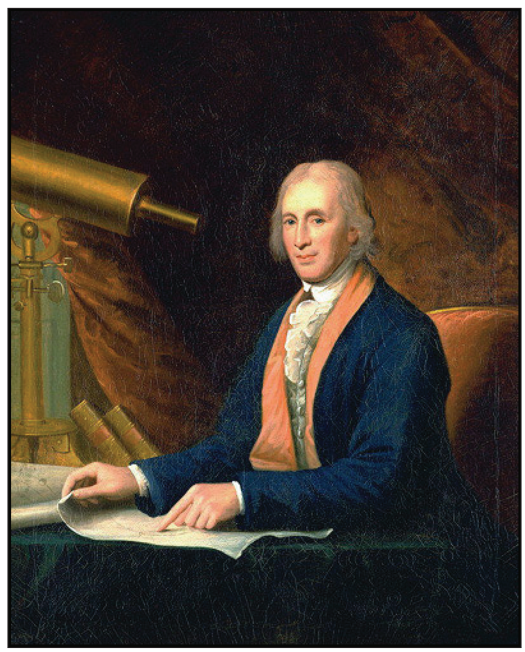
David Rittenhouse: Modern Mathematician

David E. Zitarelli

On the surface it seems implausible that a colonial scientist could be characterized as a mathematician by today's standards. Yet the single aim of this note is to provide evidence that two of David Rittenhouse's papers from 220 years ago qualify him as a modern analyst.

Rittenhouse (1732–1796) had no formal education and never earned a degree. Except for a brief appointment as professor of astronomy at the University of Pennsylvania, he never held an academic position. Yet he became one of America's leading colonial scientists.

Rittenhouse can be regarded as a modern mathematician in three ways. For one, his research on the transit of Venus of 1769 was financed by a governmental agency (the Colony of Pennsylvania) for the first time in American history, setting an early precedent for NSF grants. For another, much like Oswald Veblen in World War I, he aided a war effort by refining the ballistics of rifles and cannons as well as locating forts to maximize defenses for General Washington. But the main reason is because of two papers he read at scientific meetings and subsequently published in that society's *Transactions*. A close reading of these papers would label him an analyst—even, perhaps, a numerical analyst—today.



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David Rittenhouse

Analyst

Rittenhouse read the paper "A method of finding the sum of the several powers of the sines" at a monthly meeting of the American Philosophical Society (APS) in May 1792, a month after turning sixty. Its introduction reveals the source of the project he was investigating: to determine the times of vibration of a pendulum. However, he stated a philosophy embraced by many mathematicians today: "I was induced to attempt the means of doing this solely by its usefulness, but in prosecuting the enquiry I found much of that pleasing regularity, the discovery of which the geometrician often thinks a sufficient reward for his labours" [2, p. 155]. The "pleasing regularity" can be seen in the bottom half of Figure 1. The first two cases reveal formulas for the sums of sin and $\sin^2$ in the first quadrant. Here the old English way of writing the letter "s" is suggestive because the first letter

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resembles an integral sign. Thus by "the sum of the sines" Rittenhouse was appealing to areas under their curves which, in modern terms, become the two formulas

$$\int_0^{\pi/2} \sin x\, dx = 1 \tag{1}$$

and

$$\int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}. \tag{2}$$

Throughout this section I restrict the radius of the circle to be $r = 1$; it is a simple exercise to extend these formulas to Rittenhouse's expression, with some care. For instance, in all even-numbered cases "the arch of 90°" means the circumference of the first quadrant of a circle, so $\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2} \times \frac{\pi}{2}$.

Rittenhouse indicated his methods on the right side of Figure 1. Thus he proved the first two cases "By Demonstration," meaning he was in possession of proofs by synthetic geometry. Without saying so, he left the proofs as an exercise for the reader.

Similarly, Rittenhouse stated that he obtained formulas for the next four cases "By Infinite Series." It is therefore impressive that he was able to evaluate $\int_0^{\pi/2} \sin^n x\, dx$ for $n = 3, 4, 5, 6$ by this method, but it is equally tantalizing because, once again, he provided no proofs. Moreover, Rittenhouse was a taciturn, introverted individual who corresponded infrequently and left few notes behind, so we have nary a hint as to how he proved, for example, that

$$\int_0^{\pi/2} \sin^6 x\, dx = \frac{5\pi}{32}.$$

Neither his file in the APS archives nor correspondence with Thomas Jefferson reveals any evidence.

If we are left feeling empty for lack of proofs, Rittenhouse felt equally frustrated, though for a different reason. He wrote, "I have not been able strictly to demonstrate any more than the first two cases." This statement places him squarely in the eighteenth century, a time when proofs by Euclidean geometry set the standard.

