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Schwartz, Chancellor’s Professor of Mathematics at Brown, is a Guggenheim Fellow, a Clay Research Scholar, and a Simons Fellow. Among his many publications are several popular math books for children, including *You Can Count on Monsters* and *Really Big Numbers*. In this lecture, Schwartz will show some of his graphical user interfaces that are designed to help to understand problems in geometry and dynamics and explain the mathematics behind them. The interfaces let you discover patterns and organize information in a way that would be practically impossible using pencil and paper, and sometimes also facilitate computational proofs that are a kind of deal with the devil: They give a rigorous proof for an appealing result without offering conventional insight into why the result is true.
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In 1611 Johannes Kepler published a conjecture on the tightest way to pack unit spheres in 3-D. In April 2000, Thomas Hales described his proof here in the Notices. Our cover story reports that last year Maryna Viazovska proved the 8-D case, promptly followed by a collaborative proof in 24-D. Meanwhile, as described in our second feature article, Stanley's Partitionability Conjecture has been disproved by a counterexample. The Graduate Student Section features an interview with Tom Grandine, senior technical fellow at Boeing Company, and "WHAT IS...Benford’s Law?" A new Mathematical Moment on "Maintaining a Balance" vs. global environmental catastrophe has an accompanying deeper explanation by MIT climate scientist Daniel H. Rothman. This issue also includes an article on active learning, a report on The Bridges Conference—the world’s largest interdisciplinary conference on mathematics and art, a new BookShelf, a book review examining recreational math, and a firsthand account of a Fulbright Specialist’s time in Qatar. The BackPage has a special comic on refereeing and the Super Bowl. —Frank Morgan, Editor-in-Chief

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Delivered and written during Robert Steinberg's sabbatical visit to Yale University in 1967, these lectures present the status of the theory of Chevalley groups as it was in the mid-1960s. This posthumous edition incorporates additions and corrections prepared by the author during his retirement, including a new introductory chapter, bibliography, and editorial notes.


This is a great unsurpassed introduction to the subject of Chevalley groups that influenced generations of mathematicians. I would recommend it to anybody whose interests include group theory.

— Efim Zelmanov, University of California, San Diego

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A Conceptual Breakthrough in Sphere Packing

Henry Cohn

Maryna Viazovska solved the sphere packing problem in eight dimensions.

On March 14, 2016, the world of mathematics received an extraordinary Pi Day surprise when Maryna Viazovska posted to the arXiv a solution of the sphere packing problem in eight dimensions [15]. Her proof shows that the $E_8$ root lattice is the densest sphere packing in eight dimensions, via a beautiful and conceptually simple
argument. Sphere packing is notorious for complicated proofs of intuitively obvious facts, as well as hopelessly difficult unsolved problems, so it’s wonderful to see a relatively simple proof of a deep theorem in sphere packing. No proof of optimality had been known for any dimension above three, and Viazovska’s paper does not even address four through seven dimensions. Instead, it relies on remarkable properties of the $E_8$ lattice. Her proof is thus a notable contribution to the story of $E_8$, and more generally the story of exceptional structures in mathematics.

One measure of the complexity of a proof is how long it takes the community to digest it. By this standard, Viazovska’s proof is remarkably simple. It was understood by a number of people within a few days of her arXiv posting, and within a week it led to further progress: Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and I worked with Viazovska to adapt her methods to prove that the Leech lattice is an optimal sphere packing in twenty-four dimensions [4]. This is the only other case above three dimensions in which the sphere packing problem has been solved.

The new ingredient in Viazovska’s proof is a certain special function, which enforces the optimality of $E_8$ via the Poisson summation formula. The existence of such a function had been conjectured by Cohn and Elkies in 2003, but what sort of function it might be remained mysterious despite considerable effort. Viazovska constructs this function explicitly in terms of modular forms by using an unexpected integral transform, which establishes a new connection between modular forms and discrete geometry.

A landmark achievement like Viazovska’s deserves to be appreciated by a broad audience of mathematicians, and indeed it can be. In this article we’ll take a look at how her proof works, as well as the background and context. We won’t cover all the details completely, but we’ll see the main ideas and how they fit together. Readers who wish to read a complete proof will then be well prepared to study Viazovska’s paper [15] and the follow-up work on the Leech lattice [4]. See also de Laat and Vallentin’s survey article and interview [13] for a somewhat different perspective, as well as [1] and [7] for further background and references.

**Sphere Packing**

The sphere packing problem asks for the densest packing of $\mathbb{R}^n$ with congruent balls. In other words, what is the largest fraction of $\mathbb{R}^n$ that can be covered by congruent balls with disjoint interiors?

Pathological packings may not have well-defined densities, but we can handle the technicalities as follows. A sphere packing $\mathcal{P}$ is a nonempty subset of $\mathbb{R}^n$ consisting of congruent balls with disjoint interiors. The upper density of $\mathcal{P}$ is

$$
\limsup_{r \to \infty} \frac{\text{vol}(B^r(0) \cap \mathcal{P})}{\text{vol}(B^r(0))},
$$

where $B^r(x)$ denotes the closed ball of radius $r$ about $x$, and the sphere packing density $\Delta_{\mathbb{R}^n}$ in $\mathbb{R}^n$ is the supremum of all the upper densities of sphere packings. In other words, we avoid technicalities by using a generous definition of the packing density. This generosity does not cause any harm, as shown by the theorem of Groemer that there exists a sphere packing $\mathcal{P}$ for which

$$
\lim_{r \to \infty} \frac{\text{vol}(B^r(x) \cap \mathcal{P})}{\text{vol}(B^r(x))} = \Delta_{\mathbb{R}^n}
$$

uniformly for all $x \in \mathbb{R}^n$. Thus, the supremum of the upper densities is in fact achieved as the density of some packing, in the nicest possible way. Of course the densest packing is not unique, since there are any number of ways to perturb a packing without changing its overall density.

Why should we care about the sphere packing problem? Two obvious reasons are that it’s a natural geometric problem in its own right and a toy model for granular materials. A more surprising application is that sphere packings are error-correcting codes for a continuous communication channel. Real-world communication channels can be modeled using high-dimensional vector spaces, and thus high-dimensional sphere packings have practical importance.
Instead of justifying sphere packing by aspects of the problem or its applications, we’ll justify it by its solutions: a question is good if it has good answers. Sphere packing turns out to be a far richer and more beautiful topic than the bare problem statement suggests. From this perspective, the point of the subject is the remarkable structures that arise as dense sphere packings.

To begin, let’s examine the familiar cases of one, two, and three dimensions. The one-dimensional sphere packing problem is the interval packing problem on the line, which is of course trivial: the optimal density is 1. The two- and three-dimensional problems are far from trivial, but the optimal packings, shown in Figure 1, are exactly what one would expect. In particular, the sphere packing density is $\pi/\sqrt{12} = 0.9068 \ldots$ in $\mathbb{R}^2$ and $\pi/\sqrt{18} = 0.7404 \ldots$ in $\mathbb{R}^3$. The two-dimensional problem was solved by Thue. Giving a rigorous proof requires a genuine idea, but there exist short, elementary proofs [8]. The three-dimensional problem was solved by Hales [9] via a lengthy and complex computer-assisted proof, which was extraordinarily difficult to check but has since been completely verified using formal logic [10].

In both two and three dimensions, one can obtain an optimal packing by stacking layers that are packed optimally in the previous dimension, with the layers nestled together as closely as possible. Guessing this answer is not difficult, nor is computing the density of such a packing. Instead, the difficulty lies in proving that no other construction could achieve a greater density. Unfortunately, our low-dimensional experience is poor preparation for understanding high-dimensional sphere packing. Based on the first three dimensions, it appears that guessing the optimal packing is easy, but this expectation turns out to be completely false in high dimensions. In particular, stacking optimal layers from the previous dimension does not always yield an optimal packing. (One can recursively determine the best packings in successive dimensions under such a hypothesis [6], and this procedure yields a suboptimal packing by the time it reaches $\mathbb{R}^{10}$.)

The sphere packing problem seems to have no simple, systematic solution that works across all dimensions. Instead, each dimension has its own idiosyncracies and charm. Understanding the densest sphere packing in $\mathbb{R}^n$ tells us only a little about $\mathbb{R}^2$ or $\mathbb{R}^3$, and hardly anything about $\mathbb{R}^{10}$.

Aside from $\mathbb{R}^2$ and $\mathbb{R}^3$, our ignorance grows as the dimension increases. In high dimensions, we have absolutely no idea how the densest sphere packings behave. We do not know even the most basic facts, such as whether the densest packings should be crystalline or disordered. Here “do not know” does not merely mean “cannot prove,” but rather the much stronger “cannot predict.”

A simple greedy argument shows that the optimal density in $\mathbb{R}^n$ is at least $2^{-n}$. To see why, consider any sphere packing in which there is no room to add even one more sphere. If we double the radius of each sphere, then the enlarged spheres must cover space completely, because any uncovered point could serve as the center of a new sphere that would fit in the original packing. Doubling the radius multiplies volume by $2^n$, and so the original packing must cover at least a $2^{-n}$ fraction of $\mathbb{R}^n$.

That may sound appallingly low, but it is very nearly the best lower bound known. Even the most recent bounds, obtained by Venkatesh [14] in 2013, have been unable to improve on $2^{-n}$ by more than a linear factor in general and an $n \log \log n$ factor in special cases. As for upper bounds, in 1978 Kabatianskii and Levenshtein [11] proved an upper bound of $2^{-0.599 \ldots + o(1) n}$, which remains essentially the best upper bound known in high dimensions. Thus, we know that the sphere packing density decreases exponentially as a function of dimension, but the best upper and lower bounds known are exponentially far apart.

Table 1 lists the best packing densities currently known in up to 36 dimensions, and Figure 2 shows a logarithmic plot. The plot has several noteworthy features:

1. The curve is jagged and irregular, with no obvious way to interpolate data points from their neighbors.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sphere_packings.png}
\caption{Fragments of optimal sphere packings in two and three dimensions, with density $\pi/\sqrt{12} = 0.9068 \ldots$ in $\mathbb{R}^2$ and $\pi/\sqrt{18} = 0.7404 \ldots$ in $\mathbb{R}^3$.}
\end{figure}
(2) The density is clearly decreasing exponentially, but the irregularity makes it unclear how to extrapolate to estimate the decay rate as the dimension tends to infinity.

(3) There seem to be parity effects. Even dimensions look slightly better than odd dimensions, multiples of four are better yet, and multiples of eight are the best of all.

(4) Certain dimensions, most notably 24, have packings so good that they seem to pull the entire curve in their direction. The fact that this occurs is not so surprising, since one expects cross sections and stackings of great packings to be at least good, but the effect is surprisingly large.

**Lattices and Periodic Packings**

How can we describe sphere packings? Random or pathological packings can be infinitely complicated, but the most important packings can generally be given a finite description via periodicity.

Recall that a lattice in \( \mathbb{R}^n \) is a discrete subgroup of rank \( n \). In other words, it consists of the integral span of a basis of \( \mathbb{R}^n \). Equivalently, a lattice is the image of \( \mathbb{Z}^n \) under an invertible linear operator.

A sphere packing \( \mathcal{P} \) is periodic if there exists a lattice \( \Lambda \) such that \( \mathcal{P} \) is invariant under translation by every element of \( \Lambda \). In that case, the translational symmetry group of \( \mathcal{P} \) must be a lattice, since it is clearly a discrete group, and \( \mathcal{P} \) consists of finitely many orbits of this group. A lattice packing is a periodic packing in which the spheres form a single orbit under the translational symmetry group (i.e., their centers form a lattice, up to translation). See Figure 3 for an illustration.

It is not known whether periodic packings attain the optimal sphere packing density in each dimension, aside from the five cases in which the sphere packing problem has been solved. They certainly come arbitrarily close to the optimal density: given an optimal packing, one can approximate it by taking the spheres contained in a large box and repeating them periodically throughout space, and the density loss is negligible if the box is large enough. However, there seems to be no reason why periodic packings should reach the exact optimum, and perhaps they don’t in high dimensions.

By contrast, lattices probably do not even come arbitrarily close to the optimal packing density in high dimensions. For example, the best periodic packing known in \( \mathbb{R}^{10} \) is more than 8% denser than the best lattice packing known.

Seen in this light, the optimality of lattices in \( \mathbb{R}^8 \) and \( \mathbb{R}^{24} \) is not a foregone conclusion, but rather an indication that sphere packing in these dimensions is particularly simple.
To compute the density of a lattice packing, it’s convenient to view the lattice as a tiling of space with parallelepipeds (the \(n\)-dimensional analogue of parallelograms). Given a basis \(v_1, \ldots, v_n\) for a lattice \(\Lambda\), the parallelepiped
\[
\{x_1v_1 + \cdots + x_nv_n : 0 \leq x_i < 1 \text{ for } i = 1, 2, \ldots, n\}
\]
is called the \textit{fundamental cell} of \(\Lambda\) with respect to this basis. Translating the fundamental cell by elements of \(\Lambda\) tiles \(\mathbb{R}^n\), as in Figure 3. From this perspective, a lattice sphere packing amounts to placing spheres at the vertices of such a tiling. On a global scale, there is one sphere for each copy of the fundamental cell. Thus, if the packing uses spheres of radius \(r\) and has fundamental cell \(C\), then its density is the ratio
\[
\frac{\text{vol}(B^n_r)}{\text{vol}(C)}.
\]

Both factors in this ratio are easily computed if we are given \(r\) and \(C\). The volume of a fundamental cell is just the absolute value of the determinant of the corresponding lattice basis; we will write it as \(\text{vol}(\mathbb{R}^n/\Lambda)\), the volume of the quotient torus, to avoid having to specify a basis. Computing the volume of a ball of radius \(r\) in \(\mathbb{R}^n\) is a multivariate calculus exercise, whose answer is
\[
\frac{\pi^{n/2}}{(n/2)!} r^n,
\]
where of course \((n/2)!\) means \(\Gamma(n/2 + 1)\) when \(n\) is odd. We can therefore compute the density of any lattice packing explicitly. The density of a periodic packing is equally easy to compute: if the packing consists of \(N\) translates of a lattice \(\Lambda\) in \(\mathbb{R}^n\) and uses spheres of radius \(r\), then its density is
\[
\frac{N \cdot \text{vol}(B^n_r)}{\text{vol}(\mathbb{R}^n/\Lambda)}.
\]

Of course the density of a packing depends on the radius of the spheres. Given a lattice with no radius specified, it is standard to use the largest radius that does not lead to overlap. The \textit{minimal vector length} of a lattice \(\Lambda\) is the length of the shortest nonzero vector in \(\Lambda\), or equivalently the shortest distance between two distinct points in \(\Lambda\). If the minimal vector length is \(r\), then \(r/2\) is the largest radius that yields a packing, since that is the radius at which neighboring spheres become tangent.

\textbf{The \(E_8\) and Leech Lattices}

Many dimensions feature noteworthy sphere packings, but the \(E_8\) root lattice in \(\mathbb{R}^8\) and the Leech lattice in \(\mathbb{R}^{24}\) are perhaps the most remarkable of all, with connections to exceptional structures across mathematics. In this section, we’ll construct \(E_8\) and prove some of its basic properties. It was discovered by Korkine and Zolotareff in 1873, in the guise of a quadratic form they called \(W_8\). We’ll give a construction much like Korkine and Zolotareff’s but more modern. The Leech lattice \(\Lambda_{24}\), discovered by Leech in 1967, is similar in spirit, but more complicated. In lieu of constructing it, we will briefly summarize its properties.

To specify \(E_8\), we just need to describe a lattice basis \(v_1, \ldots, v_8\) in \(\mathbb{R}^8\). Furthermore, only the relative positions of the basis vectors matter, so all we need to specify is their inner products with each other. All this information will be encoded by the \textit{Dynkin diagram}

of \(E_8\). In this diagram, the eight nodes correspond to the basis vectors, each of squared length 2. The inner product between distinct vectors is \(-1\) if the nodes are joined by an edge, and 0 otherwise. Thus, if we number the nodes

then the Gram matrix of inner products for this basis is given by

\[
(\langle v_i, v_j \rangle)_{1 \leq i, j \leq 8} = \\
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}.
\]

Before we go further, we must address a fundamental question: how do we know there really are vectors \(v_1, \ldots, v_8\) with these inner products? All we need is for the matrix in (1) to be symmetric and positive definite, and indeed it is, although it’s not obviously positive definite. That can be checked in several ways. We’ll take the pedestrian approach of observing that the characteristic polynomial of this matrix is

\[
t^8 -16t^7 + 105t^6 - 364t^5 + 714t^4 - 784t^3 + 440t^2 - 96t + 1,
\]

which clearly has no roots when \(t < 0\) because every term is then positive.

We can now define the \(E_8\) root lattice to be the integral span of \(v_1, \ldots, v_8\). We will use this definition to derive several fundamental properties of \(E_8\). These properties will let us determine its packing density, and they will also be essential for Viazovska’s proof.

The \(E_8\) lattice is an \textit{integral lattice}, which means all the inner products between vectors in \(E_8\) are integers. This follows immediately from the integrality of the inner products of the basis vectors \(v_1, \ldots, v_8\). Even more importantly, \(E_8\) is an \textit{even lattice}, which means the squared length of every vector is an even integer. Specifically, for \(m_1, \ldots, m_8 \in \mathbb{Z}\) the vector \(m_1v_1 + \cdots + m_8v_8\) has squared length

\[
|m_1v_1 + \cdots + m_8v_8|^2 = 2m_1^2 + \cdots + 2m_8^2 + \sum_{1 \leq i < j \leq 8} 2m_im_j\langle v_i, v_j \rangle,
\]

which is visibly even. Thus, the distances between distinct points in \(E_8\) are all of the form \(\sqrt{k}\) with \(k = 1, 2, \ldots,\) and in fact each of those distances does occur.
In particular, the distance between neighboring points in $E_8$ is $\sqrt{2}$, so we can form a packing with spheres of radius $\sqrt{2}/2$ and density

$$\frac{\text{vol}(B^S_{\sqrt{2}/2})}{\text{vol}(\mathbb{R}^S/E_8)} = \frac{\pi^4}{384 \text{vol}(\mathbb{R}^S/E_8)}.$$  

To compute the density of the $E_8$ packing, all we need to compute is $\text{vol}(\mathbb{R}^S/E_8)$.

To compute this volume, recall that it’s the absolute value of the determinant of the basis matrix:

$$\text{vol}(\mathbb{R}^S/E_8) = |\det \begin{bmatrix} v_1 & v_2 & \cdots & v_8 \end{bmatrix}|.$$  

However, we can write the Gram matrix $(\langle v_i, v_j \rangle)_{1 \leq i, j \leq 8}$ as the product

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_8 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_8 \end{bmatrix}^\top$$

of the basis matrix with its transpose, and thus

$$\det (\langle v_i, v_j \rangle)_{1 \leq i, j \leq 8} = \text{vol}(\mathbb{R}^S/E_8)^2.$$  

Computing the determinant of the matrix in (1) then shows that $\text{vol}(\mathbb{R}^S/E_8) = 1$. In other words, $E_8$ is a unimodular lattice.

Putting together our calculations, we have proved the following proposition:

**Proposition 1.** The $E_8$ lattice packing in $\mathbb{R}^8$ has density $\pi^4/384 = 0.2536 \ldots$.

Our calculations so far have led us to what turns out to be the densest sphere packing in $\mathbb{R}^8$, but it’s not obvious from this construction that $E_8$ is an especially interesting lattice. The $E_8$ lattice is in fact magnificently symmetrical, far more so than one might naively guess based on its lopsided Dynkin diagram. Its symmetry group is the $E_8$ Weyl group, which is generated by reflections in the hyperplanes orthogonal to each of $v_1, \ldots, v_8$. We will not make use of this group, but it’s important to keep in mind that the lattice itself is far more symmetrical than its definition. This is a common pattern when defining highly symmetrical objects.

Our density calculation for $E_8$ was based on its being an even unimodular lattice. In fact, $E_8$ is the unique even unimodular lattice in $\mathbb{R}^8$, up to orthogonal transformations. Even unimodular lattices exist only when the dimension is a multiple of eight, and they play a surprisingly large role in the theory of sphere packing.

The last property of $E_8$ we will need for Viazovska’s proof is that it is its own dual lattice, a concept we will define shortly. Given a lattice $\Lambda$ with basis $v_1, \ldots, v_n$, let $v_1^*, \ldots, v_n^*$ be the dual basis with respect to the usual inner product. In other words,

$$\langle v_i, v_j^* \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the dual lattice $\Lambda^*$ of $\Lambda$ is the lattice with basis $v_1^*, \ldots, v_n^*$. It is not difficult to check that $\Lambda^*$ is independent of the choice of basis for $\Lambda$; one basis-free way to characterize it is that

$$(2) \quad \Lambda^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda \}.$$  

The self-duality $E_8^* = E_8$ is a consequence of the following lemma:

**Lemma 2.** Every integral unimodular lattice $\Lambda$ satisfies $\Lambda^* = \Lambda$.

**Proof.** Let $v_1, \ldots, v_n$ be a basis of $\Lambda$, and $v_1^*, \ldots, v_n^*$ the dual basis of $\Lambda^*$. By construction, the basis matrix formed by $v_1^*, \ldots, v_n^*$ is the inverse of the transpose of that formed by $v_1, \ldots, v_n$, and hence $\text{vol}(\mathbb{R}^n/\Lambda^*) = 1/\text{vol}(\mathbb{R}^n/\Lambda)$. If $\Lambda$ is an integral lattice, then $\Lambda \subseteq \Lambda^*$, and the index of $\Lambda$ in $\Lambda^*$ is given by

$$[\Lambda^* : \Lambda] = \text{vol}(\mathbb{R}^n/\Lambda)/\text{vol}(\mathbb{R}^n/\Lambda^*) = \text{vol}(\mathbb{R}^n/\Lambda)^2.$$  

If $\Lambda$ is unimodular as well, then $[\Lambda^* : \Lambda] = 1$ and hence $\Lambda^* = \Lambda$. $\square$

As mentioned above, the Leech lattice $\Lambda_{24}$ is similar to $E_8$ but more elaborate. It’s an even unimodular lattice in $\mathbb{R}^{24}$, but this time with no vectors of length $\sqrt{2}$, and it’s the unique lattice with these properties, up to orthogonal transformations. The nonzero vectors in $\Lambda_{24}$ have lengths $\sqrt{2k}$ for $k = 2, 3, \ldots$, and of course $\Lambda_{24}^* = \Lambda_{24}$ because $\Lambda_{24}$ is integral and unimodular. One noteworthy property of $\Lambda_{24}$ is that it’s chiral: all of its symmetries are orientation-preserving, and the Leech lattice therefore occurs in left-handed and right-handed variants, which are mirror images of each other. (By contrast, the symmetry group of $E_8$ is generated by reflections, so $E_8$ is certainly not chiral.)
Noam Elkies developed the linear programming bounds for sphere packing with Henry Cohn.

The sphere packing density of the Leech lattice is

\[
\frac{\text{vol}(\mathbb{B}^{24})}{\text{vol}(\mathbb{R}^{24}/\Lambda_{24})} = \frac{\pi^{12}}{12!} = 0.001929\ldots,
\]

which looks awfully low, but keep in mind that the optimal density decreases exponentially as a function of dimension. In fact, the density of the Leech lattice is remarkably high, as one can see from Figure 2 and Table 1. For comparison, the best density known in \(\mathbb{R}^{23}\) is 0.001905 \ldots, which is lower than the density of the Leech lattice, and this is the only case in which the density increases from one dimension to the next in Table 1.

### Linear Programming Bounds

The underlying technique used in Viazovska’s proof is linear programming bounds for the sphere packing density in \(\mathbb{R}^n\). These upper bounds were developed by Cohn and Elkies [2], based on several decades of previous work initiated by Delzant and extended by numerous mathematicians. In this approach to sphere packing, one uses auxiliary functions with certain properties to obtain density bounds. Viazovska’s breakthrough consists of a new technique for constructing these auxiliary functions, but before we turn to her proof let’s examine the general analysis.

Linear programming bounds are based on harmonic analysis. That may sound surprising, since sphere packing is a problem in discrete geometry, which at first glance seems to have little to do with the continuous problems studied in harmonic analysis. However, there is a deep connection between these fields, because the Fourier transform is essential for understanding the action of the additive group \(\mathbb{R}^n\) on itself by translation, so much so that one can’t truly understand lattices without harmonic analysis.

### The key technical tool behind linear programming bounds is the Poisson summation formula.

Define the Fourier transform \(\hat{f}\) of an integrable function \(f: \mathbb{R}^n \to \mathbb{R}\) by

\[
\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} \, dx.
\]

Fourier inversion tells us that if \(\hat{f}\) is integrable as well, then one can similarly recover \(f\) from \(\hat{f}\):

\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i \langle x, y \rangle} \, dy
\]

almost everywhere. In other words, the Fourier transform gives the unique coefficients needed to express \(f\) in terms of complex exponentials.

To avoid analytic technicalities, we will focus on Schwartz functions. Recall that \(f: \mathbb{R}^n \to \mathbb{R}\) is a Schwartz function if \(f\) is infinitely differentiable,

\[
f(x) = O((1 + |x|)^{-k})
\]

for all \(k = 1, 2, \ldots\), and the same holds for all the partial derivatives of \(f\) (of every order). Schwartz functions behave particularly well, well enough to justify everything we’d like to do with them, and they are closed under the Fourier transform. We could get by with weaker hypotheses, but in fact Viazovska’s construction produces Schwartz functions, so we might as well focus on that case.

The significance of the Fourier transform in sphere packing is that it diagonalizes the operation of translation by any vector. Specifically, (3) implies that

\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i \langle x, y \rangle} \, dy.
\]

which means that translating the input to the function \(f\) by \(t\) amounts to multiplying its Fourier transform \(\hat{f}(y)\) by \(e^{2\pi i \langle t, y \rangle}\). Simultaneously diagonalizing all these translation operators makes the Fourier transform an ideal tool for studying periodic structures.

The key technical tool behind linear programming bounds is the Poisson summation formula, which expresses a duality between summing a function over a lattice and summing the Fourier transform over the dual lattice, as defined in (2). Poisson summation says that if \(f\) is a Schwartz function, then

\[
\sum_{x \in \Lambda} f(x) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y).
\]

In other words, summing \(\hat{f}\) over \(\Lambda^*\) is almost the same as summing \(f\) over \(\Lambda\), with the only difference being a factor of \(\text{vol}(\mathbb{R}^n/\Lambda)\). When expressed in this form, Poisson summation looks mysterious, but it becomes far more transparent when written in the translated form

\[
\sum_{x \in \Lambda} f(x + t) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) e^{2\pi i \langle y, t \rangle}.
\]
This equation reduces to (4) when \( t = 0 \), and it has a simple proof. As a function of \( t \), the left side of (5) is periodic modulo \( \Lambda \), while the right side is its Fourier series. In particular, the right side uses exactly the complex exponentials \( t \mapsto e^{2\pi i (y, t)} \) that are periodic modulo \( \Lambda \), namely those with \( y \in \Lambda^* \) (as follows easily from (2)). Orthogonality lets us compute the coefficient of such an exponential, and some manipulation yields \( \hat{f}(y) / \text{vol}(\mathbb{R}^n/\Lambda) \).

Now we can state and prove the linear programming bounds, which show how to convert a certain sort of auxiliary function into a sphere packing bound. Specifically, we will use functions \( f: \mathbb{R}^n \to \mathbb{R} \) such that \( f \) is eventually nonpositive (i.e., there exists a radius \( r \) such that \( f(x) \leq 0 \) for \( |x| \geq r \)) while \( \hat{f} \) is nonnegative everywhere.

**Theorem 3** (Cohn and Elkies [2]). Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a Schwartz function and \( r \) a positive real number such that \( f(0) = \hat{f}(0) > 0 \), \( \hat{f}(y) \geq 0 \) for all \( y \in \mathbb{R}^n \), and \( f(x) \leq 0 \) for \( |x| \geq r \). Then the sphere packing density in \( \mathbb{R}^n \) is at most \( \text{vol}(B_{r/2}^n) / \text{vol}(\mathbb{R}^n/\Lambda) \).

The name “linear programming” refers to optimizing a linear function subject to linear constraints. The optimization problem of choosing \( f \) so as to minimize \( r \) can be rephrased as an infinite-dimensional linear program after a change of variables, but we will not adopt that perspective here.

**Proof.** The proof consists of applying the contrasting inequalities \( f(x) \leq 0 \) and \( \hat{f}(y) \geq 0 \) to the two sides of Poisson summation. We will begin by proving the theorem for lattice packings, which is the simplest case.

Suppose \( \Lambda \) is a lattice in \( \mathbb{R}^n \), and suppose without loss of generality that the minimal vector length of \( \Lambda \) is \( r \) (since the sphere packing density is invariant under rescaling). In other words, the packing uses balls of radius \( r/2 \), and its density is

\[
\frac{\text{vol}(B_{r/2}^n)}{\text{vol}(\mathbb{R}^n/\Lambda)}.
\]

Proving the desired density bound \( \text{vol}(B_{r/2}^n) / \text{vol}(\mathbb{R}^n/\Lambda) \) for \( \Lambda \) amounts to showing that \( \text{vol}(\mathbb{R}^n/\Lambda) \geq 1 \). By Poisson summation,

\[
\sum_{x \in \Lambda} f(x) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y).
\]

Now the inequality \( f(x) \leq 0 \) for \( |x| \geq r \) tells us that the left side of (6) is bounded above by \( f(0) \), and the inequality \( \hat{f}(y) \geq 0 \) tells us that the right side is bounded below by \( \hat{f}(0) / \text{vol}(\mathbb{R}^n/\Lambda) \). It follows that

\[
f(0) \geq \frac{\hat{f}(0)}{\text{vol}(\mathbb{R}^n/\Lambda)},
\]

which yields \( \text{vol}(\mathbb{R}^n/\Lambda) \geq 1 \) because \( f(0) = \hat{f}(0) > 0 \).

