A Trip from Classical to Abstract Fourier Analysis

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Classical Fourier analysis began with Fourier series, i.e., the study of periodic functions on the real line $\mathbb{R}$. Because trigonometric functions are involved, we will focus on 2π-periodic functions, which are determined by functions on $[0, 2\pi)$. This is a group under addition modulo 2π, and this group is isomorphic to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ under the map $t \to e^{it}$. Classical results can be viewed as results on the compact abelian group $\mathbb{T}$, and we will do so.

Similarly, Fourier analysis on $\mathbb{R}^n$ can be viewed as analysis on the locally compact[1] abelian (LCA) group $\mathbb{R}^n$.

Our general setting will be a locally compact group G. Every such group has a (left) translation-invariant measure, called Haar measure because Alfred Haar proved this statement in 1932. For $G = \mathbb{R}^n$, this is Lebesgue measure. Haar measure for $G = \mathbb{T}$ is also Lebesgue measure, either on $[0, 2\pi)$ or transferred to the circle group $\mathbb{T}$. Haar measure will always be the underlying measure in our $L^p(G)$ spaces. In particular, $L^1(G)$ is the space of all integrable functions on G. Functions in $L^1(G)$ can be convolved:

$$f * g(x) = \int_G f(x+y)g(-y)\,dy$$

or

$$\int_G f(xy)g(y^{-1})\,dy \quad \text{for} \quad f, g \in L^1(G).$$

Under this convolution, $L^1(G)$ is a Banach algebra.

Question. Which functions h in $L^1(G)$ can be written as $f * g$? That is, which h in $L^1(G)$ factors as the convolution of two L^1-functions?

Theorem ([32, Raphaël Salem, 1939])**.** $L^1(\mathbb{T}) * L^1(\mathbb{T}) = L^1(\mathbb{T})$*. Also* $L^1(\mathbb{T}) * C(\mathbb{T}) = C(\mathbb{T})$*, where* $C(\mathbb{T})$ *is the space of continuous functions on* $\mathbb{T}$*.* [2]

What year did Salem publish this result? According to Zygmund's book [37] and Salem's published works [33, page 90], this paper was published in 1939. According to Hewitt & Ross [17], this paper was published in 1945. The paper was reviewed in March 1940, so 1939 is surely correct. The story is more complicated than a simple error in [17]. The journal had received papers for that issue in 1939, but no more issues were published until 1945 because of World War II. Unfortunately, Hewitt and I used the publication date on the actual journal.

Who was Salem? He was a very talented banker in France who did mathematics on the side. When World War II broke out, he went to Canada and ended up a professor at M.I.T. Antoni Zygmund gives a brief account of Salem's life in the Preface to Salem's complete works (*Oeuvres Mathèmatiques*) [33]. Edwin Hewitt told me that Salem was a real gentleman who treated graduate students well. Salem died in 1963.[3]

In 1957, Walter Rudin announced[4] that $L^1(\mathbb{R}^n) * L^1(\mathbb{R}^n) = L^1(\mathbb{R}^n)$. This prompted the eminent French mathematician and leading co-author of Nicholas Bourbaki, Jean Dieudonné, to write to

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[1] *For us, locally compact groups are always Hausdorff.*

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[2] *See the Appendix for more about Salem's theorems.*

[3] *Starting in 1968, the Salem Prize has been awarded almost every year to a young mathematician judged to have done outstanding work in Fourier series, very broadly interpreted. The prize is considered highly prestigious and many recipients have also been awarded the Fields Medal later in their careers.*

[4] Bull. Amer. Math. Soc. *abstract 731t, p. 382.*

Walter Rudin, on December 17, 1957, as follows [30, Chapter 23]:

