective nonsingular varieties, the role played by the action of Galois is similar to the role played by the Hodge decomposition in the complex case. For instance, the Hodge conjecture, expressed using the Hodge decomposition, has as counterpart the Tate conjecture, expressed using the action of Galois. In the l-adic case, cohomology and the action of Galois remain defined for singular or noncompact varieties.

This forces us to ask: what is the analogue in the complex case? One clue is given by the existence, in l-adic cohomology, of an increasing filtration, the weight filtration W, for which the i-th quotient W_i/W_{i-1} is a subquotient of the cohomology of a projective nonsingular variety. We hence expect in the complex case a filtration W such that the i-th quotient has a Hodge decomposition of weight i. Another clue, coming from works of Griffiths and Grothendieck, is that the Hodge filtration is more important than the Hodge decomposition. Both clues force the definition of mixed Hodge structures, suggest that they form an abelian category, and suggest also how to construct them.

Raussen and Skau: *What about the Langlands program? Have you been involved in it?*

Deligne: I have been very interested in it, but I have contributed very little. I have only done some work on $GL(2)$, the linear group in two variables. I tried to understand things. A somewhat remote application of the Weil conjecture has been used in Ngo's recent proof of what is called the fundamental lemma. I didn't do a lot of work myself, though I had a lot of interest in the Langlands program.

French, American, and Russian Mathematics

Raussen and Skau: *You have already told us about the two institutions you mainly have worked for, namely the IHÉS in Paris and then, since 1984, the IAS in Princeton. It would be interesting for us to hear what your motives were for leaving IHÉS and moving to Princeton. Moreover, we would like to hear what unites the two institutions and how they differ, in your opinion.*

Deligne: One of the reasons I left was that I don't think it's good to spend all of one's life in the same place. Some variation is important. I was hoping to have some contact with Harish-Chandra, who had done some beautiful work in representation theory and automorphic forms. That was a part of the Langlands program that I am very interested in, but unfortunately Harish-Chandra died shortly before I arrived at Princeton.

Another reason was that I had imposed on myself to give seminars, each year on a new subject, at the IHÉS in Bures. That became a little too much. I was not really able to both give the seminars and to write them down, so I did not impose the same obligation on myself after I came to Princeton. These are the main reasons why I left the IHÉS for IAS in Princeton.

Concerning the difference between the two institutions, I would say that the Institute for Advanced Study is older, bigger, and more stable. Both are very similar in the way that there are many young visitors who come there. So they are not places where you can fall asleep since you will always be in contact with young people who will tell you that you are not as good as you think you are.

In both places there are physicists, but I think the contact with them was more fruitful for me in Princeton than it was in Bures. In Princeton, there have been common seminars. One year was very intense, with both mathematicians and physicists participating. This was due mainly to the presence of Edward Witten. He has received the Fields Medal even though he is a physicist. When Witten asks me questions, it's always very interesting to try to answer them, but it can be frustrating as well.

Princeton is also bigger in the sense that it has not only math and physics, but also the School of Historical Studies and the School of Social Sciences. There is no real scientific interaction with these schools, but it is pleasant to be able to go and hear a lecture about, for instance, ancient China. One good feature about Bures which you do not have in Princeton is the following: In Bures, the cafeteria is too small. So you sit where you can and you don't get to choose the people you are sitting with. I was often sitting next to an analyst or a physicist, and such random informal interactions are very useful. In Princeton, there is one table for the mathematicians, another for the astronomers, the ordinary physicists, and so on. You will not be told to go away if you sit down at the wrong table, but still there is segregation.

The Institute for Advanced Study has a big endowment, while the IHÉS had none, at least when I was there. This didn't affect the scientific life. Sometimes it created instability, but the administration was usually able to hide the difficulties from us.

Raussen and Skau: *Apart from your connections with French and U.S. mathematics, you have also had a very close contact with Russian mathematics for a long time, even from long before the fall of the Iron Curtain. In fact, your wife is the daughter of a Russian mathematician. How did your contact with Russian mathematics develop?*

Deligne: Grothendieck or Serre told Manin, who was in Moscow at the time, that I had done some interesting work. The Academy [Russian Academy of Sciences] invited me to a conference for I. M. Vinogradov, a terribly anti-Semitic person, by the way. I came to Russia, and I found a beautiful culture for mathematics. At that time mathematics was one of the few subjects where the Communist Party could not meddle, as it did not understand it at all, and this turned it into a space of freedom.

We would go to somebody's home and sit by the kitchen table to discuss mathematics over a cup of tea. I fell in love with the atmosphere and this enthusiasm for mathematics. Moreover, Russian mathematics was one of the best in the world at that time. Today there are still good mathematicians in Russia, but there has been a catastrophic emigration. Furthermore, among those wanting to stay, many need to spend at least half of the time abroad, just to make a living.

Raussen and Skau: *You mentioned Vinogradov and his anti-Semitism. You talked to somebody and asked whether he was invited?*

Deligne: It was Piatetskii-Shapiro. I was completely ignorant. I had a long discussion with him. For me it was obvious that someone like him should be invited by Vinogradov, but I was told that that was not the case.

After this introduction to Russian mathematics, I still have some nostalgia for the beautiful memories of being in Moscow and speaking with Yuri Manin and Sergey Bernstein or being at the Gelfand seminar. There was a tradition, which still exists, of a strong connection between the university and the secondary education. People like Andrey Kolmogorov had a big interest in secondary education (perhaps not always for the best).

They have also the tradition of Olympiads, and they are very good at detecting promising people in mathematics early on in order to help them. The culture of seminars is in danger because it's important that the head of the seminars is working full-time in Moscow, and that is not always the case. There is a whole culture which I think is important to preserve. That is the reason why I used half of the Balzan Prize to try to help young Russian mathematicians.

Raussen and Skau: *That was by a contest that you arranged.*

Deligne: Yes. The system is falling apart at the top because there is no money to keep people, but the infrastructure was so good that the system continues to produce very good young mathematicians. One has to try to help them and make it possible for them to stay somewhat longer in Russia so that the tradition can continue.

Competition and Collaboration in Mathematics

Raussen and Skau: *Some scientists and mathematicians are very much driven by the aim to be the first to make major discoveries. That seems not to be your main driving force?*

Deligne: No. I don't care at all.

Raussen and Skau: *Do you have some comments on this culture in general?*

Deligne: For Grothendieck it was very clear: he once told me that mathematics is not a competitive sport. Mathematicians are different, and some will want to be the first, especially if they are working on very specific and difficult questions. For me it's more important to create tools and to understand the general picture. I think mathematics is much more a collective enterprise of long duration. In contrast to what happens in physics and biology, mathematical articles have long and useful lives. For instance, the automatic evaluation of people using bibliographic criteria is particularly perverse in mathematics, because those evaluation methods take account only of papers published during the last three or five years. This does not make sense in mathematics. In a typical paper of mine, I think at least half of the papers cited can be twenty to thirty years old. Some will even be two hundred years old.

Raussen and Skau: *You like to write letters to other mathematicians?*

Deligne: Yes. Writing a paper takes a lot of time. Writing it is very useful, to have everything put together in a correct way, and one learns a lot doing so, but it's also somewhat painful. So in the beginning of forming ideas, I find it very convenient to write a letter. I send it, but often it is really a letter to myself. Because I don't have to dwell on things the recipient knows about, some short-cuts will be all right. Sometimes the letter, or a copy of it, will stay in a drawer for some years, but it preserves ideas, and when I eventually write a paper, it serves as a blueprint.

Raussen and Skau: *When you write a letter to someone and that person comes up with additional ideas, will that result in a joint paper?*

Deligne: That can happen. Quite a lot of my papers are by me alone, and some are joint work with people having the same ideas. It is better to make a joint paper than having to wonder who did what. There are a few cases of genuine collaborations where different people have brought different intuitions. This was the case with George Lusztig. Lusztig had the whole picture of how to use *l*-adic cohomology for group representations, but he did not know the techniques. I knew the technical aspect of *l*-adic cohomology, and I could give him the tools he needed. That was real collaboration.

A joint paper with Morgan, Griffiths, and Sullivan was also a genuine collaboration. Also with Bernstein, Beilinson, and Gabber: we put together our different understandings

Work Style, Pictures, and Even Dreams

Raussen and Skau: *Your CV shows that you haven't taught big classes of students a lot. So, in a sense, you are one of the few full-time researchers in mathematics.*

Deligne: Yes. And I find myself very lucky to have been in this position. I never had to teach. I like very much to speak with people. In the two institutions where I have worked young people come to speak with me. Sometimes I answer their questions, but more often I ask them counter-questions that sometimes are interesting, too. So this aspect of teaching with one-to-one contact, trying to give useful information and learning in the process, is important to me.

I suspect it must be very painful to teach people who are not interested, but are forced to learn math because they need the grade to do something else. I would find that repulsive.

Raussen and Skau: *What about your mathematical work style? Are you most often guided by examples, specific problems and computations, or are you rather surveying the landscape and looking for connections?*

Deligne: First I need to get some general picture of what should be true, what should be accessible, and what tools can be used. When I read papers I will not usually remember the details of the proofs, but I will remember which tools were used. It is important to be able to guess what is true and what is false in order not to do completely useless work. I don't remember statements that are proved, but rather I try to keep a collection of pictures in my mind—more than one picture, all false but in different ways, and knowing in which way they are false. For a number of subjects, if a picture tells me that something should be true, I take it for granted and will come back to the question later on.

Raussen and Skau: *What kind of pictures do you have of these very abstract objects?*

Deligne: Sometimes very simple things! For instance, suppose I have an algebraic variety, and hyperplane sections, and I want to understand how they are related, by looking at a pencil of hyperplane sections. The picture is very simple. I draw it in my mind something like a circle in the plane and a moving line that sweeps it. Then I know how this picture is false: the variety is not one-dimensional, but higher-dimensional, and when the hyperplane section degenerates, it is not just two intersection points coming together. The local picture is more complicated, like a conic that becomes a quadratic cone. These are simple pictures put together.

When I have a map from some space to another I can study properties it has. Pictures can then convince me that it is a smooth map. Besides having a collection of pictures, I also have a collection of simple counter-examples, and statements that I hope to be true have to be checked against both the pictures and the counter-examples.

Raussen and Skau: *So you think more in geometric pictures than algebraically?*

Deligne: Yes.

Raussen and Skau: *Some mathematicians say that good conjectures, or even good dreams, are at least as important as good theorems. Would you agree?*

Deligne: Absolutely. The Weil conjectures, for instance, have created a lot of work. Part of the conjecture was the existence of a cohomology theory for algebraic systems with some properties. This was a vague question, but that is all right. It took over twenty years of work, even a little more, in order to really get a handle on it. Another example of a dream is the Langlands program, which has involved many people over fifty years, and we have now only a slightly better grasp of what is happening.

Another example is the philosophy of motives of Grothendieck, about which very little is proved. There are a number of variants taking care of some of the ingredients. Sometimes, such a variant can be used to make actual proofs, but more often the philosophy is used to guess what happens, and then one tries to prove it in another way. These are examples of dreams or conjectures that are much more important than specific theorems.

Raussen and Skau: *Have you had a "Poincaré moment" at some time in your career where you, in a flash, saw the solution of a problem you had worked on for a long time?*

Deligne: The closest I have been to such a moment must have been while working on the Weil conjecture when I understood that perhaps there was a path using Rankin against Grothendieck. It took a few weeks after that before it really worked, so it was a rather slow development. Perhaps also for the definition of mixed Hodge structures, but also in this case, it was a progressive process. So it was not a complete solution in a flash.

Raussen and Skau: *When you look back on fifty years of doing mathematics, how have your work* *and your work style changed over the years? Do you work as persistently as you did in your early years?*

Deligne: I am not as strong as I was earlier, in the sense that I cannot work as long or as intensively as I did. I think I have lost some of my imagination, but I have much more technique that can act as a substitute to some extent. Also the fact that I have contact with many people gives me access to some of the imagination I am lacking myself. So when I bring my technique to bear, the work can be useful, but I'm not the same as when I was thirty.

Raussen and Skau: *You have retired from your professorship at IAS rather early…*

Deligne: Yes, but that's purely formal. It means I receive retirement money instead of a salary, and no school meetings for choosing next year's members. So that's all for the best. It gives me more time for doing mathematics.

Hopes for the Future

Raussen and Skau: *When you look at the development of algebraic geometry, number theory, and the fields that are close to your heart, are there any problems or areas where you would like to see progress soon? What would be particularly significant, in your opinion?*

Deligne: Whether or not it's within reach in ten years, I have absolutely no idea; as it should be… But I would very much like to see progress in our understanding of motives. Which path to take and what are the correct questions, is very much in the air. Grothendieck's program relied on proving the existence of algebraic cycles with some properties. To me this looks hopeless, but I may be wrong.

The other kind of question for which I would really like to see some progress is connected with the Langlands program, but that is a very long story…

In yet another direction, physicists regularly come up with unexpected conjectures, most often using completely illegal tools. But, so far, whenever they have made a prediction, for instance a numerical prediction on the number of curves with certain properties on some surface—and these are big numbers, in the millions perhaps—they were right! Sometimes previous computations by mathematicians were not in accordance with what the physicists were predicting, but the physicists were right. They have put their fingers on something really interesting, but we are, so far, unable to capture their intuition. Sometimes they make a prediction, and we work out a very clumsy proof without real understanding. That is not how it should be. In one of the seminar programs that we had with the physicists at IAS, my wish was not to have to rely on Ed Witten but instead to be able to make conjectures myself. I failed! I did not understand enough of their picture to be able to do that, so I still have to rely on Witten to tell me what should be interesting.

Raussen and Skau: *What about the Hodge conjecture?*

Deligne: For me, this is a part of the story of motives, and it is not crucial whether it is true or false. If it is true, that's very good, and it solves a large part of the problem of constructing motives in a reasonable way. If one can find another purely algebraic notion of cycles for which the analogue of the Hodge conjecture holds, and there are a number of candidates, this will serve the same purpose, and I would be as happy as if the Hodge conjecture were proved. For me it is motives, not Hodge, that is crucial.

Private Interests—and an Old Story

Raussen and Skau: *We have the habit of ending these interviews by asking questions that are outside of mathematics. Could you tell us a little bit about your private interests outside your profession? We know about your interest in nature and in gardening, for example.*

Deligne: These are my main interests. I find the earth and nature so beautiful. I don't like just to go and have a look at a scenery. If you really want to enjoy the view from a mountain, you have to climb it on foot. Similarly, to see nature, you have to walk. As in mathematics, in order to take pleasure in nature—and nature is a beautiful source of pleasure—one has to do some work.

I like to bicycle because that's also a way to look around. When distances are a little bigger than what is convenient on foot, this is another way of enjoying nature.

Raussen and Skau: *We heard that you also build igloos?*

Deligne: Yes. Unfortunately, there's not enough snow every year and even when there is, snow can be tricky. If it's too powdery, it's impossible to do anything; likewise if it's too crusty and icy. So there is maybe just one day, or a few hours each year when building an igloo is possible, and one has to be willing to do the work of packing the ice and putting the construction together.

Raussen and Skau: *And then you sleep in it?*

Deligne: And then I sleep in the igloo, of course.

Raussen and Skau: *You have to tell us what happened when you were a little child.*

Deligne: Yes. I was in Belgium at the seaside for Christmas, and there was much snow. My brother and sister, who are much older than me, had the nice idea to build an igloo. I was a little bit in the way. But then they decided I might be useful for one thing: if they grabbed me by my hands and feet, I could be used to pack the snow.

Raussen and Skau: *Thank you very much for granting us this interview. These thanks come also on behalf of the Norwegian, the Danish, and the European mathematical societies that we represent. Thank you very much!*

Deligne: Thank you.

An American Mathematician in Moscow, or How I Destroyed the Soviet Union

Melvyn B. Nathanson

Dedicated to I. M. Gel'fand on the 100th anniversary of his birth

The great Soviet mathematician Israel Moiseevich Gel'fand was born on September 2, 1913, in Okny (later Krasni Okny, or Red Okny) near Odessa in Ukraine and died on October 5, 2009, in New Brunswick, New Jersey. The Russian Revolutions of 1917 led to the approval of the Treaty of Creation of the USSR on December 29, 1922. The Soviet Union ceased to exist on December 26, 1991. Gel'fand was born before the Soviet Union and outlived it.

I was indirectly introduced to Gel'fand in 1970. I was a visiting research student in the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge during the Lent and Easter terms. One of my friends was Béla Bollobás, a Hungarian who had received his Ph.D. at Oxford and had decided to remain in England and not return to Hungary. In the terminology of the cold war, Béla had defected to the capitalists. Before being allowed to study in the West, the Hungarians had required him to study in the East, that is, in the USSR, and Béla had spent a year at Moscow State University, where he worked with Gel'fand. Professor Gel'fand had impressed him deeply as a man and as a mathematician, and Béla often told me how extraordinary he was.

In the summer of 1970, at the end of my study in Cambridge, I made a short trip to the Soviet Union. Another American, a biologist from MIT, had just completed a postdoctoral year at Cambridge and had posted a note on a bulletin board that he was looking for someone to accompany him on a driving tour through the USSR. The plan was to enter the Soviet Union by train from Helsinki to Leningrad, rent a car, and drive south through Moscow to the Caucasus. The biologist had been invited to lecture in Moscow as a guest of the Soviet Academy of Sciences. His official host was a distinguished Soviet biochemist, David Gold'farb. I also met Gold'farb and told him that I would like to visit Gel'fand. Gold'farb contacted Gel'fand, who was too busy or (more likely) too prudent to rendezvous with an unknown American. When Gold'farb told Gel'fand that I was interested in number theory, Gel'fand gave him to give to me a copy of his book *Representation Theory and Automorphic Functions*, written with M. I. Graev and I. I. Pyatetskii-Shapiro, the sixth volume of the series of monographs Generalized Functions.

Gold'farb had lost a leg fighting in World War II. He had wanted to be a historian, but history is a dangerous profession in totalitarian regimes. In Stalin's Russia, history was particularly life-threatening, so Gold'farb went to medical school and did research in molecular genetics. "A stomach is always a stomach," he told me.

My biologist traveling companion was very leftwing politically. Like most academics, I was against the war in Vietnam, but he was so far to my left that by comparison I seemed to be on the right. That immediately endeared me to Soviet scientists, many of whom acted in public as if they were loyal followers of the Communist Party line, but inwardly were strongly antitotalitarian and, indeed, unlike most American scientists, supported American intervention in Vietnam. They believed in killing

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Commies. This was one of the first lessons in irony that I learned in Moscow.

Many senior Communist Party officials pulled strings to get their children into scientific careers. They had done what they needed to do to survive, they understood the dangers of politics, and they wanted safer lives for their own children. Many graduate students and researchers in the Soviet Academy of Sciences and in Soviet universities were relatives of high-ranking party functionaries. Of course, being a scientist with ties to the Communist Party brought its own benefits. Many distinguished non-party Soviet researchers collaborated with colleagues who were much weaker scientists, but politically powerful, in order to shelter themselves and their students. "Collaboration" often meant nothing more than adding the names of mediocre party members to their papers as coauthors. Of course, this kind of scientific politics also occurs in nontotalitarian regimes.

It was because of Béla Bollobás that I got the idea of trying to spend a year studying mathematics with Gel'fand. This would not be easy to arrange. During the cold war it was almost impossible for an American to study or do research in Moscow. To study in Cambridge or Paris was trivial. Just get on a plane and fly to England or France. But the only way an ordinary American could enter the Soviet Union was on a brief and expensive tourist visit. There were, however, two formal academic exchanges. One was between the National Academy of Sciences in Washington and the Soviet Academy of Science, but this usually provided only short-term visits for senior scientists, not young scholars.

The other program was part of a broad cultural affairs treaty between the US and the USSR. We would send the New York Philharmonic to Moscow, and they would send the Bolshoi Ballet to New York. One small part of this treaty was a university-level exchange for graduate students and postdocs. Each year the Americans selected forty American scholars, and the Soviets selected forty Soviet scholars. Each country's choices had to be approved by the other. The US program was administered by IREX, the International Research and Exchanges Board, an organization based in New York and associated with the American Council of Learned Societies.

Typically, the Russians sent forty engineers and computer scientists to MIT, and the Americans sent forty students to Moscow to study Dostoevsky and Rasputin. IREX had sent very few scientists to the USSR. The logic on the American side was reasonable: Because there were so few opportunities to do research in the USSR, an American exchange student should have a research project that could not be carried out anywhere else in the world. Science is everywhere, so it would be hard to argue that a scientific research problem could only be solved in the USSR. On the other hand, if your scholarly work were in Russian or Soviet literature or history and if the archives you needed were in the Soviet Union, then you clearly had a research proposal that would justify a trip to Moscow.

In 1972–73 I became the first American mathematician to participate in the IREX exchange program. In my application to IREX, I wrote that the Soviet Union had many of the greatest mathematicians in the world, that they were concentrated in Moscow and not allowed to travel outside the country, and that it would be extremely valuable to be able to meet and work with them. The argument had merit and was accepted by IREX. My research proposal was to work with Gel'fand. He had to agree to supervise me, and he did. I got my visa and went.

The Americans on the IREX exchange were supposed to rendezvous in Paris in August and fly together to Moscow. I got to Paris a few days early with a suitcase full of math books and an Olivetti portable typewriter, stayed in a cheap hotel on Rue des Écoles, and worked desperately hard to finish what would become my first joint paper with Paul Erdős. I did not have time to mail the manuscript from France, but IREX exchange students had certain privileges at the American Embassy in Moscow, and one of the most valuable was the use of the diplomatic pouch for sending letters out of the USSR. The Soviet postal system was, to put it politely, "unreliable" for manuscripts being sent to the United States, but I was able to submit my paper to the *Proceedings of the AMS* in a mailsack hand-carried by a US Marine to Vienna.