The general case is almost as simple, but the algebraic manipulations are a little trickier. Because periodic packings come arbitrarily close to the optimal sphere packing density, without loss of generality we can consider a periodic packing using balls of radius \( r/2 \), centered at the translates of a lattice \( \Lambda \subseteq \mathbb{R}^n \) by vectors \( t_1, \ldots, t_N \). The packing density is

\[
\frac{N \text{vol}(B_{r/2}^n)}{\text{vol}(\mathbb{R}^n/\Lambda)},
\]

and so we wish to prove that \( \text{vol}(\mathbb{R}^n/\Lambda) \geq N \).

We will use the translated Poisson summation formula (5), which after a little manipulation implies that

\[
\sum_{j,k=1}^N f(t_j - t_k + x) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) \left( \sum_{j=1}^N e^{2\pi i (y, t_j)} \right)^2.
\]

Again we apply the contrasting inequalities on \( f \) and \( \hat{f} \) to the left and right sides, respectively. On the left, we obtain an upper bound by throwing away every term except when \( j = k \) and \( x = 0 \); on the right, we obtain a lower bound by throwing away every term except when \( y = 0 \). Thus,

\[
NF(0) \geq \frac{N^2}{\text{vol}(\mathbb{R}^n/\Lambda)} \hat{f}(0),
\]

which implies that \( \text{vol}(\mathbb{R}^n/\Lambda) \geq N \) and hence that the density is at most \( \text{vol}(B_{r/2}^n) / \text{vol}(\mathbb{R}^n/\Lambda) \), as desired. \( \square \)

![Figure 4. The logarithm of sphere packing density as a function of dimension. The upper curve is the numerically optimized linear programming bound, while the lower curve is the best packing currently known. The truth lies somewhere in between.](image)

**Table 2. The linear programming bound for the sphere packing density in \( \mathbb{R}^n \) with \( 1 \leq n \leq 36 \). All numbers are rounded up.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>upper bound</th>
<th>( n )</th>
<th>upper bound</th>
<th>( n )</th>
<th>upper bound</th>
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</thead>
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<td>13</td>
<td>0.062417002</td>
<td>25</td>
<td>0.001394190723</td>
</tr>
<tr>
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<td>14</td>
<td>0.046364893</td>
<td>26</td>
<td>0.000912389</td>
</tr>
<tr>
<td>3</td>
<td>0.27946762</td>
<td>15</td>
<td>0.034248262</td>
<td>27</td>
<td>0.000708229796</td>
</tr>
<tr>
<td>4</td>
<td>0.647704966</td>
<td>16</td>
<td>0.025194131</td>
<td>28</td>
<td>0.00050524217</td>
</tr>
<tr>
<td>5</td>
<td>0.324980022</td>
<td>17</td>
<td>0.018464094</td>
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</tr>
<tr>
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<td>30</td>
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<tr>
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<td>0.009817955</td>
<td>31</td>
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</tr>
<tr>
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<td>20</td>
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<td>24</td>
<td>0.0019295744</td>
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This proof technique may look absurdly inefficient. We start with Poisson summation, which expresses a deep duality, and then we recklessly throw away all the nontrivial terms, leaving only the contributions from the origin. One practical justification is that we have little choice in the matter, since we don’t know what the other terms are (they all depend on the lattice). A deeper justification is that the omitted terms are generally small, and sometimes zero, so omitting them is not as bad as it sounds.

To apply Theorem 3, we must choose an auxiliary function \( f \). The theorem then shows how to obtain a density bound from \( f \), but it says nothing about how to choose \( f \) so as to minimize \( r \) and hence minimize the density bound. Sadly, optimizing the auxiliary function remains an unsolved problem, and the best possible choice of \( f \) is known only when \( n = 1, 8, \) or 24.

As a first step towards solving this problem, note that we can radially symmetrize \( f \), so that \( f(x) \) depends only on \( |x| \), because all the constraints on \( f \) are linear and rotationally invariant. Then \( f \) is really a function of one radial variable, as is \( \hat{f} \). Functions of one variable feel like they should be tractable, but this optimization problem turns out to be impressively subtle.

If we can’t fully optimize the choice of \( f \), then what can we do? Several explicit constructions are known, but in general we must resort to numerical computation. For this purpose, it’s convenient to use auxiliary functions of the form \( f(x) = p(|x|^2)e^{-\pi|x|^2} \), where \( p \) is a polynomial. These functions are flexible enough to approximate arbitrary radial Schwartz functions, but simple enough to be tractable. Numerical optimization then yields a high-precision approximation to the linear programming bound, which is shown in Figure 4 and Table 2.

The Hunt for the Magic Functions

The most striking property of Figure 4 is that the upper and lower bounds in \( \mathbb{R}^n \) seem to touch when \( n = 8 \) or 24. In other words, there should be magic auxiliary functions that solve the sphere packing problem in these dimensions, by achieving \( r = \sqrt{2} \) in Theorem 3 when \( n = 8 \) and \( r = 2 \) when \( n = 24 \). (These values of \( r \) are the minimal vector lengths in \( E_8 \) and \( \Lambda_{24} \), respectively.) This is exactly what has now been proved, and the proof simply amounts to constructing an appropriate auxiliary function. Linear programming bounds do not seem to be sharp for any other \( n > 2 \), which makes these two cases truly remarkable.

The existence of these magic functions was conjectured by Cohn and Elkies [2] on the basis of numerical evidence and analogies with other problems in coding theory. Further evidence was obtained by Cohn and Kumar [3] in the course of proving that the Leech lattice is the densest lattice in \( \mathbb{R}^{24} \), while Cohn and Miller [5] carried out an even more detailed study of the magic functions. These calculations left no doubt that the magic functions existed: one could compute them to fifty decimal places, plot them, approximate their roots and power series coefficients, etc. They were perfectly concrete and accessible functions, amenable to exploration and experimentation, which indeed uncovered various intriguing patterns. All that was missing was an existence proof.

However, proving existence was no easy matter. There was no sign of an explicit formula, or any other characterization that could lead to a proof. Instead, the magic functions seemed to come out of nowhere.

The fundamental difficulty is explaining where the magic comes from. One can optimize the auxiliary function in any dimension, but that will generally not produce a sharp bound for the packing density. Why should eight and twenty-four dimensions be any different? The numerical results show that the bound is nearly sharp in those dimensions, but why couldn’t it be exact for a hundred decimal places, followed by random noise? That’s not a plausible scenario for anyone with faith in the beauty of mathematics, but faith does not amount to a proof, and any proof must take advantage of special properties of these dimensions.

For comparison, the answer is far less nice in sixteen dimensions. By analogy with \( r = \sqrt{2} \) when \( n = 8 \) and \( r = 2 \) when \( n = 24 \), one might guess that \( r = \sqrt{3} \) when \( n = 16 \), but that bound cannot be achieved. Instead, numerical optimization seems to converge to \( r^2 = 3.0252593116828820 \ldots \), which is close to 3 but not equal to it. This number has not yet been identified exactly.

Despite the lack of an existence proof, the proof of Theorem 3 implicitly describes what the magic functions must look like:

Lemma 4. Suppose \( f \) satisfies the hypotheses of the linear programming bounds for sphere packing in \( \mathbb{R}^n \), with \( f(x) \leq 0 \) for \( |x| \geq r \), and suppose \( \Lambda \) is a lattice in \( \mathbb{R}^n \) with minimal vector length \( r \). Then the density of \( \Lambda \) equals the bound \( \text{vol}(B^r_\Lambda) \) from Theorem 3 if and only if \( f \) vanishes on \( \Lambda \setminus \{0\} \) and \( \hat{f} \) vanishes on \( \Lambda^* \setminus \{0\} \).

Proof. Recall that the proof of Theorem 3 for a lattice \( \Lambda \) amounted to dropping all the nontrivial terms in the Poisson summation formula, to obtain the inequality

\[
\hat{f}(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) \geq \frac{\hat{f}(0)}{\text{vol}(\mathbb{R}^n/\Lambda)}.
\]

The only way this argument could yield a sharp bound is if all the omitted terms were already zero. In other words, \( f \) proves that \( \Lambda \) is an optimal sphere packing if and only if \( f \) vanishes on \( \Lambda \setminus \{0\} \) and \( \hat{f} \) vanishes on \( \Lambda^* \setminus \{0\} \). \( \square \)

As discussed in the previous section, without loss of generality we can assume that \( f \) is a radial function, as is \( \hat{f} \). We know exactly where the roots of \( f \) and \( \hat{f} \) should be, since \( E_8 = E_8^* \) with vector lengths \( \sqrt{2k} \) for \( k = 1, 2, \ldots \), while \( \Lambda_{24} = \Lambda_{24}^* \) with vector lengths \( \sqrt{2k} \) for \( k = 2, 3, \ldots \). These roots should have order two, to avoid sign changes,
except that the first root of $f$ should be a single root. See Figure 5 for a diagram.

Thus, our problem is simple to state: how can we construct a radial Schwartz function $f$ such that $f$ and $\hat{f}$ have the desired roots and no others? Note that Poisson summation over $F_8$ or $A_{24}$ then implies that $f(0) = \hat{f}(0)$, and flipping the sign of $f$ if necessary ensures that all the necessary inequalities hold.

Unfortunately it’s difficult to take advantage of this characterization. The problem is that it’s hard to control a function and its Fourier transform simultaneously: it’s easy to produce the desired roots in either one separately, but not at the same time. Our inability to control $f$ without losing control of $\hat{f}$ is at the root of the Heisenberg uncertainty principle, and it’s a truly fundamental obstacle.

One natural way to approach this problem is to carry out numerical experiments. Cohn and Miller used functions of the form $f(x) = p(|x|^2)e^{-\pi |x|^2}$ to approximate the magic functions, where $p$ is a polynomial chosen to force $f$ and $\hat{f}$ to have many of the desired roots. Such an approximation can never be exact, since it has only finitely many roots, but it can come arbitrarily close to the truth. This investigation uncovered several noteworthy properties of the magic functions, which showed that they had unexpected structure. For example, if we normalize the magic functions $f_8$ and $f_{24}$ in 8 and 24 dimensions so $f_8(0) = f_{24}(0) = 1$, then Cohn and Miller conjectured that their second Taylor coefficients are rational:

\[
\begin{align*}
 f_8(x) &= 1 - \frac{27}{10} |x|^2 + O(|x|^4), \\
 \hat{f}_8(x) &= 1 - \frac{3}{2} |x|^2 + O(|x|^4), \\
 f_{24}(x) &= 1 - \frac{14347}{5460} |x|^2 + O(|x|^4), \\
 \hat{f}_{24}(x) &= 1 - \frac{205}{156} |x|^2 + O(|x|^4).
\end{align*}
\]

If all the higher-order coefficients had been rational as well, then it would have opened the door to determining these functions exactly, but frustratingly it seems that the other coefficients are far more subtle and presumably irrational. The magic functions retained their mystery, and this Taylor series behavior went unexplained until the exact formulas for the magic functions were discovered.

Given the difficulty of controlling $f$ and $\hat{f}$ simultaneously, one natural approach is to split them into eigenfunctions of the Fourier transform. By Fourier inversion, every radial function $f$ satisfies $\hat{\hat{f}} = f$. Thus, if we set $f_+ = (f + \hat{f})/2$ and $f_- = (f - \hat{f})/2$, then $f = f_+ + f_-$ with $\hat{f}_+ = f_+$ and $\hat{f}_- = -f_-$. Because $f$ and $\hat{f}$ vanish at the same points, they share these roots with $f_+$ and $f_-$. Our goal is therefore to construct radial eigenfunctions of the Fourier transform with prescribed roots. The advantage of this approach is that it conveniently separates into two distinct problems, namely constructing the $+1$ and $-1$ eigenfunctions, but these problems remain difficult.

**Modular Forms**

Ever since the Cohn-Elkies paper in 2003, number theorists had hoped to construct the magic functions using modular forms. The reasoning is simple: modular forms are deep and mysterious functions connected with lattices, as are the magic functions, so wouldn’t it make sense for them to be related? Unfortunately, they are entirely different sorts of functions, with no clear connection between them. That’s where matters stood until Viazovska discovered a remarkable integral transform, which enabled her to construct the magic functions using modular forms. We’ll get there shortly, but first let’s briefly review how modular forms work.

We’ll start with some examples. Every lattice $\Lambda$ has a theta series $\Theta_\Lambda$, defined by

\[
\Theta_\Lambda(z) = \sum_{x \in \Lambda} e^{\pi i |x|^2 z}.
\]

This series converges when $3z > 0$, and it defines an analytic function on the upper half-plane $\mathfrak{h} = \{z \in \mathbb{C} : \Re z > 0\}$. To motivate the definition, think of the theta series as a generating function, where the coefficient of $e^{\pi i t z}$ counts the number of $x \in \Lambda$ with $|x|^2 = t$. However, there’s one aspect not explained by the generating function interpretation: why write this function in terms of $e^{\pi i t z}$? Doing so may at first look like a gratuitous nod to Fourier series, but it leads to an elegant transformation law based on applying Poisson summation to a Gaussian:

**Proposition 5.** If $\Lambda$ is a lattice in $\mathbb{R}^n$, then

\[
\Theta_\Lambda(z) = \frac{1}{\text{vol}(\mathbb{R}^n/\Lambda)} \left( \frac{i}{z} \right)^{n/2} \Theta_{\Lambda^+}(-1/z)
\]

for all $z \in \mathfrak{h}$.

**Proof.** One of the most important properties of Gaussians is that the set of Gaussians is closed under the Fourier transform: the Fourier transform of a wide Gaussian is a
narrow Gaussian, and vice versa. More precisely, for \( t > 0 \) the Fourier transform of the Gaussian \( x \mapsto e^{-tm|x|^2} \) on \( \mathbb{R}^n \) is \( x \mapsto t^{-n/2}e^{-n|x|^2/t} \). In fact, the same holds whenever \( t \) is a complex number with Re \( t > 0 \), by analytic continuation. Then Poisson summation tells us that

\[
\sum_{x \in \Lambda} e^{-tn|x|^2} = \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} t^{-n/2} e^{-n|y|^2/t}.
\]

Setting \( z = it \), we find that

\[
\Theta_\Lambda(z) = \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \left( \frac{i}{z} \right)^{n/2} \Theta_{\Lambda^*}(-1/z)
\]

whenever \( \Im z > 0 \), as desired. \( \square \)

If we set \( \Lambda = \mathcal{E}_8 \), then \( \Lambda^* = \mathcal{E}_8 \) as well, and we find that

\[
\Theta_{\mathcal{E}_8}(-1/z) = z^4 \Theta_{\mathcal{E}_8}(z).
\]

Furthermore, \( \mathcal{E}_8 \) is an even lattice, and hence the Fourier series (7) implies that

\[
\Theta_{\mathcal{E}_8}(z + 1) = \Theta_{\mathcal{E}_8}(z).
\]

These two symmetries are the most important properties of \( \Theta_{\mathcal{E}_8} \). For exactly the same reasons, the theta series of the Leech lattice \( \Lambda_{24} \) satisfies

\[
\Theta_{\Lambda_{24}}(-1/z) = z^{12} \Theta_{\Lambda_{24}}(z) \quad \text{and} \quad \Theta_{\Lambda_{24}}(z + 1) = \Theta_{\Lambda_{24}}(z).
\]

The mappings \( z \mapsto z + 1 \) and \( z \mapsto -1/z \) generate a discrete group of transformations of the upper half-plane, called the modular group. It turns out to be the same as the action of the group \( \text{SL}_2(\mathbb{Z}) \) on the upper half-plane by linear fractional transformations, but we will not need this fact except for naming purposes.

A modular form of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \) is a holomorphic function \( \varphi : \mathbb{H} \to \mathbb{C} \) such that \( \varphi(z + 1) = \varphi(z) \) and \( \varphi(-1/z) = z^k \varphi(z) \) for all \( z \in \mathbb{H} \), while \( \varphi(z) \) remains bounded as \( \Im z \to \infty \). (The latter condition is called being holomorphic at infinity, because it means the singularity there is removable.) It’s not hard to show that the weight of a nonzero modular form must be nonnegative and, the only modular forms of weight zero are the constant functions.

We have seen that \( \Theta_{\mathcal{E}_8} \) and \( \Theta_{\Lambda_{24}} \) satisfy the transformation laws for modular forms of weight 4 and 12, respectively, and it is easy to check that they are holomorphic at infinity. Thus, these theta series are modular forms.

There are a number of other well-known modular forms. For example, the Eisenstein series \( E_k \) defined by

\[
E_k(z) = \frac{1}{2\zeta(k)} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mz + n)^k}
\]

is a modular form of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \) whenever \( k \) is an even integer greater than 2 (while it vanishes when \( k \) is odd). The proofs of the required identities \( E_k(z + 1) = E_k(z) \) and \( E_k(-1/z) = z^k E_k(z) \) simply amount to rearranging the sum. Here \( \zeta \) denotes the Riemann zeta function, and \( 2\zeta(k) \) is a normalizing factor. The advantage of this normalization is that it leads to the Fourier expansion

\[
E_k(z) = 1 + \frac{2}{\zeta(1 - k)} \sum_{m=1}^{\infty} \alpha_{k-1}(m) e^{2\pi imz},
\]

where \( \alpha_{k-1}(m) \) is the sum of the \( (k - 1) \)-st powers of the divisors of \( m \) and \( \zeta(1 - k) \) turns out to be a rational number.

The notational conflict between the Eisenstein series \( E_k \) and the \( E_8 \) lattice is unfortunate, but both notations are well established. Fortunately, we will never need to set \( k = 8 \), and the context should easily distinguish between Eisenstein series and lattices.

Modular forms are highly constrained objects, which makes coincidences commonplace. For example, \( \Theta_{\mathcal{E}_8} \) is the same as \( E_4 \), because there is a unique modular form of weight 4 for \( \text{SL}_2(\mathbb{Z}) \) with constant term 1. Equivalently, for \( m = 1, 2, \ldots \) there are exactly \( 240 \sigma_3(m) \) vectors \( x \in \mathcal{E}_8 \) with \( |x|^2 = 2m \). The theta series \( \Theta_{\Lambda_{24}} \) is not an Eisenstein series, but it can be written in terms of them as

\[
\Theta_{\Lambda_{24}} = \frac{7}{12} E_4 + \frac{5}{12} E_6.
\]

More generally, let \( \mathcal{M}_k \) denote the space of modular forms of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \). Then \( \bigoplus_{k=0} \mathcal{M}_k \) is a graded ring, because the product of modular forms of weights \( k \) and \( \ell \) is a modular form of weight \( k + \ell \). This ring is isomorphic to a polynomial ring on two generators, namely \( E_4 \) and \( E_6 \). In other words, the set

\[
\left\{ E_4 E_6^j : i, j \geq 0 \text{ and } 4i + 6j = k \right\}
\]

is a basis for the modular forms of weight \( k \). In particular, there is no modular form of weight 2 for \( \text{SL}_2(\mathbb{Z}) \), because the weights of \( E_4 \) and \( E_6 \) are too high to generate such a form.

One cannot obtain a modular form of weight 2 by setting \( k = 2 \) in the double sum definition of \( E_k \). The problem is that rearranging the terms is crucial for proving modularity, but when \( k = 2 \) the series converges only conditionally, not absolutely. Instead, we can define \( E_2 \) using (8). That defines a merely quasimodular form, rather than an actual modular form, because one can show that \( E_2(-1/z) = z^2 E_2(z) - 6iz/\pi \) rather than \( z^2 E_2(z) \). This imperfect Eisenstein series will play a role in constructing the magic functions.

By default all modular forms are required to be holomorphic, but we can of course consider quotients that are no longer holomorphic. A meromorphic modular form is the quotient of two modular forms, and it is weakly holomorphic if it is holomorphic on \( \mathbb{H} \) (but not necessarily at infinity). Unlike the holomorphic case, there is an infinite-dimensional space of weakly holomorphic modular forms of each even weight, positive or negative. Allowing a pole at infinity offers tremendous flexibility.

On the face of it, modular forms seem to have little to do with the magic functions. In particular, it’s not clear what modular forms have to do with the radial Fourier transform in \( n \) dimensions. One hint that they may be relevant comes from the Laplace transform.
As we saw when we looked at theta series, Gaussians are a particularly useful family of functions for which we can easily compute the Fourier transform. It’s natural to define a function $f$ as a continuous linear combination of Gaussians via

$$f(x) = \int_0^\infty e^{-\pi|x|^2} g(t) \, dt,$$

where the weighting function $g(t)$ gives the coefficient of the Gaussian $e^{-\pi|x|^2}$. This formula is simply the Laplace transform of $g$, evaluated at $\pi|x|^2$.

Assuming $g$ is sufficiently well behaved, we can compute $\hat{f}$ by interchanging the Fourier transform with the integral over $t$, which yields

$$\hat{f}(y) = \int_0^\infty e^{-\pi|y|^2/t} g(t) \, dt = \int_0^\infty e^{-\pi|y|^2} t^{n/2-2} g(1/t) \, dt.$$  

In other words, taking the Fourier transform of $f$ amounts to replacing $g$ with $t \mapsto t^{n/2-2}g(1/t)$.

As a consequence, if $g(1/t) = \varepsilon t^{2-n/2}g(2t)$ with $\varepsilon = \pm 1$, then $\hat{f} = sf$. Thus, we can construct eigenfunctions of the Fourier transform by taking the Laplace transform of functions satisfying a certain functional equation. What’s noteworthy about this functional equation is how much it looks like the transformation law for a modular form on the imaginary axis. If we set $g(t) = \varphi(it)$, then the modular form equation $\varphi(-1/z) = z^k \varphi(z)$ with $z = it$ corresponds to $g(1/t) = t^{k^2}g(t)$. If $\varphi$ is a meromorphic modular form of weight $k = 2 - n/2$ that vanishes at $i\infty$ and has no poles on the imaginary axis, then $f$ is a radial eigenfunction of the Fourier transform in $\mathbb{R}^n$ with eigenvalue $i^k$.

Of course this is far from the only way to construct Fourier eigenfunctions, but it’s a natural way to construct them from modular forms. As stated here, it’s clearly not flexible enough to construct the magic functions, because it produces only one eigenvalue. If we take $n = 8$ and weight $k = 2 - n/2 = -2$, then $i^k = -1$, so we can construct a $-1$ eigenfunction but not a $+1$ eigenfunction for the same dimension. This turns out not to be a serious obstacle: there are many variants of modular forms (for other groups or with characters), and it’s not hard to produce eigenfunctions with both eigenvalues. However, there’s a much worse problem. If we build an eigenfunction this way, then there’s no obvious way to control the roots of the eigenfunction using the Laplace transform. Given that our goal is to prescribe the roots, this approach seems to be useless. What’s holding us back is that we have not taken full advantage of the modular form: we are using only the identity $\varphi(-1/z) = z^k \varphi(z)$, and not $\varphi(z+1) = \varphi(z)$.

**Viazovska gets around this difficulty by a bold construction**

Viazovska’s Proof

The fundamental problem with the Laplace transform approach in the previous section is that it seems to be impossible to achieve the desired roots. Viazovska gets around this difficulty by a bold construction: she simply inserts the desired roots by brute force, by including an explicit factor of $\sin^2(\pi|x|^2/2)$, which vanishes to second order at $|x| = \sqrt{2k}$ for $k = 1, 2, \ldots$ and fourth order at $x = 0$. In her construction for eight dimensions, both eigenfunctions have the form

$$\sin^2(\pi|x|^2/2) \int_0^\infty g(t)e^{-\pi|x|^2t} \, dt$$

for some function $g$.

One obvious issue with this approach is that $\sin^2(\pi|x|^2/2)$ vanishes more often than we would like. Specifically, it vanishes to fourth order when $x = 0$ and second order when $|x| = \sqrt{2}$, whereas we wish to have no root when $x = 0$ and only a first-order root when $|x| = \sqrt{2}$. To avoid this difficulty, the integral in (9) must have poles at 0 and $\sqrt{2}$ as a function of $|x|$, which cancel the unwanted roots. The integral will converge only for $|x| > \sqrt{2}$, but the function defined by (9) extends to $|x| \leq \sqrt{2}$ by analytic continuation.

Which choices of $g$ will produce eigenfunctions of the Fourier transform in $\mathbb{R}^8$? This is not clear, because the factor of $\sin^2(\pi|x|^2/2)$ disrupts the straightforward Laplace transform calculations from the end of the previous section. Instead, Viazovska writes the sine function in terms of complex exponentials and carries out elegant contour integral arguments to show that (9) gives an eigenfunction whenever $g$ satisfies certain transformation laws. Identifying the right conditions on $g$ is not at all obvious, and it’s the heart of her paper.

To get a $+1$ eigenfunction, Viazovska shows that it suffices to take $g(t) = t^2 \varphi(i/t)$, where $\varphi$ is a weakly holomorphic quasimodular form of weight 0 and depth 2 for $SL_2(\mathbb{Z})$. Here, a quasimodular form of depth 2 is a quadratic polynomial in $E_2$ with modular forms as coefficients, where $E_2$ is the Eisenstein series of weight 2. Recall that $E_2$ fails to be a modular form because of the strange transformation law $E_2(-1/z) = z^2 E_2(z) - 6iz/\pi$, but that functional equation works perfectly here.

To get a $-1$ eigenfunction, Viazovska shows that it suffices to take $g(t) = \psi(it)$, where $\psi$ is a weakly holomorphic modular form of weight $-2$ for a subgroup of $SL_2(\mathbb{Z})$ called $\Gamma(2)$ and $\psi$ satisfies the additional functional equation

$$\psi(z) = \psi(z+1) + z^2 \psi(-1/z).$$

We have not discussed modular forms for other groups such as $\Gamma(2)$, but they are similar in spirit to those for $SL_2(\mathbb{Z})$. In particular, the ring of modular forms for $\Gamma(2)$ is generated by two forms of weight 2, namely $\Theta_2^4$ (the fourth power of the theta series of the one-dimensional integer lattice) and its translate $z \mapsto \Theta_2^4(z + 1)$.

These conditions for $\varphi$ and $\psi$ are every bit as arcane as they look. It’s far from obvious that they lead to eigenfunctions, but Viazovska’s contour integral proof shows that they do. Even once we know that this method
gives eigenfunctions, it’s unclear how to choose $\varphi$ and $\psi$ to yield the magic eigenfunctions, or whether this is possible at all.

Fortunately, one can write down some necessary conditions, and then the simplest functions satisfying those conditions work perfectly. In particular, we can take

$$\varphi = \frac{4\pi(E_2 E_4 - E_6)^2}{5(E_6 - E_4)}$$

and

$$\psi = \frac{-32\Theta_2^2|T|(5\Theta_2^2 - 5\Theta_4^2 + 2\Theta_6^2)|T|}{15\pi\Theta_2^2(\Theta_2^2 - \Theta_4^2)^2},$$

where $f|_T$ denotes the translate $z \mapsto f(z + 1)$ of a function $f$.

Thus, to obtain the magic function for $E_8$ we set

$$f(x) = \sin^2(\pi|x|^2/2) \int_0^\infty (t^2 \varphi(i/t) + \psi(it)) e^{-\pi|x|^2t} dt$$

for the specific $\varphi$ and $\psi$ identified by Viazovska. Because the $\varphi$ and $\psi$ terms yield eigenfunctions of the Fourier transform, we find that

$$\hat{f}(y) = \sin^2(\pi|y|^2/2) \int_0^\infty (t^2 \varphi(it) - \psi(it)) e^{-\pi|y|^2t} dt.$$ 

The integral in the formula for $f(x)$ converges only when $|x| > \sqrt{2}$, but the one in the formula for $\hat{f}(y)$ turns out to converge whenever $|y| > 0$, because the problematic growth of the integrand cancels in the difference $t^2 \varphi(i/t) - \psi(it)$.

These formulas define Schwartz functions that have the desired roots, and one can check that $f(0) = \hat{f}(0) = 1$, but it’s not obvious that they satisfy the inequalities $f(x) \leq 0$ for $|x| \geq \sqrt{2}$ and $\hat{f}(y) \geq 0$ for all $y$, because there might be additional sign changes. In fact, these inequalities hold for a fundamental reason:

$$t^2 \varphi(i/t) + \psi(it) < 0 \quad \text{and} \quad t^2 \varphi(i/t) - \psi(it) > 0$$

for all $t \in (0, \infty)$. In other words, the inequalities already hold at the level of the quasimodular forms, with no need to worry about the Laplace transform except to observe that it preserves positivity. Note that the restriction of the inequality $f(x) \leq 0$ to $|x| \geq \sqrt{2}$ fits perfectly into this framework, because the integral in (10) diverges for $|x| < \sqrt{2}$ and thus we do not obtain $f(x) \leq 0$ there. All that remains is to prove the inequalities (11). Unfortunately, no simple proof of these inequalities is known at present, but one can verify them by reducing the problem to a finite calculation.

Thus, Viazovska’s formula (10) defines the long-sought magic function for $E_8$ and solves the sphere packing problem in eight dimensions. What about twenty-four dimensions? The same basic approach works, but choosing the quasimodular forms requires more effort. Fortunately, the conjectures by Cohn and Miller can be used to help pin down the right choices. Once the magic function has been identified, there are additional technicalities involved in verifying the inequality for $\hat{f}$, but these challenges can be overcome, which leads to a solution of the sphere packing problem in twenty-four dimensions.