> Dear Professor Rudin:
>
> In the last issue of the *Bulletin of the AMS*, I see that you announce in abstract 731t, p. 382, that in the algebra $L^1(\mathbb{R}^n)$, any element is the convolution of two elements of that algebra. I am rather amazed at that statement, for a few years ago I had made a simple remark which seemed to me to disprove your theorem [9].[5] I reproduce the proof for your convenience:
>
> Suppose f, g are in L^1 and ≥ 0, and for each n consider the usual "truncated" functions $f_n = \inf(f, n)$ and $g_n = \inf(g, n)$; f (resp. g) is the limit of the increasing sequence (f_n) (resp. (g_n)), hence, by the usual Lebesgue convergence theorem, $h = f * g$ is a.e. the limit of $h_n = f_n * g_n$, which is obviously an increasing sequence. Moreover, f_n and g_n are both in L^2, hence it is well known that h_n can be taken *continuous* and bounded. It follows that h is a.e. equal to a Baire function of the *first class*. However, it is well known that there are integrable functions which *do not* have that property, and therefore they cannot be convolutions.
>
> I am unable to find any flaw in that argument, and if you can do so, I would very much appreciate if you can tell me where I am wrong.
>
> Sincerely yours, J. Dieudonné

As Rudin notes, there are nonnegative L^1-functions that cannot be written as $f * g$ where f, g are nonnegative.[6] Rudin wrote to Dieudonné explaining this subtle error. In fact, in a separate letter to me in 1979, now lost, Rudin commented, "It's a very subtle error, and I would certainly have believed it if I hadn't proved the opposite." Later, in his memoir [30], Rudin wrote, "Needless to say, I was totally amazed. Here was Dieudonné, a world-class mathematician and one of the founders of Bourbaki, not telling me, a young upstart, 'you are wrong, because here is what I proved a few years ago' but asking me, instead, to tell him what he had done wrong! Actually, it took me a while to find the error, and if I had not proved earlier that convolution-factorization is always possible in L^1, I would have accepted his conclusion with no hesitation, not because he was famous, but because his argument was simple and perfectly correct, *as far as it went.*"

Here is Dieudonné's response of January 17, 1958:

> Dear Professor Rudin:
>
> Thank you for pointing out my error; as it is of a very common type, I suppose I should have been able to detect it myself, but you know how hard it is to see one's own mistakes, when you have once become convinced that some result must be true!!
>
> Your proof is very ingenious; I hope you will be able to generalize that result to arbitrary locally compact abelian groups, but I suppose this would require a somewhat different type of proof.
>
> With my congratulations for your nice result and my best thanks, I am
>
> Sincerely yours, J. Dieudonné

In fact, Rudin proved:

Theorem (Walter Rudin [27, 1957], [28, 1958]). *$L^1(G) * L^1(G) = L^1(G)$ for G abelian and locally Euclidean. In particular, this equality holds for $\mathbb{R}$ and $\mathbb{R}^n$.*

I have heard stories that Rudin heard Salem or Zygmund talk about this theorem in Chicago and then ran off and stole the result for himself. But the timing and context are all wrong, and Rudin's proof is quite different from Salem's. When I wrote to Rudin in 1979, I did not of course allude to the stories I'd heard. Nevertheless, his response began:

> First of all, let me say that I regret very much that I never got around to acknowledging Salem's priority in print. I was unaware of his factoring—when I did my stuff, and later there never seemed to be a good opportunity to do anything about it.

The Cohen Factorization Theorem

Walter Rudin did his work at the University of Rochester. His first book, his very successful undergraduate analysis text [26], fondly known as "baby Rudin" or "blue Rudin," was published in 1953 and lists Rudin as at Rochester. However, he wrote this book while at M.I.T. In 1957, Paul Cohen went to the University of Rochester without finishing his Ph.D. at the University of Chicago under Antoni Zygmund. Neither Zygmund nor Cohen were impressed with his thesis, but Rudin and others finally persuaded Cohen to write up his thesis and return to Chicago for his final exam.

