WHAT IS…

Tropical Geometry?

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This note was written to answer the question, What is tropical geometry? That question can be interpreted in two ways: Would you tell me something about this research area? and Why the unusual name 'tropical geometry'? To address the second question, tropical geometry is named in honor of Brazilian computer scientist Imre Simon. This naming is complicated by the fact that he lived in São Paolo and commuted across the Tropic of Capricorn. Whether his work is tropical depends on whether he preferred to do his research at home or in the office.

Tropical geometry transforms questions about algebraic varieties into questions about polyhedral complexes.

The main goal of tropical geometry is transforming questions about algebraic varieties into questions about polyhedral complexes. A process called *tropicalization* attaches a polyhedral complex to an algebraic variety. The polyhedral complex, a combinatorial object, encodes some of the geometry of the original algebraic variety. There are other ways of constructing similar polyhedral complexes, and the polyhedral complexes that arise can be studied in their own right. We will discuss three approaches: the synthetic, the valuative, and the degeneration-theoretic.

Synthetic Approach

Tropical geometry originally arose from considerations of tropical algebra, itself motivated by questions in computer science. Here, tropical geometry can be considered as algebraic geometry over the tropical semifield $(\mathbb{R}\cup\{\infty\},\oplus,\otimes)$ with operations given by

$$a\oplus b=\min(a,b),\quad a\otimes b=a+b.$$

One can then find tropical analogues of classical mathematics and define tropical polynomials, tropical hypersurfaces, and tropical varieties. For example, a degree 2 polynomial in variables x, y would be of the form

$$\min(a_{20}+2x, a_{11}+x+y, a_{02}+2y, a_{10}+x, a_{01}+y, a_{00})$$

for constants $a_{ij}\in\mathbb{R}\cup\{\infty\}$. The zero locus of a tropical polynomial is defined to be the set of points where the minimum is achieved by at least two entries. In the above example, it would be the set of points (x,y) where there are distinct indices $(i_1,j_1),(i_2,j_2)$ such that

$$a_{i_1j_1}+i_1x+j_1y=a_{i_2j_2}+i_2x+j_2y\le a_{ij}+ix+jy$$

for all pairs (i,j). These objects do not look like their classical counterparts and instead are polyhedral complexes of differing combinatorial types. For example, Figure 1 shows a tropical line and two tropical curves cut out by degree 2 polynomials in x and y.

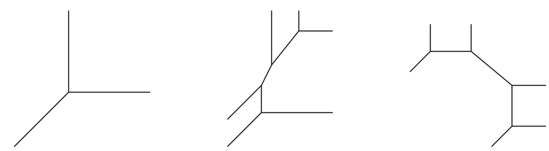

Figure 1. Tropical curves, such as this tropical line and two tropical conics, are polyhedral complexes.

Recall that a polyhedral complex in $\mathbb{R}^n$ is a union of polyhedra (that is, sets cut out by linear equations and inequalities) such that any set of polyhedra intersects in a common (possibly empty) face of each member. More is true about the polyhedral complexes that arise in this fashion; they are tropical varieties, which are defined to be integral, weighted, balanced polyhedral complexes. Here *integral* means that their defining linear equations and inequalities have integer coefficients; *weighted* means that their top-dimensional polyhedra are assigned positive

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integer weight (in the above example, all weights are 1); and *balanced* means that the weights around a codimension 1 face satisfy a particular balancing relation, which in the 1-dimensional case says that the primitive integer vectors along edges emanating from a given vertex, multiplied by the weight of the edges, sum to zero. This can be thought of as a zero tension condition as in physical dynamics.

A combinatorial object encodes some geometry of the algebraic variety.

Tropical geometry's first major result was Grigory Mikhalkin's proof (2005) that the number of plane curves of degree d and genus g passing through $3d-1+g$ points in general position could be computed by counting tropical curves with multiplicity. This led to the definition of an abstract tropical curve as a graph equipped with additional data. Much of the early development of tropical geometry involved finding tropical analogues of theorems about algebraic curves and their enumerative geometry.

Valuation-Theoretic Approach

Another approach to tropical geometry, which was described in an unpublished manuscript of Mikhail Kapranov from the early 1990s but dates back to work of George Bergman (1971) and Robert Bieri and J. R. J. Groves (1984), is to define a tropical variety as a shadow of an algebraic variety. We will first discuss a more familiar, analytic version of this approach involving logarithmic limit sets. For $t > 0$, consider the map $\mathrm{Log}_t\colon (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by

$$(z_1, \ldots, z_n) \mapsto (\log_t(|z_1|), \ldots, \log_t(|z_n|)).$$

The image of an algebraic subvariety $X \subset (\mathbb{C}^*)^n$ is called an *amoeba*. Under the Hausdorff limit, $\lim_{t\to\infty} \mathrm{Log}_t(X)$, it becomes a piecewise linear object called a polyhedral fan. When one studies a family of varieties $X_t \subset (\mathbb{C}^*)^n$ parameterized by t and considers $\lim_{t\to\infty} \mathrm{Log}_t(X_t)$, one obtains a richer polyhedral complex.