**Future Prospects**

Nobody expects Viazovska’s proof to generalize to any other dimensions above two. Why just eight and twenty-four? At one level, we really don’t know why. Nobody has been able to find a proof, or even a compelling heuristic argument, that rules out similar phenomena in higher dimensions. We can’t even rule out the possibility that linear programming bounds might solve the sphere packing problem in every sufficiently high dimension, although that’s clearly ridiculous.

Despite our lack of understanding, the special role of eight and twenty-four dimensions aligns with our experience elsewhere in mathematics. Mathematics is full of exceptional or sporadic phenomena that occur in only finitely many cases, and the $E_8$ and Leech lattices are prototypical examples. These objects do not occur in isolation, but rather in constellations of remarkable structures. For example, both $E_8$ and the Leech lattice are connected with binary error-correcting codes, combinatorial designs, spherical designs, finite simple groups, etc.

Each of these connections constrains the possibilities, especially given the classification of finite simple groups, and there just doesn’t seem to be room for a similar constellation in higher dimensions.

Instead, solving the sphere packing problem in further dimensions will presumably require new techniques. One particularly attractive case is the $D_4$ root lattice, which is surely the best sphere packing in $\mathbb{R}^4$. This lattice shares some of the wonderful properties of $E_8$ and the Leech lattice, but not enough for the four-dimensional linear programming bound to be sharp. It would be a plausible target for any generalization of this bound, and in fact such a generalization may be emerging.

Building on work of Schrijver, Bachoc and Vallentin, and other researchers, de Laat and Vallentin have generalized linear programming bounds to a hierarchy of semidefinite programming bounds [12]. Linear programming bounds are the first level of this hierarchy, which means that $E_8$ and the Leech lattice have the simplest possible proofs from this perspective. What about $D_4$? Perhaps this case can be solved at one of the next few levels of the hierarchy. Much work remains to be done here, and it’s unclear what the prospects are for any particular dimension, but it is not beyond hope that four dimensions could someday join eight and twenty-four among the solved cases of the sphere packing problem.
Maryna Viazovska solved the sphere packing problem in $\mathbb{R}^8$ while holding the Dirichlet Postdoctoral Fellowship at the Berlin Mathematical School and the Humboldt University of Berlin, and she will begin a faculty position at the École Polytechnique Fédérale de Lausanne in 2017. She received her PhD in 2013 at the University of Bonn under the supervision of Don Zagier, with a dissertation entitled Modular functions and special cycles. In 2016, she was awarded the Salem Prize "for her breakthrough work on densest sphere packings in dimensions 8 and 24 using methods of modular forms."

Although the solution of the sphere packing problem in $\mathbb{R}^8$ was Viazovska's first publication on sphere packing per se, she has long had an interest in discrete geometry and optimization. While she was still in graduate school, Viazovska and her coauthors Andrei Bondarenko and Danylo Radchenko published an important paper on the theory of spherical designs, namely their 2013 *Annals* paper “Optimal asymptotic bounds for spherical designs.” This paper analyzes how uniformly one can distribute points over the surface of a sphere. A spherical $t$-design on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ is a finite subset $D$ of $S^{n-1}$ such that for every polynomial $p: \mathbb{R}^n \to \mathbb{R}$ of total degree at most $t$, the average of $p$ over $D$ is the same as its average over the entire sphere $S^{n-1}$. In other words, the distribution of points cannot be distinguished from the uniform distribution by averaging a polynomial of degree at most $t$. Even the existence of spherical designs with arbitrarily large $t$ on a fixed sphere $S^{n-1}$ is not obvious, and was first proved by Seymour and Zaslavsky in 1984. Viazovska and her coauthors showed that when $n$ is fixed, spherical $t$-designs exist with at most $O(t^n)$ points, which matches a 1977 lower bound of Delsarte, Goethals, and Seidel. 

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**References**


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**Higher-Dimensional Sphere Packing**

Learn about recent breakthroughs in our understanding of hyperspheres in the first episode of PBS’s *Infinite Series*, a show written and hosted by 2016 AMS Mass Media Fellow Kelsey Houston-Edwards that tackles the mystery and the joy of mathematics. [https://www.youtube.com/watch?v=cI16wigZK0w/](https://www.youtube.com/watch?v=cI16wigZK0w/)
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In 1979 Richard Stanley made the following conjecture: Every Cohen-Macaulay simplicial complex is partitionable. Motivated by questions in the theory of face numbers of simplicial complexes, the Partitionability Conjecture sought to connect a purely combinatorial condition (partitionability) with an algebraic condition (Cohen-Macaulayness). The algebraic combinatorics community widely believed the conjecture to be true, especially in light of related stronger conjectures and weaker partial results. Nevertheless, in a 2016 paper [DGKM16], the three of us (Art, Carly, and Jeremy), together with Jeremy’s graduate student Bennet Goeckner, constructed an explicit counterexample. Here we tell the story of the significance and motivation behind the Partitionability Conjecture and its resolution. The key mathematical ingredients include relative simplicial complexes, nonshellable balls, and a surprise appearance by the pigeonhole principle. More broadly, the narrative of the Partitionability Conjecture highlights a general theme of modern algebraic combinatorics: to understand discrete structures through algebraic, geometric, and topological lenses.

History and Motivation
As basic discrete building blocks, simplicial complexes arise throughout mathematics, whether as surface meshes, simplicial polytopes, or abstract models of multiway relations. An abstract simplicial complex $\Delta$ is simply a family of subsets of a finite vertex set $V$ that is closed under taking subsets: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of $\Delta$, which are called faces or simplices, admit a natural partial order by containment, and we typically regard a simplicial complex as equivalent to its face poset (partially ordered set of faces). The maximal faces are called facets. A simplicial complex can be realized topologically by representing each face $\sigma$ by a geometric simplex of dimension $|\sigma| - 1$, as in Figure 1.

For this reason, the standard convention is to write the dimension of $\Delta$ (that is, the largest dimension of a face of $\Delta$) as $d - 1$. Here we are concerned with the problem of decomposing the face poset into intervals, i.e., subsets of the form $[\sigma, \tau] := \{\rho \in \Delta \mid \sigma \subseteq \rho \subseteq \tau\}$.

The $f$-vector of a complex records the face numbers, dimension by dimension. Specifically, the $f$-vector is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i$ is the number of $i$-dimensional faces in $\Delta$. Understanding the $f$-vectors of various classes of complexes is a major area of algebraic combinatorics; the standard source is [Sta96]. The Partitionability Conjecture began as an attempt to gain a combinatorial understanding of the face numbers of Cohen-Macaulay complexes.

The classical problem in this subject is to characterize the $f$-vectors of convex polytopes. For example, $(1, 5, 9, 6)$ is realized by the polytope in Figure 1, and it is not hard to convince yourself that $(1, 5, 9, 87)$ is impossible, but what about $(1, 15, 36, 12)$? What can be said about this combinatorial invariant of inherently geometric objects?
This problem dates back at least to Euler and remains unsolved in general, though many special cases are understood.

It often turns out to be more convenient to work with a linear transformation of the $f$-vector called the $h$-vector, $h(\Delta) = (h_0, \ldots, h_d)$, defined by the identity

$$\sum_{i=0}^d h_i x^{d-i} = \sum_{i=0}^d f_{i-1} (x-1)^{d-i}.$$ 

The formula can easily be inverted to express the $f$-vector in terms of the $h$-vector, so the two carry the same information. The $h$-vector is a key thread in our narrative, and we will see that it carries simultaneous interpretations in algebra, combinatorics, and topology. Many relations on the face numbers, particularly those arising in algebraic contexts, can be expressed naturally in terms of the $h$-vector. A spectacular example is the Dehn-Sommerville equations, which state that $h_i = h_{d-i}$ for every simplicial $d$-polytope. (This symmetry reflects Poincaré duality on the associated toric variety.) What do the $h$-numbers count?

In general they need not even be positive, but for certain special classes such as shellable simplicial complexes they admit an elementary combinatorial interpretation. A shellable complex can be assembled facet by facet so that the new faces added at each step form an interval in the face poset; the number $h_i$ counts the intervals of height $d-i$. The boundary complexes of convex polytopes are shellable, as proven by a beautiful geometric argument of Heinz Bruggesser and Peter Mani in 1970. Shellability of polytopes is an essential ingredient in a cornerstone of the theory of face numbers, Peter McMullen’s upper bound theorem, which describes an upper bound that simultaneously maximizes all entries of the $h$-vectors of polytopes.

A condition weaker than shellability is partitionability, which first appears in the theses of Michael O. Ball and J. Scott Provan. Ball had been working on network reliability, and Provan on diameters of polytopes. A partitionable complex can be decomposed into disjoint intervals, each one topped by a facet. That is, a partitioning matches each facet to one of its subfaces (possibly itself), namely, the minimum element of the corresponding interval. Unlike a shelling, there is no restriction on how the intervals of a partitioning fit together. On the other hand, when $X$ is partitionable, the number $h_i$ counts the intervals of height $d-i$, just as for a shellable complex. (See Figure 2.) Hence the $h$-vectors of partitionable complexes have a simple combinatorial interpretation.

Constructible and Cohen-Macaulay simplicial complexes also have nonnegative $h$-vectors. Constructibility is a purely combinatorial recursive condition, while Cohen-Macaulayness arises in commutative algebra. Cohen-Macaulay rings have long enjoyed great importance in algebra and algebraic geometry; Mel Hochster is often quoted as saying that “life is really worth living” in a Cohen-Macaulay ring. Their significance in combinatorics, via what is now known as Stanley-Reisner theory, was established by Hochster, Gerald Reisner, and Richard Stanley in the early 1970s. In this theory combinatorial questions about a complex are translated to algebraic questions about a ring. Specifically, the Stanley-Reisner ring (or face ring) of a simplicial complex $\Delta$ on the vertex set $\{1, 2, \ldots, n\}$ (over a field $k$) is defined as $k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta$ is the monomial ideal generated by nonfaces of $\Delta$. The construction is bijective: every quotient of a polynomial ring by a square-free monomial ideal gives rise to a simplicial complex. The two sides of the correspondence are tightly linked: for instance, the $h$-vector of $\Delta$ corresponds naturally to the Hilbert series of $k[\Delta]$ via the formula

$$\sum_{i \geq 0} (\dim k[I_\Delta]) t^i = \sum_j h_j(\Delta) \frac{t^j}{(1-t)^d},$$

where $k[I_\Delta]_i$ denotes the $i$th graded piece of $k[\Delta]$.

Two fundamental algebraic invariants of a commutative ring $R$ are its (Krull) dimension and its depth. In all
cases \( \dim R \geq \text{depth} R \), and if equality holds, then the ring is \textit{Cohen-Macaulay}. Likewise, a simplicial complex is Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay. In the setting of Stanley-Reisner theory, the dimension of \( k[\Delta] \) is simply \( d \), but depth is a subtler invariant of \( \Delta \), so it is not easy to translate this definition from algebra to combinatorics. Fortunately, Reisner found an equivalent criterion for Cohen-Macaulayness in terms of simplicial homology: \( \Delta \) is Cohen-Macaulay if and only if for every face \( \sigma \), the subcomplex \( \text{link}_\Delta(\sigma) = \{ \tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta \} \) has the homology type (over \( k \)) of a wedge of \((d - 1) - \dim \sigma \) spheres. Reisner’s criterion is the working definition of the Cohen-Macaulay property for many combinatorialists.

The Cohen-Macaulay property provided the algebraic bridge between topology and combinatorics in Stanley’s celebrated extension of McMullen’s upper bound theorem from polytopes to simplicial spheres. Specifically, Cohen-Macaulayness played the role in bounding the \( h \)-vectors of spheres that shellability did for convex polytopes. Furthermore, the hierarchy of strict implications

\[
\text{shellable} \implies \text{constructible} \implies \text{Cohen-Macaulay}
\]

was well known since Hochster’s 1973 work on face rings. (Of these properties, only Cohen-Macaulayness is topological, as proven by James Munkres in 1984.) On the other hand, Stanley proved in 1977 that the \( h \)-vectors of Cohen-Macaulay, shellable, and constructible complexes are precisely the same. The Partitionability Conjecture sought to explain this equality: perhaps partitionability sat at the base of the hierarchy above.

**Conjecture 1** (The Partitionability Conjecture). Every Cohen-Macaulay simplicial complex is partitionable.

The Partitionability Conjecture is one of many interrelated conjectures about the structure of simplicial complexes. Adriano Garsia made the same conjecture in 1980 in the more restricted setting of order complexes of Cohen-Macaulay posets. Motivated by the theory of algebraic shifting, Gil Kalai conjectured that every simplicial complex admits a more general decomposition into intervals. In 1993 Stanley conjectured that every \( k \)-acyclic complex—one for which \( \text{link}_\Delta(\sigma) \) is acyclic for every face \( \sigma \) of dimension less than \( k \)—admits a partitioning into intervals of rank at least \( k + 1 \), with each interval topped by a face of dimension at least depth \( \Delta \). Each of Kalai’s and Stanley’s conjectures, if true, would imply the Partitionability Conjecture. Meanwhile, Masahiro Hachimori proved in 2000 that a certain strengthening of constructibility \textit{does} in fact imply partitionability. The Partitionability Conjecture was widely believed to be true within the combinatorics community, and the works of Garsia, Kalai, Stanley, and Hachimori provided many tantalizing approaches, both combinatorial and algebraic.

The Partitionability Conjecture gained additional significance in the study of \textit{Stanley depth}, a purely combinatorial counterpart to the depth of a graded module over a polynomial ring, introduced by Stanley in 1982. This invariant has attracted substantial attention in combinatorial commutative algebra over the last twenty years; the article [PSFTY09] by Mohammed Pournak, Seyed A. Seyed Fakhari, Massoud Tousi, and Siamak Yassemi in the Notices is an excellent introduction. The focal point of this area has been Stanley’s conjecture that the Stanley depth of every module is an upper bound for its (algebraic) depth. Jürgen Herzog, Ali Soleyman Jahan, and Yassemi proved in 2008 that Stanley’s Depth Conjecture implies the Partitionability Conjecture.

On the other hand, as with so many areas of mathematics, combinatorics is full of surprises, and the study of simplicial complexes is rife with counterintuitive examples. Shellability does not depend only on the topology of the underlying space: Mary Ellen Rudin famously constructed a nonshellable triangulation of the 3-dimensional ball in 1955. There are also nonshellable 3-spheres; the first was constructed in 1991 by William Lickorish. It can get even more wild: obstructions to shellability and constructibility can take the form of nontrivial knots embedded on a sphere. So perhaps the Partitionability Conjecture would also turn out to be false.

**Building a Counterexample**

We started by trying to prove the Partitionability Conjecture, not to disprove it. Our previous joint work on simplicial and cellular trees and follow-up work of Carly and Olivier Bernardi had revealed unexpected connections to the Cohen-Macaulay condition and to Stanley’s conjectures about decompositions of simplicial complexes. At first, we had hoped to use the theory of trees to approach partitionability. However, we eventually arrived at two turning points, key realizations that persuaded us to look for a counterexample instead of a proof.

When we first began to investigate partitionability, we wanted to experiment with a Cohen-Macaulay complex that is \textit{not} shellable, because as we have seen, every shelling induces a partitioning. However, many Cohen-Macaulay complexes that arise in combinatorics, such as convex simplicial spheres and order complexes of certain posets, are naturally shellable. Rudin’s nonshellable 3-ball would have met our requirements, but we chose the one constructed by Günter Ziegler in 1998 [Zie98], since it has fewer faces and is thus easier to work with. It has \( f \)-vector

\[
(1, 10, 38, 50, 21)
\]

and is known to be partitionable.

Almost any approach to working with decompositions of simplicial complexes also requires working with \textit{relative simplicial complexes}. A relative simplicial complex \( Q \) on vertex set \( V \) is a family of subsets of \( V \) that is convex with respect to the natural partial order by inclusion: if \( \rho \) and \( \tau \) are faces of \( Q \) with \( \rho \subseteq \sigma \subseteq \tau \), then \( \sigma \) is a face of \( Q \) as well. Equivalently, the face poset of \( Q \) is of the form \( X \setminus A \), where \( X \) is a simplicial complex and \( A \) is a subcomplex (see Figure 3). The concept of a pair of spaces is familiar from algebraic topology; in particular, \( Q \) can be regarded as a combinatorial model of the quotient space \( X/A \). The combinatorics of simplicial complexes, including Cohen-Macaulayness and partitionability, carries over well to the relative setting.
Furthermore, the ability to remove subcomplexes makes it the right setting to study decompositions.

Our first turning point arose from one of our experiments on Ziegler’s 3-ball. We used a simple greedy algorithm to remove intervals one at a time, producing a sequence of smaller and smaller relative complexes, each of which remains Cohen-Macaulay (by a Mayer-Vietoris argument). In one trial, the final output was the relative complex \( Q_5 \), whose face poset is shown in Figure 4. It is not hard to check directly that \( Q_5 \) is nonpartitionable. Therefore, the Partitionability Conjecture is false for relative simplicial complexes.

We next wanted to find a general method for turning a relative counterexample \( Q = (X, A) \), such as \( Q_5 \), into
a nonrelative counterexample. Our first idea was to construct a complex $C_2$ by gluing two isomorphic copies of $X$ together along the common subcomplex $A$, as in Figure 5; this complex is Cohen-Macaulay (again, by a Mayer-Vietoris argument). The single extra copy of $Q$ does not, however, create an obstruction to partitionability, since it may be possible to partition $C_2$ by pairing faces of $A$ with facets in different copies of $Q$.

But now the pigeonhole principle comes into play. Construct a Cohen-Macaulay complex $C_N$ by gluing $N$ copies of $X$ together along their common subcomplex $A$. Suppose that $N$ is greater than the total number of faces of $A$. Then, in every partitioning of $C_N$, there must be at least one copy of $Q$ whose faces are matched with faces in that same copy. That is, $Q$ is partitionable! This contradiction implies that $C_N$ cannot be partitionable. Here is a formal statement:

**Theorem 2.** Let $Q = (X, A)$ be a relative complex such that

(i) $X$ and $A$ are Cohen-Macaulay,
(ii) $A$ has codimension at most 1,
(iii) every minimal face of $Q$ is a vertex, and
(iv) $Q$ is not partitionable.

Let $k$ be the total number of faces of $A$, let $N > k$, and let $C = C_N$ be the simplicial complex constructed from $N$ disjoint copies of $X$ identified along the subcomplex $A$. Then $C$ is Cohen-Macaulay and not partitionable.

Condition (iii) is a technical requirement to ensure that $C$ is a simplicial complex, not merely a cell complex. For instance, the relative counterexample $Q_5$ does not satisfy condition (iii).

It was not clear that Theorem 2 would produce an actual counterexample: finding a relative complex satisfying the conditions of the theorem might be just as difficult as proving or disproving Conjecture 1 through other means. But by considering the special case that $A$ is a single facet, we realized that the conjecture implied a stronger version of itself: If $X$ is a Cohen-Macaulay complex, then $X$ has a partitioning including the interval $[\emptyset, \sigma]$ for any facet $\sigma$. This flexibility struck us as suspiciously strong: for example, the corresponding statement fails for shellability. Here was our second turning point.

Now we were determined to find a complex satisfying the conditions of Theorem 2. Once again, Ziegler’s ball $Z$ provided the answer. Consider the subcomplex $B$ consisting of all faces supported on the vertex set $\{0, 2, 3, 4, 6, 7, 8\}$, so that the minimal faces of the relative complex $Q = Z/B$ are vertices 1, 5, and 9. We proved that $B$ is Cohen-Macaulay and that $Q$ is not partitionable. These were exactly the ingredients we needed to construct a counterexample!

We can describe $Q$ most simply as the relative complex $(X, A)$, where $X$ is the smallest simplicial complex containing $Q$. In fact $X$ is a 3-ball with 10 vertices and 14 tetrahedra (see Figure 6), and $A$ is a topological disk on the boundary of $X$, with $f$-vector $(1, 7, 11, 5)$. Therefore, gluing $(1 + 7 + 11 + 5) + 1 = 25$ copies of $X$ along $A$ produces a counterexample $C_{25}$ to the Partitionability Conjecture. The complex $C_{25}$ is Cohen-Macaulay (in fact, constructible) for the usual Mayer-Vietoris reasons, but it is nonpartitionable by the pigeonhole argument.

In fact, the argument that $Q$ is nonpartitionable revealed that the full power of Theorem 2 was actually not necessary. Gluing just three copies of $X$ together along $A$ gives a nonpartitionable complex $C_3$. The complex $C_3$
is the smallest counterexample we know: its $f$-vector is $f(A) + 3f(Q) = (1, 16, 71, 98, 42)$. It is contractible but not homeomorphic to a ball. (See Figure 7.)

The counterexample disproves several stronger conjectures, as discussed above: that constructible complexes are partitionable, Kalai’s conjecture about partitions indexed by algebraic shifting, and the Depth Conjecture. Yet the counterexample is not the end of the story, and many questions remain. For instance, does the Partitionability Conjecture hold in dimension two? Are simplicial balls partitionable? What about Garsia’s version of the conjecture for Cohen-Macaulay posets? What are the consequences for the theory of Stanley depth? What does the $h$-vector of a Cohen-Macaulay complex count?

The story of the Partitionability Conjecture has many facets. Shellability, partitionability, constructibility, and the Cohen-Macaulay property come from different but overlapping areas of mathematics: combinatorics, commutative algebra, topology, and discrete geometry. The hierarchy of these structural properties turned out to be more complicated than we had anticipated, just as Rudin’s and Ziegler’s examples demonstrate that even the simplest spaces can have intricate combinatorics. Even though statements like the Partitionability Conjecture can seem too beautiful to be false, we should remember to keep our minds open about the mathematical unknown—the reality might be quite different, with its own unexpected beauty.

References


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Figure 6 is courtesy of Jennifer Wagner.
The photo of Caroline Klivans, Art Duval, and Jeremy Martin is courtesy of Matt Macauley.

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COMMUNICATION

What Does Active Learning Mean For Mathematicians?

Benjamin Braun, Priscilla Bremser, Art M. Duval, Elise Lockwood, and Diana White

In August 2016 fifteen presidents of member societies of the Conference Board of the Mathematical Sciences (CBMS), an umbrella organization consisting of the American Mathematical Society and sixteen other professional societies in the mathematical sciences, released a statement on active learning [1] with the following call to action:

We call on institutions of higher education, mathematics departments and the mathematics faculty, public policy-makers, and funding agencies to invest time and resources to ensure that effective active learning is incorporated into post-secondary mathematics classrooms.

This call is part of a broad movement to increase the use of active and student-centered teaching techniques across science, technology, engineering, and mathematics (STEM) disciplines. A landmark 2014 meta-analysis published in the Proceedings of the National Academy of Sciences [2] highlighted the efficacy of active learning techniques across STEM disciplines. In mathematics specifically, a comprehensive study of student outcomes for inquiry-based learning [3] has further established that active learning methods have a strong positive impact on women and members of other underrepresented groups in mathematics. This movement extends beyond the academic community—for example, at the federal level the White House STEM-for-All initiative [4] includes active learning as one of its three areas of emphasis for the 2017 budget.

While robust support from education researchers, funding agencies, public policymakers, and institutions is a critical component of effective active learning implementation, at the end of the day these techniques and methods are put into practice by mathematics faculty leading classes of students. Thus, mathematics faculty need to be well informed about active learning and related topics. Our goal in this article is to provide a foundation for productive discussions about the use of active learning in postsecondary mathematics. We will focus on topics that frequently arise at the department level, namely: definitions of active learning, examples of active learning techniques and environments used by individual faculty or teams of faculty, things to expect when using active learning methods, and common concerns. An extended discussion of these issues and a substantial bibliography can be found in the six-part series on active learning [5] written by the authors for the AMS blog On Teaching and Learning Mathematics.

What Is Active Learning?

A frequently asked question is, what is active learning? We base our discussion on the definition given in the CBMS statement [1]: Active learning refers to classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote higher-order thinking. Using a broad definition such as this increases the risk of faculty, administrators, and other stakeholders “talking past” one another, as much is left to the imagination regarding what actually happens with such methods. However, it also acknowledges that active learning can and does involve a wide variety of specific activities in diverse settings, with instructors of varied
Another approach to defining active learning is more useful in local settings, such as internal department discussions or conversations between department leaders and administrators. In this approach, one focuses discussion on a specific course and defines active learning as a task that students will complete during class time. This helps ensure that everyone in the discussion has a similar vision of what methods are actually being proposed or discussed in the context of explicit course goals and student-learning outcomes.

Examples of Active Learning Techniques and Environments

In contemporary college and university courses, lecturing remains the dominant teaching technique used by mathematics faculty. While active learning and lecture are sometimes viewed as two diametrically opposed teaching options, this is a misconception, as the following examples illustrate. We begin with examples that primarily involve individual faculty, and we end with examples that require collective buy-in and support from faculty, departments, and institutions.

Think-Pair-Share. One of the simplest examples of an active learning technique suitable for use in lectures is “think-pair-share.” In this technique, the instructor provides students with a short task such as doing a computation, completing a step in a proof, generating one or more examples, or forming a hypothesis or conjecture. After providing the students with two to three minutes of time to independently consider the task (“think”), students take two minutes to compare their answers with other students sitting nearby (“pair”). Finally, some or all of the students are asked to share their answers with the entire class (“share”). Giving students time to think about and discuss mathematics mid-lecture encourages their active participation in the class. This task has no implications for departments or institutions and serves as an effective comprehension check in which students are able to refocus their attention during a lecture.

Classroom Response Systems (“Clickers”). In addition to think-pair-share, there are many related examples of “classroom voting” systems and techniques that can be used to increase student engagement. These systems are often useful when scaling up think-pair-share and related techniques to large-lecture environments. While some systems are entirely Web and mobile phone based, others require students to rent or purchase a response device. Thus, depending on the choice of system, there can be implications for departments when clicker systems are widely used, and often it is helpful to implement clicker use with a team of faculty rather than individually.

Inverted (or “Flipped”) Classes. In an inverted (or “flipped”) classroom environment, instructor presentations of basic definitions, examples, proofs, and heuristics are provided to students in videos or in assigned readings that are completed prior to attending class. As a result, class time becomes available for active learning tasks that directly engage students. The type of task that instructors use during this time ranges from using think-pair-shares with complex problems or examples to having students work in small groups on a sequenced activity worksheet with occasional instructor or teaching assistant feedback. The inverted model of teaching has been used as the structure for entire courses, as an occasional event for handling topics that are less amenable to lecture presentations, as the basis for review sessions or problem-solving sessions, and more. Depending on the method used for flipping individual class periods or entire courses, department and/or institutional support (in the form of technical assistance) may often be key ingredients in this model.

Inquiry-Based Learning. One of the most well-known active learning methods in mathematics is inquiry-based learning (IBL). In IBL courses, class time is spent with students working on problem sets individually or in groups, presenting solutions and/or proofs to the class, and receiving feedback from peers and faculty. IBL courses are not based on pure, unguided student discovery; instead, faculty design a series of carefully scaffolded (i.e., sequenced in a structured way) activities, some for individuals, some for pairs, some for small groups, and some for the whole class, including mini-lectures as appropriate. Because faculty using IBL need to develop facility with a range of teaching strategies and need to develop
familiarity with many “teaching moves” that are not typically used in lecture environments, IBL is a more ambitious active learning environment. There are various opportunities for professional development with IBL, including the workshops offered by the Academy for Inquiry-Based Learning at [www.inquirybasedlearning.org](http://www.inquirybasedlearning.org).

**Math Emporium.** The math emporium model uses a large room filled with computer workstations at which students progress through self-paced online courses. Unlike inverted classes, many emporium models do not include a lecture component at all, and most have been developed to handle remediation issues and low-level courses such as developmental mathematics and college algebra. An emporium usually has tables at which students can work collaboratively and is staffed by a large number of teaching assistants and tutors. Because the work of students is self-paced, students spend most of their time actively engaging with course content through a range of tasks. Because of the significant investment in classroom space and technological resources required, a math emporium is typically launched as a collaborative venture among faculty, departments, and administrators.

**Modeling and Computer Laboratories.** Modeling is a rich arena for increasing student engagement, one that is often augmented with computer labs. Since the 1990s many mathematics courses have included computer lab activities for exploration using programs such as Mathematica, Maple, MATLAB, and Sage. Recent years have seen a growth in the number of support networks for faculty using lab and modeling components, such as the SIMIODE.org project for differential equations. The 2016 SIAM report *Guidelines for Assessment and Instruction in Mathematical Modeling Education* provides examples of modeling activities across the undergraduate curriculum that actively engage students and discusses related issues such as assessment. Incorporating modeling and laboratory components into postsecondary courses can be done at many levels, ranging from stand-alone activities in a single class to program-wide implementations supported at the institutional level.

**Things to Expect with Active Learning**
Faculty using active learning for the first time need a realistic expectation of what impact these techniques will have. Because there are so many different active learning techniques and because different techniques often influence students in unique ways, it is not always possible to clearly say what will happen when we use a new active learning method. However, there do seem to be a few things faculty can typically expect. Here are five of them.

- **Expect to gain insights about your students.** For many faculty using active learning, these techniques inspire richer discussions with students and provide a window into the reality of students’ mathematical experiences. This allows faculty to be more responsive to students’ misunderstandings, which in turn causes students to feel more supported in the course, frequently leading to increased engagement. Even in 200-student lectures, where student-faculty dialogue might be heavily moderated by clicker systems, faculty often report that active learning methods provide a clearer sense of what their students understand than with traditional lecture alone.