About this time, Rudin showed Cohen his work on factorization. Cohen saw Féjer and Riesz kernels and the like and said, "aha, approximate identities." An approximate unit is a sequence or net $\{e_\alpha\}$ such that $x = \lim_\alpha x e_\alpha = \lim_\alpha e_\alpha x$ for all x in the topological algebra. Soon Cohen found an elegant, completely elementary (using only the definitions) short proof of:

[5] *The remark was in a footnote, and the footnote includes the following assertion: "Il suffit pour le montrer de considérer le cas où f et g sont positives;"*

[6] *See the preceding footnote.*

Theorem ([5, Paul J. Cohen, 1959]). *If A is a Banach algebra with a bounded approximate unit, then $AA = A$, i.e., each element of A can be factored as a product of two elements in A.*[7]

Paul J. Cohen was a brilliant mathematician who went on to do substantial work in my field, harmonic analysis, for a couple of years. In fact, his other three papers in the subject are the basis for the main results in Chapters 3 and 4 of Rudin's book [29, *Fourier Analysis in Groups*, 1962]. Then in 1961, starting from scratch, he moved into the field of foundations. By 1963 or 1964 he had done the first really fundamental work in foundations since Gödel. [Among many other things, he showed that neither the Axiom of Choice nor the Continuum Hypothesis can be proved in standard Zermelo-Fraenkel set theory.]

In 1968, I presented the Cohen Factorization Theorem in a graduate course I was teaching, probably on the second day, since the proof is self-contained. Bill Paschke, a student in the class, wondered whether the converse might be true and found an example of a Banach algebra A, without a bounded approximate unit, satisfying $AA = A$ [24].

Since every $L^1(G)$ is a Banach algebra with bounded approximate unit, we have the following stunning extension of Rudin's theorem:

Theorem. *$L^1(G) * L^1(G) = L^1(G)$ for all locally compact groups G, including all non-abelian groups.*

Cohen's factorization theorem was the real breakthrough. In fact, he also proved that $L^1(G) * C(G) = C(G)$ for compact G, and module generalizations appeared soon after, created independently by Edwin Hewitt [15] and Philip C. Curtis Jr. and Alessandro Figà-Talamanca [6]. Neither used module language, but Hewitt's axioms were basically module axioms.

Given a Banach algebra A, a left Banach A-module is a Banach space L that the algebra A acts on. We use the notation $(a, x) \to a \bullet x$ for the action of A on L, so that $(a, x) \to a \bullet x$ is a mapping from $A \times L$ into L. In addition, we require the inequality $\| a \bullet x \|_L \leq \| a \|_A \cdot \| x \|_L$ for all $(a, x) \in A \times L$.

The Banach Module Factorization Theorem. ([17, Theorem 32.22]). *If A is a Banach algebra with a bounded approximate unit, and if L is a left Banach A-module, then $A \bullet L$ is a closed linear subspace of L. In particular, if $A \bullet L$ is dense in L (which is usually obvious when true), then $A \bullet L = L$.*

Note. If $A \bullet L = M$, then $A \bullet M = A \bullet (A \bullet L) = (AA) \bullet L = A \bullet L = M$.

[7] *Note that this is obvious if A has a unit.*

Hewitt's main application was:

Theorem ([15, Edwin Hewitt, 1964]). *For $1 \leq p < \infty$, we have $L^1(G) * L^p(G) = L^p(G)$ for any locally compact group G.*

Philip Curtis and Alessandro Figà-Talamanca gave this application and other applications to harmonic analysis and to function algebras. In particular, they proved:

Theorem ([6, 1965, Curtis & Figà-Talamanca, Theorem 3.1]). *For any locally compact group,*

$$(1) \quad L^1(G) * L^\infty(G) = C_{ru}(G) = L^1(G) * C_{ru}(G),$$

where $C_{ru}(G)$ is the space of all bounded right uniformly continuous functions on G. The last equality follows from the Note after the Banach Module Factorization Theorem.

We also have $L^1(G) * C_0(G) = C_0(G)$, where $C_0(G)$ is the space of continuous functions on G that vanish at infinity; see [17, 32.44f].