For the algebraic approach, let $\mathbb{K}$ be an algebraically closed field equipped with a nontrivial non-Archimedean valuation $v\colon \mathbb{K}^* \to G \subseteq \mathbb{R}$ where G is an additive subgroup of $\mathbb{R}$. Here $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$ is the set of units in $\mathbb{K}$. The valuation is said to be non-Archimedean if

$$\begin{aligned} v(xy) &= v(x) + v(y), \\ v(x+y) &\geq \min(v(x), v(y)). \end{aligned}$$

An example to keep in mind is $\mathbb{K} = \mathbb{C}\{\{t\}\}$, the field of formal Puiseux series, which is the algebraic closure of the field of Laurent series. Here, an element $x \in \mathbb{K}$ can be written as a formal power series with complex coefficients and fractional exponents with bounded denominator:

$$x = \sum_{i=l}^{\infty} c_i t^{i/N},\ c_l \neq 0.$$

The valuation is defined by $v(x) = l/N$, the smallest exponent with nonzero coefficient.

An algebraic torus $(\mathbb{K}^*)^n$ is the Cartesian product of finitely many copies of $\mathbb{K}^*$ and should be thought of as analogous to $(S^1)^n$. A subvariety of $(\mathbb{K}^*)^n$ is the common zero set of a system of Laurent polynomial equations in the coordinates $x_1, x_2, \ldots, x_n$. The tropicalization of X is defined to be $\mathrm{Trop}(X) = \overline{v(X)}$, the topological closure of the image of X under the product of valuation maps $v\colon (\mathbb{K}^*)^n \to \mathbb{R}^n$. This approach does specialize to the synthetic approach, a fact that we illustrate with an example. Consider the subvariety X in $(\mathbb{K}^*)^2$ defined by $x + y + 1 = 0$. For a point (x, y) to belong to X, what must be true of its valuations? Let us express the defining equation of X in terms of the Puiseux series of x, y, and 1:

$$\sum_{i=l}^{\infty} c_i t^{i/N} + \sum_{i=m}^{\infty} d_i t^{i/N} + t^0 = 0.$$

For this equality to be satisfied, the coefficients of any exponent must sum to zero. In particular, the coefficients of the smallest power of t must sum to 0, so there must be at least two such nonzero coefficients. This tells us that $\min(v(x), v(y), v(1))$ must be achieved by at least two entries. It follows that $\mathrm{Trop}(X)$ is contained in the tropical hypersurface of $x \oplus y \oplus 0$ (where $0 = v(1)$). It is a theorem of Kapranov that the reverse containment is true, in fact, for all hypersurfaces.

This valuative approach allows one to speak of tropical varieties, not just tropical hypersurfaces. The tropical varieties that arise are integral, weighted, balanced polyhedral complexes by a result of David Speyer (2005). Moreover, they capture some of the geometry of the original variety X by reflecting X's class in intersection theory and, under suitable smoothness conditions, some of X's cohomology. In this sense, $\mathrm{Trop}(X)$ is a shadow of X. Computing tropicalizations of algebraic varieties is an interesting problem making use of Gröbner basis techniques.

Degeneration-Theoretic Approach

Tropical geometry is very closely related to the study of degenerations of algebraic varieties. In the classical situation, one may have a family of varieties $Y_t \subset (\mathbb{C}^*)^n$ depending on a parameter t varying in a small disc $\mathbb{D}$ around the origin. This arises, for instance, if one is given an algebraic variety $\mathcal{Y} \subset (\mathbb{C}^*)^n \times \mathbb{D}$ with projection $p\colon (\mathbb{C}^*)^n \times \mathbb{D} \to \mathbb{D}$. The varieties Y_t are fibers of the projection restricted to $\mathcal{Y}$. One usually studies semistable families so that $\mathcal{Y}$ is nonsingular, the fibers Y_t for $t \neq 0$ are smooth, and the central fiber Y_0 has very mild (so-called normal crossing) singularities. The irreducible components of Y_0 may intersect in a combinatorially interesting fashion encoded by a polyhedral complex Γ, called the dual complex. The central fiber Y_0 is called a degeneration of the generic fiber Y_t for $t \neq 0$.