- **Expect your students to surprise you.** Active learning provides opportunities for faculty-student interaction not present in courses focused on direct instruction.
Active learning methods can reach and excite some students who might not typically be vocal or engaged in class—students who are quiet and reserved by nature frequently demonstrate their full potential when provided with the right opportunity. On the other hand, active learning methods can uncover deep misconceptions about mathematics, even from straight-A students, that homework and exams do not reveal. Further, students often respond to active learning tasks with interesting observations and thought-provoking questions, infusing standard courses like calculus with fresh energy.

**Expect resistance from some students.** For many reasons, it is common for some students to resist active learning methods, especially at the beginning of a course. Some students are not particularly interested in mathematics and do not want to engage at a deeper level. Other students have experienced significant success in traditional mathematics courses and feel threatened by an unfamiliar environment. With all students, instructors need to clearly articulate the value of the active learning methods they use and maintain high expectations for student participation and engagement. Often, students who are initially resistant find themselves surprised at the end of a course by how much they appreciate active learning.

**Expect to learn from your mistakes.** Much like learning mathematics, learning how to effectively use a new pedagogical technique, especially one of the more complex active learning techniques, involves a process of persistence and error-correction through small failures. Mathematics faculty need to be prepared to start small and develop gradually and consistently. Almost every faculty member the authors have spoken with who uses active learning describes the development of his or her teaching as a sequence of mixed successes and failures. If you are implementing an active learning technique that is new to you, it is often helpful to first discuss with your department chair how teaching in that course will be evaluated for merit reviews. Many colleges have policies to support faculty as they build experience with new teaching techniques, especially if the techniques are evidence based.

**Expect long-term impacts.** When used in combination with a foundation of good general teaching practices, active learning often has a particularly positive impact on student persistence and sense of belonging in mathematics. This in turn can lead students to be more engaged in their studies and pursue more mathematics over the long term. Because many active learning techniques emphasize communication and collaboration, faculty often report that using these techniques is a catalyst for building strong student communities. These peer networks persist through subsequent courses, contributing to students’ experience throughout their mathematical studies. Many of these impacts of active learning become fully visible only after a course ends and thus can be hard to measure or even identify with standard course evaluation instruments.

**Common Concerns about Active Learning**

While active learning has many advocates among mathematicians, there are also responsible teachers who have reasonable concerns about active learning methods. We address four of them here.

**How will students learn the mathematics if we do not clearly tell them everything about it?** The historical dominance of the lecture format rests on the belief that learning occurs as a result of transmission of information from instructor to student and that students learn by a process of taking in bits of information that their instructors say or write. Further, because of our passion and love for mathematics, a natural human impulse is for mathematics faculty to tell students about the ways we have come to understand our discipline, to shed light on the subtleties that surround most mathematical ideas, and to explain the fundamental insights of our field. Our common experience, supported by research, demonstrates that learning is not this simple. For example, almost every teacher has experienced telling a student a certain mathematical fact—such as \((a+b)^2 \neq a^2 + b^2\)—only to have them demonstrate on a test that they have not learned it. Such experiences suggest that it is not enough for students simply to be told information if we want to produce deep and meaningful learning. Thus, the key is to find an effective balance between direct instruction and active learning, wherein instructors provide guidance through a combination of explanations and active learning tasks.

**What if I can't cover the same amount of material?**

Direct instruction alone can be an efficient way of getting through material. However, the example of students not knowing that \((a+b)^2 \neq a^2 + b^2\) should not be far from our minds: lecturing in order to cover more material is not always effective for students. By exclusively considering course content coverage and responding to content coverage with telling, we risk forgetting the many other elements of student learning that active learning addresses, such as the cognitive goals for students outlined in the 2015 MAA *CUPM Curriculum Guide* [6], including:

- recognize and make mathematically rigorous arguments,
- communicate mathematical ideas clearly and coherently both verbally and in writing,
- work creatively and self-sufficiently,
- assess the correctness of solutions,
- create and explore examples,
- carry out mathematical experiments, and
• devise and test conjectures.

In addition to the recognition that content topics are not the exclusive subject of coverage, recent research suggests that coverage of material is less important for student persistence and achievement in mathematics than the use of teaching techniques that address these other types of learning goals.

How do I know if I’m doing a good job with my teaching? The crafting of rich lectures contributes to mathematicians’ feelings of efficacy in their discipline. There are, however, a number of other ways in which teachers may gain efficacy while balancing traditional lecture with active learning in their classrooms. These include activities such as choosing problems, predicting student reasoning, generating and directing discussion, pushing students for high-quality explanations, asking for questions that extend student knowledge, and obtaining immediate feedback from students regarding what they just learned. Reflecting in this manner shifts the way we measure our own teaching away from the quality of our presentations and toward the quality of the tasks we provide students. Further, many mathematicians who implement active learning report that they have a deeper understanding of student progress and can observe changes in students more clearly than in their previous courses.

I didn’t need active learning, why do my students? Although this is changing, many mathematicians have not personally experienced undergraduate teaching environments that include active learning components. Thus, for many mathematicians and graduate students, their first experience with active learning techniques will be as teachers rather than as students. However, we should be careful when comparing our own experiences with those of our students; as Carl Lee [7] has written:

I often engaged in math classes at a high cognitive level merely as a result of a teacher’s direct instruction (“lecture”). As a teacher I quickly learned that I engaged few of my students by this process. Not all developed their “mathematical habits of mind” or “mathematical practices” through my in-class lectures and out-of-class homework (often worked on individually). I now better appreciate the significant role of personal context and informal education in the development of students’ capacity.

Research [2], [3] suggests that active learning has a strong positive impact on a wide range of students, not only those who enter our courses ready to independently engage with math at a high cognitive level. That research also suggests that active learning does not harm, and may further benefit, already high-achieving students. Reflecting on our own educations, the authors agree that we would likely have built a firmer mathematical foundation had we experienced more active learning environments and that active learning would have prompted us an earlier understanding of mathematics as an inquiry-based discipline.

Conclusion

New instructional techniques cannot be effectively implemented overnight. We must start small and develop gradually and consistently, ideally implementing changes as part of a team that can provide feedback and support. For experienced faculty, this is something we need to do not only for ourselves but with an eye toward training the next generation of mathematicians.

Those of us who work at master’s- and doctoral-granting institutions should provide graduate students with training and experience in using active learning techniques, whether as part of recitation duties or in situations where graduate students serve as independent instructors for courses. Given the many demands of graduate school, it is unreasonable to expect that every graduate student in math will emerge as an expert teacher, but we should provide as many opportunities as possible for graduate students to build their skill in using a combination of direct instruction and active learning techniques. For early-career faculty, long-standing professional development programs such as Project NExT provide a valuable service to the mathematical community.

There is a fundamental way in which our training as mathematicians can help us develop as teachers: mathematicians are expert problem solvers. As a community of mathematicians and educators, we are in the process of solving the problem of how best to teach mathematics, and we are working together toward that end. As with all complex real-world problems, the challenge for us is that there is not an exact solution but rather a collection of approximate solutions. Nevertheless, our mathematical training has prepared us as problem solvers to hone our intelligence, our diligence, our spirit of curiosity, and our love of learning in order to develop meaningful and effective ways of teaching. These qualities are directly related to who we are as mathematicians, and they give us hope for success in our continued endeavor of improving mathematics teaching and learning for all.

References


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Photo of the SAIL class is courtesy of University of Pennsylvania.
Photo of the classroom at Middlebury College is courtesy of Middlebury College.
Photo of the Math Emporium is courtesy of Virginia Tech Math Emporium.
Photo of the Merit program is courtesy of the Department of Mathematics, University of Illinois at Urbana-Champaign.
Diaz-Lopez: When/how did you know you wanted to be a mathematician?

Grandine: Although I had been an above-average mathematics student throughout most of grade school and middle school, I became a great one when I learned algebra in ninth grade. At the time, learning algebra felt like being handed the keys to a magical world in which really interesting knowledge could be extracted from limited information, and I was spellbound by the possibilities that offered. I don’t think I ever seriously considered doing anything else after that.

Diaz-Lopez: Who encouraged or inspired you (mathematically or otherwise)?

Grandine: I was really lucky to have a great collection of teachers at every turn, from a truly amazing second grade teacher to incredible professors in graduate school and at every other level along the way. I learned so much from all of them, both about math and about having the confidence to tackle such a difficult subject, explore its possibilities, and leverage it to solve many of life’s problems.

Diaz-Lopez: How would you describe your work to a graduate student?

Grandine: The Boeing Company has 35,000 engineers, and they have an uncanny ability to generate large numbers of very interesting math problems, all of which need to be solved to design, build, and maintain airplanes. Although many of these problems can be solved by the engineers themselves, there are always a few that require more mathematical training and skill than engineers are typically equipped with. That’s where we mathematicians come in. Our most successful contributions result when we partner with our engineering colleagues.
Alexander Diaz-Lopez, having earned his PhD at the University of Notre Dame, is now visiting assistant professor at Swarthmore College. Diaz-Lopez was the first graduate student member of the Notices Editorial Board.

Grandine: Enjoy the rich, energetic campus atmosphere that graduate school offers in a university environment. Outside of academia, the atmosphere is often less enthusiastic and energetic.

Diaz-Lopez: All mathematicians feel discouraged occasionally. How do you deal with discouragement?

Grandine: One of the most effective strategies for dealing with discouragement I have ever found is to engage in professional activities beyond my job. Being able to discuss what I do with graduate students in a professional society setting or engage with company interns or participate in mentoring and coaching activities all serve to remind me that I have one of the coolest jobs in the world and that I really enjoy what I do and am grateful to have found such a rewarding and fulfilling career.

Diaz-Lopez: If you were not a mathematician, what would you be?

Grandine: If I were starting over, I think it would be very exciting to get into the clean energy field. The opportunity to participate in something as exciting as a near complete rebuild of the energy production and distribution system for the second half of the twenty-first century is very exciting to me.

Diaz-Lopez: If you could recommend one lecture (book, paper, article, etc.) to graduate students, what would it be?

Grandine: It's now an older book, but I found Winning the PhD Game by Richard W. Moore to be thoroughly outstanding. It offers a comprehensive discussion of all the issues facing graduate students, from academic disputes to family and emotional issues. It was a great comfort and help to me during my last two years in graduate school.

Diaz-Lopez: You finished your PhD and shortly after you started working for Boeing. What message would you give to those doctoral students and professional mathematicians thinking about having a career in industry?

Grandine: I was really fortunate to be equipped with excellent (at least at that time) computing skills, and they came in handy right away. Because the most effective means of transitioning mathematical technologies into engineering processes is via software, the importance of this skill cannot be overstated, but there are many other important skills that are needed as well. One is the ability to work effectively on teams, something that mathematicians often struggle with because of a culture that promotes individual work starting at a very young age. Also important are the ability to communicate mathematical material to a non-mathematical audience, a willingness to dive in and tackle problems that don’t necessarily satisfy all the preconditions of the relevant theory, and a tolerance for solving problems that aren’t always mathematically pretty.

Diaz-Lopez: What advice do you have for graduate students?
Benford’s Law?

Arno Berger and Theodore P. Hill

Communicated by Cesar E. Silva

Benford’s law quantifies the surprising fact that in many datasets, such as populations of counties, numbers on the World Wide Web, or incomes and expenses on tax returns, the numbers are much more likely to start with small digits like 1 or 2 than with large digits like 8 or 9. The law actually provides a specific probability distribution on the (significant) digits, telling exactly how likely each sequence of digits is. The law depends on the raw data \( x \) only via the significand \( S(x) \), which is obtained by moving the decimal point immediately to the right of the first nonzero digit and by ignoring signs; e.g., \( S(2017) = S(-0.02017) = 2.017 \), with \( D_1(2017) = 2 \) and \( D_2(2017) = 0 \) being the first and second significant (decimal) digit, respectively.

A real-valued random variable \( X \) is said to be Benford (base 10) if

\[
\mathbb{P}(S(X) \leq s) = \log_{10} s \quad \text{for all} \quad 1 \leq s < 10.
\]

Note that every Benford random variable also satisfies the first-digit law,

\[
\mathbb{P}(D_1(X) = d) = \log_{10} \frac{d + 1}{d} \quad \text{for all} \quad d = 1, \ldots, 9,
\]

so in particular the probability of \( D_1(X) = 1 \) is more than six times the probability of \( D_1(X) = 9 \); see Figure 1.

The first known reference to the logarithmic distribution (1) is a two-page note in the American Journal of Mathematics in 1881 by renowned Canadian-American mathematician and astronomer Simon Newcomb, future president of the American Mathematical Society. Newcomb’s discovery went largely unnoticed, and it came to be known as Benford’s law after its rediscovery and popularization by physicist Frank Benford in 1938.

**Stochastic Processes**

One key to the analysis of (1) is the **significand** \( \sigma \)-algebra \( \mathcal{S} \), the sub-\( \sigma \)-algebra of the Borel sets on \( \mathbb{R}^+ \) generated by the significand function \( S \) (or, equivalently, generated by the significant digit functions \( D_1, D_2, D_3, \ldots \)). This \( \sigma \)-algebra has several interesting properties: Every nonempty set \( A \in \mathcal{S} \) is infinite with accumulation points at 0 and \( +\infty \); \( \mathcal{S} \) is self-similar with respect to multiplication by integral powers of 10, i.e., \( 10^k A = A \) for all \( k \in \mathbb{Z} \) and \( A \in \mathcal{S} \); and \( \mathcal{S} \) is closed under (positive) scalar multiplication and integral roots, i.e.,

\[
a A^{1/n} \in \mathcal{S} \quad \text{for all} \quad a > 0, A \in \mathcal{S}, n \in \mathbb{N},
\]

but is not closed under integral powers.

The closure property (2), together with tools from Fourier analysis and ergodic theory, yields that Benford’s
law (1) is the unique probability distribution on the significant digits or, equivalently, on $\mathbb{R}^+$, that is scale-invariant (i.e., the distributions of $S(X)$ and $S(aX)$ are identical for all $a > 0$), is the unique continuous distribution that is base-invariant, and is the unique distribution that is sum-invariant.

None of the classical probability distributions—uniform, normal, exponential, gamma, Cauchy, Poisson, etc.—are exactly Benford. Some Pareto and lognormal distributions are close to being Benford and some are not, say, with respect to the Kolmogorov (sup norm) distance between the cumulative distribution functions, whereas other families such as uniform distributions are bounded strictly away from being Benford.

Sums of random variables are generally not Benford, not even if the summands are independent and Benford. The product $XY$ of two independent positive random variables, on the other hand, is Benford if either $X$ or $Y$ is Benford. The sequence $(X, X^2, X^3, \ldots)$ of powers of a random variable $X$ is a Benford sequence (see below) with probability one if and only if
\begin{equation}
P(|X| \text{ is rational}) = 0.
\end{equation}
The sequence $(X_1, X_1X_2, X_1X_2X_3, \ldots)$ of products of independent and identically distributed copies $X_j$ of $X$ is Benford with probability one if and only if
\[ P\left(\log_{10}|X| \in m^{-1}\mathbb{Z}\right) < 1 \quad \text{for every } m \in \mathbb{N}. \]

Note that the latter follows from (3). Both conditions are satisfied whenever $X$ is absolutely continuous, in which case $S(X^n)$ and $S(\prod_{j=1}^n X_j)$ also converge in distribution to a Benford random variable; see Figure 2.

In a central-limit-like context, taking random samples from random distributions in an “unbiased” manner will entail convergence to Benford’s law with probability one in the sense that the empirical distributions of the significands of the mixed random samples will converge almost surely to the logarithmic distribution (1). This perhaps helps explain why numbers appearing in newspapers or on the Web have been found to be good fits to (1). How- ever, rates of convergence in these settings, or even good rules of thumb analogous to those for the Central Limit Theorem, have yet to be discovered.

Deterministic Processes
In direct analogy to (1), a sequence $(x_n) = (x_1, x_2, x_3, \ldots)$ of real numbers is said to be Benford if, for every $1 \leq s < 10$, the natural density of the set $\{n \in \mathbb{N} : S(x_n) \leq s\}$ exists and equals $\log_{10} s$. Many familiar sequences of real numbers, including the sequences of positive integers and prime numbers, are not Benford, but many others are, including the sequences of Fibonacci and Lucas numbers, factorials, and powers of 2.

To see, for instance, that $(2^n)$ is Benford, check first that a sequence $(x_n)$ of real numbers is Benford if and only if the sequence $(\log_{10}|x_n|)$ is uniformly distributed mod 1, note that $\log_{10} 2$ is irrational, and apply a theorem of Weyl about the uniform distribution under irrational rotations on the circle. Since $(2^n)$ is Benford, the scale invariance of Benford’s law implies that the orbit under the map $f(x) = 2x$, i.e., the sequence $(f^n(x_0)) = (2^nx_0)$, is Benford for all $x_0 \neq 0$.

A more subtle result, which can be established utilizing finer uniform distribution tools and the dynamical systems technique of shadowing, is that the orbit under any map $f(x) = ax + g(x)$ with $g = o(x/ \log x)$ as $x \to +\infty$ is Benford for all sufficiently large $x_0$ if and only if $a > 1$ is not a rational power of 10. The orbit under a smooth map $f(x) = ax^2 + g(x)$ with $g' = o(x/ \log x)$, on the other hand, is Benford without any restriction on $a > 0$, but only for Lebesgue-almost all sufficiently large $x_0$; the set of exceptional $x_0$ is nonempty (as illustrated by Figure 3) and in fact uncountable.

Even when a map produces Benford orbits for almost all starting points, determining exactly for which $x_0$ it does so can be a challenge. For example, the orbit under $f(x) = x^2 + 1$ is Benford for Lebesgue-almost all $x_0$, but it is not known whether the orbit of $x_0 = 0$, i.e., the integer sequence $(x_n) = (1, 2, 5, 26, 677, \ldots)$, is Benford. (Incidentally, $x_n$ for each $n \in \mathbb{N}$ equals the number of binary trees of height less than $n$.)

Figure 2. For absolutely continuous $X$, the distribution for the significands of powers (top) and i.i.d. products of $X$ approach the logarithmic distribution (1).

Analogous to the notion of a Benford sequence, a real-valued function $f$ on $\mathbb{R}^+$ is Benford if
\[ T^{-1}\text{length}\{0 < t \leq T : S(f(t)) \leq s\} \]
converges to $\log_{10} s$ as $T \to +\infty$ for every $1 \leq s < 10$ or, equivalently, if $\log_{10} |f|$ is (continuously) uniformly distributed. For instance, all functions $f(t) = e^{at}p(t)$, with $a \neq 0$ and any polynomial $p \neq 0$, are Benford. Note that these functions are solutions of autonomous linear differential equations. Perhaps not too surprisingly, the solutions of many other differential equations also turn out to be Benford functions.

Current Status. (Warning: Benford’s law may be addictive!) Many questions regarding Benford’s law are natural variants of famous open problems. For instance, the question of which starting points generate Benford orbits is closely related to the problem of deciding which real numbers are normal (i.e., have all finite strings of digits equally represented in their decimal expansion). Even though a theorem of Borel guarantees that almost all numbers are normal, this remains a hard problem in general. The question of whether Benford random variables are the only continuous random variables for which $S(X), S(X^2), S(X^3)$ all have the same distribution is a variant of Furstenberg’s question regarding invariant measures for $2x \mod 1$ and $3x \mod 1$. And the answer to the question of whether any solution $(x_n)$ of the linear difference equation $x_n + 2x_{n-1} + 3x_{n-2} = 0$ is Benford hinges on Schanuel’s conjecture on transcendence degrees.

There is no known back-of-the-envelope argument, not even a heuristic one, that explains the appearance of Benford’s law across the board—in data that is pure or mixed, deterministic or stochastic, discrete or continuous-time, real-valued or multidimensional. The main mathematical challenge is to establish theories that help explain and predict when data will follow Benford’s law and when it will not.

If useful conclusions can be drawn from huge datasets by looking at only a few significant digits, as has been reported to be the case for detecting financial fraud and alterations of digital images, as well as for detecting natural phenomena such as earthquakes and phase changes in quantum processes, then these same techniques may well provide useful tools for many other applications in the future.

References

Photo Credits
All illustrations are courtesy of Arno Berger.

Applications
Knuth’s classic The Art of Computer Programming noted the ubiquity of Benford’s law in scientific calculations and its consequent use in analyzing the average behavior of floating point arithmetic algorithms. In the mid-1990s, accountant Mark Nigrini found that Benford goodness-of-fit tests can be very effective red-flag tests, since true tax data often is a close fit to Benford’s law, whereas fabricated data is not. The statistical phenomenon of Benford’s law has since seen widespread interest across a broad range of fields, from accounting and physics to digital-image forensics, biology, and medicine. Of the nearly thousand entries in the database www.benfordonline.net more than two thirds have appeared since 2000.

ABOUT THE AUTHORS
Arno Berger is interested in dynamical systems, analysis, and probability theory. When not immersed in more mundane matters, he enjoys marvelling at the big Alberta sky from his office window.

Ted Hill spends much of his spare time in a cabin overlooking the Pacific Central Coast, and in addition to fair division, optimal stopping, and general probability, he loves hiking, diving, and mountain biking.
A Game With Mirrors

by DJ Bruce, University of Wisconsin

Throughout my grad school experience, from conference registration forms and university-wide surveys to actual grad school applications themselves, I have often run into the following question:

Gender? *

- Male
- Female

Always making me think: Why is this still a thing?

Okay, certainly there are legitimate reasons to collect demographic information, including information regarding gender. ... However, why must this information be collected in such biased fashion?

... As phrased above, the question “Gender?” takes complexities and nuances and smashes them; smashes them into two extremely narrow little circles. It takes individuals and smashes their agency, forcing them to squeeze themselves into a narrow space predefined by someone else.

Moreover, to some who are “gender-expansive,” this question is yet another instance of someone—a colleague, a peer, a fellow mathematician—neither recognizing nor accepting part of who they are.

... As the Human Rights Campaign notes, the least restrictive and most preferable option is to allow individuals to self-identify. For example, by re-phrasing the question as follows:

Gender? *

Your answer

ABOUT THE AUTHOR

DJ Bruce is a third-year graduate student at the University of Wisconsin and new managing editor of the Graduate Student Blog.
Beginning each February 1, the AMS will accept applications for the AMS-Simons Travel Grants program. Each grant provides an early-career mathematician with $2,000 per year for two years to reimburse travel expenses related to research. Individuals who are not more than four years past the completion of their PhD are eligible. The department of the awardee will also receive a small amount of funding to help enhance its research atmosphere.

The deadline for applications is March 31 of each year.

Applicants must be located in the United States or be US citizens. For complete details of eligibility and application instructions, visit:

www.ams.org/programs/travel-grants/AMS-SimonsTG
Maintaining a Balance

Tipping points are important—and well-named—features of many complex systems, including financial and ecological systems. At a tipping point, a small change in conditions in the system can result in drastic changes overall (just as a small change in the weight of one end of a see-saw can reverse positions). Most such systems are studied using mathematical models based on collections of differential equations. The variables in the equations are related, leading to feedback in the system and potentially substantial changes, such as economic collapse. Research is now being done to recognize tipping points in hopes that something can be done before it is too late.
Two hundred and fifty-two million years ago, life on Earth nearly vanished [1]. So many marine animal species disappeared—more than 90 percent—that the event, known as the end-Permian extinction, qualifies as the most severe mass extinction in the geologic record. Unlike the later demise of the dinosaurs, the end-Permian extinction is not linked to a meteor impact. Yet it is unquestionably associated with major environmental change, including a strong perturbation of Earth’s carbon cycle. Recently, an additional piece of the puzzle fell into place. Massive Siberian volcanism, long thought to coincide roughly with the extinction, is now known to have preceded it and continued beyond it [2].

A simple narrative emerges. Carbon dioxide degassed from millions of cubic kilometers of Siberian magma perturbed the carbon cycle, leading to an extreme form of our own contemporary concerns: global warming and ocean acidification. Life then nearly perished. It is an attractive story, but it fails. Geochemical studies reveal an influx of carbon dioxide that is much greater than Siberian volcanism would have provided on its own [3]. Something else must have happened.

Mathematics now enters the picture. The same advances that allow us to identify the chronological order of magma in Siberia and fossils elsewhere also make possible mathematical analyses of the temporal evolution of geochemical signals. These analyses then aid the development and testing of hypotheses, which in turn help to identify underlying mechanisms.

The best-understood geochemical signals are time series of the isotopic composition of carbon derived from sedimentary rocks (Figure 1). Carbon occurs as two stable isotopes, $^{12}$C and $^{13}$C. The total mass of each isotope is conserved. But if we divide the world’s carbon into two reactive global “reservoirs”—inorganic and organic carbon—the mass of $^{12}$C and $^{13}$C in each reservoir may vary. For example, the production of organic carbon from carbon dioxide by photosynthesis slightly favors the lighter isotope, so that organic carbon contains a smaller fraction of $^{13}$C than the carbon dioxide from which it was produced. Geochemists measure the abundance ratio $R_x = (^{13}C/^{12}C)_x$ in carbon of type $x$. Such isotopic data are reported in terms of the departure of this ratio from a standard ratio $R_{std}$ as the quantity

$$\delta_x = \frac{R_x - R_{std}}{R_{std}} \times 1000,$$

Figure 1: Sedimentary rocks located in Meishan, China. The layers represent a succession of marine sediments deposited before, during, and after the end-Permian extinction. The rocks contain chemical signatures of environmental conditions at the time of deposition. The arrow of time points upward, to the right.
where multiplication by 1000 means that the units are parts per thousand, or per mil. Time series for inorganic (carbonate) carbon ($\delta_a$) and organic carbon ($\delta_o$) are shown in Figure 2a. The evolution of $\delta_o(t)$ is consistent with a growing perturbation in which isotopically light carbon increasingly fills the inorganic pool. In the following, I describe a recent collaborative effort to characterize this evolution mathematically and to interpret the results [4].

First, equations for the conservation of $^{12}$C and $^{13}$C and their exchange via photosynthesis and respiration (i.e., the oxidation of organic carbon to carbon dioxide) are inverted to estimate the time-dependent flux of isotopically light organic carbon consistent with the signals in Figure 2a. Integrating that flux, we obtain the accumulated mass $M(t)$ of carbon added to the inorganic pool, normalized with respect to the initial mass of that pool. The result, in Figure 2b, suggests that $M(t)$ grows like an incipient singular blowup of the form

$$M(t) \propto \frac{1}{t_c - t},$$

where $t_c$ is the onset of the extinction. Noting that equation (2) solves $dM/dt \propto M^2$, we conclude that we have detected a nonlinear interaction within the carbon cycle. But what is that interaction? And how does it lead to equation (2)?

It has long been recognized that population dynamics can exhibit growth like equation (2)[5]. Exponential growth is more common, but if per capita growth rates increase with population size (due, e.g., to improved technology) and resources are effectively unlimited, superexponential growth ensues. We know from geochemical analyses that late-Permian sediments contain unusually high concentrations of organic matter. Such organic matter is usually metabolized by microbes deep beneath the seafloor, but its decay was somehow hindered. Now suppose that a new metabolic pathway evolved to consume this stockpile. Microbes endowed with the new pathway would then proliferate.

To pursue this line of reasoning, we hypothesized that a previously identified genomic innovation was favored by the end-Permian environment. The new pathway allowed a particular genus of microbes, *Methanosarcina*, to convert acetate, a form of organic carbon, to methane more efficiently than before. The oxidation of the methane to carbon dioxide would then provide the required flux of isotopically light carbon. Using methods of computational genomics, we found that the new pathway evolved 240±41 million years ago. The date is propitious, but its degree of uncertainty is undesirable. One would like more evidence.

Further support follows from two facts: *Methanosarcina* requires nickel to make methane. And the world’s largest economic concentration of nickel is located in Siberia, having been deposited during the period of massive Siberian volcanism. If the nickel entered the oceans, then elevated nickel concentrations would have removed a significant barrier to the expansion of *Methanosarcina*. Our group measured the concentration of nickel in the same rocks that yielded the carbon isotopic data of Figure 2a. We found that nickel concentrations increase abruptly just before the blowup. Moreover, subsequent dating of Siberian nickel deposits confirms that nickel-rich lavas erupted prior to the extinction [2].
A mathematical story is now straightforwardly written. Let the total production of methane by the new pathway be \( A \propto M \). Note that methane producers live where there is no free oxygen and that the conversion of methane to carbon dioxide consumes oxygen. Consequently the carrying capacity \( K \)—i.e., the maximum population supported by the environment—for \( \text{Methanosarcina} \) grows as its population grows, because its anoxic niche becomes larger while its resources (acetate and nickel) remain abundant. In other words, \( K \) grows with \( A \), like
\[
K(t) = k_0 + k_1 A(t),
\]
where \( k_0 \) is the initial carrying capacity and \( k_1 > 0 \) is a constant. The rate at which methane is produced scales like the carrying capacity:
\[
dA/dt = \beta K,
\]
where \( \beta > 0 \) is related to the metabolic rate of an individual organism. If \( \beta \) is constant, \( A(t) \) grows exponentially. However, it may not be constant. For example, new species of \( \text{Methanosarcina} \) may evolve to take advantage of specialized niches, or the expansion of \( \text{Methanosarcina} \) may bring it closer to nutrients that limit its growth. Thus it is reasonable to imagine \( \beta \) growing with \( A \). The simplest relation would be linear:
\[
\beta(t) = \beta_0 + \beta_1 A(t),
\]
where \( \beta_0 \) and \( \beta_1 \) are positive constants. Inserting (5) and (3) into (4), we find that \( dA/dt \) grows quadratically with \( A \) for large \( A \); thus, to leading order, \( A \propto (t_c - t)^{-1} \) as \( t \rightarrow t_c \).