My first meeting with Gel'fand was in the lobby of Moscow State University. I remember two things that he told me. The first was his famous mantra: "There is only one mathematics." Then, after reciting a short list of the best young mathematicians in the Soviet Union, he said, "They know much more mathematics than I, but my intuition is better." Gel'fand suggested that I attend courses by Pyatetskii-Shapiro and Manin, but the most important part of my mathematical education in Moscow was participation in Gel'fand's famous Monday night seminar. I don't recall the official starting time of the seminar. People would show up early and talk mathematics in the hall, the seminar would eventually begin, and there would be a series of speakers, lasting long after the seminar was supposed to end, until finally we were evicted by a cleaning lady who had to do her job.

It was common in the seminar for Gel'fand to interrupt a talk and ask someone in the audience to explain what was going on. The first time I went to the seminar, in the middle of a lecture, he asked, "Melvyn, do you understand?"

"No," I replied.

"Why not?"

"Because they're speaking Russian."

He then assigned Dima Fuks the task of sitting next to me and translating the lecture from Russian into English. In a short time my Russian improved enough that I could understand the talks, and the language excuse was lost.

Gel'fand would decide that someone needed to learn something and present it in the seminar. For example, he asked Arnol'd to give a series of talks on p-adic numbers. Arnol'd seemed to find this difficult. A master of the real and complex domains, he had trouble understanding non-Archimedean absolute values. Of course, this is something every young number theorist knows. To see a great mathematician like Arnol'd struggling with p-adic analysis teaches that you are not an idiot if you don't understand some piece of mathematics that "everyone" finds trivial.

Walking is a Russian tradition. The winters are cold, but there is little wind and the effective temperature is certainly bearable. After the seminars, a group of people would often leave with Gel'fand and walk and talk late into the night while writing mathematics in the snowdrifts along the sidewalks. Outside you could talk more freely than in rooms where the walls had ears. In the course of the year, many mathematicians would ask me to go for a walk, and in the privacy of the streets would ask, "What is it like in America?" "How much anti-Semitism is in America?" "How hard is it to get a job in an American university?" In a few years, as soon as emigration became possible, they all emigrated.

Gel'fand immigrated to the United States in 1989. He was a visiting professor at Harvard and MIT and then distinguished professor at Rutgers University. The biochemist David Gold'farb also left Russia for New York.

American students on the IREX exchange lived in the dormitory of Moscow State University. We were told that Americans were always assigned the same rooms, not on the same floor, but on different floors, one room directly above the other, because it was easier to bug them by dropping wires vertically through the building. A standard joke was "If you need something in your room fixed, speak into the lightbulb." I had many friends who were active in the university Komsomol, the youth division of the Communist Party. They were, presumably, assigned to befriend Americans. One of them told me, "They can't identify all the voices on the tapes from your room."

The Komsomol mirrored life outside the university, where Communist Party leaders had perks and privileges not available to the hoi poloi. For example, there were private parties in the university for the Komsomol elite only. I attended a party with music provided by a rock band brought in from Estonia. When "ordinary students" tried to crash the party, the Komsomol called the police.

I flew back to Philadelphia during the Christmas break to visit my mother, who was sick. When I returned, Gel'fand asked, "How did it feel to be in the Soviet Union, then back in the US, then back in Moscow?" I replied, "OK, but for the first week at home I was afraid to use the phone."

Gel'fand urged me to read Russian literature, especially Pushkin. He gave me a recording of Pushkin's poem, *Mozart and Salieri*, and novels by Ilf and Petrov. He also gave me various mathematics books, including his book with Minlos and Shapiro, *Representations of the Rotation Group and Lorentz Group*, and the English edition of Weil's *Basic Number Theory*. Gel'fand had an enormous capacity to create friendships. André Weil visited him in Moscow, and they became close. After I returned to the US, Weil invited me to spend a year as his assistant at the Institute for Advanced Study. I do not know, but always assumed, that Gel'fand had recommended me, and his friend Weil obliged.

You learned in Moscow to keep your Soviet friends in disjoint circles. Knowing an American was dangerous, informants were ubiquitous, and some of your acquaintances were undoubtedly reporting on you to the "competent organs", which wanted to know everyone with whom an American was in contact. There was no reason for me to be paranoid, only careful. I always felt completely safe because I held an American passport. The Russians would not want to create an international incident. I might be arrested and threatened, but if I kept cool I would only be deported, which was no big deal. But Soviet citizens could really be endangered, expelled from universities, fired from jobs, their lives seriously impacted. Even though the United States was intensely waging the Vietnam war and we were bombing Hanoi, Kissinger's policy of a multitrack foreign policy with the stick in Southeast Asia and the carrot in USSR, namely, the allure of American trade concessions and exports to Russia, convinced me that an American in Moscow on an official academic exchange program whose only crime was "acting like an American", not espionage, was perfectly safe.

In Moscow State University, as in all Soviet universities, there was the "First Department" (in Russian, the *Pervii Otdel*), which was the KGB office within the university. After Gel'fand had emigrated and was a professor at Rutgers, he recounted the following story. "I could not tell you this when you were in Moscow," he said, "but during your stay here I was visited by someone from the Pervii Otdel. The KGB officer told me, 'You have an American student, Nathanson. I know Americans are independent, but Nathanson is too independent even for an American, and we have to expel him from the country.'"

Gel'fand, who possessed great political savvy, replied, "Of course, you should expel him if you have to, but I know that Nathanson has many important friends in America, and, if you expel him, there will be an international furor. It might be better to let him finish the year and then not let him return." That's what happened.

Gel'fand thought it would be good for my mathematical education to spend another year with him in Moscow, but it would clearly be impossible for me to return to Moscow State University. The other US–USSR scientific exchange program was with the Soviet Academy of Sciences. Gel'fand told me to apply to the Academy exchange, but not to request placement in the Stekhlov Institute, which was the notoriously anti-Semitic mathematics institute in the Soviet Academy. Instead, Gel'fand recommended that I apply to the Institute for Problems in the Transmission of Information, where several first-rate Soviet Jewish mathematicians found safe haven. In 1977 I applied and was accepted, but at the last minute the Soviet Foreign Ministry refused to issue me a visa and I could not go. This action did, in fact, become an international incident, with coverage in the *New York Times* and news media around the world, as a Soviet violation of the human rights for scientists provisions of the Helsinki Accords.

Under Communism, whether in the Soviet Union or in Eastern Bloc countries, there was a strange and tense separation of one's inner life and outer life, between what one had to say and do in front of strangers and how one thought and acted with friends you really trusted. It was, as Russians liked to say, "sloznii", that is, "complicated". Academic jobs in Moscow typically went to those who were well connected and acceptable to the Communist Party. There was no great monetary reward for studying mathematics or, more precisely, for living mathematics. It was done for free, for love, for intellectual and emotional enrichment, and not, as often in the West, for professional advancement. With the collapse of the USSR, Russian mathematics lost some of its purity and became more, in the American and European sense, "professional". On the other hand, now you can buy meat in Irkutsk, so we in the West, who never experienced Soviet-scale deprivation, should not be disparaging about this. To an American in Moscow during the cold war, the intellectual quality of life in mathematical circles was awesome.

Kazhdan once said that when he got to know me, he had never met anyone with my attitude, a kind of unfrightened, relaxed approach to life. In Russia everyone was constantly on guard, alert to danger, afraid of saying something that could unintentionally, or malignantly intentionally, be misinterpreted, reported to the "competent authorities", and cause expulsion from school, exile, imprisonment, or death. One had to be always vigilant. One had to make decisions: Who do I trust? How much can I trust this person? How open can I be? Is this guy reporting on me to the KGB? Will this person lie about me for no obvious reason? Americans don't understand this pressure. We have grown up unpersecuted and without fear of persecution. Our country is rich, even if we are not, and there is a sense of fairness. Even an older generation, in the bad but brief period of communist witch hunts and McCarthyism, never had to fear what Soviet citizens feared.

It would be hard to overestimate the brittleness of the former USSR. An American could endanger its political system by going to Moscow and being American. Not being terrified. Not being cowed. Not being blackmailed by the threat of exclusion from libraries and archives. Soviets could sense the huge psychological difference between living in a free country and living in a totalitarian one. Soviet authorities were correct to want to keep Americans away from ordinary Russians. We were a threat to the state. We were dangerous.

The NSA Back Door to NIST

Thomas C. Hales

Use once. Die once.
—Activist saying about insecure communication

We give a brief mathematical description of the NIST standard for cryptographically secure pseudo-random number generation by elliptic curves, the back door algorithm discovered by Ferguson and Shumow, and finally the design of the back door based on the Diffie-Hellman key exchange algorithm.

NIST (the National Institute for Standards and Technology) of the U.S. Department of Commerce derives its mandate from the U.S. Constitution through the congressional power to "fix the standard of weights and measures." In brief, NIST establishes the basic standards of science and commerce. Whatever NIST says about cryptography becomes implemented in cryptographic applications throughout U.S. government agencies. Its influence leads to the widespread use of its standards in industry and the broad adoption of its standards internationally.

Through the Snowden disclosures, the NIST standard for pseudo-random number generation has fallen into disrepute. Here I describe the back door to the NIST standard for pseudo-random number generation in elementary and mathematically precise terms. The NIST standard offers three methods for pseudo-random number generation [1]. My remarks are limited to the third of the three methods, which is based on elliptic curves.

Random number generators can either be truly random (obtaining their values from randomness in the physical world, such as a quantum mechanical process) or pseudo-random (obtaining their values from a deterministic algorithm, yet displaying a semblance of randomness). The significance of random number generation within the theory of algorithms can be gauged by Knuth's multivolume book *The Art of Computer Programming.* It devotes a massive 193 pages (half of volume two) to the subject! A subclass of pseudo-random number generators are *cryptographically secure*, intended for use in cryptographic applications such as key generation, one-way hash functions, signature schemes, private key cryptosystems, and zero knowledge interactive proofs [3].

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Elliptic Curves as Pseudo-Random Number Generators

The NIST standard gives a list of explicit mathematical data (E, p, n, f, P, Q) to be used for pseudo-random number generation [1]. Here E is an elliptic curve defined over a finite field $\mathbb{F}_p$ of prime order p. The group $E(\mathbb{F}_p)$ has order n, which is prime for all of the curves that occur in the NIST standard. The elements of the group $E(\mathbb{F}_p)$ consist of the set of points on an affine curve, together with a *point at infinity* which serves as the identity element of the group. The affine curve is defined by an equation $y^2 = f(x)$ for some explicit cubic polynomial f in $\mathbb{F}_p[x]$. Finally, P and Q are given points on the affine curve.

NIST gives a few sets of data, and in each case the prime number p is large. (The smallest is greater than 10^{77}.) No explanation is given of the particular choices (E, p, n, f, P, Q). We are told to use these data and not to question why. The standard stipulates that "one of the following NIST approved curves with associated points shall be used in applications requiring certification under FIPS-140 [U.S. government computer security accreditation]."

When A is any point other than the identity in $E(\mathbb{F}_p)$, we may evaluate the coordinate function x at A to obtain $x(A) \in \mathbb{F}_p$. By further lifting $\mathbb{F}_p$ to a set of representatives in $\mathbb{Z}$, we obtain a function by composition

$$x1 : E(\mathbb{F}_p) \setminus \{0\} \to \mathbb{F}_p \to \mathbb{Z}.$$

Write $(n, A) \mapsto n * A$ for the $\mathbb{Z}$-module action of $\mathbb{Z}$ on E. (We write powers of the group element A using multiplicative rather than exponential notation.)

The pseudo-random bit generator is initialized with a random integer seed s obtained by some different process such as a separate random number generator. What is important for us is that the number s represents the hidden internal state of the algorithm. The hidden state must be kept secret for the pseudo-randomness to be effective. (Once the state is disclosed, a pseudo-random sequence becomes predictable and useless for many cryptographic purposes.)

The essence of the pseudo-random bit generator can be written in the Objective Caml language as follows. In the syntax of this language, each phrase (`let x = a in` …) defines the value of x to be a. The last line of the block of code gives the output of the function.

```
let pseudo_random s =
  let r = x1 (s * P) in
  let s' = x1 (r * P) in
  let t = x1 (r * Q) in
  let b = extract_bits t in
    (s',b);
```

That is, we successively apply the integer s or r to the point P or the point Q and take the $x1$ coordinate of the resulting point, then extract some bits from the number t. The integer s' becomes the new secret internal state to be fed into the next iteration of the function. The output `b` is passed to the consumer of pseudo-random bits. This output may become publicly known. The function `extract_bits` operates by converting t to a list of bits, discarding the 16 most significant bits (for reasons that do not matter to this discussion), and giving the remaining bits as output. According to NIST standards, by iterating this function, updating the internal state at each iteration, a cryptographically secure stream `b`…of pseudo-random bits is obtained.

The Back Door

This algorithm is fatally flawed, as Ferguson and Shumow have pointed out [5]. Since P and Q are nonidentity elements of a cyclic group of prime order, each is a multiple of the other. Write $P = e * Q$ for some integer e. We show that, once we have e in hand, it is a simple matter to determine the secret internal state s of the pseudo-random bit generator by observing the output `b` and thus to compromise the entire system.

The function `extract_bits` discards 16 bits. Given the output `b`, we take the 2^{16} (a small number of) possible preimages t of `b` under `extract_bits`. For each t, the coordinate x is known, and solving a quadratic, there are at most two possibilities for the coordinate y of a point A on the elliptic curve such that $t = x1(A)$. One such A is $r * Q$. For each A, we compute $e * A$. One of the small number of possibilities for $e * A$ is

$$e * (r * Q) = r * (e * Q) = r * P. \tag{1}$$

Finally $s' = x1(r * P)$. In short, the internal state s' can be narrowed down to a small number of possibilities by an examination of the pseudo-random output bitstream. Shumow and Ferguson state that in experiments, "32 bytes of output was sufficient to uniquely identify the internal state of the PRNG [pseudo-random number generator]."

The back door to the algorithm is the number e such that $P = e * Q$. To use the back door, one must know the value of e. The NIST standard does not disclose e (of course!), and extensive cryptographic experience suggests that it is hard to compute e from the coordinates of P and Q (unless you happen to own a quantum computer). This is the problem of *discrete logarithms*. But, starting with e, there is no difficulty in creating a pair P and Q. The back door is universal: a single number e gives back door access to the internal state of the algorithm of all users worldwide.

It is a matter of public fact that the NSA was tightly involved in the writing of the standard. Indeed, NIST is required by law to consult with the NSA in creating its standard. According to the *New York Times*, "classified NSA memos appear to confirm that the fatal weakness, discovered by two Microsoft cryptographers in 2007, was engineered by the agency" [4]. The news article goes on to say that "eventually, NSA became the sole editor" and then pushed aggressively to make this the standard for the 163 member countries of the International Organization for Standardization. Further historical and social context appears in [6]. The NSA had facile access to the crown jewel e and motive to seize it. Draw your own conclusions.

Observations

1. The back door to this algorithm is extremely elementary from a mathematical perspective. We wrote the essential algorithm in six lines of computer code, even if more supporting code is needed to make it industrial strength. The algorithm could be explained to undergraduate math majors or sufficiently advanced high school students. The story also has the spy agency intrigue to make a good math club talk or a special lecture in an elementary abstract algebra course. We essentially just need to understand that an elliptic curve is an abelian group whose elements (other than the identity element) are determined by two numbers x and y, that y is the root of a quadratic when x is given, and that every nonidentity element of a cyclic group of prime order is a generator. Easy stuff.

2. Without prior knowledge of the back door, how difficult would it be to rediscover the possible existence of a back door? An analysis of the argument shows the required level of creativity is that of an undergraduate homework problem. We

must think to write the element P as a multiple of the generator Q in a cyclic group of prime order. This a student learns in the first weeks of undergraduate algebra.

The rest of the process of inverting the pseudo-random number generator is determined by the definition of the function itself: simply take each step defining the function and reverse the steps, asking for the preimage of the function at each step of its definition, working from the output back to the secret state s'. Once the question of inverting the function is asked, it is easy to do the group theory, even if it is computationally difficult to write e explicitly.

One-way functions are a standard tool in the cryptographer's bag. Every professional who has been trained to analyze cryptographic algorithms knows to ask the question of invertibility. It is unsettling that NIST and others do not seem to have asked this basic question.

Diffie-Hellman Key Exchange

In what follows, let us assume that someone, whom we will call *the Spy*, has access to the back door e. How is it possible for the Spy and the end user (*the User*) of the NIST algorithm to come into possession of the same shared secret (the internal state of the pseudo-random number generator) when all communication between them is public? Information flows from the Spy to the User through the published NIST standard, and from the User back to the Spy through the public output of the pseudo-random generator. The back door must have a remarkable cryptographic design to permit a secret to pass across these public channels yet prevent the secret from becoming known to a third party.

As we now explain, the design of the back door to NIST is based on a well-known algorithm in cryptography called the Diffie-Hellman key exchange [2]. This is an algorithm to share a secret between two parties when there is a possibility that the channel of communication is being monitored. In the current context, the Spy has full knowledge of the Diffie-Hellman key exchange for what it is. However, the User participates in the exchange innocently and unwittingly by blindly following the rules of the NIST protocol.

The Diffie-Hellman key exchange requires a group, which we will take to be a cyclic group E of order n (to preserve notation). The group E, its order n, and a generator Q are made public. To share a secret, the first party (the Spy) picks a random number e, which is kept secret, and publishes $P = e * Q$ to the world. The second party (the User) picks a random number r, which is kept secret, and publishes $r * Q$. Then, by equation (1), the Spy, who knows e and $r * Q$, and the User, who knows r and $e * Q$, can both compute $(re) * Q = r * P$, which is the shared secret. (In our context, the shared secret determines the internal state s' of the pseudo-random number generator.) If E is a group in which the public knowledge of E, n, Q, $P = e * Q$, $r * Q$ does not allow the easy computation of $(re) * Q$, then the shared secret is protected from public disclosure by the difficulty of the computation. In this way, the only two who learn the internal state of the pseudo-random number generator are the Spy and the User.

What we have described here is not an imaginary scenario: NIST documents do in fact publish the data E, n, Q, and P needed to initiate the Diffie-Hellman exchange. A user, when making public the output from the pseudo-random number generator, does in fact complete the exchange. Diffie-Hellman is Diffie-Hellman, whether it has been advertised as such or not.

To say that the Diffie-Hellman key exchange algorithm is well known is a vast understatement. This algorithm is a significant lesson in virtually every first course in cryptography everywhere in the world. Building on Merkle, the Diffie-Hellman paper, by starting the entire field of public key cryptography, is one of the most important papers in cryptography ever written.

What is the significance of all this? It is no secret that the NSA employs some of the world's keenest cryptographic minds. They all know Diffie-Hellman. In my opinion, an algorithm that has been designed by NSA with a clear mathematical structure giving them exclusive back door access is no accident, particularly in light of the Snowden documents.

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Musings on MOOCs

Jim Fowler and Tara Smith

MOOCs (massive open online courses) are causing a revolution in higher education today. What will be the impact of this revolution on mathematics teaching in colleges and universities? The *Notices* is hosting a discussion of MOOCs, which began in the November 2013 issue with the Opinion column "MOOCs and the future of mathematics" by Robert Ghrist of the University of Pennsylvania. The first installment of the discussion appeared in the January 2014 issue and continues in the present issue. The *Notices* invites readers to submit short pieces (800 words or less) on the subject of MOOCs in mathematics. Please send contributions to `notices-mooc@ams.org`.

James Fowler

With a team at Ohio State, I've created two MOOCs, namely Calculus One (which first ran in the spring of 2013) and Calculus Two (which first ran in the fall of 2013). More are on the way. Both MOOCs debuted on Coursera, but much of the content is also available on iTunes U and YouTube and has been used to "flip the classroom" at Ohio State. MOOC content can be deployed for a variety of purposes.

I agree with what Robert Ghrist wrote in "MOOCs and the future of mathematics" [*Notices*, November 2013]. Ghrist emphasizes that MOOCs make possible experimentation with the exposition of mathematics; I'll emphasize that MOOCs are also experiments with assessment. The basic question is this: how do we get more people to do more homework? For our MOOC, we built an adaptive learning platform called MOOCulus—a play on "cow-culus". MOOCulus provides randomly generated interactive calculus exercises with hints. Correct and incorrect responses and requests for hints are used to estimate the student's present level of mastery so that, as the student masters a particular topic, the student progresses to a new challenge. The hope then is that each student is actively working on problems that are at the appropriate level to help that student grow.

All these students doing all these problems means we have a lot of data on student learning. Armed with this data, Bart Snapp, David Lindberg, and I are examining how a student's experience with MOOCulus relates to that student's performance on traditional in-class assessments.

That we are collaborating on MOOCulus is significant. MOOCs are usually said to be "open" in the sense of open enrollment, but "open" might also mean "open source". The source code and other materials for our MOOCs are available in a public repository, so anyone can look behind the scenes to see how we've built what we've built. Improvements to our code have come not just from faculty elsewhere, but also from our students. In short, MOOCs make teaching collaborative and public—just like research.

Collaboration is the whole game. MOOCs are only about technology insofar as technology facilitates the development of communities. Those communities are not just communities of learners. They also include communities—like this one facilitated by the *Notices*—of teachers and researchers. In the past, Ghrist's innovations might have been known only to the people at his home institution, the University of Pennsylvania, but Ghrist has, in a sense, published his teaching, and that publication makes possible a discussion about his innovations.

Tara Smith

My daily professional routine is not markedly different from that of my advisor, or indeed of his advisor. I teach classes, mentor graduate students, and immerse myself in my research and writing, collaborating with others or on my own. Classroom instruction, supported by recitation sections often led by a TA for large classes, has continued to be the norm for most of us. We've embraced pedagogical changes (or tried and rejected them in some instances): inquiry-based learning, cooperative learning, graphing calculators, computer algebra

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systems, tablets, clickers, etc. Still, collegiate math instruction has continued primarily to consist of an instructor on site delivering content to his or her students, running problem sessions, assessing mastery via completion of homework and performance on exams, and interacting directly with the students three to five times per week.

Lately, however, as I converse with current and prospective doctoral students in mathematics, I have begun to wonder what the rapidly expanding menu of mechanisms for delivering mathematics content—most notably MOOCs—portend for the future of our profession, our students, and our community. How substantially will the careers of this next generation of mathematicians differ from those of us who have been in the field for twenty, thirty, or forty years? What will the daily professional life of the academic mathematician look like in another five, ten, or twenty years?