Rates of Decrease of Fourier Coefficients and Transforms

We now consider an LCA group G. As in the classical setting, Fourier transforms are the key tool. These are functions defined on the character group $\widehat{G}$, which consists of all characters χ on G, i.e., continuous homomorphisms of G into the circle group. Thus

$$\widehat{\mathbb{R}} = \{\chi_y : y \in \mathbb{R}\}, \text{ where } \chi_y(x) = e^{ixy} \text{ for all } x \in \mathbb{R},$$

and

$$\widehat{\mathbb{T}} = \{\chi_n : n \in \mathbb{Z}\}, \text{ where } \chi_n(z) = z^n \text{ for } z \in \mathbb{T}.$$

On $[0, 2\pi)$, χ_n looks like $\chi_n(t) = e^{int}$. It is not an accident that the index sets for $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{T}}$ are familiar groups. It is not obvious in general but, with a suitable topology, $\widehat{G}$ is always an LCA group.

Now, for a function f in $L^1(G)$, its Fourier transform on $\widehat{G}$ is defined by $\widehat{f}(\chi) = \int_G f\overline{\chi}$. On $\mathbb{R}$, except for a mildly controversial constant, the Fourier transform is defined by

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx.$$

On $\mathbb{T}$, the Fourier transform is defined by

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) \overline{z}^n\, dz \quad \text{or} \quad \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt.$$

Since $e^{int} = \cos nt + i \sin nt$, there is a simple intimate connection between our Fourier transforms on $\mathbb{T}$ and the classical Fourier coefficients of Fourier series. So it is almost trivial to translate theorems back and forth; you will see that I did this to Salem's theorem if you look at his original paper.

A basic fact is:

Riemann-Lebesgue Lemma. *If f is in $L^1(G)$, then its Fourier transform $\hat{f}$ is in $C_0(\hat{G})$, i.e., $\hat{f}$ is a continuous function on $\hat{G}$ and vanishes at infinity.*

Thus the Fourier transform of any function in $L^1(\mathbb{R})$ vanishes at infinity. Likewise, for f in $L^1(\mathbb{T})$, the sequence $(\hat{f}(n))_{n\in\mathbb{Z}}$ vanishes at infinity, and the same is true for the Fourier coefficients in the Fourier series corresponding to f.

Here is a natural question.

Question. Can $\hat{f}$ vanish at infinity as slowly as we wish?

For the circle group $\mathbb{T}$, the answer is "yes" in the strongest sense. Indeed, if (a_n) is any sequence of nonnegative numbers, on $\mathbb{Z}$, and if $\lim_{|n|\to\infty} a_n = 0$, then there exists f in $L^1(\mathbb{T})$ so that $\hat{f}(n) \geq a_n$ for all $n \in \mathbb{Z}$. This was stated and proved by A. N. Kolmogorov [21, 1923]. This result is true because every sequence (a_n) on $\mathbb{Z}$ that vanishes at $\pm\infty$ is dominated by a symmetric convex sequence (c_n) of positive numbers, and every such sequence is the Fourier transform of a function f in $L^1(\mathbb{T})$. In symbols, $\hat{f}(n) = c_n \geq |a_n|$ for all $n \in \mathbb{Z}$. The last assertion, that such sequences (c_n) are Fourier transforms, goes back to W. H. Young [35, 1913].[8]

Back in 1967 or 1968, when Edwin Hewitt and I were writing our book [17], Ed "assigned" me the task of generalizing these results. I focused on compact abelian groups and got nowhere. Mostly I foolishly worked on the task of generalizing convex functions to more general compact groups. Then I had one of my very rare good ideas. By the Banach-module version of the factorization theorem,

$$\widehat{L^1(G)} \cdot C_0(\hat{G}) = C_0(\hat{G}) \tag{2}$$

for all LCA groups. Here the module action is $f \bullet \psi = \hat{f}\psi$ for $(f, \psi) \in L^1(G) \times C_0(\hat{G})$.