Our story has the virtue of fitting the observations, but it is presumably not unique. It nevertheless contains two valuable lessons. First, it provides new insight into the ways in which microbial evolution impacts not only the global environment but also the evolution of macroscopic life. Second, it shows how mathematics can play a foundational role in developing such insight. Manifestations of mathematics in nature are not new. But when one can look back a quarter of a billion years and see a mathematical expression of Earth’s greatest natural catastrophe, one feels a renewed sense of awe. Can it all really be so simple? Indeed it can.

**Acknowledgments.** I thank the coauthors of [4] for their valuable collaboration. This work was supported by NASA Astrobiology grant NNA13AA90A.

**References**


Leroy P. Steele Prizes

The selection committee for these prizes requests nominations for consideration for the 2018 awards. Further information about the prizes can be found in the November 2015 Notices pp.1249–1255 (also available at www.ams.org/profession/prizes-awards/ams-prizes/steele-prize).

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2018 the prize for Seminal Contribution to Research will be awarded for a paper in Discrete Mathematics/Logic.

Further information and instructions for submitting a nomination can be found at the Leroy P. Steele Prizes website:

www.ams.org/profession/prizes-awards/ams-prizes/steele-prize

Nominations for the Steele Prizes for Lifetime Achievement and for Mathematical Exposition will remain active and receive consideration for three consecutive years.

For questions contact the AMS Secretary at secretary@ams.org.

The nomination period is February 1, 2017 through March 31, 2017.
In the elections of 2016 the Society elected a vice president, a trustee, five members at large of the Council, four members of the Nominating Committee, and two members of the Editorial Boards Committee.

**Vice President**

**David Jerison**  
*Massachusetts Institute of Technology*  
Term is three years  

**Trustee**

**Ralph L. Cohen**  
*Stanford University*  
Term is five years  

**Members at Large of the Council**  
Terms are three years (February 1, 2017–January 31, 2020).

- **Nathan M. Dunfield**  
  *University of Illinois, Urbana–Champaign*
- **Gregory F. Lawler**  
  *University of Chicago*
- **Irina Mitrea**  
  *Temple University*
- **Ravi Vakil**  
  *Stanford University*
- **Talitha M. Washington**  
  *Howard University*
Nominating Committee
Terms are three years (January 1, 2017–December 31, 2019).

Linda Chen
Swarthmore College

Laura De Carli
Florida International University

Shelly Harvey
Rice University

Bjorn Poonen
Massachusetts Institute of Technology

Editorial Boards Committee
Terms are three years (February 1, 2017–January 31, 2020).

Hélène Barcelo
MSRI and Arizona State University

Scott Sheffield
Massachusetts Institute of Technology
From the AMS Secretary

2017

AMS Election

Nomination by Petition

VICE PRESIDENT OR MEMBER AT LARGE

One position of vice president and member of the Council ex officio for a term of three years is to be filled in the election of 2017. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and procedures, which are described below.

EDITORIAL BOARDS COMMITTEE

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

NOMINATING COMMITTEE

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

RULES AND PROCEDURES

Use separate copies of the form for each candidate for vice president, member at large, member of the Nominating or Editorial Boards Committees.

1. To be considered, petitions must be addressed to Carla D. Savage, Secretary, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA, and must arrive by 24 February 2017.

2. The name of the candidate must be given as it appears on the “American Mathematical Society” entry in the Combined Membership List (www.ams.org/cml). If the name does not appear in the list, as in the case of a new AMS member or by error, it must be as it appears in the mailing lists, for example on the mailing label of the Notices. If the name does not identify the candidate uniquely, append the member code, which may be obtained from the candidate’s mailing label or by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).

3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.

4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.

5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.

6. The signature may be in the style chosen by the signer. However, the printed name and address will be checked against the AMS entry in the Combined Membership List and on the mailing lists. No attempt will be made to match variants of names with the form of name in the AMS CML entry. A name neither in the CML nor on the mailing lists is not that of a member. (Example: The name Carla D. Savage is that of a member. The name C. Savage appears not to be.)

7. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.
The undersigned members of the American Mathematical Society propose the name of

as a candidate for the position of (check one):

☐ Vice President (term beginning 02/01/2018)
☐ Member at Large of the Council (term beginning 02/01/2018)
☐ Member of the Nominating Committee (term beginning 01/01/2018)
☐ Member of the Editorial Boards Committee (term beginning 02/01/2018)

of the American Mathematical Society.

Return petitions by February 24, 2017 to:
Carla D. Savage, Secretary, AMS, 201 Charles Street, Providence, RI
02904-2294 USA

Name and address (printed or typed)

______________________________
Signature

______________________________
Signature

______________________________
Signature

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Signature

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Signature

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Signature

______________________________
Signature
mathematics

LANGUAGE OF THE SCIENCES

- engineering
- astronomy
- robotics
- genetics
- medicine
- biology
- climatology
- forensics
- statistics
- finance
- computer science
- physics
- neuroscience
- chemistry
- geology
- biochemistry
- ecology
- molecular biology
Voisin Awarded CNRS Gold Medal

Claire Voisin of the Collège de France has been awarded the 2016 CNRS Gold Medal of the Centre National de la Recherches Scientifiques (CNRS). The prize citation reads in part: “By virtue of her remarkable intuition and rigorousness, Claire Voisin has developed abstract mathematics at the interface between several fields. She has devoted her research to algebraic geometry (the study of the properties of sets defined by a system of algebraic equations). She is particularly keen on the interaction between the three highly different mathematical fields of topology, complex geometry, and algebraic geometry. Claire Voisin continues to dedicate an important part of her research to the Hodge conjecture, which is one of the seven Millennium Prize Math problems, as well as to its version generalized by Grothendieck.

“Her most important achievement involves the Kodaira theorem for surfaces. In 2005, Claire Voisin demonstrated that it could not be generalized for all dimensions: topology makes it possible to distinguish in dimensions greater than four the projective varieties of compact Kähler manifolds. This finding opened new avenues of research in mathematics. More recently, Claire Voisin has also played a pioneering role in the discovery and study of new birational invariants, which have led to significant advances in the analysis of the Lüroth problem and its variants.”

Voisin’s honors include the European Mathematical Society Prize (1992), the Ruth Lyttle Satter Prize (2007), a Clay Research Award (2008), and the Heinz Hopf Prize (2015). In 2016 she was appointed chair of algebraic geometry at the Collège de France after thirty years at CNRS.

—From a CNRS announcement

Hamilton and Hopper Awarded Presidential Medal of Freedom

Margaret H. Hamilton of Hamilton Technologies, Inc., and Rear Admiral Grace Hopper are among twenty-one recipients of the Presidential Medal of Freedom announced by President Obama.

Hamilton is a computer scientist who studied mathematics at the University of Michigan and Earlham College, then joined the Massachusetts Institute of Technology (MIT) as a programmer, eventually becoming director of the Software Engineering Division at the MIT Instrumentation Laboratory. She led the team that created the onboard flight software for NASA’s Apollo command modules and lunar modules. Hamilton contributed to concepts of asynchronous software, priority scheduling and priority displays, and human-in-the-loop decision capability, which set the foundation for modern, ultrareliable software design and engineering. In 1986 she founded Hamilton Technologies in Cambridge, Massachusetts. Among her other awards are the Augusta Ada Lovelace Award (1986) of the Association for Women in Computing and the NASA Exceptional Space Act Award for scientific and technical contributions.

Rear Admiral Grace Murray Hopper was honored posthumously. Known as “Amazing Grace” and “the first lady of software,” she was at the forefront of computers and programming development from the 1940s through the 1980s. She received a BA in mathematics from Vassar College and a PhD in mathematics from Yale University (1934). Hopper’s work helped make coding languages more practical and accessible, and she created the first compiler, which translates source code from one language into another. She taught math-
Anne Broadbent

Anne Broadbent of the University of Ottawa has been named the recipient of the 2016 André Aisenstadt Prize by the International Scientific Advisory Committee of the Centre de Recherches Mathématiques (CRM).

According to the prize citation, Broadbent “is a leader in the field of quantum information and cryptography. In 2009, she and her coauthors introduced the concept of blind quantum computation—roughly, using quantum properties to permit third parties to perform extensive computations on data without jeopardizing the secrecy of the data. These highly cited papers launched new important research directions in quantum information processing, including her current groundbreaking work on quantum homomorphic encryption. Other significant contributions she has made to this field include characterizing quantum one-time programs and presenting a novel automated technique for parallelizing quantum circuits.”

Broadbent earned her PhD from the Université de Montréal in 2008, under the joint supervision of Alain Tapp and Gilles Brassard. Her doctoral thesis, “Quantum nonlocality, cryptography and complexity,” was awarded the NSERC Doctoral Prize by the Natural Sciences and Engineering Research Council of Canada. She received the John Charles Polanyi Prize in Physics in 2010. She has been affiliated with the Institute for Quantum Computing of the University of Waterloo as an NSERC Postdoctoral Fellow and as a Canadian Institute for Advanced Research Global Scholar (2011–2013). She joined the Department of Mathematics and Statistics at the University of Ottawa in 2014, where she holds the University Research Chair in Quantum Information. She and her husband welcomed daughter Emily in October 2016, joining her brothers, three and seven years old.

—From a CRM announcement

Amlendu Krishna

AMLENDU KRISHNA of the Tata Institute of Fundamental Research (TIFR) and NAVIEN GARG of the Indian Institute of Technology Delhi have been selected as the winners of the Bhatnagar Prize in Mathematical Sciences. Krishna was honored for his work in algebraic geometry. He received his PhD in 2001 from TIFR and was the recipient of the ICTP-IMU Ramanujan Prize in 2005. Garg was honored for his work in algorithms and complexity. He has received a number of awards, including the Young Scientist Medal of the Indian National Science Academy (2006) and the Teaching Excellence Award, IIT Delhi (2012).

—From a Bhatnagar Prize announcement

Freddy Cachazo

FREDDY CACHazo of the Perimeter Institute for Theoretical Physics, Waterloo, Canada, has been awarded the CAP-CRM Prize in Theoretical and Mathematical Physics by the Canadian Association of Physicists (CAP) and the Centre de Recherches Mathématiques (CRM). According to the prize citation, he was honored for his work “introducing elegant new mathematical ideas and methods that have led to unexpected insights in the way scattering amplitudes are calculated in supersymmetric Yang-Mills theory. Inspired in part by twistor-string theory, the Cachazo-Svrcek-Witten (CSW) and Britto-Cachazo-Feng-Witten (BCFW) recursion relations revolutionized the field, making it possible to perform previously impossible calculations analytically in a few lines using explicit integral formulae. These results turned out to be in remarkable correspondence with structures explored concurrently by mathematicians for completely different purposes, establishing a suggestive link with the modern theory of integrable systems.

With collaborators, Cachazo has creatively drawn upon a variety of elegant mathematical ideas to develop entirely new methods for studying scattering processes in gauge theories and gravity. Cachazo’s contributions to
Rhodes Scholarships Awarded

Thirty-two young men and women in the United States have been selected as Rhodes Scholars for 2017. Rhodes Scholarships provide all expenses for two or three years of study at the University of Oxford in England and may allow funding in some instances for four years.

CHRISTIAN E. NATTEL of Madeira Beach, Florida, is a senior at the US Military Academy, where he is double majoring in mathematical sciences and philosophy. He is interested in narratives of struggle, social justice, and self-determination. Christian is president of the Cultural Affairs Seminar, which champions diversity and inspires cadets through mentorship and tutoring. He was awarded the Superintendent’s Award for Achievement, the Distinguished Cadet Award, and the Black Engineer of the Year Award for Military Leadership. Christian is a member of the men’s handball team. At Oxford, he will pursue an M.Sc. in Comparative Social Policy, followed by the Master of Public Policy.

—From a Rhodes Trust announcement

Dickenstein Awarded TWAS Prize in Mathematics

Alicia Dickenstein of the University of Buenos Aires, Argentina, has been awarded the TWAS Prize in Mathematics for her “outstanding contribution to the understanding of discriminants.” Dickenstein says, “Discriminants...are polynomials with a fascinating combinatorial structure that provide a key tool when examining singularities of systems of multivariate polynomial equations. Their theoretical study is a thriving and fruitful domain today, and they are also very useful in a variety of applications, in particular to hypergeometric systems of differential equations, to geometric modeling problems and to the determination of multistationarity of biochemical reaction networks.”

Dickenstein tells the Notices: “I started my research in the realm of several complex variables, in a subject with different ingredients, and it was hard for me to find my way due to the mathematical and geographical isolation I experienced in Argentina. At some moment, I felt that my weakness was that I knew a little bit of many subjects but not much of any one. But then, I managed to make this my strength. I was not working in any established general area, and I learned that, as the Spanish poet Antonio Machado said: ‘se hace camino al andar,’ which means that the path is traced as one walks.”

The TWAS Prizes are awarded every year in nine fields, including mathematics. Each prize carries a cash award of US$15,000.

—From a TWAS announcement

Blackwell-Tapia Conference 2016

More than one hundred researchers, from undergraduates to retired professionals, gathered to witness MARIEL VAZQUEZ, professor of mathematics and microbiology and molecular genetics at the University of California, Davis, receive the 2016 Blackwell-Tapia Prize, the nation’s highest research award for minority mathematicians, at the conclusion of the biennial Blackwell-Tapia Conference, October 28–29, 2016, in Knoxville, Tennessee. (See Notices, Vol. 63, Number 6, p. 671, for the announcement of the prize to Vazquez.)

The ceremony was the capstone of two days of talks by an inspiring lineup of mathematicians and statisticians on topics from using math to move robots quickly to conducting statistical research in industrial, government, and academic settings. The event included a poster session, networking, and a preconference event for undergraduates at which Vazquez and Jose Perea, assistant professor of mathematics from Michigan State University, shared
Hillel Gauchman (1937–2016)

Hillel Gauchman was a distinguished differential geometer. He was a faculty member in the Department of Mathematics and Computer Science at Eastern Illinois University and a member of the AMS for almost 30 years. Born in Moscow in 1937, Hillel was an outstanding mathematical student and problem solver as an adolescent and a young man. Overcoming the many obstacles placed in the way of Jewish students in Russia throughout the 1950s, he managed to earn a master’s degree from the Moscow Pedagogical Institute in 1959 and a PhD from the University of Moscow in 1962. His thesis, “Affine Connections on Almost Complex Manifolds,” was written under the direction of G. F. Laptev.

Hillel spent the next ten years as a faculty member in the Department of Mathematics at the Institute of Civil Engineering in Moscow. In 1972 he and his family emigrated to Israel, where he took up an appointment at the Ben Gurion University of the Negev. During his ten years there, he even did a short stint as the department chair. Although he would always say that he was a hopeless administrator, those who knew him during that time remember him fondly because of his warmth, his humanity, his sense of humor, and his willingness to help both students and colleagues at all levels.

After spending 1984–1986 as a visiting faculty member at North Carolina State and the University of Illinois at Urbana–Champaign, the Gauchmans relocated permanently to the United States. Hillel accepted a tenure-track position at Eastern Illinois University beginning in the fall of 1986, continuing there until his retirement in 2005, thus completing forty-three years as a mathematics professor on three continents.

While most of Hillel’s almost fifty published papers are related to differential geometry, his mathematical interests were both broad and deep. His last paper, written with his son-in-law, who was a graduate student in engineering at the time, was on numerical analysis. Hillel was always willing to talk to colleagues about their own work—often with insightful suggestions—and he had a true passion for problem solving. Rare was the Monday morning after the Putnam exam when he didn’t come in to the department with all the problems solved. Once he was almost late for a class on such a Monday morning because he drove right past his exit off the Interstate while he was solving the trickiest of that year’s problems!

In spite of his mathematical accomplishments and talents, Hillel will be remembered most for his humility and his deep humanity. Everyone who knew him felt like he was a friend. He had a delightful, self-effacing sense of humor and was a marvelous storyteller, even in English or Hebrew—neither of which was his native tongue and neither of which he spoke without a strong Russian accent. He was incredibly supportive of and inspirational to so many Russian mathematicians, especially in the United States, but young mathematicians, wherever he met them, benefitted from his interest and encouragement.

—From an obituary by Peter Andrews, Eastern Illinois University

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Carlo H. Séquin

For the last decade, the international Bridges conference has showcased mathematical connections in art, architecture, music, and many other cultural domains. It regularly attracts a few hundred participants—artists, mathematicians, computer scientists, teachers, etc.—from dozens of countries.

This year the annual Bridges conference was held at the beautiful University of Jyväskylä in Finland. The conference comprised four days of talks and interactions plus an optional excursion day. The formal, refereed part of the conference entailed ten plenary presentations, forty regular papers, sixty-one short papers, and seventeen “hands-on” workshops. There was also a curated art exhibition, a festival of short mathematical movies, a session of mathematical poetry, and an informal theatre event performed by conference participants. More-
over, participants who did not want to formally present a refereed paper could give ten-minute summaries or display their works in a large “Show and Tell” area.

Sadly, this was the first conference where Reza Sarhangi, the founder of this conference series and the president of the Bridges organization, could no longer be with us and infect us with his boundless enthusiasm. The conference started with a touching memorial session, where Bridges board members and friends shared fond memories of interactions with Reza. We all miss him very much as a person, but it was also clear that he was present in spirit and that he lives on through his various legacies: the annual Bridges conferences, the mini-Bridges symposia, and a whole shelf full of beautiful and inspiring Bridges Proceedings.

The memorial session was followed by two particularly inspiring plenary papers. Marjorie Wikler Senechal’s “The subtlety of influence: Math, art, and Black Mountain College” was an illuminating set of reminiscences about the role of this college in the hills above Asheville, North Carolina, promoting a spirit and interactions similar to those fostered by the Bridges conferences. Henry Segerman then showed us what astounding pictures a true 4π-spherical camera can produce when coupled with the right image-processing software. On the second day, Judy Holdener talked about “Immersion in Mathematics,” explaining, among other things, her black and white submission to the Art Exhibit, which won the prize for Most Effective Use of Mathematics.

In this year’s Art Exhibit, the most astounding piece was “Gödel, Escher, Bach: just another Braid” by Hans Kuiper and Walt van Ballegooijen. The central piece consists of 256 cubelets connected in a fractal tetrahedral lattice. The black plastic struts of each cubelet were individually modulated in thickness to change the gray-tone density when the lattice was viewed from a particular direction. A frame was set up to show this lattice from the front, from one side, and from the bottom with two suitably placed mirrors. As a result, from the proper vantage point, one could see simultaneously the portraits of Gödel, Escher, and Bach. This piece won the Best of Show award.

Other awards were given to Kyoko Urata for “Icosahedral Temari” (Best Craftsmanship) and to Albrecht Wintterlin for “Projective Plane without Crossings” (Most Innovative). High-quality pictures of all exhibited artwork can be seen online in the “Bridges Galleries.” Two additional special exhibits featured works by Rinus Roelofs, Lajos Szilassi, and István Orosz.

One of the four conference days was open to the general public. It is typically referred to as “Family Day,” since it attracts a large number of
children. The great lobby of the main conference building was teeming with multiple activities: sculpture constructions, dozens of “Show & Tell” tables, 3D-printing of mathematical pancakes, and occasional music performances. The unusual group photo on page 155 was taken by Henry Segerman with his $4\pi$-spherical camera and manipulated with some special code he developed to produce a stereographic projection of the scene.

The traditional Music Night featured an exquisite concert by the duo Corey Cerovsek and Paavali Jumppanen.

The next Bridges conference will be held in Waterloo, Ontario, July 27–31, 2017. Check the Bridges website for the Sarhangi memorial and material on past and future conferences.

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ABOUT THE AUTHOR

For the last thirty years, Carlo H. Séquin has connected mathematics and art in graduate courses on solid modeling, computer-aided design, and rapid prototyping, as well as in freshman seminars “Symmetry and Topology.” Since 1994 he has collaborated with Brent Collins to create a few large sculptures such as “Pax Mundi” (at the H&R Block Headquarters, Kansas City) or “Music of the Spheres” and “Evolving Trefoil” (at the Agenstein Science Center of the Missouri Western State University in Saint Joseph, MO). He has attended all nineteen Bridges conferences so far and over those years has presented twenty-seven papers.
Mathematics Opportunities

Call for Nominations for Ramanujan Prize

The Ramanujan Prize for young mathematicians from developing countries is awarded annually by the International Centre for Theoretical Physics (ICTP), the International Mathematical Union, and the Department of Science and Technology of the Government of India to mathematicians under the age of forty-five who have conducted outstanding research. The deadline for nominations for the 2017 prize is February 1, 2017. See https://www.ictp.it/about-ictp/prizes-awards/the-ramanujan-prize/call-for-nominations.aspx.

—From an ICTP announcement

ACM Prize in Computing

The ACM Prize in Computing recognizes an early to mid-career fundamental, innovative contribution in computing. (Candidates are typically within 8–12 years of their terminal degrees.) All recipients become laureates of the Heidelberg Laureate Forum and are invited to the HLF (September 24–29, 2017; see www.heidelberg-laureate-forum.org for more information). The deadline for nominations is January 31, 2017. See acm.org/acmprize/nominations.cfm.

—From HLF and ACM announcements

Call for Nominations for CRM Aisenstadt Prize

The Centre de Recherches Mathématiques (CRM) solicits nominations for the André Aisenstadt Mathematics Prize, awarded to recognize talented young Canadian mathematicians. The deadline for nominations is March 1, 2017. See www.crm.umontreal.ca/prix/prixAndreAisenstadt/prix_attributionAA_an.shtml.

—From a CRM announcement

News from the Mathematical Biosciences Institute

The Mathematical Biosciences Institute (MBI) offers the Online National Mathematical Biology Colloquium. The series is available as an online interactive event or through online streaming afterward; individuals can watch talks and ask questions interactively. The colloquia are held on Wednesdays at 12:00 pm EST beginning January 11, 2017.

January 11, 2017: Leah Edelstein-Keshet, University of British Columbia, "Navigating Biochemical Pathways for Cell Polarization and Motility (A Personal Journey)."

February 15, 2017: Joel Cohen, Rockefeller University, "The Variation Is the Theme: Taylor’s Law from Chagas Disease Vector Control to Tornado Outbreaks."


April 12, 2017: James Keener, University of Utah, "Cell Physiology: Making Diffusion Your Friend."

To connect to the talks, go to https://mbi.osu.edu/go/pm/colloquium. Previous colloquia are available by streaming at https://mbi.osu.edu/programs/mbi-colloquium.

—From an MBI announcement

The Budapest Semesters in Mathematics

The Budapest Semesters in Mathematics (BSM) Director’s Mathematician in Residence Program (DMiR): Spend three weeks as the BSM Director’s Mathematician in Residence in beautiful Budapest, and enjoy a unique opportunity for professional development, networking, and collaboration with renowned Hungarian mathematicians. The DMiR program funds travel to Hungary and housing in Budapest. Office space, internet access, and a math library will be available. Faculty members accepted to the program (DMiR scholars) will be expected to be available to BSM students.
two to three hours per week and to give a short lecture series, targeted to BSM students, on their research area.

The first step in the application process is to contact either Professor Kristina Garrett, garrettk@stolaf.edu or Professor Kendra Killpatrick, Kendra.Killpatrick@pepperdine.edu to inquire about eligibility and to obtain a required letter of invitation from the BSM Directors. Each application will be reviewed in terms of:

• the significance of the proposed research project
• its contribution or significance to the applicant’s field
• project design
• dissemination plans

For more information: www.budapestsemesters.com

The Summer 2018 program will be available pending funding. Applications will be available September 2017; application deadline will be December 1, 2017. Decisions will be made by January 15, 2018. DMiR 2018 Program dates will be three weeks beginning in late June–early July.

The Editor-in-Chief invites all readers, from students to retired folks, to get more involved with Notices as authors, editors, guest editors, writers of Letters to the Editor, and so on.

E-mail your interests, ideas, or nominations of others to Frank.Morgan@williams.edu.
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doceamus ... let us teach

The Fulbright Specialist Program

Noureen A. Khan

Abstract: The Fulbright Specialist Program sends US faculty and professionals to serve as expert consultants on curriculum, faculty development, institutional planning, and related subjects at academic institutions abroad. This manuscript describes the process and the author's assignment on remedial courses in Qatar.

The Fulbright program is the US government’s flagship program in international educational exchange, administered by the Council for the International Exchange of Scholars in over 155 countries.

The Fulbright Specialist Program provides two- to six-week grant opportunities for US faculty and professionals. Typically Fulbright Specialists collaborate with their international counterparts on curriculum and faculty development, institutional planning, and lecture delivery. If selected, a faculty applicant becomes a “candidate” and stays on a roster for up to five years. Meanwhile, the Fulbright staff match the candidate’s expertise with an overseas host institution. Some scholars may get selected from the roster once or even twice while others may not be selected at all.

When an opportunity arises, appropriate candidates are requested to develop proposals for the specific job. My host institution, Qatar University, interviewed two finalists via Skype before selecting the winner. In my case, the Fulbright office arranged the travel and the host institution graciously took care of everything and made me feel at home. I gained invaluable work experience and came to love Middle Eastern culture.

The project objective was to improve the teaching of remedial mathematics, mainly noncredit courses in college algebra and pre-calculus. The faculty participated in workshops, shared their teaching challenges during a one-on-one discussion hour, and expressed their concerns about program strengths and weaknesses. I visited classes to gauge program effectiveness and possible areas of improvement. My biggest challenge was how to approach the traditional mindset of mathematics teachers in a Middle Eastern country where straight lecturing, rote memorization, and repetition were the rule. The teachers gave the students formulas and steps for solving algebra problems to memorize. Students copied notes from the board and did algebra drill exercises.

My open-door discussion hours had a relaxed atmosphere, where faculty felt free to share their feelings, achievements, and concerns. Because of this relaxed atmosphere, we were able to also discuss possible solutions to address many of their concerns.

I designed a strategic process to follow “best practices” and created an online forum to share ideas and teaching challenges. I encouraged a move away from the one-size-fits-all model to group instruction in order to accommodate the varying levels of learner, some faster some slower, and toward self-paced learning during lab hours. My biggest accomplishment was building trust and long-lasting relationships with faculty,

Noureen A. Khan is associate professor of mathematics at University of North Texas at Dallas. Her email address is Noureen.Khan@unt.dallas.edu.

1See www.cies.org/program/fulbright-specialist-program.

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DOI: http://dx.doi.org/10.1090/noti1476

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giving me the memories of a lifetime. At a final meeting, the Vice President and Chief Academic Officer of Qatar University told me, “We appreciate your dedication and hard work on this project. You brought motivation to teach to our faculty. More importantly, for the first time, nobody felt offended by your class visits. We want you on our team.” I am invited for a second Fulbright visit next year, and I can’t wait to go back.

Meanwhile, back at home, I have become very invested in innovative methods of teaching remedial courses. And to my surprise, I have learned ways to improve my own teaching and gained a new appreciation for the support innovation regularly receives in the United States.

Photo Credits
Photo of Noureen Khan is courtesy of UNT Dallas. All other article photos are courtesy of the author.

ACKNOWLEDGMENT. This project was supported by Fulbright Global Program, the Council for International Exchange of Scholars, and the Foundation Program, Qatar University.

ABOUT THE AUTHOR
After studies at Bahaud-din Zakariya University, Pakistan, Khan got her PhD from the University of Texas at Dallas, where she is now on the faculty. In 2016, she spent four weeks in Qatar as a Fulbright Specialist.

Remote Access is a mechanism by which users can pair various web browsing devices (smartphones, tablets, laptops, desktops) with an institution’s network, allowing users to access AMS electronic products when not connected to a host institution’s network. The AMS permits authorized users (faculty, staff, students, and visiting faculty) to pair mobile devices with a host institution’s subscriptions.

For more information, visit:
www.ams.org/RemoteAccess
For Your Information/Inside the AMS

Colliander Appointed PIMS Director

James Colliander of the University of British Columbia has been appointed the director of the Pacific Institute for the Mathematical Sciences (PIMS). His appointment began on July 1, 2016, and will run for five years.

Colliander works at the interface of partial differential equations, harmonic analysis, and dynamical systems. He received his PhD from the University of Illinois in 1997. He has held postdoctoral positions at the Mathematical Sciences Research Institute and the University of California Berkeley. He was full professor at the University of Toronto from 2007 to 2015. His honors include a Sloan Foundation Fellowship (2003), the McLean Award (2007), and the Outstanding Teaching Award in Arts and Science (2010) at the University of Toronto. He is the founder and CEO of Crowdmark, an education technology company that was awarded the Connaught Seed Stage Startup Award at Toronto in 2013.

PIMS is a consortium of ten universities in the Pacific Northwest and Western Canada. For twenty years PIMS has been promoting research in and applications of the mathematical sciences, facilitating the training of highly qualified personnel, enriching public awareness of and education in the mathematical sciences, and creating robust mathematical partnerships.

—From a PIMS announcement

Photo Credit

Photo of James Colliander is courtesy of Denise Grant.

AMS E-mail Support for Frequently Asked Questions

A number of e-mail addresses have been established for contacting the AMS staff regarding frequently asked questions.