All we know for sure is that it is likely to be different in substantive ways. The differences will be driven by the rapid expansion of alternative ways to deliver content as well as the harsh economic realities facing higher education. We might find ourselves confronting the critical question Robert Ghrist posed in his piece about MOOCs in the November 2013 *Notices*, "Why do mathematicians exist?"; surely we will at least face the question, "Why should mathematicians be hired by a university?" We need to answer compellingly if academic mathematicians are to continue to exist in significant numbers across many institutions.

What of value do we offer? At many colleges, the justification for sizable math departments is the need for faculty to teach service courses that deliver basic, fairly low-level mathematics content and skill instruction to students in other disciplines who need to have some facility with mathematics as a tool. What is it that our physical presence on campus and in the classroom provides that cannot be provided, perhaps substantially better, by having a student watch a YouTube video starring a skilled lecturer and subsequently be evaluated by a computer-generated assessment? If you believe, as I do, that something magical happens in the personal interaction between instructor and student, something that takes learning and understanding to a deeper level, then how do we demonstrate that? How do we ensure that it happens consistently in our work with students, and how do we persuade those who pay the bills that the added value is worth the greater cost? What formats make sense for faculty-student instruction in light of the ability to get content delivered inexpensively or for free via online sources? Do we move away from lectures and toward a system of tutorials? Flipped classrooms? Can we teach students more efficiently and cheaply by taking advantage of new resources for mass instruction and content? If so, how? And if so, how do we continue to justify the same number of faculty, or will we inevitably be downsized? What would downsizing do to the research climate and opportunities for graduate students?

With the potential for excellent delivery of mathematical content for very low cost, the opportunities for those who previously had no access to such instruction have grown dramatically, which is exciting from any perspective. However, if there is value in more personal delivery, will some who previously had access to that now lose access because the cost of providing it is so much greater than the inexpensive options? We might be moving toward two distinct systems of instruction in higher education. In the first would be students who are prepared for and can afford access to the elite institutions whose faculty are well-funded research mathematicians with some degree of teaching expectation; they would be instructed via MOOCs and other online options, supplemented by classroom and tutorial instruction provided by active researchers and their graduate students. In the second system, students would have their math instruction provided solely by online sources and math tutoring offices staffed by adjunct faculty, perhaps overseen by one or two regular faculty charged with maintaining standards and quality.

I have no answers, but there is no shortage of questions. We do indeed live in interesting times.

Mathematics People

Ye Tian Awarded 2013 ICTP/IMU Ramanujan Prize

Ye Tian of the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, has been named the recipient of the 2013 Ramanujan Prize for Young Mathematicians from Developing Countries, awarded by the International Centre for Theoretical Physics (ICTP) and the International Mathematical Union (IMU). According to the prize citation, he was honored for "his outstanding contributions to number theory. These include the completion of the proof of a multiplicity one conjecture for local theta correspondences and important work related to Heegner points and to the Birch and Swinnerton-Dyer conjecture: the nonexistence of points on twisted Fermat curves, and recently remarkable progress on the congruent number problem, showing the existence of infinitely many congruent numbers with arbitrarily many prime factors."

The Ramanujan Prize is awarded annually to a researcher from a developing country who is younger than forty-five years of age on December 31 of the year of the award and who has conducted outstanding research in a developing country. Researchers working in any branch of the mathematical sciences are eligible. The prize carries a cash award of US$15,000, and the winner is invited to deliver a lecture at ICTP.

Tian received his Ph.D. in mathematics from Columbia University in 2003. He has been affiliated with the Institute for Advanced Study and McGill University. Earlier in 2013 he was awarded the Morningside Medal of Mathematics at the Sixth International Congress of Chinese Mathematicians.

—From an ICTP announcement

Avila Awarded TWAS Prize

Artur Avila of the Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, has been awarded the 2013 TWAS Prize in Mathematics of the Academy of Sciences for the Developing World (TWAS). He was recognized "for his fundamental contributions to the theory of renormalization in low-dimension dynamical systems, to the theory of one-dimensional Schrödinger operators and related co-cycles, to the theory of Teichmüller flow, interval exchange transformations and translation flows." The TWAS Prizes honor individual scientists who have been working and living in a developing country for at least ten years. The prize carries a cash award of US$15,000.

Avila received his Ph.D. from IMPA in 2001. He has also been affiliated with the Collège de France and the Centre National de la Recherche Scientifique (CNRS) and has been a research fellow of the Clay Mathematics Institute. He was awarded the Salem Prize in 2006. Among his other honors are the European Mathematical Society (EMS) Prize (2008), the Grand Prix Jacques Herbrand of the French Academy of Sciences (2009), the Wolff Memorial Lectures (2008), and the International Association of Mathematical Physics (IAMP) Early Career Award (2012). He gave a plenary address at the 2010 International Congress of Mathematicians. Avila will present a lecture at the TWAS general meeting in 2014.

—From a TWAS announcement

2013 Hopf Prizes Awarded

Yakov Eliashberg of Stanford University and Helmut Hofer of the Institute for Advanced Study have been selected recipients of the 2013 Heinz Hopf Prize by ETH Zurich. Eliashberg received his Ph.D. in 1972 from Leningrad University and has been at Stanford since 1989. He received a Guggenheim Fellowship in 1995 and the Oswald Veblen Prize in Geometry in 2001. He is a fellow of the AMS. Hofer received his Ph.D. from the University of Zurich in 1981. He is a founder of the field of symplectic topology, and his work has led to a new area of mathematics known as Hofer geometry. He has been an Alfred P. Sloan Fellow (1987–1989) and is a member of the National Academy of Sciences and is a fellow of the AMS. He received the Ostrowski Prize in 1999.

The Hopf Prize is awarded every two years for outstanding scientific work in the field of pure mathematics. It carries a cash award of 30,000 Swiss francs (approximately US$33,000). Eliashberg and Hofer presented the Heinz Hopf lectures in December 2013 entitled "From Dynamical Systems to Geometry and Back".

—Elaine Kehoe

2013 CMS G. de B. Robinson Award Announced

KENNETH DAVIDSON of the University of Waterloo and ALEX WRIGHT of the University of Chicago have been awarded the 2013 G. de B. Robinson Award of the Canadian Mathematical Society (CMS) for their paper titled "Operator algebras with unique preduals", published in the *Canadian Mathematical Bulletin* **54** (2011), 411–421; `http://cms.math.ca/10.4153/CMB-2011-036-0`. The award is given in recognition of outstanding contributions to the *Canadian Journal of Mathematics* or the *Canadian Mathematical Bulletin*.

—From a CMS announcement

Ghate Awarded 2013 Bhatnagar Prize

EKNATH PRABHAKAR GHATE of the Tata Institute of Fundamental Research has been awarded the 2013 Shanti Swarup Bhatnagar Prize for Science and Technology in the mathematical sciences. The prize is awarded by the Council of Scientific Research and Industrial Development to recognize outstanding Indian work in science and technology. Shanti Swarup Bhatnagar was the founding director of the Council. It is the highest award for science in India. The prize carries a cash award of 500,000 rupees (approximately US$8,000).

—Council of Scientific Research and Industrial Development, India

2013 Infosys Prize Awarded

RAHUL PANDHARIPANDE of ETH Zurich has been awarded the 2013 Infosys Prize in mathematical sciences by the Infosys Science Foundation. He was recognized "for his profound work in algebraic geometry, in particular, for his work on Gromov-Witten theory for Riemann surfaces, for predicting the connection between Gromov-Witten and Donaldson-Thomas theories, and for his recent work with Aaron Pixton that establishes this connection for Calabi-Yau 3-folds." The prizewinners are chosen based on significant progress showcased in their chosen spheres, as well as for the impact their research will have on the specific field. The prize carries a cash award of Rs. 55 lakhs (approximately US$87,000). In addition to the prize purse, each category award includes a gold medallion and a citation certificate.

—From an Infosys Science Foundation announcement

2013 Prix de Recherches Awarded

BENJAMIN JOURDAIN of Université Paris-Est and ENPC, SYLVIE MÉLÉARD of Ecole Polytechnique, and WOJBOR WOYCZYNSKI of Case Western Reserve University have been chosen the recipients of the 2013 Prix de Recherches Award for their joint article "Lévy flights in evolutionary ecology", published in the *Journal of Mathematical Biology*. Given by the French magazine *La Recherche*, the award highlights research at the crossroads of science and technology.

—From a La Recherche *announcement*

CAREER Awards Presented

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) has honored a number of young mathematicians in fiscal year 2013 with Faculty Early Career Development (CAREER) awards. The NSF established the awards to support promising scientists, mathematicians, and engineers who are committed to the integration of research and education. The grants provide funding of at least US$400,000 over a five-year period. The 2013 CAREER grant awardees and the titles of their grant projects follow.

ETHAN ANDERES, University of California Davis, Deformations in Statistics, Cosmology and Image Analysis; ARAVIND ASOK, University of Southern California, Vector Bundles, Rational Points, and Homotopy Theory; LYDIA BIERI, University of Michigan, Ann Arbor, Geometric-Analytic Investigations of Spacetimes and their Nonlinear Phenomena; ANDREA BONITO, Texas A&M University, Explicit Adaptive Methods for Coupled Problems; CHING-SHAN CHOU, Ohio State University, Spatial Modeling and Computation of Cell Signaling in Cell-to-Cell Communication; MARK CULP, West Virginia University, Statistical Methodology in Multi-View Learning with Large Data; LAURENT DEMANET, Massachusetts Institute of Technology, Super-Resolution and Subwavelength Imaging; MOON DUCHIN, Tufts University, Finer Coarse Geometry; AMANDA FOLSOM, Yale University, Maass Forms, Modular Forms, and Applications in Number Theory; MARK HOEFER, North Carolina State University, Solitary Waves and Wavetrains in Dispersive Media; ADRIAN IOANA, University of California San Diego, Classification and Rigidity for von Neumann Algebras; SAMUEL ISAACSON, Boston University, Numerical Methods for Stochastic Reaction Diffusion Equations; GAUTAM IYER, Carnegie Mellon University, Anomalous Diffusion, Homogenization, and Averaging; Todd Kemp, University of California San Diego, Free Probability and Connections to Random Matrices, Stochastic Analysis, and PDEs; Kay Kirkpatrick, University of Illinois, Urbana-Champaign, Mechanics of Superconductors and Other Macroscopic Phenomena; Alex Kontorovich, Yale University, Local-Global Phenomena and Sieves in Thin Orbits; Aaron Lauda, University of Southern California, Interactions between Knot Homology and Representation Theory; Radu Laza, State University of New York, Stony Brook, Advances in Hodge Theory and Moduli; Tai Melcher, University of Virginia, Heat Kernel Measures in Infinite Dimensions; Karin Melnick, University of Maryland, Frontiers of Rigidity in Pseudo-Riemannian, Conformal, and Parabolic Geometries; Debashis Mondal, University of Chicago, New Directions in Spatial Statistics; Yi Ni, California Institute of Technology, Heegaard Floer Homology and Low-Dimensional Topology; Jessica Purcell, Brigham Young University, Hyperbolic Geometry and Knots and Links; Andrew Putman, Rice University, The Topology of Infinite Groups; Brian Rider, Temple University, Random Matrices, Random Schroedinger, and Communication; Ralf Schiffler, University of Connecticut, Cluster Algebras, Combinatorics and Representation Theory; Karl Schwede, Pennsylvania State University, Test Ideals and the Geometry of Projective Varieties in Positive Characteristic; James Scott, University of Texas at Austin, Bringing Richly Structured Bayesian Models into the Discrete-Data Realm via New Data-Augmentation Theory and Algorithms; Luis Silvestre, University of Chicago, Regularity Estimates for Elliptic and Parabolic Equations; Wenguang Sun, University of Southern California, Simultaneous and Sequential Inference of High-Dimensional Data with Sparse Structure; Rachel Ward, University of Texas at Austin, Sparsity-Aware Sampling Theorems and Applications; Daniela Witten, University of Washington, Flexible Network Estimation from High-Dimensional Data; Jianlin Xia, Purdue University, Structured Matrix Computations: Foundations, Methods, and Applications; Lexing Ying, Stanford University, Fast Algorithms for Oscillatory Integrals; Ming Yuan, Morgridge Institute for Research, Sparse Modeling and Estimation with High-Dimensional Data; Hao Zhang, University of Arizona, Nonparametric Models Building, Estimation, and Selection with Applications to High-Dimensional Data Mining.

—Elaine Kehoe

2013 Professors of the Year Chosen

Three college professors whose work involves the mathematical sciences are among the 2013 Professors of the Year, selected by the Carnegie Foundation for the Advancement of Teaching and the Council for Advancement for Support of Education (CASE). Robert Chaney, a professor of mathematics at Sinclair Community College in Dayton, Ohio, was named Outstanding Community Colleges Professor of the Year. He uses hands-on learning projects with his students, such as teaching them to program a robot using algebraic functions. Gintaras Duda, an associate professor of physics at Creighton University, was chosen Outstanding Master's Universities and Colleges Professor of the Year. He teaches courses that have no lecture component but are problem-based, and he often coauthors articles with his undergraduate students. Steven Pollock, a professor of physics at the University of Colorado at Boulder, was named Outstanding Doctoral and Research Universities Professor of the Year. He considers himself more of a coach than a teacher, letting his students make sense of ideas by themselves. In his research he studies how students' mathematical skills help them with physics concepts.

—From a Carnegie Foundation announcement

Mathematics Opportunities

Call for Applications for the Second Heidelberg Laureate Forum

The Second Heidelberg Laureate Forum (HLF) will be held September 21–26, 2014. It will bring together winners of the Abel Prize and the Fields Medal, both in mathematics, as well as the Turing Award and the Nevanlinna Prize, both in computer science, in Heidelberg, Germany. The Heidelberg Laureate Forum Foundation (HLFF) is looking for outstanding young mathematicians and computer scientists from all over the world who would like to get the chance to personally meet distinguished experts from both disciplines and find out how to become leading scientists in their fields. Applications will be accepted until **February 28, 2014**. Applications must be submitted online at `http://application.heidelberg-laureate-forum.org`. The forum is being organized by the Heidelberg Laureate Forum Foundation in cooperation with the forum's founders, as well as the Association for Computing Machinery (ACM), the International Mathematical Union (IMU), and the Norwegian Academy of Science and Letters. For more information, see `www.heidelberg-laureate-forum.org`.

—From an HLF announcement

Call for Nominations for Second Stephen Smale Prize

The second Stephen Smale Prize will be awarded at the Foundations of Computational Mathematics (FoCM) meeting in Montevideo, Uruguay, December 11–20, 2014. The goal of the Smale Prize is to recognize major achievements in furthering the understanding of the connections between mathematics and computation, including the interfaces between pure and applied mathematics, numerical analysis, and computer science. To be eligible for the prize a candidate must be in his or her early to mid-career, meaning, typically, removed by at most ten years from his or her (first) doctoral degree by the first day of the FoCM meeting (December 11, 2014). Eligible candidates should be nominated by email to the secretary of FoCM, `Antonella.Zanna@math.uib.no`, no later than **March 10, 2014**. Each nomination should be accompanied by a brief case for support. The recipient of the prize will be expected to give a lecture at the meeting. A written version of this lecture (tagged as the Smale Prize Lecture) will be included in the volume of plenary talks. For more information, see `http://focm-society.org/smale_prize.php`.

—From a FoCM announcement

Summer Program for Women Undergraduates

The 2014 George Washington University Summer Program for Women in Mathematics (SPWM) is open for applications. The program will take place in Washington, D.C., during the summer of 2014 on dates to be determined. This is a five-week intensive program for mathematically talented undergraduate women who are completing their junior years and may be contemplating graduate study in mathematical sciences. The goals of this program are to communicate an enthusiasm for mathematics, to develop research skills, to cultivate mathematical self-confidence and independence, and to promote success in graduate school. A number of seminars will be offered, led by active research mathematicians with the assistance of graduate students. The seminars will be organized to enable the students to obtain a deep understanding of basic concepts in several areas of mathematics, to learn how to do independent work, and to gain experience in expressing mathematical ideas orally and in writing. There will be panel discussions on graduate schools, career opportunities, and the job market. Weekly field trips will be organized to facilities of mathematical interest around the Washington area.

Applicants must be U.S. citizens or permanent residents studying at a U.S. university or college who are completing their junior years or the equivalent and have mathematical experience beyond the typical first courses in calculus and linear algebra. Sixteen women will be selected. Each will receive a travel allowance, campus room and board, and a stipend of US$1,750. The deadline for applications is **February 28, 2014**. Early applications are encouraged. Applications are accepted only by mail. For further information and the exact dates of the program, please contact the director, Murli M. Gupta, email: `mmg@gwu.edu`; telephone: 202-994-4857; or visit the program's website at `http://www.gwu.edu/~spwm/`. Application material is available on the website.

—From an SPWM announcement

Call for Nominations for 2014 IBC Prize

This annual prize is for outstanding achievement in information-based complexity. The prize consists of US$3,000 and a plaque. The achievement can be based on work done in a single year, a number of years, or over a lifetime; it can be published in any journal, number of journals, or monographs. Nominations may be sent to Joseph Traub at `traub@cs.columbia.edu`. However, a person does not have to be nominated to win the award. The deadline for nomination is **March 31, 2014**.

—Joseph Traub
Columbia University

CRM-PISA Junior Visiting Program

The "Centro di Ricerca Matematica Ennio De Giorgi" (CRM) invites applications for three two-year Junior Visiting positions for the Academic Year 2014/15.

Successful candidates will be new or recent PhD's in mathematics with an exceptional research potential. Ph.D. students can also apply, provided they obtain their Ph.D. no later than October 2014. The annual gross compensation is 32,000 Euros, corresponding to a monthly salary of approximately 2,000 Euros (after tax), plus a research allowance of 1,000 Euros that can be used for exchange visits. Junior Visitors are expected to start their research activity at CRM no later than October 2014.

Deadline for application is January 10, 2014. The full announcement is available at `http://www.sns.it/en/servizi/job/assegnidiricerca/assegno545/`.

Hosting over 4,000 visitors since its foundation in 2001, CRM has been devoted to promoting excellence in a vast spectrum of research fields, from pure mathematics to mathematics applied to the natural and social sciences. As a consequence, CRM provides a thriving international and interdisciplinary research environment. Junior Visitors can take part in a great variety of scientific activities including intensive research periods, workshops, and seminars. Moreover, Junior Visitors have a unique opportunity to interact with top-class scientists who visit the CRM as part of our Senior Visiting Programme.

Please view our website for detailed information about our scientific activity `http://www.crm.sns.it`. (See also `http://www.crm.sns.it/news/102/`.)

—From CRM-PISA announcement

Inside the AMS

AMS Email Support for Frequently Asked Questions

A number of email addresses have been established for contacting the AMS staff regarding frequently asked questions. The following is a list of those addresses together with a description of the types of inquiries that should be made through each address.

abs-coord@ams.org for questions regarding a particular abstract or abstracts questions in general.

acquisitions@ams.org to contact the AMS Acquisitions Department.

ams@ams.org to contact the Society's headquarters in Providence, Rhode Island.

amsdc@ams.org to contact the Society's office in Washington, D.C.

amsfellows@ams.org to inquire about the Fellows of the AMS.

amsmem@ams.org to request information about membership in the AMS and about dues payments or to ask any general membership questions; may also be used to submit address changes.

ams-simons@ams.org for information about the AMS Simons Travel Grants Program.

ams-survey@ams.org for information or questions about the Annual Survey of the Mathematical Sciences or to request reprints of survey reports.

bookstore@ams.org for inquiries related to the online AMS Bookstore.

classads@ams.org to submit classified advertising for the *Notices*.

cust-serv@ams.org for general information about AMS products (including electronic products), to send address changes, place credit card orders for AMS products, to correspond regarding a balance due shown on a monthly statement, or conduct any general correspondence with the Society's Sales and Member Services Department.

development@ams.org for information about charitable giving to the AMS.

eims-info@ams.org to request information about Employment Information in the Mathematical Sciences (EIMS). For ad rates and to submit ads go to `http://eims.ams.org`.

emp-info@ams.org for information regarding AMS employment and career services.

eprod-support@ams.org for technical questions regarding AMS electronic products and services.

gradprg-ad@ams.org to inquire about a listing or ad in the Find Graduate Programs online service.

mathcal@ams.org to send information to be included in the "Mathematics Calendar" section of the *Notices*.

mathjobs@ams.org for questions about the online job application service `Mathjobs.org`.

mathprograms@ams.org for questions about the online program application service `Mathprograms.org`.

mathrev@ams.org to submit reviews to *Mathematical Reviews* and to send correspondence related to reviews or other editorial questions.

meet@ams.org to request general information about Society meetings and conferences.

mmsb@ams.org for information or questions about registration and housing for the Joint Mathematics Meetings (Mathematics Meetings Service Bureau).

msn-support@ams.org for technical questions regarding MathSciNet.

notices@ams.org to send correspondence to the managing editor of the *Notices*, including items for the news columns. The editor (`notices@math.wustl.edu`) is the person to whom to send articles and letters. Requests for permission to reprint from the *Notices* should be sent to `reprint-permission@ams.org` (see below).

notices-ads@ams.org to submit electronically paid display ads for the *Notices*.

notices-booklist@ams.org to submit suggestions for books to be included in the "Book List" in the *Notices*.

notices-letters@ams.org to submit letters and opinion pieces to the *Notices*.

notices-whatis@ams.org to comment on or send suggestions for topics for the "WHAT IS…?" column in the *Notices*.

nsagrants@ams.org for information about the NSA-AMS Mathematical Sciences Program.

paoffice@ams.org to contact the AMS Public Awareness Office.

president@ams.org to contact the president of the American Mathematical Society.

prof-serv@ams.org to send correspondence about AMS professional programs and services.

publications@ams.org to send correspondence to the AMS Publication Division.

pub-submit@ams.org to submit accepted electronic manuscripts to AMS publications (other than Abstracts). See `http://www.ams.org/submit-book-journal` to electronically submit accepted manuscripts to the AMS book and journal programs.

reprint-permission@ams.org to request permission to reprint material from Society publications.

sales@ams.org to inquire about reselling or distributing AMS publications or to send correspondence to the AMS Sales and Member Services Department.

secretary@ams.org to contact the secretary of the Society.

student-serv@ams.org for questions about AMS programs and services for students.

tech-support@ams.org to contact the Society's typesetting Technical Support Group.

textbooks@ams.org to request examination copies or inquire about using AMS publications as course texts.

webmaster@ams.org for general information or for assistance in accessing and using the AMS website.