Rittenhouse obtained the two cases $n = 7$, $n = 8$ "By the Law of Continuation," called induction today. Table 1 summarizes the eight cases. It is not particularly straightforward to see how to induce the results for $n = 7$ and $n = 8$ from the first six cases. (Spoiler alert: I am about to state the rule. Inquisitive readers are encouraged to attempt this problem independently.) Rittenhouse wrote: "The law is this, make a fraction whose denominator is the index of the given power, and its numerator the same index, diminished by unit…; by this fraction multiply the sum of the next but one lower power, and we have the form of the given power."

Table 1.

N	Sum
1	1
2	$\pi/2$
3	$2/3$
4	$3\pi/16$
5	$8/15$
6	$5\pi/32$
7	$16/35$
8	$35\pi/256$

Based on today's methods for evaluating $\int \sin^n x\, dx$ by trigonometric identities, we know that these cases must be taken in pairs, because those identities differ according to the parity of n. In symbols, for the sum of $\sin^n$ the index is n, so the multiplier is $\frac{n-1}{n}$. By interpreting "the next but one lower power" as $n - 2$, we arrive at the formula

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx.$$

Rittenhouse, it seems, had discovered this recursion formula unaware that "Wallis's formulas" were discovered by the English mathematician John Wallis around 1655.

The title of the printed version of the paper Rittenhouse delivered begins "Dr. Rittenhouse to Mr. Patterson." Rittenhouse concluded by beseeching University of Pennsylvania mathematician Robert Patterson, "Should your leisure permit you to give any attention to this subject I shall be glad to see you furnish a demonstration for the 3^d, or any subsequent case abovementioned [*sic*]."

The discovery of this recursion formula shows that Rittenhouse was a modern mathematician in practice as well as in spirit. His only other paper devoted strictly to pure mathematics looks to contain little more than brute-force arithmetic computations, but a closer examination reveals a much deeper algorithm.

Numerical Analyst

David Rittenhouse was sufficiently inspired by his progress to investigate other mathematical problems for their own sake. In August 1795, after succeeding Benjamin Franklin as president of the APS, he read the paper "Method of raising [evaluating] the common logarithm of any number immediately" at an APS meeting. It was published posthumously in 1799 [3].

The late Jesuit priest Frederick A. Homann (1929–2011) described this paper in [1], illustrating the Rittenhouse algorithm by evaluating $\log N$ for $N = 20$. A cursory glance at the massive columns of calculations in Rittenhouse's paper suggests why Homann chose $N = 20$—it simplifies

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the ultimate ratio which the ſum of the given power of the ſines bears to a known power of the radius.

Having proceeded ſo far as the 6th power the law of continuation became evident; ſo that, ſhould any problem in mathematical philoſophy require it, we may proceed as far as we pleaſe in ſumming the powers of the ſines. The law is this,

Make a fraction whoſe denominator is the index of the given power, and its numerator the ſame index, diminiſhed by unity, and multiplied by the ſquare of the radius; by this fraction multiply the ſum of the next but one lower power, and we have the ſum of the given power.

Thus 1ſt, the ſum of the 1ſt power of the ſines is $=$ rr, or the ſquare of the radius
2d, ſum of the 2d, power or ſquares is $= \frac{1}{2}$ rr $\times$ by the arch of 90°.
By Demonſtration.

3d, ſum of the 3d, power or cubes is $\frac{2}{3}$ rr of the 1ſt, or $= \frac{2}{3} r^4$.
4th. ſum of 4th powers is $= \frac{3}{4}$ rr of the 2d or $= \frac{3}{8} r^4 \times$ by the arch of 90°.
5th, ſum of 5th. powers is $= \frac{4}{5}$ rr of the 3d, or $= \frac{8}{15} r^6$.
6th, ſum of 6th. powers is $= \frac{5}{6}$ rr of the 4th or $= \frac{5}{16} r^6 \times$ by arch of 90°.
By Infinite Series.

7th, ſum of 7th. powers is $= \frac{6}{7}$ rr of the 5th, or $= \frac{16}{35} r^8$.
8th, ſum of 8th. powers is $= \frac{7}{8}$ rr of the 6th, or $= \frac{35}{128} r^3 \times$ by the arch of 90°.
By the Law of Continuation.

&c. &c.

Should your leiſure permit you to give any attention to this ſubject I ſhall be glad to ſee you furniſh a demonſtration for the 3d, or any ſubſequent caſe abovementioned.

I am, Sir,
Your moſt obedient humble ſervant,
DAVID RITTENHOUSE.

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Figure 1

calculations enormously. Yet this simplification belies Rittenhouse's numerical dexterity, so I will adapt Homann's approach to approximate $\log 99$, the case Rittenhouse exhibited in his paper.