The following is a list of those addresses together with a description of the types of inquiries that should be made through each address:

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- bookstore@ams.org for inquiries related to the online AMS Bookstore.
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- classads@ams.org to submit classified advertising for the Notices.
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- development@ams.org for information about charitable giving to the AMS.
Inside the AMS, cont'd.

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*emp-info@ams.org* for information regarding AMS employment and career services.

*eprod-support@ams.org* for technical questions regarding AMS electronic products and services.

*gradprg-ad@ams.org* to inquire about a listing or ad in the Find Graduate Programs online service.

*mathjobs@ams.org* for questions about the online job application service MathJobs.org.

*mathprograms@ams.org* for questions about the online program application service Mathprograms.org.

*mathrev@ams.org* to send correspondence to Mathematical Reviews related to reviews or other editorial questions.

*meet@ams.org* to request general information about Society meetings and conferences.

*mmsb@ams.org* for information or questions about registration, housing, and exhibits for any Society meetings or conferences (Mathematics Meetings Service Bureau).

*mr-exec@ams.org* to contact the Executive Editor of Mathematical Reviews regarding editorial and related questions.

*mr-librarian@ams.org* to contact Mathematical Reviews regarding the acquisition and cataloging of the mathematical literature indexed in MathSciNet®.

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A man is known by the books he reads. —Emerson

New and Noteworthy Titles on Our BookShelf
February 2017


The books in the Very Short Introductions series of Oxford University Press have an attractive design and are small enough to slip easily into a backpack or purse, or even a coat pocket. Launched in 2000, the series now has more than 550 titles covering a huge range of subjects, from Accounting to Navigation to Zionism. The authors are chosen for their expertise, their writing skill, and their ability to communicate to a general audience. The Very Short Introductions website has a classification for “mathematics” that lists eighteen books (including a few that perhaps don’t belong there, such as Science and Religion, by Thomas Dixon, and omitting a few that perhaps do, such as the book about Bertrand Russell, by A. C. Grayling). One of the early books in the series is Mathematics, by Timothy Gowers, which appeared in 2002. Numbers, by Peter M. Higgins (2011), was reviewed by Rafe Jones in the January 2012 issue of the Notices, and a second title by Higgins, Algebra, appeared in 2015. The newest mathematics titles are Combinatorics, by Robin Wilson (2016), and Leibniz, by Maria Rosa Antognazza (2016). Written by one of the foremost combinatorialists, who possesses a wide knowledge of the area and its history, Combinatorics uses a problem-solving approach to bring readers into the subject and treats a wide range of classic and modern problems. The book on Leibniz, written by the author of the landmark Leibniz: An Intellectual Biography (Cambridge University Press, 2011), provides a panoramic view of the massive oeuvre of this deep and wide-ranging thinker who made contributions to mathematics, physics, metaphysics, and philosophy. Antognazza shows that Leibniz, far from the naïve optimist satirized by Voltaire in Candide, held as the ultimate goal of his work the improvement of the human condition. Master expositor Ian Stewart wrote Symmetry, which came out in 2013, and his next one, Infinity, will appear in 2017.


Zombies and Calculus is a classic of mathematical horror. You might even imagine it is the classic of mathematical horror, with this book being the one and only exemplar of this peculiar genre. But there you would be wrong, as author Colin Adams has mixed horror and mathematics before in his collection of stories, Riot at the Calc Exam and Other Mathematically Bent Stories (AMS, 2009). (And Adams is not the only author to bring the zombie theme into a book about mathematics; in 2010 Penguin Books published The Calculus Diaries: How Math Can Help You Lose Weight, Win in Vegas, and Survive a Zombie Apocalypse, by Jennifer Ouellette.) Originally published in 2014 and now out in paperback, Zombies and Calculus tells the story of a zombie attack that hits the fictional Roberts College in Massachusetts. The zany tale has it all—blood and guts, drama and suspense, romance and humor, and, yes, calculus. The book’s hero is math professor Craig Williams, who uses calculus to battle the ever-increasing army of zombies. The book slips in rigorous discussions of calculus concepts as they apply to problems the characters suddenly find they need to solve in a hurry, such as modeling the paths zombies take as they come at you, estimating the force needed to crack a zombie’s skull, and calculating the rate of spread of the zombie virus. More-technical details of the mathematics can be found in two appendices. The book’s go-for-broke humor and excellent exposition of the mathematics have attracted many fans. Blogger John Dupuis called it “one of the coolest, funniest, most creative science books I’ve read in a very long time.” Colin Adams was interviewed in the Graduate Student section of the November 2016 issue of the Notices.

Suggestions for the BookShelf may be sent to notices-booklist@ams.org.

We try to feature items of broad interest. Appearance of a book in the Notices BookShelf does not represent an endorsement by the Notices or by the AMS. For more, visit the AMS Reviews webpage www.ams.org/news/math-in-the-media/reviews.
Recreational mathematics is a gateway drug. Give the novices a taste of logic puzzles, magic tricks, and perplexing geometric patterns, then, once they’re hooked, move them along to the harder stuff—complex analysis, algebraic geometry, interuniversal Teichmüller theory. As a recruiting tool this strategy has been highly successful. Many a mathematician credits Martin Gardner and his “Mathematical Games” column for inspiring a lifetime of mathematical inquiry.

The Mathematics of Various Entertaining Subjects (MOVES) is recreational math for a different audience. The puzzles and problems may be the same, but in this presentation they are meant to engage the attention of research mathematicians. But that’s certainly not to say there’s no fun here. The volume consists of selected papers from a symposium of the same name held in 2013 at the National Museum of Mathematics in New York.

Of course recreational problems have been inspiring “real” math for a long time. Jennifer Beineke and Jason Rosenhouse, the editors of this volume, point out in their preface that probability theory began with games of chance and that graph theory has roots in puzzles such as Euler’s tour of the Königsberg bridges and Hamilton’s Icosian game. Going farther back, “Among the oldest mathematical documents to have survived to the present is the Egyptian Rhind papyrus, which is largely a collection of ancient brainteasers. The isoperimetric problem is discussed by Virgil in the Aeneid.”

The book’s seventeen chapters survey quite a broad spectrum of problems and puzzles. Robert Bosch, Tim Chartier, and Michael Rowan describe schemes for converting any grayscale image into a maze by solving an optimization problem, such as the traveling salesman problem. Classic coin-weighing puzzles (“Find the counterfeit in no more than $k$ weighings”) are revisited and updated by Tanya Khovanova, who in recent years has shown that this thoroughly mined genre still has a multitude of subtle variations. Julie Beier and Carolyn Yackel find fresh results strewn along another well-trodden path—the study of flexagons, which were the subject of Martin Gardner’s first proto-column in Scientific American sixty years ago. Maureen T. Carroll and Steven T. Dougherty try playing tic-tac-toe on a finite affine plane, where a finite set of points define the positions of $X$s and $O$s, and a finite set of lines (not necessarily straight) mark the potential winning paths.

A review can’t do justice to all these varied topics and the mathematics that lies behind them. I am therefore going to focus on just three selected chapters.

Random Towers of Hanoi

Max A. Alekseyev and Toby Berger consider a variant of the Towers of Hanoi puzzle (introduced in 1883 by Édouard Lucas, a significant figure in both serious and recreational math). The original problem is challenging enough: 64 perforated disks are stacked on a peg, with the largest on the bottom and the diameter diminishing step by step up to the top. Generations of monks labor to transfer the entire heap to another peg, moving one disk at a time. They use a third peg as a temporary holding
area, but at no stage in the process do they ever place a
larger disk above a smaller one on any of the pegs. When
the task is complete, according to Lucas, the world ends.
The minimum number of moves to complete the transfer
is $2^n - 1$, or about $10^{19}$. Alekseyev and Berger ask: Sup-
pose that instead of applying the most efficient algorithm,
the monks shuffle the disks at random—though always
choosing only legal moves. What then is the expected
length of the solution?

It turns out the graph of the Towers problem is a slight
modification of the Sierpiński gasket, a fractal mesh of
triangles within triangles. Each node of this graph repre-
sents a state of the puzzle—a list of which disks occupy
which pegs—and each edge of the graph represents a le-
gal move shifting one disk between pegs. The nodes at the
tree corners of the gasket represent end states where all
the disks are on a single peg. Hence the most efficient
strategy is to follow the sequence of states along a side
of the gasket or, in other words, the shortest path between
two corners.

The random-move strategy corresponds to a random
walk on the graph. If the walk begins at the corner node
with all disks on peg 1, is it guaranteed to someday reach
the corner where all disks are on peg 3? Yes, with proba-
bility 1. But how long does it take on average? Alekseyev
and Berger prove that for a tower of $n$ rings, the answer
is not $2^n - 1$ but rather:

$$E_{1\to3}(n) = \frac{(3^n - 1)(5^n - 3^n)}{2 \cdot 3^{n-1}}.$$  

In the case of $n = 64$, this works out to about $5^{64} \sim 10^{145}$
moves. Should we be surprised by how large this number
is or by how small it is? Alekseyev and Berger remark:

[R]eplacing the minimum-moves strategy with a
random walk forestalls the end of the world by a
factor of roughly $(\frac{5}{2})^{64} > 2.9 \times 10^{25}$ on average.
Although this is reassuring, it would provide further
comfort to know that the coefficient of varia-
tion of the random number of steps...is small (i.e.,
that its standard deviation is many times smaller

than its mean). Exact determination of said coeffi-
cient of variation is an open problem that we may
address in future research.

an adventurous transposition of
the problem into quite a different realm

Alekseyev and Berger give two proofs of the
mean random walk length. One proof is a conven-
tional, recursive calculation of probabilities. The
other is an adventurous transposition of the prob-
lem into quite a different realm. They model the
gasket as a network of electrical resistors, an idea imported
from electrical engineering by Peter G.
Doyle and J. Laurie Snell in the 1980s. Suppose every
edge of the Sierpiński graph is a 1 ohm resistor and
you apply a voltage to two corners of the graph, node
1 and node 3. If you then calculate or measure the cur-
rent flowing through each resistor, the result indicates
the relative frequency with which a random walker will
pass along that edge of the graph. In itself, this strategy
is no improvement over the more straightforward calcu-
lus of probabilities. But a rearrangement of the graph
known to electrical engineers as the delta-wye transforma-
tion greatly simplifies the problem. As the name suggests,
the transformation changes a delta, or triangle $\Delta$,
into a Y-shaped network. When the delta-wye transfor-
mation is applied recursively, the entire network is reduced to one
big Y, and calculating the resistance between two corners
is a simple summation.

Heartless Poker

In poker, hands are valued in inverse order of expected
frequencies. Roughly 42 percent of all possible 5-card
hands feature a single pair (i.e., 2 cards of the same rank and nothing else of value). Fewer than 5 percent have two pairs, and about 2 percent have three of a kind. Based on this ranking, even novice players quickly learn that three of a kind beats two pair, which in turn beats one pair. The ordering of some higher-ranking hands is less obvious. For one thing, those hands are so rare that most players have little chance to develop much intuition about them. Dominic Lanphier and Laura Taalman point out another reason for occasional confusion about the ranking of the straight, the flush, and the full house: their frequencies are not dramatically different. A straight (5 cards in sequence) is less than twice as common as a flush (5 cards of the same suit). A flush is only 1.4 times as common as a full house (three of a kind plus a pair).

Lanphier and Taalman ask whether changes to the game—and in particular changes to the composition of the deck of cards—might alter the rankings of certain hands. A standard 52-card deck has four suits and thirteen ranks. Suppose we remove all the hearts from the deck and play a game of “heartless poker” with just 39 cards. Eliminating a suit makes it much easier to draw a flush, enough so that a straight becomes rarer and more valuable. Going in the other direction, a “fat pack” of cards is double the size of a normal deck, with eight suits and the usual thirteen ranks. Playing with that 104-card deck makes a flush rarer and more valuable than a full house. Lanphier and Taalman go on to show that all six possible orderings of those three hands can be achieved by some variation on the standard card deck.

Then comes the question that takes us slightly beyond the usual turf of recreational mathematics: Is there any deck for which two of these hands (or all three) have the same probability and thus should be valued equally? Taking the number of ranks \(r\) and the number of suits \(s\) to be continuous variables, they define three curves on the \(rs\) plane where the straight, the flush, and the full house are pairwise equal in frequency. Furthermore, they show that the three curves have a unique point of intersection. In other words, there exists a pair of \(r\) and \(s\) values for which the straight, flush, and full house are all equally probable. However, those values of \(r\) and \(s\) are not integers, so we can’t actually create a physical deck of cards with this property. Further Diophantine analysis shows that there are no integer numbers of ranks and suits where any two of the three hands are of equal value.

Critical Phenomena in Crossword Puzzles

Crossword puzzles are a pleasant diversion for many of us, but they seem to draw mainly on linguistic skills and general knowledge, without much mathematical content. John K. McSweeney demonstrates that even if mathematical analysis won’t help you solve the crossword in Sunday’s newspaper, it can illuminate the inner structure of the puzzle, help measure its difficulty, and explain why completing it is satisfying (or not). McSweeney writes:

What distinguishes a crossword puzzle from a simple list of trivia questions is that the answers are entered into a grid in crossing fashion, and therefore each correct answer obtained provides partial information about others...Indeed, even if there are only a few easy answers that can be found immediately, these may trigger further answers, and, in such a cascading fashion, many or all of the answers in the puzzle may be found.

McSweeney represents the puzzle as a bipartite graph. The nodes of the graph are the answers to the puzzle clues. Two nodes are connected by an edge if the corresponding answers cross within the puzzle diagram and hence have a letter in common. The graph is bipartite because only Across and Down answers can intersect.

**Remove all the hearts from the deck for a game of “heartless poker.”**
Knowing some of the letters in an answer should generally make it easier to fill in the rest. This fact leads to a nonlinearity in the puzzle-solving process that McSweeney investigates in detail. He models the solver’s task as a purely probabilistic process. Each clue $x$ is assigned a difficulty threshold $\varphi_x \in \mathbb{R}$. The answer to clue $x$ is revealed only if the proportion of letters already known from cross-clues is at least $\varphi_x$. If $\varphi_x \leq 0$, the answer is known immediately. If $\varphi_x = \frac{1}{2}$, the answer will become clear when half the letters have been revealed. When $\varphi_x \geq 1$, the answer is a stumper that you can’t understand even when all the letters are filled in.

Between the easy and the impossible are puzzles exhibiting random fluctuations on a large scale.

In most of his experiments and analyses McSweeney assumes that the difficulty thresholds are drawn from a Gaussian distribution, with a mean $\mu$ greater than zero and a standard deviation $\sigma$ wide enough that at least a few clues can be answered without help from orthogonal answers. The simplest and most symmetric case is a square grid without any black squares so that all Across and Down answers intersect one another. In this type of puzzle (almost never seen in the wild), there are parameter ranges for $\mu$ and $\sigma$ where most puzzles are boringly easy or frustratingly hard, but there’s also a more interesting region with a bimodal distribution: The solving algorithm may get stuck early and make very little progress, but if it completes a certain fraction of the grid, it will almost surely go on to solve the entire puzzle. I suspect this bimodal pattern has a lot to do with the pleasure of working crossword puzzles: The solver wants a challenge but also wants the satisfaction of completing the task.

McSweeney also runs simulations based on actual puzzles from the Sunday Times. In these grids, patterns of black squares break the puzzle into loosely coupled regions, so that only subsets of Across and Down answers intersect. Because of these barriers, it’s not uncommon to solve most of a puzzle but be stymied in a few corners where neither Across nor Down clues yield up their secrets. The bimodal distribution of outcomes is still in evidence, but it takes a somewhat different form. There are easy puzzles and impossible puzzles, but those in between exhibit random fluctuations on a large scale, with big blocks of the puzzle solved but others left blank. The patterns resemble those seen in fluids or magnetic materials near a critical point. In crosswords, the critical parameter values seem to be near $\mu = 0.3, \sigma = 0.23$.

**Conclusion**

Although MOVES addresses an audience of mathematical sophisticates, almost all the chapters are readily accessible to students and amateurs. Where proofs are given, they are explained in detail. Some knowledge of group theory is helpful in the chapter on flexagons; graph theory and probability turn up in many contexts. In general, though, all that’s needed to appreciate this work is a little facility in mathematical reasoning, and a dose of enthusiasm.

At a time when the public is once again debating the utility of mathematics—Do plumbers need to know algebra? Will calculus get you a better job?—it’s a relief to open this window on this less fretful cosmos, where mathematics is a source of understanding and wonder.

**Image and Photo Credits**


Figure 3 is courtesy of Dominic Lanphier and Laura Taalman.

Figures 4 and 5 are courtesy of John K. McSweeney.

Photo of Brian Hayes is courtesy of Harvard University.

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**ABOUT THE REVIEWER**

Brian Hayes’s next book, *Foolproof, and Other Mathematical Meditations*, will be published next year by MIT Press.

Brian Hayes
New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to www.ams.org/bookstore-email.

Algebra and Algebraic Geometry

**Birationally Rigid Fano Threefold Hypersurfaces**

Ivan Cheltsov, University of Edinburgh, United Kingdom, and Jihun Park, Pohang University of Science and Technology, South Korea

Contents: Introduction; Smooth points and curves; Singular points; Birational involutions; Proof of main theorem; Epilogue; Bibliography.

Memoirs of the American Mathematical Society, Volume 246, Number 1167


**Algebra in Action**

A Course in Groups, Rings, and Fields

Shahriar Shahriari, Pomona College, Claremont, CA

This text—based on the author's popular courses at Pomona College—provides a readable, student-friendly, and somewhat sophisticated introduction to abstract algebra. It is aimed at sophomore or junior undergraduates who are seeing the material for the first time. In addition to the usual definitions and theorems, there is ample discussion to help students build intuition and learn how to think about the abstract concepts. The book has over 1300 exercises and mini-projects of varying degrees of difficulty, and, to facilitate active learning and self-study, hints and short answers for many of the problems are provided.

There are full solutions to over 100 problems in order to augment the text and to model the writing of solutions. Lattice diagrams are used throughout to visually demonstrate results and proof techniques. The book covers groups, rings, and fields. In group theory, group actions are the unifying theme and are introduced early. Ring theory is motivated by what is needed for solving Diophantine equations, and, in field theory, Galois theory and the solvability of polynomials take center stage. In each area, the text goes deep enough to demonstrate the power of abstract thinking and to convince the reader that the subject is full of unexpected results.

Contents: (Mostly finite) group theory: Four basic examples; Groups: The basics; The alternating groups; Group actions; A subgroup acts on the group: Cosets and Lagrange’s theorem; A group acts on itself: Counting and the conjugation of action; Acting on subsets, cosets, and subgroups: The Sylow theorems; Counting the number of orbits; The lattice of subgroups; Acting on its subgroups: Normal subgroups and quotient groups; Group homomorphisms; Using Sylow theorems to analyze finite groups; Direct and semidirect products; Solvable and nilpotent groups; (Mostly commutative) ring theory: Rings; Homomorphisms, ideals, and quotient rings; Field of fractions and localization; Factorization, EDs, PIDs, and UFDs; Polynomial rings; Gaussian integers and (a little) number theory; Field and Galois theory: Introducing field theory and Galois theory; Field extensions; Straightedge and compass constructions; Splitting fields and Galois groups; Galois, normal, and separable extensions; Fundamental theorem of Galois theory; Finite fields and cyclotomic extensions; Radical extensions, solvable groups, and the quintic; Hints for selected problems; Short answers for selected problems; Complete solutions for selected (odd-numbered) problems; Bibliography; Index.

Pure and Applied Undergraduate Texts, Volume 27

This volume contains the proceedings of the workshop on Analysis and Geometry in Several Complex Variables, held from January 4-8, 2015, at Texas A&M University at Qatar, Doha, Qatar.

This volume covers many topics of current interest in several complex variables, CR geometry, and the related area of overdetermined systems of complex vector fields, as well as emerging trends in these areas.

Papers feature original research on diverse topics such as the rigidity of CR mappings, normal forms in CR geometry, the d-bar Neumann operator, asymptotic expansion of the Bergman kernel, and hypoellipticity of complex vector fields. Also included are two survey articles on complex Brunn-Minkowski theory and the regularity of systems of complex vector fields and their associated Laplacians.

Contents: B. Berndtsson, Real and complex Brunn-Minkowski theory; C. Campana, P. L. Dattori da Silva, and A. Meziani, Properties of solutions of a class of hypocomplex vector fields; M. Çelik and Y. E. Zeytuncu, Analysis on the intersection of pseudoconvex domains; D. Chakrabarti and R. Shafikov, Distributional boundary values; some new perspectives; G. Della Sala, B. Lamel, and M. Reiter, Infinitesimal and local rigidity of mappings of CR manifolds; M. Derridj, On some systems of real or complex vector fields and their related Laplacians; P. Ebenfelt, On the HJY gap conjecture in CR geometry vs. the SOS conjecture for polynomials; P. Gupta, Lower-dimensional Fefferman measures via the Bergman kernel; M. Kolar, I. Kossovskiy, and D. Zaitsev, Normal forms in Cauchy-Riemann geometry; S. Seto, Bergman kernel asymptotics through perturbation.

Contemporary Mathematics, Volume 681

Discrete Mathematics and Combinatorics

Game Theory, Alive
Anna R. Karlin, University of Washington, Seattle, WA, and Yuval Peres, Microsoft Research, Redmond, WA

We live in a highly connected world with multiple self-interested agents interacting and myriad opportunities for conflict and cooperation. The goal of game theory is to understand these opportunities.

This book presents a rigorous introduction to the mathematics of game theory without losing sight of the joy of the subject. This is done by focusing on theoretical highlights (e.g., at least six Nobel Prize winning results are developed from scratch) and by presenting exciting connections of game theory to other fields such as computer science (algorithmic game theory), economics (auctions and matching markets), social choice (voting theory), biology (signaling and evolutionary stability), and learning theory. Both classical topics, such as zero-sum games, and modern topics, such as sponsored search auctions, are covered. Along the way, beautiful mathematical tools used in game theory are introduced, including convexity, fixed-point theorems, and probabilistic arguments.

The book is appropriate for a first course in game theory at either the undergraduate or graduate level, whether in mathematics, economics, computer science, engineering, or statistics. The importance of game-theoretic thinking transcends the academic setting—for every action we take, we must consider not only its direct effects, but also how it influences the incentives of others.

Game theory’s influence is felt in a wide range of disciplines, and the authors deliver masterfully on the challenge of presenting both the breadth and coherence of its underlying world-view. The book achieves a remarkable synthesis, introducing the reader to the blend of economic insight, mathematical elegance, scientific impact, and counter-intuitive punch that characterizes game theory as a field.

— Jon Kleinberg, Cornell University, 2006 Nevanlinna Prize winner

A game theory textbook by people who love ... games! It covers many classic as well as recent topics of game theory. Its rigorous treatment, interspersed with illuminating examples, makes it a challenging pleasure to read.

— Sergiu Hart, The Hebrew University of Jerusalem
General Interest

**Pushing Limits**
From West Point to Berkeley and Beyond

Ted Hill, Georgia Tech, Atlanta, GA, and Cal Poly, San Luis Obispo, CA

*Pushing Limits: From West Point to Berkeley and Beyond* challenges the myth that mathematicians lead dull and ascetic lives. It recounts the unique odyssey of a noted mathematician who overcame military hurdles at West Point, Army Ranger School and the Vietnam War, and survived many civilian escapades—hitchhiking in third-world hotspots, fending off sharks in Bahamian reefs, and camping deep behind the forbidding Iron Curtain. From ultra-conservative West Point in the ’60s to ultra-radical Berkeley in the ’70s, and ultimately to genteeled Georgia Tech in the ’80s, this is the tale of an academic career as noteworthy for its offbeat adventures as for its teaching and research accomplishments. It brings to life the struggles and risks underlying mathematical research, the unparallelled thrill of making scientific breakthroughs, and the joy of sharing those discoveries around the world. Hill’s book is packed with energy, humor, and suspense, both physical and intellectual. Anyone who is curious about how a maverick mathematician thinks, who wants to relive the zaniest side of the ’60s and ’70s, who wants an armchair journey into the third world, or who seeks an unconventional viewpoint about some of our more revered institutions, will be drawn to this book.

*... captivating memoir reveals an intriguing character who is part Renaissance Man, part Huckleberry Finn. Fast-paced and often hilarious ... provides some penetrating and impious insights into some of our more revered institutions.*

—Rick Atkinson, three-time Pulitzer Prize winner, author of *The Long Gray Line*

Ted Hill is unique in having both a very exciting internal mathematical life ... and an action-filled, adventurous, external life. ... his natural gift, very rare for mathematicians, of story-telling, [makes this] a page-turner.

—Doron Zeilberger, Rutgers University, winner of MAA Ford Prize, AMS Steele Prize, and ICA Euler Medal

Thoughtful, funny, evocative, Ted Hill, takes us through a life well-lived ... an intensely personal story that will appeal to every profession—and to every generation!

—General Wesley Clark, former NATO Supreme Commander

Ted Hill is an original. Mathematician. Adventurer. Activist. His life has seen both his mind and body tested to extremes ... insightful, entertaining and—in a very good way—unlike any other book you will ever read by a mathematician.

—Alex Bellos, author of *Here’s Looking at Euclid and The Grapes of Math*

This book is co-published with the Mathematical Association of America.

*Contents:* Photo section; Day of the handshakes; The star years; Out of the gates; Preparing for war; Vietnam; Return to reason; The Fulbright interlude; *Berzerkeley*; The apprenticeship; *Eurekas*; The global math guild; The math *Ohana*; The Penn State syndrome; Permanent sabbatical; Postscript.


**Geometry and Topology**

**Manifolds and K-Theory**

Gregory Arone, University of Virginia, Charlottesville, VA, Brenda Johnson, Union College, Schenectady, NY, Pascal Lambrechts, Université Catholique de Louvain, Louvain-La-Neuve, Belgium, Brian A. Munson, United States Naval Academy, Annapolis, MD, and Ismar Volić, Wellesley College, MA, Editors

This volume contains the proceedings of the conference on Manifolds, K-Theory, and Related Topics, held from June 23–27, 2014, in Dubrovnik, Croatia.

The articles contained in this volume are a collection of research papers featuring recent advances in homotopy theory, K-theory, and their applications to manifolds. Topics covered include homotopy and manifold calculus, structured spectra, and their applications to group theory and the geometry of manifolds. This volume is a tribute to the influence of Tom Goodwillie in these fields.

*Contents:* G. Arone and M. Ching, Manifolds, K-theory and the calculus of functors; J. E. Bergner and P. Hackney, Diagrams...
encoding group actions on $\Gamma$-spaces; S. Chang, S. Weinberger, and G. Yu, Contractible manifolds with exotic positive scalar curvature behavior; E. D. Farjoun and Y. Segev, Relative Schur multipliers and universal extensions of group homomorphisms; T. G. Goodwillie, Scissors congruence with mixed dimensions; J. R. Klein and S. Tilson, On the moduli space of $\mathcal{A}_\infty$-structures; J. Noel, Nilpotence in the symplectic bordism ring; K. E. Pelatt and D. P. Sinha, A geometric homology representative in the space of knots; S. Tillmann and M. S. Weiss, Occupants in manifolds.

Contemporary Mathematics, Volume 682

Exotic Cluster Structures on $SL_n$: The Cremmer-Gervais Case

M. Gekhtman, University of Notre Dame, IN, M. Shapiro, Michigan State University, East Lansing, MI, and A. Vainshtein, University of Haifa, Israel

This item will also be of interest to those working in algebra and algebraic geometry.

Contents: Introduction; Cluster structures and Poisson-Lie groups; Main result and the outline of the proof; Initial cluster; Initial quiver; Regularity; Quiver transformations; Technical results on cluster algebras; Bibliography.

Memoirs of the American Mathematical Society, Volume 246, Number 1165

Abelian Properties of Anick Spaces

Brayton Gray, University of Illinois, Chicago

Contents: Introduction; Abelian structures; Whitehead products; Index $p$ approximation; Simplification; Constructing $\mathfrak{y}_n$; Universal properties; Appendix A. The Case $n = 1$ and the Case $p = 3$; Bibliography; List of symbols.

Memoirs of the American Mathematical Society, Volume 246, Number 1162

Logic and Foundations

New Foundations for Geometry–Two Non-Additive Languages for Arithmetical Geometry

Shai M. J. Haran, Technion-Israel Institute of Technology, Technion Haifa, Israel

This item will also be of interest to those working in geometry and topology.

Contents: Introduction; Part I. $F$-Rings: Definition of $F$-Rings; Appendix A; Examples of $F$-Rings; Appendix B; Geometry; Symmetric geometry; Pro - limits; Vector bundles; Modules; Part II. Generalized Rings: Generalized Rings; Ideals; Primes and spectra; Localization and sheaves; Schemes; Products; Modules and differentials; Appendix C; Bibliography.

Memoirs of the American Mathematical Society, Volume 246, Number 1166

Mathematical Physics

It’s About Time

Elementary Mathematical Aspects of Relativity

Roger Cooke, University of Vermont, Burlington, VT

This book has three main goals. First, it explores a selection of topics from the early period of the theory of relativity, focusing on particular aspects that are interesting or unusual. These include the twin paradox; relativistic mechanics and its interaction with Maxwell’s laws; the earliest triumphs of general relativity relating to the orbit of Mercury and the deflection of light passing near the sun; and the surprising bizarre metric of Kurt Gödel, in which time travel is possible. Second, it provides an exposition of the differential geometry needed to understand these topics on a level that is intended to be accessible to those with just two years of university-level mathematics as background. Third, it
reflects on the historical development of the subject and its significance for our understanding of what reality is and how we can know about the physical universe. The book also takes note of historical prefigurations of relativity, such as Euler’s 1744 result that a particle moving on a surface and subject to no tangential acceleration will move along a geodesic, and the work of Lorentz and Poincaré on space-time coordinate transformations between two observers in motion at constant relative velocity.