Reference and Book List

*The **Reference** section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.wustl.edu` in the case of the editor and `smf@ams.org` in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Upcoming Deadlines

January 15, 2014: Applications for AMS-AAAS Mass Media Summer Fellowships. See the website `http://www.aaas.org/programs/education/MassMedia`; or contact Dione Rossiter, Manager, Mass Media Program, AAAS Mass Media Science and Engineering Fellows Program, 1200 New York Avenue, NW, Washington, DC 20005; telephone 202-326-6645; fax 202-371-9849; email `drossite@aaas.org`. Further information is also available at `http://www.ams.org/programs/ams-fellowships/media-fellow/massmediafellow` and through the AMS Washington Office, 1527 Eighteenth Street, NW, Washington, DC 20036; telephone 202-588-1100; fax 202-588-1853; email `amsdc@ams.org`.

January 23, 2014: Full proposals for NSF Major Research Instrumentation Program. See `http://www.nsf.gov/pubs/2013/nsf13517/nsf13517.htm`.

January 31, 2014: Nominations for CAIMS/PIMS Early Career Award. See `http://www.pims.math.ca/pims-glance/prizes-awards`.

January 31, 2014: Entries for AWM Essay Contest. Contact the contest organizer, Heather Lewis, at `hlewis5@naz.edu`, or see `https://sites.google.com/site/awmmath/home`.

February 1, May 1, August 1, November 1, 2014: Applications for February, May, August, November reviews for National Academies Research Associateship Programs. See the website `http://sites.`

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email rap@nas.edu.

February 1, 2014: Applications for AWM Travel Grants, Mathematics Education Research Travel Grants, Mathematics Mentoring Travel Grants, and Mathematics Education Research Mentoring Travel Grants. See https://sites.google.com/site/awm-math/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

February 9, 2014: Applications for Los Angeles, New York, Utah, and Washington, D.C., fellowships for Math for America (MfA). See http://www.mathforamerica.org/.

February 12, 2014: Applications for Research in Industrial Projects for Students (RIPS) of the Institute for Pure and Applied Mathematics (IPAM). See www.ipam.ucla.edu.

February 15, 2014: Applications for AMS Congressional Fellowship. See http://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship or contact the AMS Washington Office at 202-588-1100, email: amsdc@ams.org.

February 15, 2014: Nominations for AWM-Joan & Joseph Birman Prize in Topology and Geometry. See the website http://www.awm-math.org.

February 28, 2014: Applications for Second Heidelberg Laureate Forum. See "Mathematics Opportunities" in this issue.

February 28, 2014: Applications for George Washington University Summer Program for Women in Mathematics (SPWM). See "Mathematics Opportunities" in this issue.

March 3, 2014: Applications for the EDGE for Women Summer Program. See the website http://www.edgeforwomen.org/.

March 10, 2014: Nominations for the second Stephen Smale Prize. See "Mathematics Opportunities" in this issue.

March 15, 2014: Nominations for PIMS Education Prize. See the website http://www.pims.math.ca/pims-glance/prizes-awards.

March 31, 2014: Nominations for Achievement in Information-Based Complexity Prize. See "Mathematics Opportunities" in this issue.

March 31, 2014: Applications for AMS-Simons Travel Grants. See www.ams.org/programs/travel-grants/AMS-SimonsTG or contact Steven Ferrucci, email: ams-simons@ams.org, telephone: 800-321-4267, ext. 4113.

April 15, 2014: Applications for fall 2014 semester of Math in Moscow. See http://www.mccme.ru/mathinmoscow, or contact: Math in Moscow, P.O. Box 524, Wynnewood, PA 19096; fax: +7095-291-65-01; email: mim@mccme.ru. Information and application forms for the AMS scholarships are available on the AMS website at http://www.ams.org/programs/travel-grants/mimoscow, or contact: Math in Moscow Program, Membership and Programs Department, American Mathematical Society, 201 Charles Street, Providence RI 02904-2294; email student-serv@ams.org.

May 1, 2014: Applications for May review for National Academies Research Associateship Programs. See the website http://sites.nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email rap@nas.edu.

May 1, 2014: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

August 1, 2014: Applications for August review for National Academies Research Associateship Programs. See the website http://sites.nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email rap@nas.edu.

October 1, 2014: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

November 1, 2014: Applications for November review for National Academies Research Associateship Programs. See the website http://sites.nationalacademies.org/PGA/RAP/PGA_050491 or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email rap@nas.edu.

MPS Advisory Committee

Following are the names and affiliations of the members of the Advisory Committee for Mathematical and Physical Sciences (MPS) of the National Science Foundation. The date of the expiration of each member's term is given after his or her name. The website for the MPS directorate may be found at www.nsf.gov/home/mps/. The postal address is Directorate for the Mathematical and Physical Sciences, National Science Foundation, 4201 Wilson Boulevard, Arlington, VA 22230.

James Berger (chair) (09/14)
Department of Statistical Science
Duke University

Daniela Bortoletto (09/14)
Department of Physics
Purdue University

Emery N. Brown (09/14)
Massachusetts Institute of Technology

Phil Bucksbaum (09/15)
Stanford University

Emily A. Carter (09/15)
Department of Mechanical and Aerospace Engineering
Princeton University

George W. Crabtree (09/15)
Materials Science Division
Argonne National Laboratory

Juan J. de Pablo (09/15)
Institute of Molecular Engineering
University of Chicago

Francis J. DiSalvo Jr. (09/14)
Department of Chemistry
Cornell University

Bruce Elmegreen (09/14)
IBM Watson Research Center

Barbara J. Finlayson-Pitts (09/14)
Department of Chemistry
University of California, Irvine

Irene Fonseca (09/14)
Department of Mathematical Sciences
Carnegie Mellon University

Elizabeth Lada (09/14)
Department of Astronomy
University of Florida

Juan C. Meza (09/15)
University of California Merced

Catherine Pilachowski (09/15)
Astronomy Department
Indiana University

Elsa Reichmanis (09/14)
School of Chemical and Biomolecular Engineering
Georgia Institute of Technology

Geoffrey West (09/14)
Santa Fe Institute

Book List

The Book List highlights recent books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. Suggestions for books to include on the list may be sent to notices-booklist@ams.org.

*Added to "Book List" since the list's last appearance.

Algorithms Unlocked, by Thomas H. Cormen. MIT Press, March 2013. ISBN-13:978-02625-188-02.

An Accidental Statistician: The Life and Memories of George E. P. Box, by George E. P. Box. Wiley, April 2013. ISBN-13: 978-1-118-40088-3.

Assessing the Reliability of Complex Models: Mathematical and Statistical Foundations of Verification, Validation, and Uncertainty Quantification, by the National Research Council. National Academies Press, 2012. ISBN-13: 978-0-309-25634-6.

**A Cabinet of Mathematical Curiosities at Teachers College: David Eugene Smith's Collection,* by Diane R. Murray. Docent Press, November 2013. ISBN-13: 978-0-9887449-1-2.

A Calculus of Ideas: A Mathematical Study of Human Thought, by Ulf Grenander. World Scientific, September 2012. ISBN-13: 978-98143-831-89. (Reviewed January 2014.)

Charles S. Peirce on the Logic of Number, by Paul Shields. Docent Press, October 2012. ISBN-13: 978-0-9837004-7-0.

Classic Problems of Probability, by Prakash Gorroochurn. Wiley, May 2012. ISBN-13: 978-1-1180-6325-5. (Reviewed November 2013.)

**Computability: Turing, Gödel, Church, and Beyond,* edited by B. Jack Copeland, Carl J. Posy, and Oron Shagrir. MIT Press, June 2013. ISBN-13: 978-02620-189-99.

Conflict in History, Measuring Symmetry, Thermodynamic Modeling and Other Work, by Dennis Glenn Collins. Author House, November 2011. ISBN-13: 978-1-4670-7641-8.

The Continuity Debate: Dedekind, Cantor, du Bois-Reymond, and Peirce on Continuity and Infinitesimals, by Benjamin Lee Buckley. Docent Press, December 2012. ISBN-13: 978-0-9837004-8-7.

The Crest of the Peacock: Non-European Roots of Mathematics, by George Gheverghese Joseph. Third edition. Princeton University Press, October 2010. ISBN-13: 978-0-691-13526-7. (Reviewed December 2013.)

Decoding the Heavens: A 2,000-Year-Old Computer—and the Century-Long Search to Discover Its Secrets, by Jo Marchant. Da Capo Press, February 2009. ISBN-13: 978-03068-174-27. (Reviewed June/July 2013).

Do I Count?: Stories from Mathematics, by Günter Ziegler (translation of *Darf ich Zahlen?: Geschichte aus der Mathematik,* Piper Verlag, 2010). CRC Press/A K Peters, July 2013. ISBN-13: 978-1466564916

Figures of Thought: A Literary Appreciation of Maxwell's Treatise on Electricity and Magnetism, by Thomas K. Simpson. Green Lion Press, February 2006. ISBN-13: 978-18880-093-16. (Reviewed October 2013.)

The Fractalist: Memoir of a Scientific Maverick, by Benoît Mandelbrot. Pantheon, October 2012. ISBN-13: 978-03073-773-57.

Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century, by the National Research Council. National Academies Press, 2012. ISBN-13: 978-0-309-25473-1.

Girls Get Curves: Geometry Takes Shape, by Danica McKellar. Plume, July 2013. ISBN-13: 978-04522-987-43.

**The Godelian Puzzle Book: Puzzles, Paradoxes and Proofs,* by Raymond M. Smullyan. Dover Publications, August 2013. ISBN-13: 978-04864-970-51.

The Golden Ticket: P, NP, and the Search for the Impossible, by Lance Fortnow. Princeton University Press, March 2013. ISBN-13: 978-06911-564-91.

Good Math: A Geek's Guide to the Beauty of Numbers, Logic, and Computation, by Mark C. Chu-Carroll. Pragmatic Bookshelf, July 2013. ISBN-13: 978-19377-853-38.

Google's PageRank and Beyond: The Science of Search Engine Rankings, by Amy Langville and Carl Meyer. Princeton University Press, February 2012. ISBN-13: 978-06911-526-60.

Gösta Mittag-Leffler: A Man of Conviction, by Arild Stubhaug (translated by Tiina Nunnally). Springer, November 2010. ISBN-13: 978-36421-167-11. (Reviewed September 2013.)

Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry, by Glen Van Brummelen. Princeton University Press, December 2012. ISBN-13: 978-06911-489-22.

How to Study As a Mathematics Major, by Lara Alcock. Oxford University Press, March 2013. ISBN-13: 978-0199661312.

I Died for Beauty: Dorothy Wrinch and the Cultures of Science, by Marjorie Senechal. Oxford University Press, December 2012. ISBN-13:978-01997-325-93.

Ibn al-Haytham's Theory of Conics, Geometrical Constructions and Practical Geometry, by Roshdi Rashed. Routledge, February 2013. ISBN-13: 978-0-415-58215-5.

If A, Then B: How the World Discovered Logic, by Michael Shenefelt and Heidi White. Columbia University Press, June 2013. ISBN-13:978-02311-610-53.

Imagined Civilizations: China, the West, and Their First Encounter, by Roger Hart. Johns Hopkins University Press, July 2013. ISBN-13:978-14214-060-60.

Invisible in the Storm: The Role of Mathematics in Understanding Weather, by Ian Roulstone and John Norbury. Princeton University Press, February 2013. ISBN-13: 978-06911-527-21. (Reviewed September 2013.)

Levels of Infinity: Selected Writings on Mathematics and Philosophy, by Hermann Weyl. Edited by Peter Pesic. Dover Publications, February 2013. ISBN-13: 978-0486489032.

The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age, by Paul J. Nahin, Princeton University Press, October 2012. ISBN-13: 978-06911-510-07. (Reviewed October 2013.)

**Magnificent Mistakes in Mathematics*, by Alfred S. Posamentier and Ingmar Lehmann. Prometheus Books, August 2013. ISBN-13: 978-16161-474-71.

Manifold Mirrors: The Crossing Paths of the Arts and Mathematics, by Felipe Cucker. Cambridge University Press, June 2013. ISBN-13: 978-05217-287-68.

The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics, by Clifford A. Pickover. Sterling. February 7, 2012. ISBN-13: 978-14027-882-91.

Math is Murder, by Robert C. Brigham. iUniverse, March 28, 2012. ISBN-13 978-14697-972-81.

Math on Trial: How Numbers Get Used and Abused in the Courtroom, by Leila Schneps and Coralie Colmez. Basic Books, March 2013. ISBN-13: 978-04650-329-21. (Reviewed August 2013.)

A Mathematician Comes of Age, by Steven G. Krantz. Mathematical Association of America, December 2011. ISBN-13: 978-08838-557-82.

A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form, by Paul Lockhart. Bellevue Literary Press, April 2009. ISBN-13: 978-1-934137-17-8. (Reviewed April 2013.)

Mathematics in Nineteenth-Century America: The Bowditch Generation, by Todd Timmons. Docent Press, July 2013. ISBN-13: 978-0-9887449-3-6.

Mathematics in Victorian Britain, by Raymond Flood, Adrian Rice, and Robin Wilson. Oxford University Press, October 2011. ISBN-13: 978-019-960139-4.

Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice, by Alexandre V. Borovik. AMS, January 2010. ISBN-13: 978-0-8218-4761-9.

Maverick Genius: The Pioneering Odyssey of Freeman Dyson, by Phillip F. Schewe. Thomas Dunne Books, February 2013. ISBN-13:978-03126-423-58.

Meaning in Mathematics, edited by John Polkinghorne. Oxford University Press, July 2011. ISBN-13: 978-01996-050-57. (Reviewed May 2013.)

My Brief History, by Stephen Hawking. Bantam Dell, September 2013. ISBN-13: 978-03455-352-83.

Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity, by Loren Graham and Jean-Michel Kantor. Belknap Press of Harvard University Press, March 2009. ISBN-13: 978-06740-329-34. (Reviewed January 2014.)

The New York Times Book of Mathematics: More Than 100 Years of Writing by the Numbers, edited by Gina Kolata. Sterling, June 2013. ISBN-13: 978-14027-932-26.

The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century, by Yvette Kosmann-Schwarzbach. Springer, December 2010. ISBN-13: 978-03878-786-76. (Reviewed August 2013.)

**Our Mathematical Universe: My Quest for the Ultimate Nature of Reality Hardcover*, by Max Tegmark. Knopf, January 2014. ISBN-13: 978-03075-998-03.

The Outer Limits of Reason: What Science, Mathematics, and Logic Cannot Tell Us, by Noson S. Yanofsky. MIT Press, August 2013. ISBN-13: 978-02620-193-54.

Paradoxes in Probability Theory, by William Eckhardt. Springer, September 2012. ISBN-13: 978-94007-513-92. (Reviewed March 2013.)

Peirce's Logic of Continuity: A Conceptual and Mathematical Approach, by Fernando Zalamea. Docent Press, December 2012. ISBN-13: 978-0-9837004-9-4.

Perfect Mechanics: Instrument Makers at the Royal Society of London in the Eighteenth Century, by Richard Sorrenson. Docent Press, September 2013. ISBN-13: 978-0-9887449-2-9.

Probably Approximately Correct: Nature's Algorithms for Learning and Prospering in a Complex World, by Leslie Valiant. Basic Books, June 2013. ISBN-13: 978-04650-327-16.

Relations between Logic and Mathematics in the Work of Benjamin and Charles S. Peirce, by Allison Walsh. Docent Press, October 2012. ISBN-13: 978-0-9837004-6-3.

The Search for Certainty: A Journey through the History of Mathematics, 1800–2000, edited by Frank J. Swetz. Dover Publications, September 2012. ISBN-13: 978-04864-744-27.

Seduced by Logic: Emilie Du Châtelet, Mary Somerville and the Newtonian Revolution, by Robyn Arianrhod. Oxford University Press, September 2012. ISBN-13: 978-01999-316-13. (Reviewed June/July 2013).

The Signal and the Noise: Why So Many Predictions Fail—But Some Don't, by Nate Silver. Penguin Press, September 2012. ISBN-13:978-15942-041-11.

**The Simpsons and Their Mathematical Secrets*, by Simon Singh. Bloomsbury, October 2013. ISBN: 978-14088-353-02.

Sources in the Development of Mathematics: Series and Products from the Fifteenth to the Twenty-first Century, by Ranjan Roy. Cambridge University Press, June 2011. ISBN-13: 978-05211-147-07. (Reviewed November 2013.)

Strange Attractors (comic book), by Charles Soule, Greg Scott, and Robert Saywitz. Archaia Entertainment, May 2013. ISBN-13: 978-19363-936-26.

Symmetry: A Very Short Introduction, by Ian Stewart. Oxford University Press, July 2013. ISBN-13: 978-01996-519-86.

**A Tale of Two Fractals, by* A. A. Kirillov. Birkhäuser, May 2013. ISBN-13: 978-08176-838-18.

**Théorème Vivant,* by Cédric Villani (in French). Grasset et Fasquelle, August 2012. ISBN-13: 978-2246798828. (Reviewed in this issue.)

Thinking in Numbers: On Life, Love, Meaning, and Math, by Daniel Tammet. Little, Brown and Company, July 2013. ISBN-13: 978-03161-873-74.

Thinking Statistically, by Uri Bram. CreateSpace Independent Publishing Platform, January 2012. ISBN-13: 978-14699-123-32.

Turbulent Times in Mathematics: The Life of J. C. Fields and the History of the Fields Medal, by Elaine McKinnon Riehm and Frances Hoffman. AMS, November 2011. ISBN-13: 978-0-8218-6914-7.

Turing's Cathedral: The Origins of the Digital Universe, by George Dyson. Pantheon/Vintage, December 2012. ISBN-13: 978-14000-759-97.

Visions of Infinity: The Great Mathematical Problems, by Ian Stewart. Basic Books, March 2013. ISBN-13: 978-04650-224-03.

A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing, edited by Benjamin Wardhaugh. Princeton University Press, April 2012. ISBN-13: 978-06911-477-58. (Reviewed March 2013.)

**William Fogg Osgood at Harvard: Agent of a Transformation of Mathematics in the United States*, by Diann R. Porter. Docent Press, November 2013. ISBN-13: 978-0-9887449-4-3.

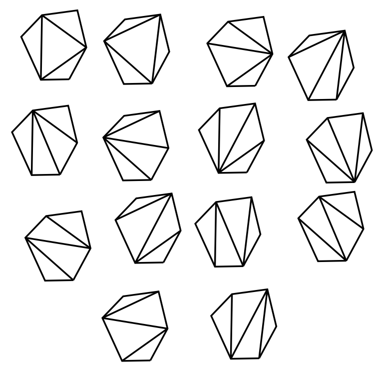
About the Cover

Remarkable combinatorics in the presence of cyclic symmetry

This month's cover illustrates in some detail one of the examples in this issue's article *What is cyclic sieving?* by Victor Reiner, Dennis Stanton, and Dennis White. If X_n is the set of triangulations of a given polygon of $n+2$ sides, then

$$|X_n| = \frac{1}{n+1} \cdot \frac{2n.2n-1.\cdots.n+1}{n.n-1.\cdots.1} .$$

On the cover, $n = 4$ and hence $|X_n| = 14$. Here is the collection of all triangulations in this case:



Inspection shows that these triangulations are not essentially distinct—some of them are combinatorially identical. There is in effect an action of the cyclic group $\mathbb{Z}/(n+2)$ on X_n, which rotates the connecting edges among the vertices. It is not a geometric symmetry since the polygon might not be regular. The cover illustrates this action.

Let R be one step of this rotation. From the formula for X_n is constructed a polynomial $X(q)$—each integer m in the formula is replaced by its q-form

$$[m]_q = \frac{q^m - 1}{q - 1} ,$$

a technique perhaps originating in Gauss' work on quadratic reciprocity. There is no reason to think it always leads to something significant. But here we have the remarkable fact that if ζ is a primitive n-th root of unity then

$$X(\zeta^k) = |\{x \in X_n \mid R^k(x) = x\}| .$$

For example, $X(1) = |X_n|$. You can verify this equation easily in the example illustrated on the cover.

This is the phenomenon called *cyclic sieving*. As Reiner et al. explain, it occurs in a myriad of contexts. Not all of them, by any means, are as simple even to formulate as this example, much less prove. And in fact there seems to be no uniform approach to the construction or verification of such occurrences, nor even a general criterion in which one would expect the phenomenon to arise. The short article in this issue gives a glimpse into the complexity of the subject.

The cover demonstrates that even some simple examples are associated with diagrams. There are in fact some extremely complicated and beautiful graphical associations in the subject. We looked through the literature to see what had been done, and the most interesting diagrams we came across were in the `arxiv` preprint *Invariant tensors and the cyclic sieving phenomenon* by Bruce Westbury. This in turn refers back to an earlier paper with exceptional graphics titled *Promotion and cyclic sieving via webs* by Kyle Petersen, Pavlo Pylyavskyy and Brendon Rhoades, which in turn is partly derived from Greg Kuperberg's *Spiders for rank two Lie algebras.*

—*Bill Casselman*
Graphics Editor
(`notices-covers@ams.org`)

2013 Election Results

In the elections of 2013 the Society elected a president, a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee.

President

Elected as the new president is **Robert Bryant** from Duke University. Term is one year as president elect (1 February 2014—31 January 2015), two years as president (1 February 2015—31 January 2017), and one year as immediate past president (1 February 2017—31 January 2018).