From (2), the answer to the question is "yes," at least for compact abelian groups. I illustrate the idea on $\mathbb{T}$ using $\widehat{L^1(\mathbb{T})} \cdot c_0(\mathbb{Z}) = c_0(\mathbb{Z})$, where $c_0(\mathbb{Z})$ consists of all sequences on $\mathbb{Z}$ that vanish at $\pm\infty$. Consider a nonnegative sequence (a_n) in $c_0(\mathbb{Z})$. Then there exist g in $L^1(\mathbb{T})$ and (b_n) in $c_0(\mathbb{Z})$ such that $a_n = \hat{g}(n)b_n$ for all $n \in \mathbb{Z}$. Thus $F = \{n \in \mathbb{Z} : |b_n| > 1\}$ is finite and

$$n \notin F \quad \text{implies} \quad |b_n| \leq 1 \quad \text{implies} \quad |\hat{g}(n)| \geq a_n.$$

So g works except possibly on F. But one can always adjust $\hat{g}$ on the finite set F, by adding a suitable trigonometric polynomial to g, to get the desired f.

I was extremely pleased, so I showed this to Ed Hewitt and also to Irving Glicksberg, who was another fine analyst at the University of Washington.[9] They both liked the proof. However, twenty-four hours later Irving returned and pointed out that the result was in Curtis and Figà-Talamanca's paper [6]. I was quite disappointed, because I thought this would be worthy of a short note.

It turns out that an easy adjustment to the argument above proves:

Theorem ([17, Hewitt & Ross, 32.47.b])**.** *Let G be an LCA group with character group $\hat{G}$. Given a nonnegative function ψ in $C_0(\hat{G})$, there is a function f in $L^1(G)$ such that $\hat{f}(\chi) \geq \psi(\chi)$ for all $\chi \in \hat{G}$.*

We credit the theorem above to Curtis and Figà-Talamanca [6], but I think we were a little too generous. Certainly the equality (2) is clear from their description of $\widehat{L^1(G)} \cdot L^\infty(\hat{G})$, but in looking through the paper more carefully I do not see this equality used to prove the theorem above. Curtis and Figà-Talamanca noted other equalities including $\widehat{L^1(G)} \cdot L^p(\hat{G}) = L^p(\hat{G})$ for $1 \leq p < \infty$. Section 32 in [17] contains many more applications, including N. Th. Varopoulos's theorem that all positive functionals on a Banach $*$-algebra with bounded approximate unit are continuous [17, Theorem 32.27].[10] His proof relied heavily on the following:

Theorem ([17, Theorem 32.23])**.** *If (a_n) is a sequence in a Banach algebra A and $\lim_n a_n = 0$ in A, then there exists b in A and a sequence (c_n) in A such that $\lim_n c_n = 0$ and $a_n = bc_n$ for all n.*

In 1969, Marc Rieffel found a simple elegant proof of this theorem, as follows. Let $c_0(A)$ be all infinite sequences $(a_1, a_2, \ldots)$ in A where $\lim_n a_n = 0$. Define

$$\| (a_1, a_2, \ldots) \|_{c_0(A)} = \sup\{\| a_n \|_A : n = 1, 2, \ldots\}.$$

Then $c_0(A)$ is a Banach A-module under the module operation $b \bullet (a_1, a_2, \ldots) = (ba_1, ba_2, \ldots)$. So the Banach Module Factorization Theorem gives $A \bullet c_0(A) = c_0(A)$, and this implies the theorem.

Kolmogorov's 1923 theorem, that $\hat{f}$ can vanish infinitely slowly for f in $L^1(\mathbb{T})$, suggests a similar-looking question. Is every sequence (a_n) in $\ell^2(\mathbb{Z})$ dominated by $\hat{f}$ for some f in $C(\mathbb{T})$? Robert E. Edwards, in Australia, and I worked on this in 1972–1973, but without success. An elegant affirmative solution was found in 1977 by Karel de Leeuw,[11] Jean-Pierre Kahane, and Yitzak Katznelson [8]. They state, in passing, that this result holds for all compact abelian groups. A detailed proof of the de Leeuw-Kahane-Katznelson theorem is given in the last chapter of Karl Stromberg's book [34].