The book is aimed at advanced undergraduate mathematics, science, and engineering majors (and, of course, at any interested person who knows a little university-level mathematics). The reader is assumed to know the rudiments of advanced calculus, a few techniques for solving differential equations, some linear algebra, and basics of set theory and groups.

Contents: The special theory: Time, space, and space-time; Relativistic mechanics; Electromagnetic theory; The general theory: Precession and deflection; Concepts of curvature, 1700–1850; Concepts of curvature, 1850–1950; The geometrization of gravity; Historical and philosophical context: Experiments, chronology, metaphysics; Bibliography; Subject index; Name index.


Number Theory

On Dwork’s $p$-Adic Formal Congruences Theorem and Hypergeometric Mirror Maps

E. Delaygue, Université Claude Bernard Lyon 1, Villeurbanne, France, T. Rivoal, CNRS and Université Grenoble Alpes, France, and J. Roques, CNRS and Université Grenoble Alpes, France

Contents: Introduction; Statements of the main results; Structure of the paper; Comments on the main results, comparison with previous results and open questions; The $p$-adic valuation of Pochhammer symbols; Proof of Theorem 4; Formal congruences; Proof of Theorem 6; Proof of Theorem 9; Proof of Theorem 12; Proof of Theorem 8; Proof of Theorem 10; Proof of Corollary 14; Bibliography.

Memoirs of the American Mathematical Society, Volume 246, Number 1163

This section contains new announcements of worldwide meetings and conferences of interest to the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. New announcements only are published in the print Mathematics Calendar featured in each Notices issue.

An announcement will be published in the Notices if it contains a call for papers and specifies the place, date, subject (when applicable). A second announcement will be published only if there are changes or necessary additional information. Asterisks (*) mark those announcements containing revised information.

In general, print announcements of meetings and conferences carry only the date, title and location of the event.

The complete listing of the Mathematics Calendar is available at: [www.ams.org/meetings/calendar/mathcal](http://www.ams.org/meetings/calendar/mathcal)

All submissions to the Mathematics Calendar should be done online via: [www.ams.org/cgi-bin/mathcal/mathcal-submit.pl](http://www.ams.org/cgi-bin/mathcal/mathcal-submit.pl)

Any questions or difficulties may be directed to mathcal@ams.org.

February 2017

27 – March 3  Spring School: From Particle Dynamics to Gradient Flows
Location: University of Kaiserslautern, Kaiserslautern, Germany.
URL: [www.mathematik.uni-kl.de/events/particleschool2017](http://www.mathematik.uni-kl.de/events/particleschool2017)

March 2017

8 – 11  51st Spring Topology and Dynamical Systems Conference (STDC 2017)
Location: NJCU (School of Business), Jersey City, NJ, USA.
URL: [https://sites.google.com/site/2017springconftopanddynamics/home](https://sites.google.com/site/2017springconftopanddynamics/home)

25 – 26  Seventh Ohio River Analysis Meeting (ORAM 7)
Location: University of Cincinnati, Cincinnati, Ohio.
URL: [homepages.uc.edu/~goldbeml/ORAM/ORAM7](http://homepages.uc.edu/~goldbeml/ORAM/ORAM7)

27 – April 7  Combinatorics on Words and Tilings
Location: Entre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt 2920, Chemin de la tour, 5th floor, Montréal (Québec) H3T 1J4 Canada.
URL: [www.crm.umontreal.ca/2017/Pavages17/index_e.php](http://www.crm.umontreal.ca/2017/Pavages17/index_e.php)

April 2017

24 – May 1  Bridges between Automatic Sequences, Algebra, and Number Theory
Location: Centre de recherches mathémátiques, Université de Montréal, Pavillon André-Aisenstadt, 2920, Chemin de la tour, 5th floor, Montréal (Québec) H3T 1J4 Canada.
URL: [www.crm.umontreal.ca/2017/Suites17/index_e.php](http://www.crm.umontreal.ca/2017/Suites17/index_e.php)

May 2017

8 – 11  The Cape Verde International Days on Mathematics 2017
Location: Cidade da Praia, Cape Verde.
URL: [sites.google.com/site/cvim2017](http://sites.google.com/site/cvim2017)

29 – June 2  Fourth International Workshop on Zeta Functions in Algebra and Geometry
Location: Center for Interdisciplinary Research, Bielefeld University Bielefeld, Germany.
URL: [www.math.uni-bielefeld.de/sfb701/2017_ZFW](http://www.math.uni-bielefeld.de/sfb701/2017_ZFW)

29 – June 5  Algebraic and Geometric Combinatorics of Reflection Groups
Location: Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, 2920, Chemin de la tour, 5th floor, Montréal (Québec) H3T 1J4 Canada.
URL: [www.crm.umontreal.ca/2017/Reflexion17/index_e.php](http://www.crm.umontreal.ca/2017/Reflexion17/index_e.php)

June 2017

5 – 9  International Conference on Special Functions: Theory, Computation, and Applications
Location: Peter Hô Lecture Theatre (LT-10), 4/F, Academic 1, City University of Hong Kong, Hong Kong.

5 – 10  Contemporary Aspects, Overview, and Outlook on Knots: International Early Summer School
Location: University of Freiburg, Freiburg, Germany.
URL: [home.mathematik.uni-freiburg.de/cookies17](http://home.mathematik.uni-freiburg.de/cookies17)

10 – 13  Equivariant Combinatorics
Location: Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, 2920, Chemin de la tour, 5th floor, Montréal (Québec) H3T 1J4 Canada.
URL: [www.crm.umontreal.ca/2017/Equivariant17/index_e.php](http://www.crm.umontreal.ca/2017/Equivariant17/index_e.php)

15 – 16  From Solid Mechanics to Mathematical Analysis: A Workshop on the Occasion of Gilles Francfort’s Sixtieth Birthday
Location: Institut Poincaré, Paris France.
URL: [gaf60.org](http://gaf60.org)
Location: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.  
URL: www.math.bas.bg/ntades

July 2017

12 – 14 Workshop on Infinity-Operads and Applications  
Location: Osnabrück, Germany.  
URL: userpage.fu-berlin.de/jcirici/infinityoperads

20 – 31 XX Summer Diffiety School  
Location: Ex Colonia Ferrarese, Lizzano in Belvedere (BO), Italy.  
URL: xx-summer-diffiety-school

August 2017

14 – 18 Topology Ecuador 2017  
Location: Galápagos Science Center, San Cristóbal, Galápagos, Ecuador.  
URL: www.usfq.edu.ec/eventos/topology/Paginas/default.aspx

September 2017

19 – 21 IMA Conference on Inverse Problems from Theory to Application  
Location: Isaac Newton Institute, Cambridge, UK.  
URL: ima.org.uk/conferences/conferences_calendar/inverse-problems.html

August 2017

19 – 21 IMA Conference on Inverse Problems from Theory to Application  
Location: Isaac Newton Institute, Cambridge, UK.  
URL: ima.org.uk/conferences/conferences_calendar/inverse-problems.html

September 2017

19 – 21 IMA Conference on Inverse Problems from Theory to Application  
Location: Isaac Newton Institute, Cambridge, UK.  
URL: ima.org.uk/conferences/conferences_calendar/inverse-problems.html

July 2018

5 – 9 The 12th AIMS Conference on Dynamical Systems, Differential Equations and Applications  
Location: National Taiwan University, Taipei, Taiwan.  
URL: aimsciences.org/conferences/2018
MEETINGS & CONFERENCES OF THE AMS

FEBRUARY TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: www.ams.org/meetings/.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 75 in the January 2017 issue of the Notices for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of \( \text{LaTeX} \) is necessary to submit an electronic form, although those who use \( \text{LaTeX} \) may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in \( \text{LaTeX} \). Visit www.ams.org/cgi-bin/abstracts/abstract.pl/. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

See www.ams.org/meetings/ for the most up-to-date information on these conferences.

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ASSOCIATE SECRETARIES OF THE AMS

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; e-mail: steve.weixintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.
Meetings & Conferences of the AMS

Charleston, South Carolina

College of Charleston

March 10–12, 2017
Friday – Sunday

Meeting #1126
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: Volume 38, Issue 2

Deadlines
For organizers: Expired
For abstracts: January 17, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Pramod N. Achar, Louisiana State University, Representations of algebraic groups via algebraic topology.
Hubert L. Bray, Duke University, The Geometry of Special and General Relativity.
Alina Chertock, North Carolina State University, Numerical methods for chemotaxis and related models.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Active Learning in Undergraduate Mathematics (Code: SS 21A), Draga Vidakovic, Georgia State University, Harrison Stalvey, University of Colorado, Boulder, and Darryl Chamberlain, Jr., Aubrey Kemp, and Leslie Meadows, Georgia State University.
Advances in Long-term Behavior of Nonlinear Dispersive Equations (Code: SS 27A), Brian Pigott, Wofford College, and Sarah Raynor, Wake Forest University.
Advances in Nonlinear Waves: Theory and Applications (Code: SS 23A), Constance M. Schober, University of Central Florida, and Andrei Ludu, Embry Riddle University.
Algebras, Lattices, Varieties (Code: SS 19A), George F. McNulty, University of South Carolina, and Kate S. Owens, College of Charleston.
Analysis and Control of Fluid-Structure Interactions and Fluid-Solid Mixtures (Code: SS 6A), Justin T. Webster, College of Charleston, and Daniel Toundykov, University of Nebraska-Lincoln.
Analysis, Control and Stabilization of PDE’s (Code: SS 13A), George Avalos, University of Nebraska-Lincoln, and Scott Hansen, Iowa State University.
Bicycle Track Mathematics (Code: SS 25A), Ron Perline, Drexel University.
Coding Theory, Cryptography, and Number Theory (Code: SS 17A), Jim Brown, Shuhong Gao, Kevin James, Felice Manganiello, and Gretchen Matthews, Clemson University.
Commutative Algebra (Code: SS 1A), Bethany Kubik, University of Minnesota Duluth, Saeed Nasrseh, Georgia Southern University, and Sean Sather-Wagstaff, Clemson University.
Bloomington, Indiana

Indiana University

April 1–2, 2017
Saturday - Sunday

Meeting #1127
Central Section

Associate secretary: Georgia Benkart

Announcement issue of Notices: February 2017

Program first available on AMS website: February 23, 2017

Issue of Abstracts: Volume 38, Issue 2

Deadlines

For organizers: Expired

For abstracts: February 7, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Ciprian Demeter, Indiana University, Title to be announced.

Sarah Koch, University of Michigan, Title to be announced.

Richard Evan Schwartz, Brown University, Modern scratch paper: Graphical explorations in geometry and dynamics (Einstein Public Lecture in Mathematics).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Enumerative Combinatorics with Applications (Code: SS 6A), Saúl A. Blanco, Indiana University, and Kyle Peterson, DePaul University.

Analysis and Numerical Computations of PDEs in Fluid Mechanics (Code: SS 20A), Gung-Min Gie, University of Louisville, and Makram Hamouda and Roger Temam, Indiana University.

Analysis of Variational Problems and Nonlinear Partial Differential Equations (Code: SS 11A), Nam Q. Le and Peter Sternberg, Indiana University.

Automorphic Forms and Algebraic Number Theory (Code: SS 2A), Patrick B. Allen, University of Illinois at Urbana-Champaign, and Matthias Strauch, Indiana University Bloomington.

Commutative Algebra (Code: SS 19A), Ela Celikbas and Olgur Celikbas, West Virginia University.

Computability and Inductive Definability over Structures (Code: SS 3A), Siddharth Bhaskar, Lawrence Valby, and Alex Kruckman, Indiana University.

Dependence in Probability and Statistics (Code: SS 7A), Richard C. Bradley and Lanh T. Tran, Indiana University.
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<td>Discrete Structures in Conformal Dynamics and Geometry</td>
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<td>Sarah Koch, University of Michigan, and Kevin Pilgrim and Dylan Thurston, Indiana University.</td>
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<td>Extreme Problems in Graphs, Hypergraphs and Other Combinatorial Structures</td>
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<td>Harmonic Analysis and Partial Differential Equations</td>
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<td>Homotopy Theory</td>
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<td>Model Theory</td>
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<td>Multivariate Operator Theory and Function Theory</td>
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<td>Hari Bercovici, Indiana University, Kelly Bickel, Bucknell University, Constanze Liaw, Baylor University, and Alan Sola, Stockholm University.</td>
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<td>Nonlinear Elliptic and Parabolic Partial Differential Equations and Their Various Applications</td>
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<td>Changyou Wang, Purdue University, and Yifeng Yu, University of California, Irvine.</td>
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<td>Probabilistic Methods in Combinatorics</td>
<td>SS 22A</td>
<td>Patrick Bennett and Andrzej Dudek, Western Michigan University.</td>
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<td>Probability and Applications</td>
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<td>Russell Lyons and Nick Travers, Indiana University.</td>
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<td>Randomness in Complex Geometry</td>
<td>SS 1A</td>
<td>Turgay Bayraktar, Syracuse University, and Norman Levenberg, Indiana University.</td>
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<tr>
<td>Representation Stability and its Applications</td>
<td>SS 23A</td>
<td>Patricia Hersh, North Carolina State University, Jeremy Miller, Purdue University, and Andrew Putman, University of Notre Dame.</td>
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<td>Representation Theory and Integrable Systems</td>
<td>SS 18A</td>
<td>Eugene Mukhin, Indiana University, Purdue University Indianapolis, and Vitaly Tarasov, Indiana University, Purdue University Indianapolis.</td>
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<td>Self-similarity and Long-range Dependence in Stochastic Processes</td>
<td>SS 10A</td>
<td>Takashi Owada, Purdue University, Yi Shen, University of Waterloo, and Yizao Wang, University of Cincinnati.</td>
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<td>Spectrum of the Laplacian on Domains and Manifolds</td>
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<td>Chris Judge and Sugata Mondal, Indiana University.</td>
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<td>Topics in Extremal, Probabilistic and Structural Graph Theory</td>
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<td>John Engbers, Marquette University, and David Galvin, University of Notre Dame.</td>
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<td>Topological Mathematical Physics</td>
<td>SS 17A</td>
<td>E. Birgit Kaufmann and Ralph M. Kaufmann, Purdue University, and Emil Prodan, Yeshiva University.</td>
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**Accommodations**

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the Indiana state hotel tax (12%). Participants must state that they are with the American Mathematical Society (AMS) Meeting at Indiana University. Several penalties exist for rate changes or for the quality of the accommodations. Hotels have varying cancellation and early checkout policies; be sure to ask for details.

**Courtyard Bloomington**

310 S. College Avenue, Bloomington, Indiana 47403; 812-335-8000; [http://www.marriott.com/hotels/travel/bmgcy-courtyard-bloomington/?scid=bb1a189a-fec3-4d19-a255-54ba596f6ebe2](http://www.marriott.com/hotels/travel/bmgcy-courtyard-bloomington/?scid=bb1a189a-fec3-4d19-a255-54ba596f6ebe2). Rates are US$169 per night, these rates are applicable for single or double occupancy. Amenities include complimentary wireless Internet; coffee; morning newspaper; indoor pool; whirlpool; complimentary parking on-site; and a well-equipped fitness room. This property is about a 13-minute walk from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **February 28, 2017**.

**Hilton Garden Inn, Bloomington**

245 North College Ave., Bloomington, IN 47404; 812-331-1335; hiltongardeninn3.hilton.com/en/hotels/indiana/hilton-garden-inn-bloomington-BMGINGI/index.html. Rates are US$189 per night for a single or double occupancy, king or double-queen room. Amenities include complimentary Wi-Fi; indoor pool; fitness center; business center; and on-site full-service restaurant. Rooms are furnished with a fridge/microwave and the hotel offers a 24-hour Pavilion Pantry for last-minute snacks and necessities. Self-parking is available at US$10 per day. This property is located approximately a half mile from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **March 1, 2017**.

**Hyatt Place Bloomington**

MEETINGS & CONFERENCES

Rates are US$159 per night for single or double occupancy room. Amenities include complimentary Wi-Fi; on-site Coffee to Cocktails Bar serving specialty Starbucks® coffee, local beer choices, wine and cocktails; and complimentary A.M. Kitchen Skillet™ featuring freshly prepared breakfast sandwiches, steel-cut oatmeal, fresh cut fruit and more, available daily in the Guest Kitchen. Self-parking is available at US$10 per day. This property is a 10-minute walk to Indiana University. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at a reduced rate is March 1, 2017.

Indiana Memorial Union Biddle Hotel, 900 E. Seventh Street, Bloomington, IN 47405; 1-800-209-8145 or 812-856-6381; https://imu.indiana.edu/hotel/. Rates for king and double queen rooms are US$189 per night and rates for single queen rooms are US$139 per night. Overnight guests of the Biddle Hotel enjoy complimentary self-parking in both pay lots adjacent to the building. Both lots are open from 7 am to midnight, seven days a week. The Indiana Memorial Union is located on the Indiana University campus and amenities include numerous on-site dining options; 24-hour fitness center; bell service; and room service. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. Individuals can make reservations by calling 800-209-8145 or 812-855-2536. For online reservations please visit www.imu.indiana.edu, click “Biddle Hotel”, click “accommodations”, scroll down to “reserve a room”, enter desired arrival / departure dates, and enter group code: AMS17. The deadline for reservations at a reduced rate is March 1, 2017.

America’s Best Value Inn, 1722 North Walnut Street, Bloomington, IN 47404; 812-339-1919; www.americasbestvalueinn.com/bestv.cfm?idp=1893. This property is offering special rates to meeting participants. Rates are US$50 per night for a room with a single bed or US$60 for a room with double beds; this rate is applicable for single or double occupancy. Amenities include free continental breakfast; free wireless high-speed Internet access; mini-fridge; cable TV with HBO and ESPN; clock radio; free local calls; free long distance within the US; and wake-up service. This property is located approximately 1.5 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation.

Super 8 Motel, 1751 N Stonelake Dr, Bloomington, IN 47404; 812-323-8000; https://www.wyndhamhotels.com/super-8/bloomington-indiana/super-8-bloomington-in/overview?brand_id=SE&hotel_id=03195&referring_brand=SE&checkin_date=12/4/2016&checkout_date=12/5/2016&useWRPoints=false&rooms=1&adults=1&children=0&radius=25&brand_code=BH, DI, RA, BU, HJ, KG, MT, SE, TL, WG, WY, WT, WP&PriceFilter=0-2147483647. Rates are US$69.99 per night for a room with two beds; this rate is applicable for single or double occupancy. All rooms include a microwave; refrigerator; hair dryer; iron and board; coffee maker; free Wi-Fi; complimentary continental breakfast; and indoor pool and spa. This property is about a 3 minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is March 17, 2017.

Food Services

On Campus: The Indiana Memorial Union offers breakfast and lunch at The Market down the hall from the Frangipani lecture room. The IMU also offers a Starbucks, Burger King and other options in the food court. Located on the lobby level of the Indiana Memorial Union, the Circle Cafe serves breakfast and lunch daily featuring bagels, sandwiches, Starbucks’ coffee and fresh pastries. The cafe is open from 6:30 am to 2:00 pm on Fridays, Saturdays and Sundays.

Located on the main level of the Indiana Memorial Union, the Sugar and Spice Bakery serves specialty cookies, cakes, and other baked goods. The bakery is open from 8:00 am to 2:00 pm on Saturdays and Sundays.

Located on the first floor of the Indiana Memorial Union, the Tudor Room serves buffet lunch on weekdays and brunch from 10:30 am to 1:30 pm on Sundays. Reservations are encouraged but not required. To make a reservation, call 812-855-9866 or e-mail tudorr@indiana.edu.

Off Campus: Downtown Bloomington is eight blocks away from campus. Kirkwood Ave and 4th Street both have numerous restaurants, including many fine ethnic restaurants. More information on restaurants and local attractions in the Bloomington area can be found at https://www.visitbloomington.com/restaurants/.

Some options for coffee include:

Hopscotch Coffee, 235 W. Dodds St, Bloomington; (812) 369-4811; https://www.visitbloomington.com/listing/hopscotch-coffee/1529/; serving ethically sourced, locally roasted coffee.

Pourhouse Cafe, 314 E. Kirkwood Ave, Bloomington; (207) 812-339-7000; https://www.visitbloomington.com/listing/pourhouse-cafe%3a9/1314/; serving locally roasted coffee from all over the world.

Some options nearby for dining include:

Big Woods Bloomington, 116 Grant St, Bloomington; 812-335-1821; https://www.visitbloomington.com/listing/big-woods-bloomington/1477/; a different kind of taproom, focused on a complete experience.

B'Town Diner, 211 N. Walnut, Bloomington; 812-822-0300; https://www.visitbloomington.com/listing/btown-diner/1483/; a traditional diner with some flair. Open all night, breakfast and lunch.

Bub’s Burgers and Ice Cream, 480 N. Morton St., Bloomington; 812-331-2827; https://www.visitbloomington.com/listing/bubs-burgers-and-ice-cream/1471/; serving quality food, burgers, and ice cream.
Advance registration for this meeting opens on January 19, 2017. Advance registration fees will be US$59 for AMS members, US$85 for nonmembers, and US$10 for students, unemployed mathematicians, and emeritus members. Participants may cancel registrations made in advance by e-mailing mmsb@ams.org. The deadline to cancel is the first day of the meeting.

On-site Information and Registration: The registration desk, AMS book exhibit, and coffee service will be located on the first floor of Ballantine Hall. The Invited Addresses, Special Sessions and Contributed Paper Sessions will take place in classrooms which will be in Ballantine Hall and Woodburn Hall. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at https://www.indiana.edu/about/map.html.


Buffalouie’s, 114 S. Indiana Ave, Bloomington; 812-333-3030; https://www.visitbloomington.com/listing/buffalouies/1467/; serving iconic wings, burgers, sandwiches, and salads in an historic setting.


Buffalouie’s, 114 S. Indiana Ave, Bloomington; 812-333-3030; https://www.visitbloomington.com/listing/buffalouies/1467/; serving iconic wings, burgers, sandwiches, and salads in an historic setting.


Schwartz from Brown University will give a talk entitled, Modern Scratch Paper: Graphical Explorations in Geometry and Dynamics. This event is open to the public. Reception to follow. Information about this lecture can be found at www.ams.org/meetings/sectional/2233_events.html

Special Needs
It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:
- Registering early for the meeting
- Checking the appropriate box on the registration form, and
- Sending an e-mail request to the AMS Meetings Department at mmsb@ams.org or meet@ams.org.

AMS Policy on a Welcoming Environment
The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity, or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Local Information and Maps
This meeting will take place on the campus of Indiana University. A campus map can be found at map.iu.edu/iub/. Information about the Indiana University Mathematics Department can be found at www.math.indiana.edu/. Please visit the Indiana University website at https://www.indiana.edu/ for additional information on the campus.

Please watch the AMS website at www.ams.org/meetings/sectional/sectional.html for additional information on this meeting.

Pricing
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Please watch the AMS website at www.ams.org/meetings/sectional/sectional.html for additional information on this meeting.

Parking
Attendees electing to stay at off campus hotels who wish to drive to campus and park their car on campus may do so in the two parking lots adjacent to the Indiana Memorial Union. There will be a fee for parking in these lots. Discounted parking passes will be available at the registration desk for conference participants. The approximate discounted parking fee is currently US$12/day.

Indiana University offers several free weekend parking options. Any vehicle may park in any CH or ST zone from 5 pm on Friday through 11 pm on Sunday, with or without a current parking permit. Parking is free in IU parking garages all day Saturday and Sunday and is free in most metered spaces from 10 pm Friday until 7 am Monday.
MEETINGS & CONFERENCES

The following three lots are free to anyone on weekday evenings after 5pm and over the weekend: Von Lee parking lot (lot #404), lot on the corner of Fourth and Dunn streets (lot #412), lot on the corner of Sixth and Dunn streets (lot #402). These locations and additional information about parking on campus can be found here https://parking.indiana.edu/parking-rules/free-parking.html.

Travel
This meeting will take place on the main campus of Indiana University, located in the heart of Bloomington, Indiana.

By Air: The Indianapolis International Airport (IND) is served by most major airlines. Please visit the Indianapolis International Airport website for a list of airlines and lists of cities with daily direct flights; www.indianapolisairport.com. There are several options available for transportation to/from the airport. Taxi service is available at the curb on the lower level of the airport terminal just outside Baggage Claim. Bloomington Shuttle Service, Inc offers nine daily trips to and from the university campus and surrounding hotels and IND Airport. Advance reservations are required at a cost of US$20.00. Please visit goexpresstravel.com/airport_shuttle_schedule for shuttle schedule information. Star of America Shuttle Service pricing starts at US$18.00 when purchased online, US$20.00 when purchased at the bus, or US$36.00 round trip when purchased in advance online. Please visit www.soashuttle.com/locations/bloomington-to-indianapolis/ for shuttle schedule information. Rental cars are also available at Indianapolis International Airport; for more details please visit business.ind.com/parking_transportation/rentalCar.aspx.

By Train: Rail service to Indiana is available through Amtrak Train. To book your trip please visit https://www.amtrak.com/servlet/ContentServer?pagename=am/rail/AmtrakTrain&lang=en. The Indianapolis train station is located at 350 South Illinois Street and is approximately 50 miles from campus.

By Bus: Bloomington Transit provides convenient local transportation to Indiana University and area hotels. Bus routes and rates are available at bloomingtontransit.com/. Please visit bloomingtontransit.com/wp-content/uploads/2015/09/SystemMap2015Front.pdf for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available on the lower level of the airport terminal just outside Baggage Claim. The main office is open Monday through Friday between 8:00 am and 4:30 pm. The Transit Center is open M-F 6:45 am to 11:00 pm and Saturdays from 8:15 am to 6:30 pm. For more information call (812) 336-5435 (TDD).

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box “I have a discount”, and type in our convention number (CV): CV#04N30007. You can also call Hertz directly at 800-654-2240 (US and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

For directions to campus, inquire at your rental car counter.

Local Transportation
Walking, biking and personal cars are recommended to get around campus and Bloomington.

By Bus: Bloomington Transit provides convenient local transportation to Indiana University and area hotels. Bus routes and rates are available at bloomingtontransit.com/. Please visit bloomingtontransit.com/wp-content/uploads/2015/09/SystemMap2015Front.pdf for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available on the lower level of the airport terminal just outside Baggage Claim. The main office is open Monday through Friday between 8:00 am and 4:30 pm. The Transit Center is open M-F 6:45 am to 11:00 pm and Saturdays from 8:15 am to 6:30 pm. For more information call (812) 336-7433, (812) 330-7853 (TDD).

Weather:
The average high temperature for April is approximately 64 degrees Fahrenheit and the average low is approximately 42 degrees Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking:
Attendees and speakers are encouraged to tweet about the meeting using the hashtags #AMSmtg.

Information for International Participants:
Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to cro@ams.org.

From the south: Take IN-37 N toward Bloomington and exit IN-37 N onto W. Bloomfield Road. Continue onto W. Second Street then turn left on S. Henderson Street. Continue straight onto S. Indiana Avenue then turn right on E. Seventh Street.

From the north: Take IN-37 S toward Bloomington and exit IN-37 S onto IN-45 N/IN-46 E. Turn right on N. College Avenue then turn left on W. Seventh Street.

From the east: Take IN-46 W toward Bloomington and continue straight onto E. Third Street. Turn right on S. Indiana Avenue then turn right on E. Seventh Street.

From the west: Take IN-45 N/IN-46 E toward Bloomington. Turn right on N. College Avenue then turn left on E. Seventh Street.

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box “I have a discount”, and type in our convention number (CV): CV#04N30007. You can also call Hertz directly at 800-654-2240 (US and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

For directions to campus, inquire at your rental car counter.

Local Transportation
Walking, biking and personal cars are recommended to get around campus and Bloomington.

By Bus: Bloomington Transit provides convenient local transportation to Indiana University and area hotels. Bus routes and rates are available at bloomingtontransit.com/. Please visit bloomingtontransit.com/wp-content/uploads/2015/09/SystemMap2015Front.pdf for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available for the full area map with all bus routes. Regular fare costs US$1 and exact change is required. Passes and tickets are available on the lower level of the airport terminal just outside Baggage Claim. The main office is open Monday through Friday between 8:00 am and 4:30 pm. The Transit Center is open M-F 6:45 am to 11:00 pm and Saturdays from 8:15 am to 6:30 pm. For more information call (812) 336-7433, (812) 330-7853 (TDD).

Weather:
The average high temperature for April is approximately 64 degrees Fahrenheit and the average low is approximately 42 degrees Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking:
Attendees and speakers are encouraged to tweet about the meeting using the hashtags #AMSmtg.
MEETINGS & CONFERENCES

Invited Addresses

Michael Hitrik, University of California, Los Angeles, *Spectra for non-self-adjoint operators and integrable dynamics.*

Andrew S. Raich, University of Arkansas, *Title to be announced.*

Daniel Rogalski, University of California, San Diego, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Analysis on the Navier-Stokes equations and related PDEs (Code: SS 9A), Kazuo Yamazaki, University of Rochester, and Litzheng Tao, University of California, Riverside.

Clustering of Graphs: Theory and Practice (Code: SS 18A), Stephen J. Young and Jennifer Webster, Pacific Northwest National Laboratory.

Combinatorial and Algebraic Structures in Knot Theory (Code: SS 5A), Sam Nelson, McKenna College, and Allison Henrich, Seattle University.