Vice President

Elected as the new vice president is **Susan Montgomery** from the University of Southern California. Term is three years (1 February 2014—31 January 2017).

Trustee

Elected as trustee is **Robert Lazarsfeld** from Stony Brook University. Term is five years (1 February 2014—31 January 2019).

Members at Large of the Council

Elected as new members at large of the Council are:

Richard Durrett from Duke University
Lisa Fauci from Tulane University
Michael Larsen from Indiana University
Kristin E. Lauter from Microsoft Research
Jennifer Taback from Bowdoin College

Terms are three years (1 February 2014—31 January 2017).

Nominating Committee

Elected as new members of the Nominating Committee are:

Peter Constantin from Princeton University
Robert L. Griess Jr. from the University of Michigan
David J. Wright from Oklahoma State University

Terms are three years (1 January 2014—31 December 2016).

Editorial Boards Committee

Elected as new members of the Editorial Boards Committee are:

Anne Schilling from the University of California, Davis
Daniel W. Stroock from the Massachusetts Institute of Technology

Terms are three years (1 February 2014—31 January 2017).

2014 AMS Election

Nominations by Petition

Vice President or Member at Large

One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2014. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations. The Council of 23 January 1979 stated the intent of the Council of nominating all persons on whose behalf there were valid petitions.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and operational considerations, which are described below.

Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, or member of the Nominating and Editorial Boards Committees.

1. To be considered, petitions must be addressed to Carla D. Savage, Secretary, American Mathematical Society, Department of Computer Science, Box 8206, North Carolina State University, Raleigh, NC 27695-8206, USA, and must arrive by 24 February 2014.
2. The name of the candidate must be given as it appears in the *Combined Membership List* (`www.ams.org/cml`). If the name does not appear in the list, as in the case of a new member or by error, it must be as it appears in the mailing lists, for example on the mailing label of the *Notices*. If the name does not identify the candidate uniquely, append the member code, which may be obtained from the candidate's mailing label or by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).
3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.
4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.
5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.
6. The signature may be in the style chosen by the signer. However, the printed name and address will be checked against the *Combined Membership List* and the mailing lists. No attempt will be made to match variants of names with the form of name in the *CML*. A name neither in the *CML* nor on the mailing lists is not that of a member. (Example: The name Carla D. Savage is that of a member. The name C. Savage appears not to be.)
7. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.

Nomination Petition

for 2014 Election

The undersigned members of the American Mathematical Society propose the name of

____

as a candidate for the position of (check one):

- ☐ **Vice President**
- ☐ **Member at Large of the Council**
- ☐ **Member of the Nominating Committee**
- ☐ **Member of the Editorial Boards Committee**

of the American Mathematical Society for a term beginning 1 February, 2015 (Nominating Committee—1 Jan.)

Return petitions by 24 February 2014 to:
Carla D. Savage, AMS Secretary, Dept. of Computer Science, Box 8206, North Carolina State University, Raleigh, NC 27695-8206 USA

Name and address (printed or typed)	
	Signature
	Signature
	Signature
	Signature
	Signature
	Signature

AMERICAN MATHEMATICAL SOCIETY

Leroy P. Steele Prizes

Call for Nominations

The selection committee for these prizes requests nominations for consideration for the 2015 awards. Further information about the prizes can be found in the November 2013 *Notices*, pp. 1372–1377 (also available at http://www.ams.org/profession/prizes-awards/ams-prizes/steele-prize).

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through Ph.D. students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2015 the prize for Seminal Contribution to Research will be awarded for a paper in algebra.

Nomination with supporting information should be submitted to www.ams.org/profession/prizes-awards/nominations. Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included. Nominations for the Steele Prizes for Lifetime Achievement and for Mathematical Exposition will remain active and receive consideration for three consecutive years. Those who prefer to submit by regular mail may send nominations to the AMS Secretary, Carla Savage, Box 8206, Computer Science Department, North Carolina State University, Raleigh, NC 27695-8206. Those nominations will be forwarded by the secretary to the prize selection committee.

Deadline for nominations is March 31, 2014.

AMS
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Mathematics Calendar

Please submit conference information for the Mathematics Calendar through the Mathematics Calendar submission form at http://www.ams.org/cgi-bin/mathcal-submit.pl. The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at http://www.ams.org/mathcal/.

February 2014

* 12–14 **International Conference on Recent Trends in Algebra and Analysis with Applications**, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.

Description: The purpose of the conference is to bring together the mathematicians from over all the world working in related areas to present their research, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to expose young researchers.

Information: http://www.amu.ac.in/newevent/event/9864.pdf.

* 17–21 **Recent Advances in PDEs and Applications (on the occasion of Professor Hugo Beirao da Veiga's 70th birthday)**, Bellavista Relax Hotel, Levico Terme, Italy.

Description: The Conference aims at presenting high level talks from leading scientists on the recent developments in the theory of partial differential equations, in particular those related to fluid dynamics. It is also a way for celebrating Hugo Beirao da Veiga's 70th birthday. The Conference will be cofinanced by CIRM–Centro Internazionale per la Ricerca Matematica, GNAMPA, DICATAM (sez. di Matematica, Università di Brescia) and Dipartimento di Matematica, Università di Trento.

Scientific organizers: Pierangelo Marcati (L'Aquila), Paolo Secchi (Brescia), Raul Serapioni (Trento), Alberto Valli (Trento).

Scientific Committee: Claude Bardos (Paris 6), Vladimir Georgiev (Pisa), Vicentiu D. Radulescu (Bucharest), Adelia Sequeira (Lisboa), Vsevolod A. Solonnikov (St. Petersburg).

Information: http://www.science.unitn.it/cirm/PDEs2014.html.

* 22–23 **21st Southern California Geometric Analysis Seminar**, University of California, Irvine, California.

Description: This annual weekend conference series gathers together mathematicians from the southern California region and throughout the United States to hear and discuss some of the recent developments in geometric analysis and related topics. The conference is supported by the National Science Foundation and there are funds available to support travel expenses of participants, especially for graduate students, recent Ph.D's, and under-represented minorities. Please see the conference website for the list of invited speakers and further information.

Information: http://www.math.uci.edu/~scgas/.

March 2014

* 8–9 **Ohio River Analysis Meeting 4**, University of Kentucky, Lexington, Kentucky 40506-0027

Description: The Fourth Ohio River Analysis Meeting (ORAM) is jointly organized by the University of Cincinnati and the University of Kentucky. The meeting highlights advances in partial differential equations and analysis.

Lectures/Speakers: It features 5-6 invited plenary lectures (confirmed speakers to date are W. Gangbo, A. Greenleaf, C. Demeter, and I. Mitrea) and contributed talks. It is anticipated that travel support will be available for young researchers, please check the website.

Information: http://www.math.uky.edu/oram14.

* 13–15 **International Workshop "Advances in Nonlinear Analysis"**, Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania.

Description: The workshop will be held in the framework of 2014 Theme Semester on Convex Integration and Analysis at the University of Pittsburgh. It brings together some leading researchers in Nonlinear Analysis and aims to present the most recent developments and techniques at the crossing of Partial Differential Equations, Geometric Analysis, Geometric Measure Theory, Harmonic Analysis and Potential Theory, and Nonlinear Analysis. Some financial assistance will be available for graduate students and young researchers. See the website for the details.

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.

An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.

In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to notices@ams.org or mathcal@ams.org.

In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.

The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.

The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: http://www.ams.org/.

Information: http://www.math.pitt.edu/~lewicka/Semester_ConvInt_14/adv_non_anal_workshop.html.

* 15–September 15 **Thematic Semester on Moduli Spaces in Rennes - Nantes - Angers - Brest in Spring/Summer 2014**, Universities of Rennes-Nantes-Brest-Angers and Ecole Normale Superieure of Rennes, Rennes, Brittany, France.
Description: In spring/summer 2014 a special algebraic geometry semester, in particular on moduli spaces, will be organized by the labex Centre Henri Lebesgue (CHL)—headed by the IRMAR dept. and the ENS in Rennes, and by the LMJL dept. in Nantes. This will consist in several activities (schools, conferences, workshops, etc.) related broadly to moduli spaces from analytic, algebraic and arithmetic points of view, to be held in Rennes, Nantes, Angers and Brest.
Subscription forms/Support: All the information and subscription forms can be found on the website http://www.lebesgue.fr/semestre2014. If you are interested in some of the events, it is also possible to get some financial support from the CHL for travel and accomodation. Please check the details on the website.
Information: http://www.lebesgue.fr/content/sem2014-espaces-de-modules.

* 20–22 **CONIAPS XVI (16th International Conference of International Academy of Physical Sciences)**, PDPM Indian Institute of Information Technology, Design & Manufacturing, Jabalpur, India.
Description: CONIAPS XVI is organized to discuss and appreciate the potentialities of the interacting ideas and researches to meet the challenges of the future. The conference will focus on established and emerging problems in the field of Physics, Chemistry, Mathematics, Statistics, Computer Science, Information Technology, and Earth-Sciences (Geophysics, Geology and Geography) as well as topics related to applications of Physical Sciences to Bio-sciences (Biophysics, Bioinformatics, Biochemistry, Bio-mathematics etc.), Electronics and Engineering Sciences. Original contributions related to these areas are invited for presentations.
Information: http://coniaps.iiitdmj.ac.in/aboutus.html.

* 24–29 **Fractal Geometry and Stochastics V**, Hotel Am Burgholz, Tabarz (Thuringian Forest), Germany.
Description: This is the fifth of a highly successful series of conferences that have been held every 4-5 years on Fractal Geometry and Stochastics. Areas covered include: Analysis and geometry on graphs and fractals; Algebraic and number theoretical methods for fractals; Fractals and dynamical systems; Random fractals and stochastic processes; Geometric measure theory in metric spaces.
Information: http://www.fgs5.uni-jena.de.

April 2014

* 5 **2nd Annual Midwest Women in Mathematics Symposium**, University of Notre Dame, Notre Dame, Indiana.
Description: The Midwest Women in Math Symposium moves to Notre Dame on April 5, 2014, for its second year. Based on the successful WIMS held in Southern California, the University of Illinois-Chicago hosted the first such event in the midwest. We hope to strengthen the network of female mathematicians in the midwest and encourage collaborations and mentoring relationships. Most of the space is dedicated to short research talks by graduate students and faculty, a keynote address, and a problem session. Lunch will be provided with time to explore issues surrounding being a woman in mathematics. We will have parallel sessions in: Algebra, Dynamical Systems, Geometry and Topology, Logic, Mathematical Biology, Partial Differential Equations Statistics.
Information: http://www3.nd.edu/~wims/.

* 10–17 **Finsler geometry and applications to hyperbolic geometry and Teichmüller spaces**, Galatasaray University, Istanbul, Turkey.
Description: A series of 5-hour lectures, on Finsler geometry and applications to hyperbolic geometry and Teichmüller spaces, by: Norbert A'Campo (Basel), Moon Duchin (Tufts University), Ludovic Marquis (University of Rennes), Constantin Vernicos (University of Montpellier), Sumio Yamada (Gakushuin University, Tokyo) and Vladimir Matveev (to be confirmed; Univ. of Iena). There will also be some individual lectures on the subject by invited researchers. The event is addressed primarily to young researchers (Ph.D. students and post-docs) but confirmed researchers are also welcome. To participate please consult the website. Contact: A. Papadopoulos, papadop@math.u-strasbg.fr.
Information: http://math.gsu.edu.tr/2014-finsler.html.

* 11–13 **29th Geometry Festival**, Stony Brook University, Stony Brook, New York.
Description: Details to be announced in December.

* 11–13 **Underrepresented Students in Topology and Algebra Research Symposium (USTARS)**, UC Berkeley, Berkeley, California.
Description: Underrepresented Students in Topology and Algebra Research Symposium (USTARS) creates a venue where Algebra and Topology graduate students from underrepresented groups present their work and form research and social support networks with other mathematicians with related research interests. This event is a two-day research symposium which will consist of underrepresented speakers giving 30-minute research talks. These talks will run in parallel sessions and will be divided by topics based on a broad definition of Algebra and Topology. In addition, two distinguished graduate students and one invited faculty mentor will each give one-hour talks and there will be a research poster session featuring invited undergraduate students. The event will close with a panel discussion addressing critical transitions of undergraduate and graduate students.
Information: http://www.ustars.org.

* 15–16 **3rd International Conference on E-Education & Learning Technologies (ICEELT 2014)**, Bangkok, Thailand.
Description: (ICEELT 2013) is designed to provide a common platform to the experts and delegates to share their experiences, research ideas and discuss various related issues and challenges.
Information: http://www.iceelt.com.

* 15–16 **3rd International Conference on Information Integration and Computing Applications (ICIICA 2014)**, Bangkok, Thailand.
Description: 3rd ICIICA 2014 mission is to provide an effective and established international forum for discussion and dissemination of recent advances and innovations in use of technology in education.
Information: http://www.iciica.com.

* 29–30 **2nd International Conference on Information, Communication and Computer Networks (ICICCN 2014)**, Kuala Lumpur, Malaysia.
Description: 2nd ICICCN 2014 mission is to provide an effective and established international forum for discussion and dissemination of recent advances and innovations in use of technology in education.
Information: http://www.iciccn.com.

* 29–30 **4th International Conference on E-Learning and Knowledge Management Technologies (ICEKMT 2014)**, Kuala Lumpur, Malaysia.
Description: ICEKMT 2014 aims at bringing together researchers and practitioners who are interested in e-Learning Technology and Knowledge Management Technology and its current applications.
Information: http://www.icekm.com.

May 2014

* 2–4 **14th Chico Topology Conference**, California State University, Chico, Chico, California.
Invited Speakers: Include Carmen Caprau (CSU, Fresno), Moon Duchin (Tufts University), Logan Hoehn (Nipissing University), and Marcus Marsh (CSU, Sacramento).

Call for Contributed Talks: Researchers at all levels are invited to present 30-minute contributed talks in any area of topology. Send Title and Abstract to: TMattman@CSUChico.edu.
Information: http://www.csuchico.edu/~tmattman/CTC.html.

* 5–9 **Eigenvectors in Graph Theory and Related Problems in Numerical Linear Algebra**, Brown University, Providence, Rhode Island.
Description: The analysis of problems modeled by large graphs is greatly hampered by a lack of efficient computational tools. The purpose of the workshop is to explore possibilities for designing appropriate computational methods that draw on recent advances in numerical methods and scientific computation. Specifically, the questions of how to form the matrices representing graph Laplacians, and how to compute the leading eigenvectors of such matrices will be addressed. It seems likely that these problems will be amenable to algorithms based on randomized projections that dramatically reduce the effective dimensionality of the underlying problems. Such techniques have recently proven highly effective for the related problems of how to find approximate lists of nearest neighbors for clouds of points in high dimensional spaces, and for constructing approximate low-rank factorizations of large matrices. In both cases, a key observation is that the problem of distortions of distances that is inherent to randomized projection techniques can be overcome by using the randomized projections only as preconditioners; they inform the algorithm of where to look, and then highly accurate deterministic techniques are used to compute the actual output. The resulting algorithms scale extra-ordinarily well on modern parallel and multicore architectures. To successfully address the enormous problems arising in the analysis of graphs, it is expected that additional machinery will be needed, such as the use of multi-resolution data structures, and more efficient scalable randomized projections.
Information: http://icerm.brown.edu/sp-s14.

* 21–24 **Progress on Difference Equations**, Izmir University of Economics, Department of Mathematics, Izmir, Turkey.
Description: This meeting continues in the line of other PODE workshops, the first two held in Laufen (Germany) PODE 2007, PODE 2008, followed by the workshops in Bedlewo (Poland) PODE 2009, Xanthi (Greece) PODE 2010, Dublin (Ireland) PODE 2011, Richmond (Virginia, USA) PODE 2012 and Bialystok (Poland) PODE 2013. The workshop aim is to provide a forum for researchers in the area of difference equations (ordinary and partial), discrete dynamical systems, and their applications, to discuss and exchange their latest works. There will be organized a special session dedicated to fractional difference equations. Registration fee includes some meal expenses (lunch and tea/coffee/cookies) in addition to the conference package. Discounted airfare to Izmir may be obtained from American/Europe, Turkish Airlines (or any OneWorld airline). Ramada Encore Hotel Izmir will give a discount for accommodations.
Information: http://dm.ieu.edu.tr/pode2014.

* 23–25 **Non-Associative & Non-Commutative Algebra and Operator Theory. Dakar's Workshop in Honor of Professor Amin Kaidi**, University Cheikh Anta Diop, Dakar, Senegal.
Description: The objective of this Conference is to bring together pure/applied mathematicians, with common interest for algebra, functional analysis and applications.
Topics: Focus on the non-commutative algebras, the non-associative algebras, the operator theory and the rings and modules theory. These themes are relevant in research and development in coding theory, cryptography and quantum mechanics.
Information: http://workshop2014.lacgaa.com/.

* 26–June 20 **Teichmüller theory and surfaces in 3-manifolds (Intensive Research Period)**, Centro di Ricerca Matematica "Ennio De Giorgi", Piazza dei Cavalieri 7, 56100 Pisa, Italy.
Main topics: Riemann surfaces, hyperbolic surfaces, CP1-structures. Geometric structures on Riemann surfaces, representation of surface groups. Teichmüller theory, special mappings between surfaces. Riemannian and Lorentzian 3-manifolds of constant curvature. Surfaces immersed in 3-manifolds, minimal surfaces. During weeks I, II, and IV there will be mini-courses, research talks, discussions. An intensive workshop will take place from June 9–13. T. Barbot (Avignon), F. Bonahon (USC), M. Burger (ETHZ), W. Goldman (College Park), A. Goncharov (Yale), J. Kahn (Brown), F. Labourie (Orsay), R. Mazzeo (Stanford) and A. Neitzke (Austin) have accepted to address thematic mini-courses.
Scientific Committee: S. Kerckhoff (Stanford Univ.), B. Martelli (Univ. di Pisa), J.-M. Schlenker, (Univ. Luxemburg), M. Wolf (Rice Univ.).
Funds: For a limited number of selected young researchers and students are available (http://www.crm.sns.it/event/291/financial.html until March 28, 2014).
Information: http://www.crm.sns.it/event/290/.

June 2014

* 9–15 **School on Nonlinear Analysis, Function Spaces and Applications 10**, Trest, Czech Republic.
Description: Following a long-standing tradition, the Institute of Mathematics of the Academy of Sciences of the Czech Republic and the Department of Mathematical Analysis of the Charles University, Faculty of Mathematics and Physics, are going to organize the 10th international conference on Nonlinear Analysis, Function Spaces and Applications (NAFSA 10). The conference will take place at the Conference Centre of the Academy of Sciences, Trest, Czech Republic.
Lectures: The scientific programme will consist of the following series of invited lectures: Giovanni Alberti (Pisa, Italy), Rectifiable measures and applications, Kari Astala (Helsinki, Finland), Holomorphic deformations and vectorial calculus of variations; Bernd Carl (Jena, Germany), TBA; Andrea Cianchi (Florence, Italy), Reduction principles for Sobolev type embeddings; Carlos Perez (Sevilla, Spain), TBA; Andreas Seeger (Wisconsin, USA), Radial Fourier multipliers and some variants.
Information: http://nafsa.cuni.cz/2014/.

* 16–20 **Conference on stochastic processes and high dimensional probability distributions**, Euler International Mathematical Institute of the Russian Academy of Sciences, Saint Petersburg, Russia.
Description: The conference will focus on several closely related directions in Probability Theory and Analysis including geometric problems about Gaussian and other linear stochastic processes; typical distributions, measure concentration and high dimensional phenomena; optimal transportation and associated Sobolev-type and information-theoretic inequalities.
Invited speakers: V. Bogachev (Moscow University), A. Dembo (Stanford), R. Dudley (MIT), W. Gangbo (Georgia Tech), N. Gozlan (Paris-Est), I. Ibragimov (Steklov Institute), S. Kwapien (Warsaw), R. Latala (Warsaw), M. Ledoux (Toulouse), R. McCann (Toronto), M. Milman (Florida), V. Milman (Tel Aviv), H. von Weizsäcker (Kaiserslautern). There will be an opportunity for contributed talks.
Information: http://www.pdmi.ras.ru/EIMI/2014/Sppd/index.html.

* 18–27 **Summer school on the Gan-Gross-Prasad conjectures**, Institut de Mathématiques de Jussieu - Paris, Rive Gauche 4, place Jussieu, Paris, France.
Description: The aim of the summer-school is to present the Gan-Gross-Prasad conjectures and survey some recent developments.
Information: http://ggp-2014.sciencesconf.org/?lang=en.

* 22–29 **The 52nd International Symposium on Functional Equations**, Innsbruck, Austria.
Topics: Functional equations and inequalities, mean values, equations on algebraic structures, Hyers-Ulam stability, regularity

properties of solutions, conditional equations, iteration theory; applications to the natural, social, and behavioral sciences.
Local Organizer: Wolfgang Förg-Rob, Univ. Innsbruck, Tech. 19a, A-6020 Innsbruck; email: wolfgang.foerg-rob@uibk.ac.at. L. Reich (Austria); R. Ger (Chair, Poland); Zs. Páles (Hungary); J. Rätz (Switzerland), J. Schwaiger (Austria), and A. Sklar (USA).
Information: Participation at these annual meetings is by invitation only. Those wishing to be invited should send details of their interest and, preferably, publications with their postal and e-mail address to: Roman Ger, Institute of Mathematics, Silesian University, Bankowa 14, PL-40-007 Katowice, Poland; email: romanger@us.edu.pl before February 15, 2014.

* 24–27 **Mathematics Meets Physics**, University of Helsinki, Finland.
Description: A four-day conference exploring the frontiers of mathematical physics on the occasion of Antti Kupiainen's 60th birthday. The aim of the conference is to foster the exchange of recent breakthroughs, new ideas and advances in methodology by bringing together world-leading experts, ranging the full spectrum from pure mathematics to physics.
Information: http://wiki.helsinki.fi/display/mathphys/mathphys2014; http://wiki.helsinki.fi/download/attachments/105154551/mmp_poster.pdf.