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[8] *For more about this, see the Appendix.*

[9] *Hewitt and I worked on* [17] *at the University of Washington, and Glicksberg's office was down the hall.*

[10] *Varopoulos was the first recipient of the Salem Prize.*

[11] *This was de Leeuw's last paper. He was murdered on August 18, 1978, by a disgruntled graduate student at Stanford University. He was 48.*

This result has been generalized in several ways. S. V. Kislyakov [20] showed that, for any nonnegative sequence (a_n) in $\ell^2(\{0,1,2,\ldots\})$, there is a function f in $C(\mathbb{T})$ such that $|\hat{f}(n)| \geq a_n$ for all $n \geq 0$ and $\hat{f}(n) = 0$ for all $n < 0$. Also, the de Leeuw-Kahane-Katznelson theorem was generalized to compact non-abelian groups and to a special class of compact metrizable abelian hypergroups. See Barbara Heiman [14] and the paper [12, Theorem 4.10, Example 2.7], co-authored by John Fournier and me.

Some Other Applications

A representation $\pi : L^1(G) \to B(H)$, where G is a locally compact group and $B(H)$ is the space of bounded operators on a Hilbert space H, is "nondegenerate" if the set $\{\pi(f)\xi : f \in L^1(G), \xi \in H\}$ spans a dense linear subspace of H.[12] See, for example, [7, just before Proposition 6.2.3]. It is worth noting that, in fact, such a set $\{\pi(f)\xi : f \in L^1(G), \xi \in H\}$ must be equal to H. To see this, we use the module operation $(f,\xi) \to f \bullet \xi$ from $L^1(G) \times H \to H$ where $f \bullet \xi = \pi(f)\xi$; then H is a Banach $L^1(G)$-module. Thus, by the Banach Module Factorization Theorem, we have

$$\{\pi(f)\xi : f \in L^1(G), \xi \in H\} = L^1(G) \bullet H = \pi(L^1(G))H = H.$$

This fact was essentially observed by Curtis and Figà-Talamanca [6, Corollary 2.4].

Consider an LCA group G with character group $\hat{G}$. Equation (1) in Curtis and Figà-Talamanca's theorem applies to $\hat{G}$:

$$(3)\quad L^1(\hat{G}) * L^\infty(\hat{G}) = C_{ru}(\hat{G}) = L^1(\hat{G}) * C_{ru}(\hat{G}).$$

$L^1(\hat{G})$ is often called the Fourier algebra of G and written $A(G)$. This object was generalized to all locally compact groups G by Pierre Eymard [11, 1964]. Its dual space is $L^\infty(\hat{G})$ if G is abelian and, in the general case, the dual of $A(G)$ is the von Neumann algebra $VN(G)$. For a suitable generalization of $C_{ru}(\hat{G})$, which he called $UCB(\hat{G})$, Ed Granirer [13, 1974, Proposition 1] showed that

$$(4)\quad A(G) \cdot VN(G) = UCB(\hat{G}) = A(G) \cdot UCB(\hat{G}) \quad \text{for amenable groups } G.$$

He applied the Cohen factorization theorem to the Banach $A(G)$-module $VN(G)$ and Horst Leptin's theorem [23, 1968, Proposition 1] stating that $A(G)$ has a bounded approximate unit if and only if G is amenable. Much later, Tony Lau and Viktor Losert [22, 1993, Proposition 7.1] showed the converse: $UCB(\hat{G}) = A(G) \cdot VN(G)$ implies that G is amenable, so that (4) can be rewritten as an "if and only if" assertion.

Local Units in Fourier Algebras

Here is another theorem in this arena. It can be found in Rudin [29, Theorem 2.6.8], Hewitt & Ross [17, Theorem 31.37] and Bourbaki [3, Ch. II, §2, Exercise 13(b)].

Theorem. *Let G be an LCA group. Let Φ be a nonempty compact subset of $\hat{G}$ and $\epsilon > 0$. Then there exists f in $L^1(G)$ such that $\hat{f}$ has compact support, $\hat{f} = 1$ on Φ and $\| f \|_1 < 1 + \epsilon$.*

This theorem was easy for G equal to $\mathbb{R}^n$, $\mathbb{Z}^m$ and $\mathbb{T}$, and in fact for all compact (abelian) groups. In the general case, the theorem is still easy except for the norm requirement. It is apparently due to Rudin [29] who proved the highly nontrivial structure theorem for compactly generated LCA groups [29, Theorem 2.4.1] just to be able to prove this useful theorem. Hewitt & Ross and Bourbaki used the same proof.

