WHAT IS...

an Ergodic Transformation?

Cesar E. Silva

The theory of ergodic transformations developed from considerations in statistical mechanics involving the distribution of orbits in phase space. Now ergodic systems arise in many areas of mathematics, and ergodic methods have contributed to the solution of problems in several fields.

We start with a concrete example, paraphrasing a question of Gelfand. Are there infinitely many powers of 6 that start with a 9? In the first 18 powers of 6 (see Table 1) there is no initial 9. Indeed, in the first 175 powers of 6 (see Table 2) there is no initial 9. The first one does not appear until

6^{176}=90078276385246202645102918820475217309622015212833705033780614505275352569662789031588822572244111398877227429324097608129063079175520256.

6^1	6	6^{10}	60466176
6^2	36	6^{11}	362797056
6^3	216	6^{12}	2176782336
6^4	1296	6^{13}	13060694016
6^5	7776	6^{14}	78364164096
6^6	46656	6^{15}	470184984576
6^7	279936	6^{16}	2821109907456
6^8	1679616	6^{17}	16926659444736
6^9	10077696	6^{18}	101559956668416

Table 1. 6^n for $n = 1, \ldots, 18$.

We know that 6^n starts with a 9 if and only if

$$9 \times 10^k \leq 6^n < 10 \times 10^k, \text{ for some } k.$$

Taking logs of both sides yields the condition

$$\log 9 \leq n \log 6 - k < \log 10, \text{ for some } k,$$

where the logarithm is base 10. This means that $n \log 6$ is in some translation of the interval $[\log 9, \log 10)$, or equivalently

$$n \log 6 \bmod 1 \in [\log 9, \log 10).$$

Let T denote translation by $\log 6$ modulo 1

$$T(x) = x + \log 6 \pmod 1,$$

and let T^n denote the composition of T with itself n times. So 6^n starts with a 9 if and only if

$$T^n(0) \in [\log 9, 1).$$

Thus the frequency of powers of 6 that start with a 9, if it exists, is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_{[\log 9,1)} \circ T^i(0), \tag{1}$$

where $\mathbb{I}_A$ is the indicator function of a set A. Figure 1 shows the first 100 points in the orbit of 0, and we see that they miss $[\log 9, 1)$.

Digit	$n \leq 175$	$n \leq 175 \cdot 10^3$	Benford
1	38.67	30.10	30.10
2	19.33	17.60	17.61
3	16.00	12.50	12.49
4	10.00	9.69	9.69
5	6.67	7.92	7.92
6	12.67	6.69	6.69
7	2.67	5.80	5.80
8	10.00	5.11	5.12
9	0.00	4.58	4.58

Table 2. Frequencies of the first digit of 6^n for $n \leq 175$, $n \leq 175 \cdot 10^3$, and Benford's law.

We have just defined a *dynamical system* consisting of a set of states $X = [0, 1)$ (the phase space of the system), and a transformation T defined on X that one can easily verify preserves Lebesgue measure μ: $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets A. If we had a set A such that $x \in A$ if and only if $T(x) \in A$ (i.e., an *invariant* set), then we could restrict the dynamics of T to A. For example, if 0 were in A and $[\log 9, 1)$ in A^c, then the limit in (1) would be 0.

A transformation T is *ergodic* if every measurable invariant set or its complement has measure 0. When a transformation T is ergodic, by the ergodic theorem, for

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Cesar Silva is the Hagey Family Professor of Mathematics at Williams College. He studied at Pontificia Universidad Católica del Perú, and at the University of Rochester with Dorothy Maharam, who introduced him to Kakutani and Oxtoby, his two other mathematical influences, and to his wife Margaret, Oxtoby's daughter. He has run the Falmouth Road Race, a good 50 percent slower than their two daughters. His email address is `csilva@williams.edu`.