* 26–July 1 **Sixth International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'14)**, Black-Sea resort, Albena, Bulgaria.
Description: The conference is organized by the Euro-American Consortium for Promoting the Application of Mathematics in Technical and Natural Sciences. It will be scheduled in plenary and keynote lectures followed by special and contributed sessions. The accents of the conference will be on Mathematical Physics, Solitons and Transport Processes, Numerical Methods, Scientific Computing, Continuum Mechanics, Applied Analysis, Applied Physics, Biomathematics, which can be complemented by some specific topics in contributed special sessions. Everybody who is interested in attending AMiTaNS.14 please prepare a short abstract within 300 words clearly stating the goal, tools, and fill out the online Application form.
Deadline: For submissions is March 31, 2014. More information on the website.
Information: http://2014.eac4amitans.eu.

* 30–July 3 **8th Annual International Conference on Statistics**, The Mathematics & Statistics Research Unit (ATINER), Athens Institute for Education and Research, Athens, Greece.
Description: The conference is soliciting papers (in English only) from all areas of Statistics and other related areas. Selected (peer-reviewed) papers will be published in a Special Volume of ATINER's book series. If you think that you can contribute, please submit a 300-word abstract.
Registration: The registration fee is 300 (euros), covering access to all sessions, two lunches, coffee breaks, and conference material. Special arrangements will be made with a local luxury hotel for a limited number of rooms at a special conference rate. In addition, a number of special events will be organized: A Greek night of entertainment with dinner, a special one-day cruise in the Greek islands, an archaeological tour of Athens, and a one-day visit to Delphi.
Information: http://www.atiner.gr/statistics.htm.

July 2014

* 3–7 **2014 International Conference on Topology and its Applications**, University of Patras and Technological Educational Institute of Western Greece, Nafpaktos, Greece.
Description: All areas of Topology and its Applications are included (General Topology, Set-Theoretic Topology, Geometric Topology, Algebraic Topology, Applied Topology. In particular, Topological Groups, Dimension Theory, Dynamical Systems and Continua Theory, Computational Topology, History of Topology).
Organizers: S. D. Iliadis (Chairman), D. N. Georgiou (University of Patras), I. E. Kougias (Technological Educational Institute of Western Greece), A. C. Megaritis (Technological Educational Institute of Western Greece), I. Boules (Mayor of the city of Nafpaktos).
Deadline for abstracts:
Information: http://www.lepantotopology.gr.

* 7–10 **International Conference "Mathematics Days in Sofia"**, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.
Description: The conference is organized by the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences. The purpose of this event is to present the current state of the art in all areas of mathematics, in particular, in the following main sections: Algebra, Logic, and Combinatorics; Analysis, Geometry, and Topology; Differential Equations and Mathematical Physics; Mathematical Modelling; Mathematical Aspects of the Computer Science. One of the main goals of the conference is to bring together Bulgarian mathematicians and computer scientists working all over the world. In this way, the event will enrich the relations between the mathematicians working in Bulgaria and the Bulgarian mathematical diaspora.
Information: http://www.math.bas.bg/mds2014.

* 21–24 **Mixed Integer Programming (MIP) workshop 2014**, The Ohio State University, Columbus, Ohio.
Description: The Mixed Integer Programming (MIP) workshop series is designed to bring together the discrete optimization research community in an annual meeting. MIP 2014 will be the eleventh workshop in the series. The workshop program is composed of a limited number of invited talks (approx. 20 to 25) on recent and typically unpublished work on mixed integer optimization. The talks are organized in a single track and scheduled to leave ample time for discussion and interaction among the participants. To encourage participation of students and junior researchers, no registration fee is charged and limited travel support is available. The workshop features a poster session, in which the attendees can present their work on mixed integer optimization related topics. For enquiries, the conference organizers' email is: mip2014@osu.edu.
Information: http://mip2014.engineering.osu.edu/home.

* 21–25 **Mathematics and Engineering in Marine and Earth Problems**, University of Aveiro, Portugal.
Description: The main goal of MEME2014 is to provide a multidisciplinary forum to engage and discuss the development of innovative scientific and technological tools for the study and exploration of the Ocean and the Earth. The conference is expected to foster a tighter cooperation between theoretical and experimental practitioners.
Information: http://meme.glocos.org.

* 21–25 **Perspectives of Modern Complex Analysis**, Banach Conference Center, Bedlewo, Poland.
Description: The goal of the conference is to bring together researchers working in complex analysis and its most important applications, with emphasis on informal interactions/contacts. The main topics reflect the influence of A. Eremenko (Purdue University) on modern complex analysis. The principal themes include: Classical complex analysis and its associated potential theory, iteration of real, rational and entire mappings, real algebraic geometry, spectral theory and mathematical physics, diverse subjects centered on analysis.
Invited plenary speakers: At present include: C. Bender (St. Louis), W. Bergweiler (Kiel), M. Bonk (UCLA), M. Lyubich (SUNYSB). M. Sodin (Tel Aviv), F. Sottile (Texas A&M), A. Volberg (MSU), K. Yamanoi (Tokyo Tech), A. Zdunik (Warsaw).
Support: Application for NSF support for US participants is pending. The organizers strongly encourage participation from members of underrepresented groups, young, and active mathematicians.
Information: http://bcc.impan.pl/14Perspectives/.

September 2014

* 2–5 **Black-Box Global Optimization: Fast Algorithms and Engineering Applications (part of the CST2014 Conference)**, Hotel Royal Continental, Naples, Italy.
Description: The aim of this session is to create a multidisciplinary discussion platform focused on new theoretical, computational and applied results in solving black-box multiextremal optimization problems. In these problems, frequently encountered in engineering design, the objective function and constraints (if any) are multidimensional functions with unknown analytical representations often evaluated by performing computationally expensive simulations. Researchers from both theoretical and applied sciences are welcome to present their recent developments concerning this important class of optimization problems. To encourage young researchers to attend these conferences a 1000 Euro Young (35 years or younger) Researcher Best Paper Prize will be awarded to the best paper presented at the conferences.
Deadlines: Submission of one-page abstracts: December 5, 2013. Notification of acceptance: December 20, 2013. Payment of the regular registration fee: April 15, 2014
Information: `http://www.civil-comp.com/conf/cstect2014/cst2014-s23.htm`.

* 2–5 **NUMAN2014 Recent Approaches to Numerical Analysis: Theory, Methods and Applications**, Chania, Crete, Greece.
Description: The themes of the conference are in the broad area of numerical analysis and applications, including numerical methods, algorithms and software; numerical and scientific computing; numerical methods and computational modeling; high-performance numerical computing. All areas of numerical analysis are considered, including numerical linear algebra; numerical solution of ODEs, PDEs and stochastic DEs. Several Workshops will be organized to highlight current mathematical, numerical and computational trends in areas of high scientific interest, including Mathematical Biology and Medicine; Enviromental Science and Engineering; Multiphysics/Multidomain Problems. We invite interested researchers to submit one-page abstracts, for lecture or poster presentations, by April 23, 2014.
Information: `http://numan2014.amcl.tuc.gr`.

October 2014

* 22–24 **28th Midwest Conference on Combinatorics and Combinatorial Computing**, University of Nevada, Las Vegas (UNLV), Las Vegas, Nevada.
Description: The Midwest Conferences on Combinatorics and Combinatorial Computing (MCCCC) are of small size (50 to 70 participants) and have been growing slowly. Papers cover a spectrum of pure and applied combinatorics, including graph theory, design theory, enumeration, and combinatorial computing. For 28th MCCCC, the invited speakers are: Brian Alspach; Saad El-Zanati; Futaba Fujie-Okamoto; Joseph Gallian; Margaret Readdy; Ian Wanless. Contributed papers (15–20 minutes talks) are very welcomed.
Information: `http://www.mcccc.info`.

December 2014

* 1–12 **Winter School on Operator Spaces, Non-commutative Probability and Quantum Groups**, Métabief, France.
Description: This two-week school will include 6 courses on quantum groups, operator spaces and non-commutative probability. The venue is located in a village in Jura mountains, France, close to the Swiss border. This school is a part of a trimester in Functional Analysis of the University of Franche-Comte (Besann). The trimester includes also other events, in particular a workshop on Non-commutative Geometry (November 27–29, Besann) and a conference on Operator spaces and Quantum Probability (December 15–19, Besann).
Information: `http://trimestres-lmb.univ-fcomte.fr/Christmas-School.html`.

* 6–31 **The Info-Metrics Annual Prize in Memory of Halbert L. White Jr.**, Washington, DC.
Description: The Info-Metrics Institute is pleased to announce the creation of the Halbert L. White Jr. prize in memory of one of the Institute's founding board members who passed away on March 31, 2012. The prize is intended to reward outstanding academic research by an early career scholar in the field of info-metrics and carries an award of $2000 to be conferred either to an individual or shared among joint recipients. A maximum of one prize will be awarded each year. The award ceremony will occur at the first Info-Metrics meeting (conference or workshop) following the announcement of the award recipient. The annual Info-Metrics prize will be given for the best recent published work, in any academic discipline, that is deemed likely to bring important advances to multiple academic disciplines in the area of info-metrics (the science and practice of inference and quantitative information processing). The first prize will be given in 2014.
Information: `http://www.american.edu/cas/economics/info-metrics/prize.cfm`.

* 11–20 **Foundations of Computational Mathematics Conference**, Universidad de la República, Montevideo, Uruguay.
Description: The conference, organized by the Society for Foundations of Computational Mathematics, is eighth in a sequence that commenced with the Park City, Rio de Janeiro, Oxford, Minneapolis, Santander, Hong Kong and Budapest FoCM meetings. The conference format consists of plenary invited lectures in the mornings and theme-centered parallel workshops in the afternoons. Each workshop extends over three days and the conference will consist of three periods, comprises of different themes. We encourage the participants to attend the full conference.
Information: `http://www.fing.edu.uy/~jana/www2/focm_2014.html`.

January 2015

* 4–6 **ACM-SIAM Symposium on Discrete Algorithms (SODA15), being held with Analytic Algorithmics and Combinatorics (ANALCO15) and Algorithm Engineering and Experiments (ALENEX15)**, The Westin Gaslamp Quarter, San Diego, California.
Description: Information on SODA, ALENEX and ANALCO will be available at `http://www.siam.org/meetings/da15/` in May 2014.
Information: `http://www.siam.org/meetings/da15/`.

The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.

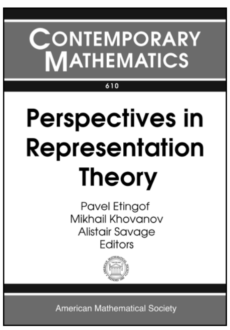
July 2015

* 13–17 **12th International Conference on Finite Fields and Their Applications (Fq12)**, Skidmore College, Saratoga Springs, New York.
Description: The bi-annual series of "Fq" conferences returns to the USA for the first time since 1993. The Fq12 conference will feature 8 invited lectures and approximately 80 contributed talks on all aspects, theoretical and applied, of mathematics and computer science which are related to finite fields. Truly an international event, recent conferences in the series have attracted researchers from about 30 different countries. See the Fq12 website for more information.
Information: `http://www.skidmore.edu/fq12`.

New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to http://www.ams.org/bookstore-email.

Algebra and Algebraic Geometry



Perspectives in Representation Theory

Pavel Etingof, *Massachusetts Institute of Technology, Cambridge, MA*, **Mikhail Khovanov**, *Columbia University, New York, NY*, and **Alistair Savage**, *University of Ottawa, ON, Canada*, Editors

This volume contains the proceedings of the conference Perspectives in Representation Theory, held from May 12–17, 2012, at Yale University, in honor of Igor Frenkel's 60th birthday.

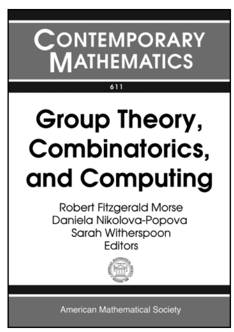
The aim of the conference was to present current progress on the following (interrelated) topics: vertex operator algebras and chiral algebras, conformal field theory, the (geometric) Langlands program, affine Lie algebras, Kac-Moody algebras, quantum groups, crystal bases and canonical bases, quantum cohomology and K-theory, geometric representation theory, categorification, higher-dimensional Kac-Moody theory, integrable systems, quiver varieties, representations of real and p-adic groups, and quantum gauge theories.

The papers in this volume present representation theory connections of numerous other subjects, as well as some of the most recent advances in representation theory, including those which occurred thanks to the application of techniques in other areas of mathematics, and of ideas of quantum field theory and string theory.

Contents: **J. Duncan**, **P. Etingof**, **I. Ip**, **M. Khovanov**, **M. Libine**, **A. Licata**, **A. Savage**, and **M. Schlosser**, On the work of Igor Frenkel; **A. Braverman**, **M. Finkelberg**, and **J. Shiraishi**, MacDonald polynomials, Laumon spaces and perverse coherent sheaves; **A. Braverman**, **H. Garland**, **D. Kazhdan**, and **M. Patnaik**, An affine Gindikin-Karpelevich formula; **M. C. N. Cheng** and **J. F. R. Duncan**, On the discrete groups of Mathieu moonshine; **G. Felder**, **D. Kazhdan**, and **T. M. Schlank**, The classical master equation with an appendix by Tomer M. Schlank; **D. Gaitsgory** and **N. Rozenblyum**, DG indschemes; **T. Kobayashi**, Special functions in minimal representations; **G. Lusztig**, Asymptotic Hecke algebras and involutions; **F. Malikov** and **V. Schechtman**, Chiral differential operators on abelian varieties; **H. Nakajima**, Refined Chern-Simons theory and Hilbert schemes of points on the plane; **C. Stroppel** and **J. Sussan**, Categorified Jones-Wenzl projectors: A comparison; **Y. Zhu**, Weil representations and theta functionals on surfaces.

Contemporary Mathematics, Volume 610

March 2014, approximately 369 pages, Softcover, ISBN: 978-0-8218-9170-4, 2010 *Mathematics Subject Classification:* 17Bxx, 22E57, **AMS members US$100.80**, List US$126, Order code CONM/610



Group Theory, Combinatorics, and Computing

Robert Fitzgerald Morse, *University of Evansville, IN*, **Daniela Nikolova-Popova**, *Florida Atlantic University, Boca Raton, FL*, and **Sarah Witherspoon**, *Texas A & M University, College Station, TX*, Editors

This volume contains the proceedings of the International Conference on Group Theory, Combinatorics and Computing held from October 3–8, 2012, in Boca Raton, Florida.

The papers cover a number of areas in group theory and combinatorics. Topics include finite simple groups, groups acting on structured sets, varieties of algebras, classification of groups generated by 3-state automata over a 2-letter alphabet, new methods for construction of codes and designs, groups with constraints on the derived subgroups of its subgroups, graphs related to conjugacy classes in groups, and lexicographical configurations. Application of computer algebra programs is incorporated in several of the papers.

This volume includes expository articles on finite coverings of loops, semigroups and groups, and on the application of algebraic structures in the theory of communications.

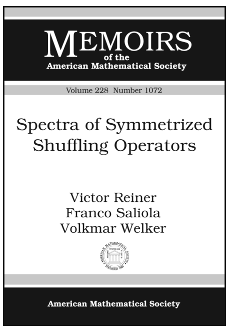
This volume is a valuable resource for researchers and graduate students working in group theory and combinatorics. The articles provide excellent examples of the interplay between the two areas.

This item will also be of interest to those working in discrete mathematics and combinatorics and applications.

Contents: **M. Bianchi**, **M. Herzog**, and **E. Pacifici**, On the regularity of a graph related to conjugacy classes of groups: Small valencies; **R. Grigorchuk** and **D. Savchuk**, Self-similar groups acting essentially freely on the boundary of the binary rooted tree; **C. Hering**, **A. Krebs**, and **T. Edgar**, Non-symmetric lexicographic configurations; **T. Hurley**, Algebraic structures for communications; **L.-C. Kappe**, Finite coverings: A journey through groups, loops, rings and semigroups; **R. Laue**, Decompositions of Kramer-Mesner matrices; **P. Longobardi**, **M. Maj**, and **D. J. S. Robinson**, Recent results on groups with few isomorphism classes of derived subgroups; **J. Moori**, Designs and codes from $PSL_2(q)$; **B. Plotkin**, Algebraic logic and logical geometry in arbitrary varieties of algebras; **P. Spiga** and **A. Zalesski**, A uniform upper bound for the character degree sums and Gelfand-Graev-like characters for finite simple groups.

Contemporary Mathematics, Volume 611

April 2014, approximately 194 pages, Softcover, ISBN: 978-0-8218-9435-4, 2010 *Mathematics Subject Classification:* 05E18, 08A99, 20B15, 20D05, 20E45, 20F14, 20F65, 20N05, 51E15, 94A99, **AMS members US$62.40**, List US$78, Order code CONM/611



Spectra of Symmetrized Shuffling Operators

Victor Reiner, *University of Minnesota, Minneapolis, Minnesota*, **Franco Saliola**, *Université du Québec à Montréal, Canada*, and **Volkmar Welker**, *Philipps-Universitaet Marburg, Germany*

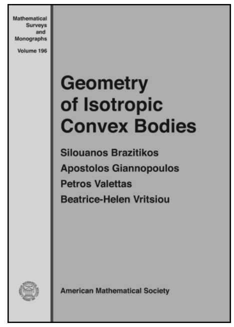
This item will also be of interest to those working in probability and statistics.

Contents: Introduction; Defining the operators; The case where $\mathcal{O}$ contains only hyperplanes; Equivariant theory of BHR random walks; The family $\nu_{(2^k,1^{n-2k})}$; The original family $\nu_{(k,1^{n-k})}$; Acknowledgements; Appendix A. $\mathfrak{S}_n$-module decomposition of $\nu_{(k,1^{n-k})}$; Bibliography; List of Symbols; Index.

Memoirs of the American Mathematical Society, Volume 228, Number 1072

March 2014, 109 pages, Softcover, ISBN: 978-0-8218-9095-0, LC 2013042563, 2010 *Mathematics Subject Classification:* 05E15, 20F55, 60J10, **AMS members US$60.80**, List US$76, Order code MEMO/228/1072

Analysis



Geometry of Isotropic Convex Bodies

Silouanos Brazitikos and **Apostolos Giannopoulos**, *University of Athens, Greece*, **Petros Valettas**, *Texas A & M University, College Station, TX*, and **Beatrice-Helen Vritsiou**, *University of Athens, Greece*

The study of high-dimensional convex bodies from a geometric and analytic point of view, with an emphasis on the dependence of various parameters on the dimension stands at the intersection of classical convex geometry and the local theory of Banach spaces. It is also closely linked to many other fields, such as probability theory, partial differential equations, Riemannian geometry, harmonic analysis and combinatorics. It is now understood that the convexity assumption forces most of the volume of a high-dimensional convex body to be concentrated in some canonical way and the main question is whether, under some natural normalization, the answer to many fundamental questions should be independent of the dimension.

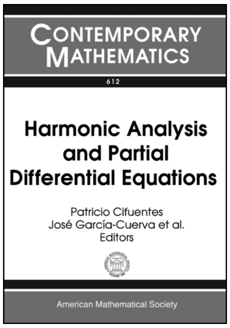
The aim of this book is to introduce a number of well-known questions regarding the distribution of volume in high-dimensional convex bodies, which are exactly of this nature: among them are the slicing problem, the thin-shell conjecture and the Kannan-Lovasz-Simonovits conjecture. This book provides a self-contained and up to date account of the progress that has been made in the last fifteen years.

This item will also be of interest to those working in geometry and topology.

Contents: Background from asymptotic convex geometry; Isotropic log-concave measures; Hyperplane conjecture and Bourgain's upper bound; Partial answers; L_q-centroid bodies and concentration of mass; Bodies with maximal isotropic constant; Logarithmic Laplace transform and the isomorphic slicing problem; Tail estimates for linear functionals; M and $M*$-estimates; Approximating the covariance matrix; Random polytopes in isotropic convex bodies; Central limit problem and the thin shell conjecture; The thin shell estimate; Kannan-Lovász-Simonovits conjecture; Infimum convolution inequalities and concentration; Information theory and the hyperplane conjecture; Bibliography; Subject index; Author index.

Mathematical Surveys and Monographs, Volume 196

April 2014, approximately 603 pages, Hardcover, ISBN: 978-1-4704-1456-6, LC 2913041914, 2010 *Mathematics Subject Classification:* 52Axx, 46Bxx, 60Dxx, 28Axx, **AMS members US$107.20**, List US$134, Order code SURV/196



Harmonic Analysis and Partial Differential Equations

Patricio Cifuentes and **José García-Cuerva**, *Universidad Autónoma de Madrid, Spain*, **Gustavo Garrigós**, *Universidad de Murcia, Spain*, **Eugenio Hernández**, *Universidad Autónoma de Madrid, Spain*, **José María Martell**, **Javier Parcet**, and **Keith M. Rogers**, *Consejo Superior de Investigaciones Científicas, Madrid, Spain*, and **Alberto Ruiz**, **Fernando Soria**, and **Ana Vargas**, *Universidad Autónoma de Madrid, Spain*, Editors

This volume contains the Proceedings of the 9th International Conference on Harmonic Analysis and Partial Differential Equations, held June 11–15, 2012, in El Escorial, Madrid, Spain.

Included in this volume is the written version of the mini-course given by Jonathan Bennett on *Aspects of Multilinear Harmonic Analysis Related to Transversality*. Also included, among other papers, is a paper by Emmanouil Milakis, Jill Pipher, and Tatiana Toro, which reflects and extends the ideas presented in the mini-course on *Analysis on Non-smooth Domains* delivered at the conference by Tatiana Toro.

The topics of the contributed lectures cover a wide range of the field of Harmonic Analysis and Partial Differential Equations and illustrate the fruitful interplay between the two subfields.

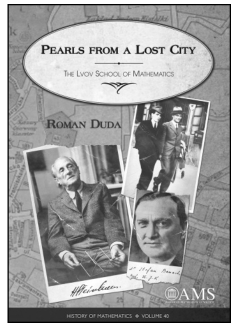
This item will also be of interest to those working in differential equations.