There is a purely algebraic analogue of the study of families of algebraic varieties over a disc, the study of schemes over a valuation ring. Here, the field $\mathbb{K}$ is the algebraic analogue of the field of germs of analytic functions near the origin on a punctured disc $\mathbb{D}^*$. A variety X over $\mathbb{K}$ is analogous to a family of varieties

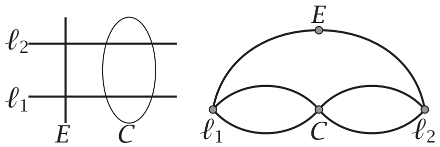

Figure 2. Associated to a curve (left) is its dual graph (right).

over the punctured disc. Under suitable conditions, one may extend X to a semistable scheme $\mathcal{X}$ over $\mathcal{O}$, the valuation ring of $\mathbb{K}$. The scheme $\mathcal{X}$ is analogous to an extension of the family over the disc. The reduction X_0 of $\mathcal{X}$ to the residue field $\mathbf{k}$ of $\mathcal{O}$ is analogous to the central fiber of the family. Under conditions on X introduced by Jenia Tevelev, $\mathrm{Trop}(X)$ is very closely related to the dual complex. The complex $\mathrm{Trop}(X)$ encodes some of the combinatorics of the components of X_0. In that sense, it is not surprising that $\mathrm{Trop}(X)$ would reflect geometric properties of X. There is a certain tension between algebraic geometry and combinatorics in tropical geometry: if the combinatorics of $\mathrm{Trop}(X)$ are simple, the algebraic geometry of the components is likely to be complicated and rich; if the components of the degeneration are simple, the combinatorics of $\mathrm{Trop}(X)$ are rich and capture the geometry of X.

One can try to make tropical geometry more intrinsic by studying the combinatorics of degenerations of abstract varieties X over $\mathbb{K}$. This approach has been developed furthest in the case of curves and has led to the theory of linear systems on graphs as pioneered by Matthew Baker and Sergei Norine. We will work with an example borrowed from the work of Baker to illustrate this theory. We examine a family of curves in $\mathbb{P}^2$ parameterized by t: consider the family $\mathcal{X}$ of quartic curves in $\mathbb{P}^2 \times \mathbb{D}$ defined by $F((X,Y,Z),t) = 0$ with

$$F((X,Y,Z),t) = (X^2 - 2Y^2 + Z^2)(X^2 - Z^2) + tY^3Z.$$

When $t \neq 0$ is small, this defines a smooth plane quartic. When $t = 0$, the curve is the union of a conic C and two lines ℓ_1 and ℓ_2. The total space of the family is singular but can be made nonsingular by blowing up the intersection point of the two lines. This introduces a new component of X_0, which is a rational curve E. The curve X_0 has at worst nodal singularities, meaning that near the intersection of components, the curve locally looks like $xy = 0$.

In Figure 2, the central fiber of the resulting family is pictured on the left. One may form its dual graph Γ by associating a vertex to each irreducible component of X_0 and associating an edge to each nodal singularity of X_0. This gives the dual graph pictured on the right.

One can define divisors on the dual graph as formal integer combinations of vertices of Γ. There is a notion of specializing a divisor $\mathcal{D}$ from the generic fiber X of $\mathcal{X}$ to a divisor D on the dual graph Γ. In fact, one can work out a rich combinatorial theory of divisors on graphs. Here, linear equivalence of divisors is generated by chip-firing moves on graphs, which have been studied in other contexts. One can define the rank $r_\Gamma(D)$ of the linear system associated with the divisor. By a semicontinuity argument, this rank provides an upper bound for the dimension of the linear system on X containing D. With this bound, one is able to use combinatorial methods to prove strong results in the theory of algebraic curves as described in the survey [1]. Recent work of Dustin Cartwright extends this theory to higher dimensions.

Research in Tropical Geometry

Research in tropical geometry is heading in several different directions. The foundations of tropical geometry are undergoing continual revision and are not yet settled. Applications of tropical geometry to enumerative geometry are still being uncovered, many of them in the direction of mirror symmetry. Tropical geometry also provides a hands-on way of studying Berkovich spaces, a theory of analytic geometry over complete non-Archimedean fields. Tropical techniques are now being employed in computational algebraic geometry. There have been many new applications of the theory of linear systems on graphs to algebraic curves. Careful use of degeneration methods has led to results in Diophantine geometry, a branch of number theory. Tropical geometry also allows one to apply geometrically motivated techniques to purely combinatorial objects through associated tropical varieties, bringing powerful new techniques to combinatorics and resolving old problems. Tropical varieties have even been studied in their own right, as they are combinatorially interesting objects.

Further Reading

[1] M. Baker and D. Jensen, Degeneration of linear series from the tropical point of view and applications, *Proceedings of the Simons Symposium on Nonarchimedean Geometry*, Simons Symposia, Springer, 2016, 365–433.

[2] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Amer. Math. Soc., Providence, RI, 2015. MR3287221

Editor's Note

See also "What is a tropical curve?" by Grigory Mikhalkin in the April 2007 *Notices*. Other related columns are "What is an amoeba?" by Oleg Viro (September 2002) and "What is a Gröbner basis?" by Bernd Sturmfels (November 2005).

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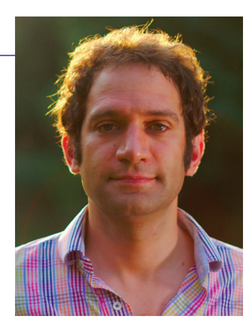
ABOUT THE AUTHOR

Eric Katz's mathematical interests include combinatorial algebraic geometry and number theory. He likes to ride bikes, run, and make outlandish statements.



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