Combinatorial and Computational Commutative Algebra and Algebraic Geometry (Code: SS 21A), Hirotachi Abo, Stefan Tohaneanu, and Alexander Woo, University of Idaho.

Commutative Algebra (Code: SS 3A), Jason Lutz and Katharine Shultis, Gonzaga University.

Inverse Problems (Code: SS 2A), Hanna Makaruk, Los Alamos National Laboratory (LANL), and Robert Owczarek, University of New Mexico, Albuquerque and Los Alamos.


Mirolocal Analysis and Spectral Theory (Code: SS 17A), Michael Hitrik, University of California, Los Angeles, and Semyon Dyatlov, Massachusetts Institute of Technology.

Noncommutative algebraic geometry and related topics

Pullman, Washington

Washington State University

April 22–23, 2017
Saturday - Sunday

Meeting #1128
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: February 2017
Program first available on AMS website: March 9, 2017
Issue of Abstracts: Volume 38, Issue 2

Deadlines
For organizers: Expired
For abstracts: February 28, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - employment contract or statement from employer stating that the position will continue when the employee returns;
* Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;
* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;
* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;
* If travel plans will depend on early approval of the visa application, specify this at the time of the application;
* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.
(Code: SS 16A), Daniel Rogalski, University of California, San Diego, and James Zhang, University of Washington.

Partial Differential Equations and Applications (Code: SS 8A), V. S. Manoranjan, C. Moore, Lynn Schreyer, and Hong-Ming Yin, Washington State University.

Recent Advances in Applied Algebraic Topology (Code: SS 14A), Henry Adams, Colorado State University, and Bala Krishnamoorthy, Washington State University.

Recent Advances in Optimization and Statistical Learning (Code: SS 19A), Hongbo Dong, Bala Krishnamoorthy, Haijun Li, and Robert Mifflin, Washington State University.

Recent Advances on Mathematical Biology and Their Applications (Code: SS 7A), Robert Dillon and Xueying Wang, Washington State University.

Several Complex Variables and PDEs (Code: SS 10A), Andrew Raich and Phillip Harrington, University of Arkansas.

Special Session on Analytic Number Theory and Automorphic Forms (Code: SS 6A), Steven J. Miller, Williams College, and Sheng-Chi Liu, Washington State University.

Theory and Applications of Linear Algebra (Code: SS 4A), Judi McDonald and Michael Tsatsomeros, Washington State University.

Undergraduate Research Experiences in the Classroom (Code: SS 13A), Heather Moon, Lewis-Clark State College.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the Washington state hotel tax (9.8%). Participants must state that they are with the American Mathematical Society (AMS) Meeting at Washington State University to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. Hotels have varying cancellation and early checkout penalties; be sure to ask for details.

Quality Inn Paradise Creek, S.E. 1400 Bishop Blvd., Pullman, WA 99163; 509-332-0500; https://www.choicehotels.com/washington/pullman/quality-inn-hotels/wa015?source=gyxt. Ask for the American Mathematical Society rates for a special discounted rate of US$109 per night for double queen beds US$119 for a king bed. This rate includes an indoor hot tub, mini fridge and microwave in each room, free Wi-Fi and complimentary hot breakfast. This property is located approximately 1.3 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is April 19, 2017.

The Hilltop Inn, 928 NW Olsen St, Pullman, WA 99163; 509-332-0928; www.hilltopinnpullman.com/. Rates are US$109.75 per night for one king bed or two queen beds. Amenities include microwave and fridge in each room, indoor pool and hot tub, free breakfast buffet and free high-speed Internet. This property has a restaurant and lounge that opens every day at 4:00 pm. This property is located approximately 2 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is April 10, 2017.

Residence Inn by Marriott, 1255 NE North Fairway Dr, Pullman, WA 99163; 509-332-4400; www.marriott.com/hotels/travel/puwri-residence-inn-pullman/?scid=bb1a189a-fec3-4d19-a255-54ba596febe2. Rates are US$125.00 per night for one king bed rooms. Amenities include indoor pool and hot tub, free breakfast buffet and free high-speed Internet. This property is located less than 1 mile from the campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at a reduced rate is March 21, 2017.

Food Services

On Campus: On Campus dining is available at the Compton Union Building between 10:00 am and 5:00 pm. This building is adjacent to the Smith Center for Undergraduate Education where most of the talks will be held.

Located on the first floor of the Compton Union Building are the Bookie Cafe, open Saturday 10:00 am–5:00 pm; Subway, open Saturday and Sunday until 7:00 pm; and Panda Express, open Saturday and Sunday until 7:00 pm.

Off Campus: Downtown Pullman is about a 15 minute walk from the campus. There are many restaurants in that area. More information on restaurants and local attractions in the Pullman area, including a map, can be found at pullmanchamber.com/wp-content/uploads/2015/04/PullmanMap08.pdf.

Some options for coffee include: Café Moro, 100A E Main St, Pullman; 509-338-3892; https://www.facebook.com/pullmancafe/oro/; a mellow coffeeshop featuring coffee drinks, frappes, baked goods, beer and occasional live music.

Daily Grind, 230 E Main St, Pullman; 509-334-3380; web.archive.org/web/20160309051325/thedailygrindespressos.com/; serving fresh coffee and baked goods.

Some downtown options for dining include: Cooky's European Deli, 215 Main Street, Pullman; 509-332-224; www.manta.com/c/mmtywrrq/cooky-s-european-deli-market-11c; a local market and delicatessen. Mandarin House, 115 N Grand Ave, Pullman; (509) 332-1888; www.pullmanchinesefood.com/; unpretentious Chinese restaurant serving standard fare such as crab rangoon and beef with broccoli.

My Office Bar & Grill, 215 S Grand Ave, Pullman; (509) 334-120; https://www.facebook.com/My-Office-Bar-Grill-115549738468097/; a basic pub with draft and bottled beers in a casual atmosphere offering darts and pool.
Old Post Office Wine Bar/Restaurant, 1245 SE Paradise St, Pullman; (509) 338-9463; www.paradisecreekbrewery.com; beers brewed in the basement of an old post office, with a spacious bar upstairs serving pub fare.

Registration and Meeting Information
Advance Registration: Advance registration for this meeting opens on January 19, 2017. Advance registration fees will be US$59 for AMS members, US$85 for nonmembers, and US$10 for students, unemployed mathematicians, and emeritus members. Participants may cancel registrations made in advance by e-mailing mmsb@ams.org. The deadline to cancel is the first day of the meeting.

On-site Information and Registration: The registration desk, AMS book exhibit, and coffee service will be located in the atrium of the Smith Center for Undergraduate Education. The Invited Addresses will be held in the Smith Center for Undergraduate Education, Room 203. Special Sessions and Contributed Paper Sessions will also take place in the Smith Center for Undergraduate Education. Please look for additional information about specific session room locations on the Web and in the printed program. For further information on building locations, a campus map is available at map.wsu.edu/. The registration desk will be open on Saturday, April 22, 7:30 am–4:00 pm and Sunday, April 23, 8:00 am–12:00 pm. The same fees apply for on-site registration, as for advance registration. Fees are payable on-site via cash, check, or credit card.

Other Activities
Book Sales: Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS members receive 40 percent off list price. Nonmembers receive a 25 percent discount. Not a member? Ask a representative about the benefits of AMS membership. Complimentary Coffee will be served courtesy of AMS membership services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

Special Needs
It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:
- Registering early for the meeting
- Checking the appropriate box on the registration form, and
- Sending an e-mail request to the AMS Meetings Department at mmsb@ams.org or meet@ams.org.

AMS Policy on a Welcoming Environment
The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity, or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Local Information and Maps
This meeting will take place on the campus of Washington State University. A campus map can be found at map.wsu.edu/. Information about the WSU Mathematics Department can be found at www.math.wsu.edu/. Please visit the Washington State University website at https://wsu.edu/ for additional information on the campus.

Please visit the local organizers’ website for this meeting at https://www.math.wsu.edu/AMS. Please watch the AMS website at www.ams.org/meetings/sectional/sectional.html for additional information on this meeting.

Parking
Parking is available under the Center for Undergraduate Education at a cost of US$3 per day on Saturday and Sunday. This is the most convenient parking as it is closest to the meeting spaces.

Day visitors are also directed to park on North Campus Drive (off of Bath Road), South Campus Drive (off of College Street), Coffin Street Parking Lot, Dayton Lot (off of Sills Drive) and Russwurm House Parking Lot (just east of Tower Drive).

Travel
This meeting will take place on the main campus of Washington State University, located in Pullman, Washington. By Air: The Pullman/Moscow Airport is two miles from the Pullman campus. Horizon Air (Alaska Airlines) services the region. Please visit their website for a list of cities with direct flights; www.flypuw.com/. Rental cars are available at the airport; for more information about rental cars and other ground transportation please visit www.flypuw.com/transportation/. Spokane International Airport is 78 miles from campus and serviced by most airlines; for information on airlines and flight schedules please visit spokaneairports.net/. There are many options available for ground transportation from Spokane International Airport. The Wheatland Express Shuttle offers service to Pullman from Spokane International Airport for US$38.00; www.wheatlandexpress.com/ariportexpres/. Rental cars are
available at the airport; for more information please visit spokaneairports.net/rental-cars/.

By Train: Rail service to Pullman is currently available from major cities through Amtrak; https://www.amtrak.com/home. The Pullman train station is located on 1205 North Grand Avenue, approximately a 20 minute walk from campus.

By Bus: Major bus routes pass through Pullman. Northwestern Trailways stops about one mile from campus and offers round-trip fare to Spokane International Airport for US$68. For more information please call 800-366-3830 or visit www.northwesterntrailways.com/Home/tabid/36/Default.aspx.

Greyhound Transit offers daily trips to Pullman. Buses stop at the Dissmore's IGA located at 1205 N Grand St, about a 20 minute walk from campus. For more information on Greyhound please call (509) 332-2918 or visit www.greyhound.com/.

By Car:

From the south: Take US-195 North to WA-27 N/S Grand Ave. Turn right onto WA-270 E/SE Paradise St (signs for Moscow/WSU). Turn left onto E Spring St. Your destination will be on the right.

From the north: Take US-195 South and turn left onto WA-270 East. In two miles turn right onto N Grand Ave. Turn left onto WA-270 E/SE Paradise St (signs for Moscow/WSU). Turn left onto E Spring St. Your destination will be on the right.

From the east: Take WA-270 West toward Airport Rd/Terre View Dr. Turn right onto E Spring St. Your destination will be on the right.

From the west: Follow I-90 East and take exit 137 to merge onto WA-26 E toward Othello/Pullman. Follow WA-26 E to US-195 S in Colfax. From US-195 South turn left onto WA-270 East. In two miles turn right onto N Grand Ave. Turn left onto WA-270 E/SE Paradise St (signs for Moscow/WSU). Turn left onto E Spring St. Your destination will be on the right.

For a map of campus with surrounding routes, go to https://visitor.wsu.edu/VisitorMap.html.

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box "I have a discount", and type in our convention number (CV): CV#04N30007. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

For directions to campus, inquire at your rental car counter.

Local Transportation

Walking, biking and personal cars are recommended to get around campus and Pullman.

Local bus transportation is available through Pullman Transit. The general public may ride Pullman Transit by paying per ride or by purchasing a Monthly, Semi-Annual or Annual pass at the WSU Visitors Center.

Weather:

The average high temperature for April is 58 degrees and the average low is 38 degrees. Historically, the probability of some precipitation in a given day is 40 percent. Weather conditions can change fairly rapidly and visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking:

Attendees and speakers are encouraged to tweet about the meeting using the hashtags #AMSmtg.

Information for International Participants:

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to cro@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

- Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - bank accounts
  - employment contract or statement from employer stating that the position will continue when the employee returns;

- Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;

- Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - bank accounts
  - employment contract or statement from employer stating that the position will continue when the employee returns;

- Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript). This list is not to be considered complete. Please visit the websites above for the most up-to-date information.
New York, New York

Hunter College, City University of New York

May 6–7, 2017
Saturday – Sunday

Meeting #1129
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: March 2017
Program first available on AMS website: March 22, 2017
Issue of Abstracts: Volume 38, Issue 2

Deadlines
For organizers: Expired
For abstracts: March 14, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Jeremy Kahn, City University of New York, Title to be announced.
Fernando Coda Marques, Princeton University, Title to be announced.
James Maynard, Magdalen College, University of Oxford, Title to be announced (Erdős Memorial Lecture).
Kavita Ramanan, Brown University, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Analysis and numerics on liquid crystals and soft matter (Code: SS 16A), Xiang Xu, Old Dominion University, and Wujun Zhang, Rutgers University.
Applications of network analysis, in honor of Charlie Saffell’s 75th birthday (Code: SS 18A), Michael Yatauro, Pennsylvania State University-Brandywine.
Banach Space Theory and Metric Embeddings (Code: SS 10A), Mikhail Ostrovskii, St John’s University, and Beata Randrianantoanina, Miami University of Ohio.
Cluster Algebras in Representation Theory and Combinatorics (Code: SS 6A), Alexander Garver, Université du Québec à Montréal and Sherbrooke, and Khrystyna Serhiyenko, University of California at Berkeley.
Cohomologies and Combinatorics (Code: SS 15A), Rebecca Patrias, Université du Québec à Montréal, and Oliver Pechenik, Rutgers University.
Common Threads to Nonlinear Elliptic Equations and Systems (Code: SS 14A), Florin Catrina, St. John’s University, and Wenxiong Chen, Yeshiva University.
Commutative Algebra (Code: SS 1A), Laura Ghezzi, New York City College of Technology-CUNY, and Jooyoun Hong, Southern Connecticut State University.
Computability Theory: Pushing the Boundaries (Code: SS 9A), Johanna Franklin, Hofstra University, and Russell Miller, Queens College and Graduate Center, City University of New York.
Computational and Algorithmic Group Theory (Code: SS 7A), Denis Serbin and Alexander Ushakov, Stevens Institute of Technology.
Cryptography (Code: SS 3A), Xiaowen Zhang, College of Staten Island and Graduate Center-CUNY.
Current Trends in Function Spaces and Nonlinear Analysis (Code: SS 2A), David Cruz-Uribe, University of Alabama.
Jan Lang, The Ohio State University, and Osvaldo Mendez, University of Texas at El Paso.
Differential and Difference Algebra: Recent Developments, Applications, and Interactions (Code: SS 12A), Omar León-Sanchez, McMaster University, and Alexander Levin, The Catholic University of America.
Geometric Function Theory and Related Topics (Code: SS 19A), Sudeb Mitra, Queens College and Graduate Center-CUNY, and Zhe Wang, Bronx Community College-CUNY.
Geometry and Topology of Ball Quotients and Related Topics (Code: SS 5A), Luca F. Di Cerbo, Max Planck Institute, Bonn, and Matthew Stover, Temple University.
Infinite Permutation Groups, Totally Disconnected Locally Compact Groups, and Geometric Group Theory (Code: SS 4A), Delaram Kahrobaei, New York City College of Technology and Graduate Center-CUNY, and Simon Smith, New York City College of Technology-CUNY.
Nonlinear and Stochastic Partial Differential Equations: Theory and Applications in Turbulence and Geophysical Flows (Code: SS 8A), Nathan Glatt-Holtz, Tulane University, Geordie Richards, Utah State University, and Xiaoming Wang, Florida State University.
Operator algebras and ergodic theory (Code: SS 17A), Genady Grabarnik and Alexander Katz, St John’s University.
Qualitative and Quantitative Properties of Solutions to Partial Differential Equations (Code: SS 20A), Blair Davey, The City College of New York-CUNY, and Nguyen Cong Phuc and Jiuyi Zhu, Louisiana State University.
Recent Developments in Automorphic Forms and Representation Theory (Code: SS 21A), Moshe Adrian, Queens College-CUNY, and Shuichiro Takeda, University of Missouri.
Representation Spaces and Toric Topology (Code: SS 13A), Anthony Bahri, Rider University, and Daniel Ramras and Mentor Stafa, Indiana University-Purdue University Indianapolis.
Montréal, Quebec Canada

McGill University

July 24–28, 2017
Monday - Friday

Meeting #1130
The second Mathematical Congress of the Americas (MCA 2017) is being hosted by the Canadian Mathematical Society (CMS) in collaboration with the Pacific Institute for the Mathematical Sciences (PIMS), the Fields Institute (FIELDS), Le Centre de Recherches Mathématiques (CRM), and the Atlantic Association for Research in the Mathematical Sciences (AARMS).

Associate secretary: Brian D. Boe
Announcement issue of Notices: To be announced
Program first available on AMS website: January 23, 2017
Issue of Abstracts: To be announced

Deadlines
For organizers: Expired
For abstracts: March 31, 2017

Denton, Texas

University of North Texas

September 9–10, 2017
Saturday - Sunday

Meeting #1131
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: June 2017
Program first available on AMS website: July 27, 2017
Issue of Abstracts: Volume 38, Issue 3

Deadlines
For organizers: February 2, 2017
For abstracts: July 18, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Mirela Çiperiani, University of Texas at Austin, Title to be announced.
Adrianna Gillman, Rice University, Title to be announced.
Kevin Pilgrim, Indiana University, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Banach Spaces and Applications (Code: SS 9A), Pavlos Motakis, Texas A&M University, and Bönyamin Sari, University of North Texas.
Commutative Algebra (Code: SS 10A), Jonathan Montano, University of Kansas, and Alessio Sammartano, Purdue University.
Differential Equation Modeling and Analysis for Complex Bio-systems (Code: SS 8A), Pengcheng Xiao, University of Evansville, and Honghui Zhang, Northwestern Polytechnical University.
Dynamics, Geometry and Number Theory (Code: SS 1A), Lior Fishman and Mariusz Urbanski, University of North Texas.
Fractal Geometry and Ergodic Theory (Code: SS 5A), Mrinal Kanti Roychowdhury, University of Texas Rio Grande Valley.
Invariants of Links and 3-Manifolds (Code: SS 7A), Charles H. Conley, University of North Texas, Dimitar Grantcharov, University of Texas at Arlington, and Natalia Rozhkovskaya, Kansas State University.
Noncommutative and Homological Algebra (Code: SS 4A), Anne Shepler, University of North Texas, and Sarah Witherspoon, Texas A&M University.
Numbers, Functions, Transcendence, and Geometry (Code: SS 6A), Willard Cherry, University of North Texas, Mirela Çiperiani, University of Texas Austin, Matt Papanikolas, Texas A&M University, and Min Ru, University of Houston.
Real-Analytic Automorphic Forms (Code: SS 2A), Olav K Richter, University of North Texas, and Martin Westerholt-Raum, Chalmers University of Technology.

Buffalo, New York

State University of New York at Buffalo

September 16–17, 2017
Saturday - Sunday

Meeting #1132
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: June 2017
Program first available on AMS website: August 3, 2017
Issue of Abstracts: Volume 38, Issue 3

Deadlines
For organizers: February 16, 2017
MEETINGS & CONFERENCES

Christopher D Sogge, Johns Hopkins University, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic Curves and their Applications (Code: SS 3A), Lubjana Beshaj, The University of Texas at Austin.

Applied Harmonic Analysis: Frames, Samplings and Applications (Code: SS 6A), Dorin Dutkay, Deguang Han, and Qiyu Sun, University of Central Florida.

Commutative Algebra: Interactions with Algebraic Geometry and Algebraic Topology (Code: SS 1A), Joseph Brennan, University of Central Florida, and Alina Iacob and Saeed Nasseh, Georgia Southern University.

Fractal Geometry, Dynamical Systems, and Their Applications (Code: SS 4A), Mrinal Kanti Roychowdhury, University of Texas Rio Grande Valley.

High Order Numerical Methods for Hyperbolic PDEs and Applications (Code: SS 2A), Jae-Hun Jung, University at Buffalo-SUNY, Fengyang Li, Rensselaer Polytechnic Institute, and Li Wang, University at Buffalo-SUNY.

Nonlinear Wave Equations, Inverse Scattering and Applications. (Code: SS 1A), Gino Biondini, University at Buffalo-SUNY.

p-adic Aspects of Arithmetic Geometry (Code: SS 3A), Ling Xiao, University of Connecticut, and Hui June Zhu, University at Buffalo-SUNY.

Riverside, California
University of California, Riverside

November 4–5, 2017
Saturday – Sunday

Meeting #1134
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: September 2017
Program first available on AMS website: September 21, 2017
Issue of Abstracts: Volume 38, Issue 4

Deadlines
For organizers: April 14, 2017
For abstracts: September 12, 2017

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Paul Balmer, University of California, Los Angeles, Title to be announced.

Pavel Etingof, Massachusetts Institute of Technology, Title to be announced.

Monica Vazirani, University of California, Davis, Title to be announced.
**MEETINGS & CONFERENCES**

**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

- **Combinatorial aspects of the polynomial ring** (Code: SS 1A), **Sami Assaf** and **Dominic Searles**, University of Southern California.
- **Ring Theory and Related Topics** (Celebrating the 75th Birthday of Lance W. Small) (Code: SS 2A), **Jason Bell**, University of Waterloo, **Ellen Kirkman**, Wake Forest University, and **Susan Montgomery**, University of Southern California.

**San Diego, California**

*San Diego Convention Center and San Diego Marriott Hotel and Marina*

**January 10–13, 2018**

*Wednesday – Saturday*

**Meeting #1135**

Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 39, Issue 1

**Deadlines**

For organizers: April 1, 2017
For abstracts: To be announced

**Nashville, Tennessee**

*Vanderbilt University*

**April 14–15, 2018**

*Saturday – Sunday*

**Meeting #1138**

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**

For organizers: To be announced
For abstracts: To be announced

**Portland, Oregon**

*Portland State University*

**April 14–15, 2018**

*Saturday – Sunday*

**Meeting #1137**

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**

For organizers: To be announced
For abstracts: To be announced
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Inverse Problems (Code: SS 2A), Hanna Makaruk, Los Alamos National Laboratory (LANL), and Robert Owczarek, University of New Mexico, Albuquerque and Los Alamos. Pattern Formation in Crowds, Flocks, and Traffic (Code: SS 1A), J. J. P. Veerman, Portland State University, Alethea Barbaro, Case Western Reserve University, and Bassam Bamieh, UC Santa Barbara.

Boston, Massachusetts
Northeastern University
April 21–22, 2018
Saturday – Sunday
Meeting #1139
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: September 21, 2017
For abstracts: March 6, 2018

Newark, Delaware
University of Delaware
September 29–30, 2018
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: February 28, 2018
For abstracts: To be announced

Fayetteville, Arkansas
University of Arkansas
October 6–7, 2018
Saturday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Shanghai, People’s Republic of China
Fudan University
June 11–14, 2018
Monday – Thursday
Meeting #1140
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadline
For organizers: To be announced

Ann Arbor, Michigan
University of Michigan, Ann Arbor
October 20–21, 2018
Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: March 20, 2018
For abstracts: August 21, 2018
San Francisco, California

San Francisco State University

October 27–28, 2018

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: August 2018
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: April 18, 2018
For abstracts: August 28, 2018

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday
Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: November 1, 2019
Issue of Abstracts: To be announced

Deadlines
For organizers: April 1, 2019
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16–19, 2019

Wednesday – Saturday
Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Steven H. Weintraub
Announcement issue of Notices: October 2018
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: April 1, 2018
For abstracts: August 28, 2018

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe
Announcement issue of Notices: October 2020
Program first available on AMS website: November 1, 2020
Issue of Abstracts: To be announced

Deadlines
For organizers: April 1, 2021
For abstracts: To be announced

Honolulu, Hawaii

University of Hawaii at Manoa

March 29–31, 2019

Friday – Sunday
Central Section
Associate secretaries: Georgia Benkart and Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: April 1, 2020
For abstracts: To be announced
Yau Mathematical Sciences Center
Tsinghua University, Beijing, China

Positions:
Professorship;
Associate Professorship;
Assistant Professorship (tenure-track).

The YMSC invites applications for the above positions in the full spectrum of mathematical sciences: ranging from pure mathematics, applied PDE, computational mathematics to statistics. The current annual salary range is between 0.25-1.0 million RMB. Salary will be determined by applicants' qualification. Strong promise/track record in research and teaching are required. Completed applications must be electronically submitted, and must contain curriculum vitae, research statement, teaching statement, selected reprints and/or preprints, three reference letters on academic research and one reference letter on teaching (Reference letters must be hand signed by referees), sent electronically to

`msc-recruitment@math.tsinghua.edu.cn`

The review process starts in December 2016, and closes by April 30, 2017. Applicants are encouraged to submit their applications before December 31, 2016.

**Positions: post-doctorate fellowship**

Yau Mathematical Sciences Center (YMSC) will hire a substantial statistics, number of post-doctorate fellows in the full spectrum of mathematical sciences. New and recent PhDs are encouraged for this position.

A typical appointment for post-doctorate fellowship of YMSC is for two-years, renewable for the third years. Salary and compensation package are determined by qualification, accomplishment, and experience. YMSC offers very competitive packages.

Completed applications must contain curriculum vitae, research, statement, teaching statement, selected reprints and/or preprints, three reference letters with referee’s signature, sent electronically to

`msc-recruitment@math.tsinghua.edu.cn`

The review process starts in December 2016, and closes by April 30, 2017. Applicants are encouraged to submit their applications before December 31, 2016.

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Tsinghua Sanya International Mathematics Forum (TSIMF)
Call for Proposal

We invite proposals to organize workshops, conferences, research-in-team and other academic activities at the Tsinghua Sanya International Mathematics Forum (TSIMF).

TSIMF is an international conference center for mathematics. It is located in Sanya, a scenic city by the beach with excellent air quality. The facilities of TSIMF are built on a 140-acre land surrounded by pristine environment at Phoenix Hill of Phoenix Township. The total square footage of all the facilities is over 28,000 square meter that includes state-of-the-art conference facilities (over 9,000 square meter) to hold two international workshops simultaneously, a large library, a guesthouse (over 10,000 square meter) and the associated catering facilities, a large swimming pool, two tennis courts and other recreational facilities.

Because of our capacity, we can hold several workshops simultaneously. We pledge to have a short waiting period (6 months or less) from proposal submission to the actual running of the academic activity.

The mission of TSIMF is to become a base for scientific innovations, and for nurturing of innovative human resource; through the interaction between leading mathematicians and core research groups in pure mathematics, applied mathematics, statistics, theoretical physics, applied physics, theoretical biology and other relating disciplines, TSIMF will provide a platform for exploring new directions, developing new methods, nurturing mathematical talents, and working to raise the level of mathematical research in China.

For information about TSIMF and proposal submission, please visit:

`http://ymsc.tsinghua.edu.cn/sanya/`

or write to Ms. Yanyu Fang

`yyfang@math.tsinghua.edu.cn`
"Somehow everything just fits perfectly together, and it's sort of a miracle."
—Henry Cohn, author of our cover story on the recent proof by Maryna Viazovska of the best sphere packing in 8-D.

"I think some of us have been hoping for this for a very long time."
—Thomas Hales, who proved the best sphere packing in 3-D.

(Both quoted in a March 2016 story by Erica Klarreich in Quanta Magazine.)

A Submitter's Worst Nightmare

Mathematicians have the least stressful high-paying jobs, according to a study by Larry Shatkin (Business Insider, September 2016). See http://bit.ly/2deZxrt

QUESTIONABLE MATHEMATICS

"Nuclear-powered vacuum cleaners will probably be a reality within ten years."
—Alexander M. Lewyt, president of the Lewyt Corporation, 1955

What crazy things happen to you? Readers are invited to submit original short amusing stories, math jokes, cartoons, and other material to: noti-backpage@ams.org.
Ad Honorem Sir Andrew J. Wiles

Leroy P. Steele Prizes

Call for Nominations

Introducing AMS Open Math Notes

This year’s JMM marked the debut of AMS Open Math Notes, a repository of freely downloadable mathematical works in progress hosted by the American Mathematical Society as a service to researchers, teachers, and students. These draft works include course notes, textbooks, and research expositions, all of which may be downloaded and used as aids in teaching or research. Each course note includes a contact link allowing visitors to send the author(s) constructive comments and suggestions. Members are encouraged to visit [www.ams.org/open-math-notes](http://www.ams.org/open-math-notes) to browse the archive or to submit their own unpublished notes.
THE ENDOSCOPIC CLASSIFICATION OF REPRESENTATIONS
ORTHOGONAL AND SYMPLECTIC GROUPS
James Arthur, University of Toronto, ON, Canada
Authored by a master on the subject, this text explores the classification of the automorphic representations of the orthogonal groups and symplectic groups, using endoscopy and the trace formula.

PARTIAL DIFFERENTIAL EQUATIONS
AN ACCESSIBLE ROUTE THROUGH THEORY AND APPLICATIONS
András Vasy, Stanford University, CA
This text is intended for readers who want to understand the theoretical underpinnings of modern PDEs in settings that are important for applications.

OPERA DE CRIBRO
John Friedlander, University of Toronto, ON, Canada, and Henryk Iwaniec, Rutgers University, Piscataway, NJ
Our understanding of sieves has improved greatly over the past 20 or 30 years, in large part due to the efforts of this book’s authors, and proofs that used to take many pages in an almost incomprehensible notation can now be done cleanly in one page. This book does a good job of keeping the notation under control.

J-HOLOMORPHIC CURVES AND SYMPLECTIC TOPOLOGY
SECOND EDITION
Dusa McDuff, Barnard College, Columbia University, New York, NY, and Dietmar Salamon, ETH, Zurich, Switzerland
This revised edition continues to serve as the definitive source of information about some areas of differential topology and applications to quantum cohomology.

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