Contents: **J. Bennett**, Aspects of multilinear harmonic analysis related to transversality; **F. Bernicot**, Multi-frequency Calderón-Zygmund analysis and connexion to Bochner-Riesz multipliers; **O. Beznosova**, **J. C. Moraes**, and **M. C. Pereyra**, Sharp bounds for t-Haar multipliers on L^2; **M. Bownik** and **J. Jasper**, Spectra of frame operators with prescribed frame norms; **J. Fan** and **T. Ozawa**, Regularity criteria for Hall-magnetohydrodynamics and the space-time monopole equation in Lorenz gauge; **T. P. Hytönen**, The A_2 theorem: Remarks and complements; **M. Junge**, **T. Mei**, and **J. Parcet**, An invitation to harmonic analysis associated with semigroups of operators; **J. M. Martell**, **D. Mitrea**, **I. Mitrea**, and **M. Mitrea**, The higher order regularity Dirichlet problem for elliptic systems in the upper-half space; **E. Milakis**, **J. Pipher**, and **T. Toro**, Perturbations of elliptic operators in chord arc domains; **A. Rosén**, Cauchy non-integral formulas.

Contemporary Mathematics, Volume 612

April 2014, 178 pages, Softcover, ISBN: 978-0-8218-9433-0, LC 2013036893, 2010 *Mathematics Subject Classification:* 31-XX, 35-XX, 42-XX, 46-XX, 47-XX, **AMS members US$62.40**, List US$78, Order code CONM/612

General Interest



Pearls from a Lost City

The Lvov School of Mathematics

Roman Duda, *University of Wrocław, Poland*
Translated by Daniel Davies

The fame of the Polish school at Lvov rests with the diverse and fundamental contributions of Polish mathematicians working there during the interwar years. In particular, despite material hardship and without a notable mathematical tradition, the school made major contributions to what is now called functional analysis. The results and names of Banach, Kac, Kuratowski, Mazur, Nikodym, Orlicz, Schauder, Sierpiński, Steinhaus, and Ulam, among others, now appear in all the standard textbooks.

The vibrant joie de vivre and singular ambience of Lvov's once scintillating social scene are evocatively recaptured in personal recollections. The heyday of the famous Scottish Café—unquestionably the most mathematically productive cafeteria of all time—and its precious *Scottish Book* of highly influential problems are described in detail, revealing the special synergy of scholarship and camaraderie that permanently elevated Polish mathematics from utter obscurity to global prominence.

This chronicle of the Lvov school—its legacy and the tumultuous historical events which defined its lifespan—will appeal equally to mathematicians, historians, or general readers seeking a cultural and institutional overview of key aspects of twentieth-century Polish mathematics not described anywhere else in the extant English-language literature.

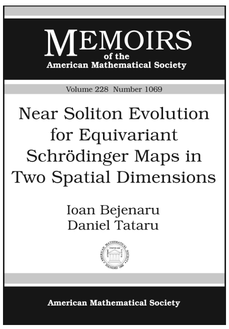
This item will also be of interest to those working in analysis.

Contents: *Background:* The University and the Polytechnic in Lvov; Polish mathematics at the turn of the twentieth century; Sierpiński's stay at the University of Lvov (1908–1914); The University in Warsaw and Janiszewski's program (1915–1920); World mathematics (active fields in Poland) around 1920; *The golden age: Individuals and community:* The mathematical community in Lvov after World War I; Mathematical studies and students; Journals, monographs, and congresses; The popularization of mathematics; Social life (the Scottish Café, the *Scottish Book*); The Polish Mathematical Society; Collaboration with other centers; In the eyes of others; *The golden age: Achievements:* Stefan Banach's doctoral thesis and priority claims; Probability theory; Measure theory; Game theory: A revelation without follow-up; Operator theory in the 1920s; Methodological audacity; Banach's monograph: Polishing the pearls; Operator theory in the 1930s: The dazzle of pearls; New perspectives for which time did not allow; On the periphery; *Oblivion:* Ukrainization the Soviet way (1939–1941); The German occupation (1941–1944); The expulsion of Poles (1945–1946); *Historical significance:* Chronological overview; Chronology of events as perceived elsewhere; Influence on mathematics of the Lvov school; A tentative summary; Mathematics in Lvov after 1945; *List of Lvov mathematicians:* Mathematicians associated with Lvov; Bibliographies; List of illustrations; Index of names.

History of Mathematics, Volume 40

May 2014, approximately 207 pages, Hardcover, ISBN: 978-1-4704-1076-6, LC 2013037212, 2010 *Mathematics Subject Classification:* 01A60, 01A70, 01A72, 01A80, **AMS members US$31.20**, List US$39, Order code HMATH/40

Mathematical Physics



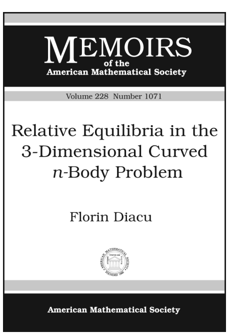
Near Soliton Evolution for Equivariant Schrödinger Maps in Two Spatial Dimensions

Ioan Bejenaru, *University of California, San Diego, La Jolla, CA*, and **Daniel Tataru**, *University of California, Berkeley, CA*

Contents: Introduction; An outline of the paper; The Coulomb gauge representation of the equation; Spectral analysis for the operators H, $\tilde{H}$; the X, LX spaces; The linear $\tilde{H}$ Schrödinger equation; The time dependent linear evolution; Analysis of the gauge elements in X, LX; The nonlinear equation for ψ; The bootstrap estimate for the λ parameter; The bootstrap argument; The $\dot{H}^1$ instability result; Bibliography.

Memoirs of the American Mathematical Society, Volume 228, Number 1069

March 2014, 108 pages, Softcover, ISBN: 978-0-8218-9215-2, LC 2013042543, 2010 *Mathematics Subject Classification:* 58J35; 35B65, **AMS members US$60.80**, List US$76, Order code MEMO/228/1069



Relative Equilibria in the 3-Dimensional Curved n-Body Problem

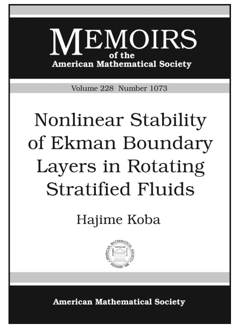
Florin Diacu, *University of Victoria, B.C., Canada*

This item will also be of interest to those working in differential equations.

Contents: Introduction; Background and equations of motion; Isometries and relative equilibria; Criteria and qualitative behaviour; Examples; Conclusions; Bibliography.

Memoirs of the American Mathematical Society, Volume 228, Number 1071

March 2014, 80 pages, Softcover, ISBN: 978-0-8218-9136-0, LC 2013042561, 2010 *Mathematics Subject Classification:* 70F10; 34C25, 37J45, **AMS members US$56.80**, List US$71, Order code MEMO/228/1071



Nonlinear Stability of Ekman Boundary Layers in Rotating Stratified Fluids

Hajime Koba, *Waseda University, Tokyo, Japan*

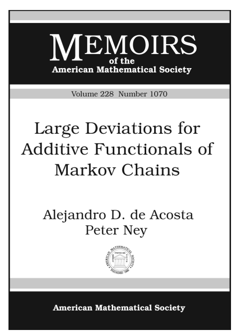
This item will also be of interest to those working in differential equations.

Contents: Introduction; Formulation and main results; Linearized problem; Existence of global weak solutions; Uniqueness of weak solutions; Nonlinear stability; Smoothness of weak solutions; Some extensions of the theory; Appendix A. Toolbox; Bibliography.

Memoirs of the American Mathematical Society, Volume 228, Number 1073

March 2014, 127 pages, Softcover, ISBN: 978-0-8218-9133-9, LC 2013042634, 2010 *Mathematics Subject Classification:* 35Q86, 76E20; 35B35, 35B40, 35B65, 76D03, 76D05, **AMS members US$63.20**, List US$79, Order code MEMO/228/1073

Probability and Statistics



Large Deviations for Additive Functionals of Markov Chains

Alejandro D. de Acosta and **Peter Ney**

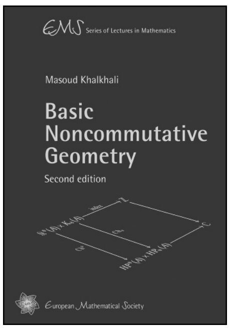
Contents: Introduction; The transform kernels K_g and their convergence parameters; Comparison of $\Lambda(g)$ and $\phi_\mu(g)$; Proof of Theorem 1; A characteristic equation and the analyticity of Λ_f: The case when P has an atom $C \in S^+$ satisfying $\lambda^*(C) > 0$; Characteristic equations and the analyticity of Λ_f: The general case when P is geometrically ergodic; Differentiation formulas for u_g and Λ_f in the general case and their consequences; Proof of Theorem 2; Proof of Theorem 3; Examples; Applications to an autoregressive process and to reflected random walk; Appendix; Background comments; References.

Memoirs of the American Mathematical Society, Volume 228, Number 1070

March 2014, 108 pages, Softcover, ISBN: 978-0-8218-9089-9, LC 2013042546, 2010 *Mathematics Subject Classification:* 60J05, 60F10, **AMS members US$60.80**, List US$76, Order code MEMO/228/1070

New AMS-Distributed Publications

Analysis



Basic Noncommutative Geometry

Second Edition

Masoud Khalkhali, *University of Western Ontario, London, Ontario, Canada*

This text provides an introduction to noncommutative geometry and some of its applications. It can be used either as a textbook for a graduate course or for self-study. It will be useful for graduate students and researchers in mathematics and theoretical physics and all those who are interested in gaining an understanding of the subject.

One feature of this book is the wealth of examples and exercises that help the reader to navigate through the subject. While background material is provided in the text and in several appendices, some familiarity with basic notions of functional analysis, algebraic topology, differential geometry and homological algebra at a first year graduate level is helpful.

Developed by Alain Connes since the late 1970s, noncommutative geometry has found many applications to long-standing conjectures in topology and geometry and has recently made headways in theoretical physics and number theory. The book starts with a detailed description of some of the most pertinent algebra geometry correspondences by casting geometric notions in algebraic terms, then proceeds in the second chapter to the idea of a noncommutative space and how it is constructed. The last two chapters deal with homological tools: cyclic cohomology and Connes–Chern characters in K-theory and K-homology, culminating in one commutative diagram expressing the equality of topological and analytic index in a noncommutative setting. Applications to integrality of noncommutative topological invariants are given as well.

Two new sections have been added to the second edition: the first new section concerns the Gauss–Bonnet theorem and the definition and computation of the scalar curvature of the curved noncommutative two torus, and the second new section is a brief introduction to Hopf cyclic cohomology. The bibliography has been extended and some new examples are presented.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: Examples of algebra-geometry correspondences; Noncommutative quotients; Cyclic cohomology; Connes–Chern character; Appendices; Bibliography; Index.

EMS Series of Lectures in Mathematics, Volume 10

December 2013, 257 pages, Softcover, ISBN: 978-3-03719-128-6, 2010 *Mathematics Subject Classification:* 58-02, 58B34, **AMS members US$38.40**, List US$48, Order code EMSSERLEC/10.R

General Interest

Jacques Tits, Œuvres–Collected Works

Volumes I–IV

Francis Buekenhout, *Université Libre de Bruxelles, Brussels, Belgium*, **Bernhard Mühlherr**, *Universität Gissen, Germany*, **Jean-Pierre Tignol**, *Université Catholique de Louvain, Belgium*, and **Hendrik Van Maldeghem**, *Ghent University, Belgium*, Editors

Jacques Tits was awarded the Wolf Prize in 1993 and the Abel Prize (jointly with John Thompson) in 2008. The impact of his contributions in algebra, group theory and geometry made over a span of more than five decades is incalculable. Many fundamental developments in several fields of mathematics have their origin in ideas of Tits. A number of Tits' papers mark the starting point of completely new directions of research. Outstanding examples are papers on quadratic forms, on Kac-Moody groups and on what subsequently became known as the Tits alternative.

These volumes contain an almost complete collection of Tits' mathematical writings. They include, in particular, a number of published and unpublished manuscripts which have not been easily accessible until now. This collection of Tits' contributions in one place makes the evolution of his mathematical thinking visible. The development of his theory of buildings and BN-pairs and its bearing on the theory of algebraic groups, for example, reveal a fascinating story. Along with Tits' mathematical writings, these volumes contain biographical data, survey articles on aspects of Tits' work, and comments by the editors on the content of some of his papers.

With the publication of these volumes, a major piece of 20th-century mathematics is being made available to a wider audience.

This item will also be of interest to those working in discrete mathematics and combinatorics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: For the table of contents, go to `www.ams.org/bookstore`.

Heritage of European Mathematics, Volume 8

November 2013, 3963 pages, Hardcover, ISBN: 978-3-03719-126-2, 2010 *Mathematics Subject Classification:* 00B60, 05-06, 17-06, 20-06, 22-06, 51-06, **AMS members US$638.40**, List US$798, Order code EMSHEM/8

Classified Advertisements

Positions available, items for sale, services available, and more

TAIWAN

NATIONAL CHENGCHI UNIVERSITY
Department of Mathematical Sciences
Open Faculty Positions (Tenure-Stream)
September 26, 2013

The Department of Mathematical Sciences at National Chengchi University at Taipei, Taiwan, anticipates openings for 1-3 tenure-stream faculty positions at any rank. Senior applicants are also welcome. We will consider outstanding candidates in Applied Mathematics or related fields. The successful candidate must hold a doctoral degree and be able to communicate effectively in Chinese. National Chengchi University is one of major leading universities in Taiwan. To apply, candidates should have evidence of, or potential for, teaching excellence and send a cover letter indicating when he/she is available to start the job, together with the following documents: 1. Current Vitae; 2. A photocopy of diploma and a photocopy of previous job contracts or appointment letters; 3. A doctoral dissertation or research publications in the past five years; 4. Three recommendation letters; 5. A list of the graduate/undergraduate courses (with syllabi) that the candidate would teach; 6. Graduate transcripts (those who have already gotten a university teaching certificate in Taiwan may only need to submit a photocopy of the certificate in lieu of the graduate transcripts); Ph.D. candidates should have a letter confirming the graduation date before May 31, 2014, by the dissertation advisor or the department head. To ensure full consideration, candidates should apply before December 30, 2013, and mail to

Faculty Search Committee
Department of Mathematical Sciences
National Chengchi University
Wen-Shan, Taipei
11605 Taiwan;
Fax: 886-2-2938-7905;
email: math@nccu.edu.tw All application materials will NOT be returned.

000013

BOOK FOR SALE

Book for sale

Book for sale on Amazon.com: D.S. Tselnik, The Function Xi*.

000012

COMMENTS SOLICITED

A Solution to the 3x + 1 Problem?

I believe I might have solved this very difficult problem. I will welcome reader comments. See "A Solution to the $3x + 1$ Problem", on http://occampress.com. Peter Schorer, peteschorer@gmail.com.

000011

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2012 rate is $3.50 per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional $10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: March 2014 issue–January 2, 2014; April 2014 issue–January 30, 2014; May 2014 issue–March 3, 2014; June/July 2014 issue–April 29, 2014; August 2014 issue–May 29, 2014; September 2014 issue–June 30, 2014.

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Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to classads@ams.org. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 20904. Advertisers will be billed upon publication.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

Knoxville, Tennessee

University of Tennessee, Knoxville

March 21–23, 2014

Friday – Sunday

Meeting #1097

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 6, 2014
Program issue of electronic *Notices*: March 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 28, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Maria Chudnovsky, Columbia University, *Coloring graphs with forbidden induced subgraphs* (Erdős Memorial Lecture).

Ilse Ipsen, North Carolina State University, *Introduction to randomized matrix algorithms*.

Daniel Krashen, University of Georgia, *Algebraic structures and the arithmetic of fields*.

Suresh Venapally, Emory University, *Quadratic forms and Galois cohomology*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Methods in Graph Theory and Combinatorics (Code: SS 7A), **Felix Lazebnik**, University of Delaware, **Andrew Woldar**, Villanova University, and **Bangteng Xu**, Eastern Kentucky University.

Arithmetic of Algebraic Curves (Code: SS 9A), **Lubjana Beshaj**, Oakland University, **Caleb Shor**, Western New England University, and **Andreas Malmendier**, Colby College.

Commutative Ring Theory (in honor of the retirement of David E. Dobbs) (Code: SS 1A), **David Anderson**, University of Tennessee, Knoxville, and **Jay Shapiro**, George Mason University.

Completely Integrable Systems and Dispersive Nonlinear Equations (Code: SS 12A), **Robert Buckingham**, University of Cincinnati, and **Peter Perry**, University of Kentucky.

Complex Analysis, Probability, and Metric Geometry (Code: SS 11A), **Matthew Badger**, Stony Brook University, **Jim Gill**, St. Louis University, and **Joan Lind**, University of Tennessee, Knoxville.

Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (Code: SS 18A), **Xiaobing Feng** and **Ohannes Karakashian**, University of Tennessee, Knoxville, and **Yulong Xing**, University of Tennessee, Knoxville, and Oak Ridge National Laboratory.

Diversity of Modeling and Optimal Control: A Celebration of Suzanne Lenhart's 60th Birthday (Code: SS 3A), **Wandi Ding**, Middle Tennessee State University, and **Renee Fister**, Murrray State University.

Fractal Geometry and Ergodic Theory (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Pan American.

Galois Cohomology and the Brauer Group (Code: SS 26A), **Ben Antieau**, University of Washington, **Daniel Krashen**, University of Georgia, and **Suresh Venapally**, Emory University.

Geometric Topology (Code: SS 21A), **Craig Guilbault**, University of Wisconsin-Milwaukee, and **Steve Ferry**, Rutgers University.

Geometric Topology and Number Theory (Code: SS 22A), **Eriko Hironaka** and **Kathleen Petersen**, Florida State University.

Geometric and Algebraic Combinatorics (Code: SS 16A), **Benjamin Braun** and **Carl Lee**, University of Kentucky.

Geometric and Combinatorial Methods in Lie Theory (Code: SS 15A), **Amber Russell** and **William Graham**, University of Georgia.

Graph Theory (Code: SS 8A), **Chris Stephens**, **Dong Ye**, and **Xiaoya Zha**, Middle Tennessee State University.

Harmonic Analysis and Nonlinear Partial Differential Equations (Code: SS 5A), **J. Denzler**, **M. Frazier**, **Tuoc Phan**, and **T. Todorova**, University of Tennessee, Knoxville.

Invariant Subspaces of Function Spaces (Code: SS 6A), **Catherine Beneteau**, University of South Florida, **Alberto A. Condori**, Florida Gulf Coast University, **Constanze Liaw**, Baylor University, and **Bill Ross**, University of Richmond.

Mathematical Modeling of the Within- and Between-Host Dynamics of Infectious Diseases (Code: SS 25A), **Megan Powell**, University of St. Francis, and **Judy Day** and **Vitaly Ganusov**, University of Tennessee, Knoxville.

Mathematical Physics and Spectral Theory (Code: SS 10A), **Roger Nichols**, The University of Tennessee at Chattanooga, and **Günter Stolz**, University of Alabama at Birmingham.

Metric Geometry and Topology (Code: SS 23A), **Catherine Searle**, Oregon State University, **Jay Wilkins**, University of Connecticut, and **Conrad Plaut**, University of Tennessee, Knoxville.

Nonlinear Partial Differential Equations in the Applied Sciences (Code: SS 19A), **Lorena Bociu**, North Carolina State University, **Ciprian Gal**, Florida International University, and **Daniel Toundykov**, University of Nebraska-Lincoln.

Randomized Numerical Linear Algebra (Code: SS 4A), **Ilse Ipsen**, North Carolina State University.

Recent Development on Hyperbolic Conservation Laws (Code: SS 20A), **Geng Chen**, **Ronghua Pan**, and **Weizhe Zhang**, Georgia Tech.

Scientific Computing, Numerical Analysis, and Mathematical Modeling (Code: SS 17A), **Vasilios Alexiades**, **Xiaobing Feng**, and **Steven Wise**, University of Tennessee, Knoxville.

Singularities and Physics (Code: SS 13A), **Mboyo Esole**, Harvard University, and **Paolo Aluffi**, Florida State University.

Stochastic Processes and Related Topics (Code: SS 14A), **Jan Rosinski** and **Jie Xiong**, University of Tennessee, Knoxville.

von Neumann Algebras and Free Probability (Code: SS 24A), **Remus Nicoara**, University of Tennessee, Knoxville, and **Arnaud Brothier**, Vanderbilt University.

Baltimore, Maryland

University of Maryland, Baltimore County

March 29–30, 2014

Saturday – Sunday

Meeting #1098

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 26, 2014
Program issue of electronic *Notices*: March 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 28, 2014

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Maria Gordina, University of Connecticut, *Stochastic analysis and geometric functional inequalities*.

L. Mahadevan, Harvard University, *Shape: Mathematics, physics, and biology*.

Nimish Shah, Ohio State University, *Dynamics of subgroup actions on homogeneous spaces and its interaction with number theory*.

Dani Wise, McGill University, *Cube complexes*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Data Assimilation Applied to Controlled Systems (Code: SS 15A), **Damon McDougall**, University of Texas at Austin, and **Richard Moore**, New Jersey Institute of Technology.

Difference Equations and Applications (Code: SS 8A), **Michael Radin**, Rochester Institute of Technology.

Discrete Geometry in Crystallography (Code: SS 14A), **Egon Schulte**, Northeastern University, and **Marjorie Senechal**, Smith College.

Harmonic Analysis and Its Applications (Code: SS 10A), **Susanna Dann**, University of Missouri, **Azita Mayeli**, Queensborough College, City University of New York, and **Gestur Olafsson**, Louisana State University.

Interaction between Complex and Geometric Analysis (Code: SS 13A), **Peng Wu**, Cornell University, and **Yuan Yuan**, Syracuse University.

Invariants in Low-Dimensional Topology (Code: SS 1A), **Jennifer Hom**, Columbia University, and **Tye Lidman**, University of Texas at Austin.