Rittenhouse began by calculating the characteristic of $\log N$, which, by definition, is the largest integer C for which $\frac{N}{10^C} \geq 1$. For $N = 99$ it is immediate that $C = 1$. The mantissa of $\log N$ is then the continued fraction $[n_0, n_1, n_2, \dots]$, where

$$Q_{-1} = 10, Q_0 = \frac{N}{10^C}, Q_{k+1} = \frac{Q_{k-1}}{Q_k^{n_k}} \text{ for } k = 0, 1, 2, \dots$$

and n_k is the largest integer for which $Q_{k+1} \geq 1$. By definition

$$Q_1 = \frac{Q_{-1}}{Q_0^{n_0}} = \frac{10}{9.9^{n_0}}.$$

Clearly $n_0 = 1$ is the largest integer for which $Q_1 \geq 1$. Therefore

$$Q_1 = \frac{10}{9.9}.$$

Now the calculations become tedious, because the next step is to find the largest integer n_1 for which $Q_2 \geq 1$, where

$$Q_2 = \frac{Q_0}{Q_1^{n_1}} = \frac{9.9}{\left(\frac{10}{9.9}\right)^{n_1}}.$$

This was easy for n_0, as we saw. But it turns out that $n_1 = 228$.

How did Rittenhouse conclude that $n_1 = 228$? His computations show a master numerical analyst at work. First, he set $a = \frac{10}{9.9}$. The approach was to calculate powers of the denominator a^n of Q_2 for $n = 2, 4, 8, 16, \ldots$ by successive squaring until a^n exceeded the numerator 9.9. He halted the process at $a^{128} = 3.619887649$ because the next term would be $a^{256} = (a^{128})^2 > 3.6^2 > 9.9$. Consequently, $128 \leq n_1 \leq 256$. At that point Rittenhouse examined prior powers of a^n to determine which of their products remained below 9.9. He concluded that $n_1 = 228 = 128+64+32+4$ because $a^{128} \cdot a^{64} \cdot a^{32} \cdot a^4 = a^{228} = 9.889521 < 9.9$.

It is especially impressive that all of these calculations were carried out by hand. Yet we wonder, how did he not become discouraged after the first several iterations of a^n? We will never know the answer because this paper, like his first, contains only the finished product, not the underlying scaffolding.

Subsequently Rittenhouse carried out similar calculations to obtain $n_2 = 9$, $n_3 = 2$, and $n_4 = 75$, producing the continued fraction approximation $[1, 228, 9, 2, 75]$. This means that the fifth Rittenhouse approximation for the mantissa of log 99 is

$$R_5 = \cfrac{1}{1 + \cfrac{1}{228 + \cfrac{1}{9 + \cfrac{1}{2 + \cfrac{1}{75}}}}}$$

$$= \frac{327103}{328537} = 0.995\,635\,194.$$

Thus, correct to nine decimal places,

$$\log 99 = 1.995\,635\,194.$$

Rittenhouse's closing statement, "3 [is] too much in the tenth [place]" reflects the style of modern numerical analysts to obtain bounds on approximations. These rather nasty computations, even when evaluated with modern software, give pause to the word "immediately" in the title of his paper.

Rittenhouse's method also suggests an easy generalization to evaluating $\log_B N$ for any base B. In this case the characteristic becomes the largest integer C for which $\frac{N}{B^C} \geq 1$. Then the mantissa is the continued fraction $[n_0, n_1, n_2, \ldots]$, where $Q_{-1} = B$, $Q_0 = \frac{N}{B^C}$, and subsequent pairs (Q_k, n_k) are defined in an identical way.

David Rittenhouse did not supply the reason why he desired logarithms to such accuracy, but undoubtedly it was due to the fact that logarithms were of great use in colonial surveying and astronomy, two areas he pursued in earnest. Although he knew much of the mathematics carried out in England in the eighteenth century, he was apparently unaware that his algorithm had already appeared in a 1717 article by Brook Taylor in the *Philosophical Transactions* of the Royal Society of London [5]. As Rittenhouse's papers on the sums of powers of sines showed, he had a masterful command of series, and thus it is surprising that he missed Taylor's work.

Ironically, in 1954 the numerical analyst Daniel Shanks (1917–1996) discovered the algorithm independently and published it in *Mathematical Tables and Aids to Computation* (*Mathematics of Computation* since 1960) without knowledge of either predecessor. That paper begins, "The method of calculating logarithms given in this paper is quite unlike anything previously known to the author and seems worth recording because of its mathematical beauty and its adaptability to high speed computing machines…[T]his algorithm is based directly upon…arithmetic continued fractions." Shanks illustrated the algorithm by computing log 2.

The colonial scientist David Rittenhouse would surely have been very happy to know of this independent discovery almost one hundred fifty years after his death.

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