In the spring of 1971, Gregory Bachelis and Bill Parker, both at Kansas State University at the time, sent me a letter saying that they had a very simple proof of this result for compact abelian groups; in fact, suitably stated for trigonometric polynomials, this result holds for all compact groups. Greg noticed this as a consequence of his study of annihilator algebras. They wanted to know whether I knew of this simple proof for the compact abelian case. I did not know their simple proof, and it took me only a few minutes to adjust their proof to prove the theorem stated above. With help from their colleague Karl Stromberg, Greg and Bill convinced me to be a co-author for such little work. The proof is published as a Shorter Note in the *Proceedings of the American Mathematical Society* [2]. The proof is six lines long, two lines for each author. Incidentally, according to the review of [2], I told the reviewer that Hans Reiter informed me that Horst Leptin also gave a structure-free proof of the theorem.

Here is the simple proof of the theorem as stated. The essence of it is that, if g and h are idempotents in a ring, so is $g + h - gh$. Using the (easy) theorem without the norm requirement, there is a function h in $L^1(G)$ so that $\hat{h}$ has compact support and $\hat{h} = 1$ on Φ. There is an approximate unit for $L^1(G)$ consisting of functions with L^1-norm 1 and with Fourier transforms having compact support. Thus there exists g in $L^1(G)$ such that $\| g \|_1 = 1$ and $\| g * h - h \|_1 < \epsilon$ and such that $\hat{g}$ has compact support. Let $f = g + h - g * h$. Then $\hat{f}$ has compact support, $\| f \|_1 \leq \| g \|_1 + \| h - g * h \|_1 < 1 + \epsilon$, and for a character χ in Φ, we obtain $\hat{f}(\chi) = \hat{g}(\chi) + \hat{h}(\chi) - \hat{g}(\chi)\hat{h}(\chi) = \hat{g}(\chi) + 1 - \hat{g}(\chi) \cdot 1 = 1$.

[12] *Of course, this definition can be extended to representations of Banach algebras by operators on a Banach space.*

How did this proof get overlooked? My guess is that Walter Rudin was a classical analyst before writing his book and saw how to prove this result for $\mathbb{R}^n$ and other special cases. Then he used the structure theorem to prove the general result. Hewitt & Ross, and apparently Bourbaki, didn't think to look for such a trivial proof.

There is more to this story. My co-authors asked me to submit this article, so I sent it to the most appropriate editor, Irving Glicksberg. I sent this on a Friday in May 1971, with a note asking him to respond to Greg Bachelis if he had a decision in less than a month, because I was going to be in India for a month. On the next Wednesday, on the way to the airport and India, I dropped by the campus office to check mail, and there was an acceptance! Later, Irving explained that he was teaching out of Rudin [29] when he got my note on that Monday in May, so he immediately recognized that our trivial note was interesting.

The theorem was shown to be valid for LCA hypergroups K for which the set $\hat{K}$ of hypergroup characters is a hypergroup under pointwise operations [4, Theorem 2.9].

The L^p Conjecture

Finally, we discuss the "The L^p Conjecture," which also concerns L^1 and L^p spaces, and which had a similar history in the sense that the conjecture seemed quite difficult until it was finally settled. If G is compact, then Haar measure on G is finite and $L^p(G) \subseteq L^1(G)$. Therefore $L^p(G) * L^p(G) \subseteq L^1(G) * L^p(G) = L^p(G)$. The following possible converse was known as the "L^p conjecture" for some time:

The L^p conjecture

If $p > 1$ and $L^p(G) * L^p(G) \subseteq L^p(G)$, then G is compact.