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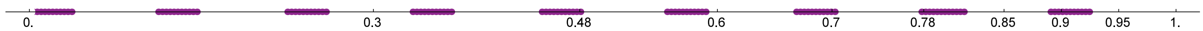

Figure 1. The orbit $T^i(0)$, for $i \in \{0, \dots, 100\}$, missing $[\log 9, 1) \approx [0.95, 1)$.

all integrable functions f and for all x outside a set of measure zero,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \int f \, d\mu. \tag{2}$$

By taking f to be $\mathbb{1}_{[\log 9, 1)}$, we get the desired density for initial 9s if we know that (2) holds for $x = 0$, which it does for this transformation. The proof begins by showing that, since $\log 6$ is irrational, for *all* x the *orbit* $\{T^n(x) : n \in \mathbb{N}\}$ is dense (Kronecker's theorem).

Thus 9s appear as first digits in powers of 6 with frequency $\mu([\log 9, 1)) = \log(10/9) \approx 4.58\%$. It also follows, for example, that 1 appears as first digit in powers of 6 with frequency $\log(2) \approx 30.10\%$. These are the frequencies predicted by Benford's law for appearances of 9 and 1 as first digit in many mathematical and real-world contexts.

When X is a compact space and T is continuous on X, the system (X, T) is *uniquely ergodic* if there is only one probability measure μ for which T is measure-preserving. Uniquely ergodic is stronger than ergodic, because if a transformation is not ergodic, using a nontrivial invariant set and its complement, it is easy to come up with other measures for which T is still measure-preserving. When T is uniquely ergodic, then (2) holds for all continuous functions f for *all* points x. We say that every orbit of x is *equidistributed* in $[0, 1)$.

There are ergodic transformations that are not uniquely ergodic. The flips of a coin where the probability of heads is p $(0 < p < 1)$ and tails is $1 - p$ can be modeled by a set Σ consisting of all infinite sequences of 0s and 1s (where we write 0 for heads and 1 for tails). The passage of time is represented by the transformation σ, which shifts each sequence x in Σ to its left. There is a natural product measure μ_p on Σ that comes from assigning the probabilities $(p, 1 - p)$ to $\{0, 1\}$. The shift is measure-preserving and ergodic for each measure μ_p. Similarly, one defines a shift that models the tosses of a possibly biased n-sided die. Ornstein in 1970 (see [1]) classified completely these Bernoulli shifts by their entropy, a rate at which nearby points typically move away from each other.

There is another interesting and remarkable construction of a measure-preserving system arising from considering a number-theoretic question. Szemerédi in 1975 answered a conjecture of Erdős and Turán by showing that a set of integers of positive upper density (for example, the even numbers have density 1/2) contains arithmetic progressions of arbitrary length. In 1977, Furstenberg showed how to associate to each set of positive upper density a measure-preserving system so that the set contains an arithmetic progression of length k precisely when the measure-preserving system is k-multiply recurrent. He then proved all finite measure-preserving systems are k-multiply recurrent for all k (see [1]).

There are other celebrated results where an equidistribution property has been shown. A transformation T on X gives an action of the group $\mathbb{Z}$ on X, where the action of $n \in \mathbb{Z}$ on $x \in X$ is $n \cdot x = T^n x$. In the 1990s Ratner proved that, for the action of unipotent matrices on finite-volume quotients of Lie groups, the orbit of *every* point is equidistributed in its closure, which is a nice manifold.

We conclude with a topic that appears in A. Eskin's invited address (see page 17). Recently Eskin, Mirzakhani, and Mohammadi proved a series of remarkable theorems that can be thought of as analogues of Ratner's results in the case of actions of the group of 2×2 real matrices with determinant 1 on the moduli space of translation surfaces (see [2]). Translation surfaces were introduced to study billiard flows. Among the many recent applications of their breakthrough results, one by Lelièvre, Monteil, and Weiss states that for billiards on polygons with angles that are rational multiples of π, from *every* point x there are billiard trajectories to all but finitely many other points.

Further Reading

[1] Karl Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1983.

[2] Alex Wright, From rational billiards to dynamics on moduli spaces, *Bull. Amer. Math. Soc.* (2015). www.ams.org/journals/bull/2016-53-01/S0273-0979-2015-01513-2/.