Knots and Applications (Code: SS 3A), **Louis Kauffman**, University of Illinois at Chicago, **Samuel Lomonaco**, University of Maryland, Baltimore County, and **Jozef Przytycki**, George Washington University.

Low-dimensional Topology and Group Theory (Code: SS 16A), **David Futer**, Temple University, and **Daniel Wise**, McGill University.

Mathematical Biology (Code: SS 6A), **Jonathan Bell** and **Brad Peercy**, University of Maryland Baltimore County.

Mathematical Finance (Code: SS 2A), **Agostino Capponi**, John Hopkins University.

Mechanics and Control (Code: SS 9A), **Jinglai Shen**, University of Maryland Baltimore County, and **Dmitry Zenkov**, North Carolina State University.

Novel Developments in Tomography and Applications (Code: SS 4A), **Alexander Katsevich**, **Alexandru Tamasan**, and **Alexander Tovbis**, University of Central Florida.

Open Problems in Stochastic Analysis and Related Fields (Code: SS 7A), **Masha Gordina**, University of Connecticut, and **Tai Melcher**, University of Virginia.

Optimization and Related Topics (Code: SS 11A), **M. Seetharama Gowda**, **Osman Guler**, **Florian Potra**, and **Jinlai Shen**, University of Maryland at Baltimore County.

Substitution and Tiling Dynamical Systems (Code: SS 17A), **Natalie Priebe Frank**, Vassar College, and **E. Arthur Robinson Jr.**, George Washington University.

Theory and Applications of Differential Equations on Graphs (Code: SS 5A), **Jonathan Bell**, University of Maryland Baltimore County, and **Sergei Avdonin**, University of Alaska Fairbanks.

Albuquerque, New Mexico

University of New Mexico

April 5–6, 2014

Saturday – Sunday

Meeting #1099

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 11, 2014

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Anton Gorodetski, University of California Irvine, *Hyperbolic dynamics and spectral properties of one-dimensional quasicrystals.*

Fan Chung Graham, University of California, San Diego, *Some problems and results in spectral graph theory.*

Adrian Ioana, University of California, San Diego, *Rigidity for von Neumann algebras and ergodic group actions.*

Karen Smith, University of Michigan, Ann Arbor, *The power of characteristic p.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Analysis and Topology in Special Geometries (Code: SS 14A), **Charles Boyer**, **Daniele Grandini**, and **Dimiter Vassilev**, University of New Mexico.

Arithmetic and Differential Algebraic Geometry (Code: SS 17A), **Alexandru Buium**, University of New Mexico, **Taylor Dupuy**, University of California, Los Angeles, and **Lance Edward Miller**, University of Arkansas.

Commutative Algebra (Code: SS 7A), **Daniel J. Hernandez**, University of Utah, **Karen E. Smith**, University of Michigan, and **Emily E. Witt**, University of Minnesota.

Descriptive Set Theory and its Applications (Code: SS 6A), **Alexander Kechris**, California Institute of Technology, and **Christian Rosendal**, University of Illinois, Chicago.

Flat Dynamics (Code: SS 8A), **Jayadev Athreya**, University of Illinois, Urbana-Champaign, **Robert Niemeyer**, University of New Mexico, Albuquerque, **Richard E. Schwartz**, Brown University, and **Sergei Tabachnikov**, The Pennsylvania State University.

Harmonic Analysis and Dispersive Equations (Code: SS 11A), **Matthew Blair**, University of New Mexico, and **Jason Metcalfe**, University of North Carolina.

Harmonic Analysis and Its Applications (Code: SS 19A), **Jens Gerlach Christensen**, Colgate University, and **Joseph Lakey** and **Nicholas Michalowski**, New Mexico State University.

Harmonic Analysis and Operator Theory (in memory of Cora Sadosky) (Code: SS 18A), **Laura De Carli**, Florida International University, **Alex Stokolos**, Georgia Southern University, and **Wilfredo Urbina**, Roosevelt University.

Hyperbolic Dynamics, Dynamically Defined Fractals, and Applications (Code: SS 22A), **Anton Gorodetski**, University of California Irvine.

Interactions in Commutative Algebra (Code: SS 4A), **Louiza Fouli** and **Bruce Olberding**, New Mexico State University, and **Janet Vassilev**, University of New Mexico.

Mathematical Finance (Code: SS 21A), **Indranil SenGupta**, North Dakota State University,.

Modeling Complex Social Processes Within and Across Levels of Analysis (Code: SS 15A), **Simon DeDeo**, Indiana University, and **Richard Niemeyer**, University of Colorado, Denver.

Nonlinear Waves and Singularities in Water Waves, Optics and Plasmas (Code: SS 23A), **Alexander O. Korotkevich** and **Pavel Lushnikov**, University of New Mexico, Albuquerque.

Partial Differential Equations in Materials Science (Code: SS 10A), **Lia Bronsard**, McMaster University, and **Tiziana Giorgi**, New Mexico State University.

Physical Knots, honoring the retirement of Jonathan K. Simon (Code: SS 13A), **Greg Buck**, St. Anselm College, and **Eric Rawdon**, University of St. Thomas.

Progress in Noncommutative Analysis (Code: SS 2A), **Anna Skripka**, University of New Mexico, and **Tao Mei**, Wayne State University.

Spectral Theory (Code: SS 12A), **Milivoje Lukic**, Rice University, and **Maxim Zinchenko**, University of New Mexico.

Stochastic Processes in Noncommutative Probability (Code: SS 20A), **Michael Anshelevich**, Texas A&M University, and **Todd Kemp**, University of California San Diego.

Stochastics and PDEs (Code: SS 5A), **Juraj Földes**, Institute for Mathematics and Its Applications, **Nathan Glatt-Holtz**, Institute for Mathematics and Its Applications and Virginia Tech, and **Geordie Richards**, Institute for Mathematics and Its Applications and University of Rochester.

The Common Core and University Mathematics Instruction (Code: SS 16A), **Justin Boyle**, **Michael Nakamaye**, and **Kristin Umland**, University of New Mexico.

The Inverse Problem and Other Mathematical Methods Applied in Physics and Related Sciences (Code: SS 1A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico and Enfitek, Inc.

Topics in Spectral Geometry and Global Analysis (Code: SS 3A), **Ivan Avramidi**, New Mexico Institute of Mining and Technology, and **Klaus Kirsten**, Baylor University.

Weighted Norm Inequalities and Related Topics (Code: SS 9A), **Oleksandra Beznosova**, Baylor University, **David Cruz-Uribe**, Trinity College, and **Cristina Pereyra**, University of New Mexico.

Lubbock, Texas

Texas Tech University

April 11–13, 2014

Friday – Sunday

Meeting #1100

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 27, 2014
Program issue of electronic *Notices*: April 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 10, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Nir Avni, Northwestern University, *To be announced.*

Alessio Figalli, University of Texas, *To be announced.*

Jean-Luc Thiffeault, University of Wisconsin-Madison, *To be announced.*

Rachel Ward, University of Texas at Austin, *To be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Geometry (Code: SS 9A), **David Weinberg**, Texas Tech University.

Analysis and Applications of Dynamic Equations on Time Scales (Code: SS 2A), **Heidi Berger**, Simpson College, and **Raegan Higgins**, Texas Tech University.

Applications of Special Functions in Combinatorics and Analysis (Code: SS 12A), **Atul Dixit**, Tulane University, and **Timothy Huber**, University of Texas Pan American.

Approximation Theory in Signal Processing (Code: SS 17A), **Rachel Ward**, University of Texas at Austin, and **Rayan Saab**, University of California San Diego.

Complex Function Theory and Special Functions (Code: SS 7A), **Roger W. Barnard** and **Kent Pearce**, Texas Tech University, **Kendall Richards**, Southwestern University, and **Alex Solynin** and **Brock Williams**, Texas Tech University.

Developments from PASI 2012: Commutative Algebra and Interactions with Related Disciplines (Code: SS 26A), **Kenneth Chan**, University of Washington, and **Jack Jeffries**, University of Utah.

Differential Algebra and Galois Theory (Code: SS 23A), **Lourdes Juan** and **Arne Ledet**, Texas Tech University, **Andy R. Magid**, University of Oklahoma, and **Michael F. Singer**, North Carolina State University.

Fractal Geometry and Dynamical Systems (Code: SS 3A), **Mrinal Kanti Roychowdhury**, The University of Texas-Pan American.

Geometry and Geometric Analysis (Code: SS 25A), **Lance Drager** and **Jeffrey M. Lee**, Texas Tech University.

Homological Methods in Algebra (Code: SS 8A), **Lars W. Christensen**, Texas Tech University, **Hamid Rahmati**, Miami University, and **Janet Striuli**, Fairfield University.

Hysteresis and Multi-rate Processes (Code: SS 19A), **Ram Iyer**, Texas Tech University.

Interactions between Commutative Algebra and Algebraic Geometry (Code: SS 11A), **Brian Harbourne** and **Alexandra Seceleanu**, University of Nebraska-Lincoln.

Issues Regarding the Recruitment and Retention of Women and Minorities in Mathematics (Code: SS 5A), **James Valles Jr.**, Prairie View A&M University, and **Doug Scheib**, Saint Mary-of-the-Woods College.

Lie Groups (Code: SS 13A), **Benjamin Harris**, **Hongyu He**, and **Gestur Ólafsson**, Louisiana State University.

Linear Operators in Representation Theory and in Applications (Code: SS 20A), **Markus Schmidmeier**, Florida

Atlantic University, and **Gordana Todorov**, Northeastern University.

Mathematical Models of Infectious Diseases in Plants, Animals and Humans (Code: SS 21A), **Linda Allen**, Texas Tech University, and **Vrushal Bokil**, Oregon State University.

Navier-Stokes Equations and Fluid Dynamics (Code: SS 14A), **Radu Dascaliuc**, Oregon State University, and **Luan Hoang**, Texas Tech University.

Noncommutative Algebra, Deformations, and Hochschild Cohomology (Code: SS 10A), **Anne Shepler**, University of North Texas, and **Sarah Witherspoon**, Texas A&M University.

Numerical Methods for Systems of Partial Differential Equations (Code: SS 27A), **JaEun Ku**, Oklahoma State Universtity, and **Young Ju Lee**, Texas State University.

Optimal Control Problems from Neuron Ensembles, Genomics and Mechanics (Code: SS 24A), **Bijoy K. Ghosh** and **Clyde F. Martin**, Texas Tech University.

Qualitative Theory for Non-linear Parabolic and Elliptic Equations (Code: SS 6A), **Akif Ibragimov**, Texas Tech University, and **Peter Polacik**, University of Minnesota.

Recent Advancements in Differential Geometry and Integrable PDEs, and Their Applications to Cell Biology and Mechanical Systems (Code: SS 4A), **Giorgio Bornia**, **Akif Ibragimov**, and **Magdalena Toda**, Texas Tech University.

Recent Advances in the Applications of Nonstandard Finite Difference Schemes (Code: SS 15A), **Ronald E. Mickens**, Clark Atlanta University, and **Lih-Ing W. Roeger**, Texas Tech University.

Recent Developments in Number Theory (Code: SS 18A), **Dermot McCarthy** and **Chris Monico**, Texas Tech University.

Statistics on Manifolds (Code: SS 22A), **Leif Ellingson**, Texas Tech University.

Topology and Physics (Code: SS 1A), **Razvan Gelca** and **Alastair Hamilton**, Texas Tech University.

Undergraduate Research (Code: SS 16A), **Jerry Dwyer**, **Levi Johnson**, **Jessica Spott**, and **Brock Williams**, Texas Tech University.

Tel Aviv, Israel

Bar-Ilan University, Ramat-Gan and Tel-Aviv University, Ramat-Aviv

June 16–19, 2014

Monday – Thursday

Meeting #1101

The Second Joint International Meeting between the AMS and the Israel Mathematical Union.

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: January 2014

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/internmtgs.html`.

Invited Addresses

Ian Agol, University of California, Berkeley, *3-manifolds and cube complexes.*

Gil Kalai, Hebrew University, *Influence, thresholds, and noise sensitivity.*

Michael Larsen, Indiana University, *Borel's theorem on word maps and some recent variants.*

Leonid Polterovich, Tel-Aviv University, *Symplectic topology: from dynamics to quantization.*

Tamar Zeigler, Technion, Israel Institute of Technology, *Patterns in primes and dynamics on nilmanifolds.*

Special Sessions

Additive Number Theory, **Melvyn B. Nathanson**, City University of New York, and **Yonutz V. Stanchescu**, Afeka Tel Aviv Academic College of Engineering.

Algebraic Groups, Division Algebras and Galois Cohomology, **Andrei Rapinchuk**, University of Virginia, and **Louis H. Rowen** and **Uzi Vishne**, Bar Ilan University.

Applications of Algebra to Cryptography, **David Garber**, Holon Institute of Technology, and **Delaram Kahrobaei**, City University of New York Graduate Center.

Asymptotic Geometric Analysis, **Shiri Artstein** and **Boaz Klar'tag**, Tel Aviv University, and **Sasha Sodin**, Princeton University.

Combinatorial Games, **A. Fraenkel**, Weizmann University, **Richard Nowakowski**, Dalhousie University, Canada, **Thane Plambeck**, Counterwave Inc., and **Aaron Siegel**, Twitter.

Field Arithmetic, **David Harbater**, University of Pennsylvania, and **Moshe Jarden**, Tel Aviv University.

Financial Mathematics, **Jean-Pierre Fouque**, University of California, and **Eli Merzbach** and **Malka Schaps**, Bar Ilan University.

Geometric Group Theory and Low-Dimensional Topology, **Ian Agol**, University of California, Berkeley, and **Zlil Sela**, Hebrew University.

History of Mathematics, **Leo Corry**, Tel Aviv University, **Michael N. Fried**, Ben Gurion University, and **Victor Katz**, University of District of Columbia.

Mirror Symmetry and Representation Theory, **Roman Bezrukavnikov**, Massachusetts Institute of Technology, and **David Kazhdan**, Hebrew University.

Nonlinear Analysis and Optimization, **Boris Mordukhovich**, Wayne State University, and **Simeon Reich** and **Alexander Zaslavski**, Technion Israel Institute of Technology.

Qualitative and Analytic Theory of ODE's, **Andrei Gabrielov**, Purdue University, and **Yossef Yomdin**, Weizmann Institute of Science.

Quasigroups, Loops and Applications, **Tuval Foguel**, Western Carolina University.

Random Matrix Theory, **Brendan Farrell**, California Institute of Technology, **Mark Rudelson**, University of Michigan, and **Ofer Zeitouni**, Weizmann Institute of Science.

Recent Trends in History and Philosophy of Mathematics, **Misha Katz**, Bar Ilan University, and **David Sherry**, Northern Arizona University.

Teaching with Mathematical Habits in Mind, **Theodore Eisenberg**, Ben Gurion University, **Davida Fishman**, California State University, San Bernardino, and **Jennifer Lewis**, Wayne State University.

The Mathematics of Menahem M. Schiffer, **Peter L. Duren**, University of Michigan, and **Lawrence Zalcman**, Bar Ilan University.

Topological Graph Theory and Map Symmetries, **Jonathan Gross**, Columbia University, and **Toufik Mansour**, University of Haifa.

Eau Claire, Wisconsin

University of Wisconsin-Eau Claire

September 20–21, 2014

Saturday – Sunday

Meeting #1102

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: June 2014
Program first available on AMS website: August 7, 2014
Program issue of electronic *Notices*: September 2014
Issue of *Abstracts*: Volume 35, Issue 3

Deadlines

For organizers: March 20, 2014
For abstracts: July 29, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Matthew Kahle, Ohio State University, *To be announced*.

Markus Keel, University of Minnesota, *To be announced*.

Svitlana Mayboroda, University of Minnesota, *To be announced*.

Dylan Thurston, Indiana University, *To be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative Ring Theory (Code: SS 3A), **Michael Axtell**, University of St. Thomas, and **Joe Stickles**, Millikin University.

Directions in Commutative Algebra: Past, Present and Future (Code: SS 1A), **Joseph P. Brennan**, University of Central Florida, and **Robert M. Fossum**, University of Illinois at Urbana-Champaign.

New Trends in Toric Varieties (Code: SS 4A), **Christine Berkesch Zamaere**, University of Minnesota, **Daniel Erman**, University of Wisconsin-Madison, and **Hal Schenck**, University of Illinois Urbana-Champaign.

Von Neumann Algebras and Related Fields (Code: SS 2A), **Stephen Avsec** and **Ken Dykema**, Texas A&M University.

Halifax, Canada

Dalhousie University

October 18–19, 2014

Saturday – Sunday

Meeting #1103

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 5, 2014
Program issue of electronic *Notices*: October 2014
Issue of *Abstracts*: Volume 35, Issue 3

Deadlines

For organizers: March 18, 2014
For abstracts: August 19, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

François Bergeron, Université du Québec à Montréal, *Title to be announced*.

Sourav Chatterjee, New York University, *Title to be announced*.

William M. Goldman, University of Maryland, *Title to be announced*.

Sujatha Ramdorai, University of British Columbia, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative Algebra and Its Interactions with Algebraic Geometry (Code: SS 2A), **Susan Marie Cooper**, Central Michigan University, **Sara Faridi**, Dalhousie University, and **William Traves**, US Naval Academy.

p-adic Methods in Arithmetic. (Code: SS 1A), **Henri Darmon**, McGill University, **Adrian Iovita**, Concordia University, and **Sujatha Ramdorai**, University of British Columbia.

San Francisco, California

San Francisco State University

October 25–26, 2014
Saturday – Sunday

Meeting #1104

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 11, 2014
Program issue of electronic *Notices*: October 2014
Issue of *Abstracts*: Volume 35, Issue 4

Deadlines

For organizers: March 25, 2014
For abstracts: September 3, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Kai Behrend, University of British Columbia, Vancouver, Canada, *Title to be announced*.

Kiran S. Kedlaya, University of California, San Diego, *Title to be announced*.

Julia Pevtsova, University of Washington, Seattle, *Title to be announced*.

Burt Totaro, University of California, Los Angeles, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Geometry (Code: SS 1A), **Renzo Cavalieri**, Colorado State University, **Noah Giansiracusa**, University of California, Berkeley, and **Burt Totaro**, University of California, Los Angeles.

Categorical Methods in Representation Theory (Code: SS 4A), **Eric Friedlander**, University of Southern California, **Srikanth Iyengar**, University of Nebraska, Lincoln, and **Julia Pevtsova**, University of Washington.

Geometry of Submanifolds (Code: SS 3A), **Yun Myung Oh**, Andrews University, **Bogdan D. Suceava**, California State University, Fullerton, and **Mihaela B. Vajiac**, Chapman University.

Polyhedral Number Theory (Code: SS 2A), **Matthias Beck**, San Francisco State University, **Martin Henk**, Universität Magdeburg, and **Joseph Gubeladze**, San Francisco State University.

Greensboro, North Carolina

University of North Carolina, Greensboro

November 8–9, 2014
Saturday – Sunday

Meeting #1105

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 25, 2014
Program issue of electronic *Notices*: November 2014
Issue of *Abstracts*: Volume 35, Issue 4

Deadlines

For organizers: April 8, 2014
For abstracts: September 16, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Susanne Brenner, Louisiana State Unviersity, *Title to be announced*.

Skip Garibaldi, Emory University, *Title to be announced*.

Stavros Garoufaldis, Georgia Institute of Technology, *Title to be announced*.

James Sneyd, University of Auckland, *Title to be announced* (AMS-NZMS Maclaurin Lecture).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Difference Equations and Applications (Code: SS 1A), **Michael A. Radin**, Rochester Institute of Technology, and **Youssef Raffoul**, University of Dayton.

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015
Saturday – Tuesday

Meeting #1106

Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America (MAA), annual meetings

of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2015
Issue of *Abstracts*: Volume 36, Issue 1

Deadlines

For organizers: April 1, 2014
For abstracts: To be announced

Washington, District of Columbia

Georgetown University

March 7–8, 2015

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 7, 2014
For abstracts: To be announced

Huntsville, Alabama

University of Alabama in Huntsville

March 27–29, 2015

Friday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 20, 2014
For abstracts: To be announced

Las Vegas, Nevada

University of Nevada, Las Vegas

April 18–19, 2015

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 18, 2014
For abstracts: To be announced

Porto, Portugal

University of Porto

June 10–13, 2015

New Dates !

Thursday – Sunday
First Joint International Meeting involving the American Mathematical Socieety (AMS), the European Mathematical Society (EMS), and the Sociedade de Portuguesa Matematica (SPM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For abstracts: To be announced

Chicago, Illinois

Loyola University Chicago

October 3–4, 2015

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2015
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 10, 2015
For abstracts: To be announced

Memphis, Tennessee

University of Memphis

October 17–18, 2015

Saturday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 17, 2015
For abstracts: August 18, 2015

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Computational Analysis (Code: SS 1A), **George Anastassiou**, University of Memphis.

Fullerton, California

California State University, Fullerton

October 24–25, 2015

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2015
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 27, 2015
For abstracts: To be announced

New Brunswick, New Jersey

Rutgers University

November 14–15, 2015

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 6–9, 2016

Wednesday – Saturday
Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2015
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2016
Issue of *Abstracts*: Volume 37, Issue 1

Deadlines

For organizers: April 1, 2015
For abstracts: To be announced

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017

Wednesday – Saturday
Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2017
Issue of *Abstracts*: Volume 38, Issue 1

Deadlines

For organizers: April 1, 2016

For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018
Wednesday – Saturday
Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 1, 2017
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16–19, 2019
Wednesday – Saturday
Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM)and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2018
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: April 2, 2018
For abstracts: To be announced

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Central Section: **Georgia Benkart**, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: **Steven H. Weintraub**, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: **Brian D. Boe,** Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** www.ams.org/meetings/.

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 99 in the January 2014 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit http://www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences in Cooperation with the AMS: (see http://www.ams.org/meetings/ for the most up-to-date information on these conferences.)

February 13–17, 2014: 2014 Annual Meeting of AAAS, Chicago, Illinois.

American Mathematical Society Distribution Center
35 Monticello Place, Pawtucket, RI 02861 USA

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