This was conjectured by M. Rajagopalan in his 1963 Yale thesis, though the conjecture had already been settled in the abelian case by W. Żelazko [36]. Various authors, including Neil Rickert who was also a graduate student at Yale, made progress. They proved this result for various choices of p and various classes of groups such as discrete, totally disconnected, nilpotent, semi-direct product of LCA groups, solvable, and amenable. The project was essentially abandoned in the early 1970s. Then in 1990, Sadahiro Saeki [31, pages 615–620], also at Kansas State University, settled the conjecture for all locally compact groups with a self-contained ingenious proof.

Appendix: A Circle of Ideas

The key to both the factorization theorem and the "rate of decrease theorem" on the circle group $\mathbb{T} = [-\pi, \pi)$ is the fact that positive symmetric sequences (c_n) on $\mathbb{Z}$ that are convex on $\{0, 1, 2, \ldots\}$, and converge monotonically to 0 on $\{0, 1, 2, \ldots\}$, are Fourier transforms of nonnegative functions in $L^1(\mathbb{T})$; this is due to W. H. Young [35, 1913, Theorem, §3]; see also [37, Theorem (1.5), Chapter V] or [10, item 7.3.1].[13] In fact, the Fourier series of the function converges uniformly to the function outside of any neighborhood of 0. Every sequence on $\mathbb{Z}$ converging to 0 is easily shown to be dominated by such a "convex" symmetric sequence; this is noted in [37, just prior to (1.11), Chapter V] and proved in [10, item 7.1.5]. The "rate of decrease theorem" for $\mathbb{T}$ follows from this observation and W. H. Young's theorem. A. N. Kolmogorov [21, 1923] independently stated and proved the "rate of decrease theorem." As we noted earlier, we needed the Banach-module version of Cohen's factorization theorem to generalize this theorem to compact abelian groups and beyond.

A similar result holds for $\mathbb{R}$. In particular, every nonnegative symmetric (i.e., even) function ψ in $C_0(\mathbb{R})$ that is convex on $(0, \infty)$ is the Fourier transform of some function in $L^1(\mathbb{R})$. For $\psi(0) = 1$, this fact is known to probabilists as "Pólya's criterion," which states that any such function is the characteristic function of an absolutely continuous distribution function; see, for example, [19, Theorem 10.5.2]. As noted in Kai Lai Chung's appendix in [1, page 193], this result is "slightly hidden" in Pólya's paper [25, 1918].

The factorization theorem for $L^1(\mathbb{T})$ was established several years later and is universally credited to Raphaël Salem [32]. In fact, his focus was a bit different. He evidently wanted to show that, given f in $L^1(\mathbb{T})$, there is a function g in $L^1(\mathbb{T})$ whose Fourier transform goes to 0 on $\mathbb{Z}$ more slowly than the transform of f. [He showed a similar result for $C(\mathbb{T})$.] Specifically, he essentially showed that, given f in $L^1(\mathbb{T})$, there is a function g in $L^1(\mathbb{T})$ and a positive symmetric sequence (λ_n) on $\mathbb{Z}$ that is concave and increases monotonically to $+\infty$ on $\{0, 1, 2, \ldots\}$ and satisfies $\hat{f}(n) \cdot \lambda_n = \hat{g}(n)$ for all n. Zygmund [37, Notes to Chapter IV, §11] noted that the factorization theorem for $L^1(\mathbb{T})$ is an easy consequence. In fact, it is very easy to check that the sequence of reciprocals $(1/\lambda_n)$ is a convex sequence satisfying the hypotheses of W. H. Young's theorem mentioned above. Thus, for some h in $L^1(\mathbb{T})$, we have $\frac{1}{\lambda_n} = \hat{h}(n)$, so that $\hat{f}(n) = \hat{h}(n)\hat{g}(n)$ for all n. Hence $f = h * g$ by the uniqueness theorem for Fourier transforms. My guess is that this was observed sometime between 1939 and 1959 by Zygmund or by Salem himself. Or both, since "from 1945 to 1959, Zygmund's

[13] *Since the sequence (c_n) is symmetric, the corresponding Fourier series is a cosine series.*

closest friend and collaborator was Salem." [18] They wrote at least ten papers and one book jointly during that period.

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