WHAT IS...

a CR Submanifold?

Phillip S. Harrington and Andrew Raich

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A real hypersurface M of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ has the following property: at every point $p \in M$, the tangent space T_pM has a $(2n-2)$-dimensional complex subspace. Whenever a submanifold M of $\mathbb{C}^n$ has the property that its tangent space T_pM admits a complex subspace whose dimension is independent of p, M is called a CR submanifold. Not all smooth submanifolds of $\mathbb{C}^n$ have this property. Consider the equator of the 2D sphere

$$S = \{(z, w) \in \mathbb{C}^2 : |(z, w)| = 1 \text{ and } \operatorname{Im} z = 0\}.$$

At $(\pm 1, 0)$, the tangent space $\{0\} \times \mathbb{C}$ is complex, but at $(0, e^{i\theta})$, the tangent space can be identified with the real span of $(1, 0)$ and $(0, ie^{i\theta})$, and this does not have a complex structure. In particular, it is not closed under multiplication by i. Hence, the real dimension of the largest complex subspace varies from 2 to 0. To motivate our definitions, we will first consider the boundary values of holomorphic functions of several complex variables.

Given a domain $\Omega \subset \mathbb{C}^n$, a continuously differentiable, or C^1, function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if it satisfies the Cauchy–Riemann equations $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all $1 \le j \le n$, where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\frac{\partial}{\partial x_j} + \frac{i}{2}\frac{\partial}{\partial y_j}$. Holomorphic functions are the fundamental objects of study in complex analysis of one and several variables. For example, a function f is equal to a convergent power series in z_j for all $1 \le j \le n$ in a neighborhood of a point if and only if it is also holomorphic in a neighborhood of that point. Consequently, a standard technique in the analysis of real power series is to complexify them and study the corresponding holomorphic function.

We begin our discussion with the following classical boundary value problem. Suppose we are given a C^1 function $g : b\Omega \to \mathbb{C}$. Does there exist a function f in $C^1(\overline{\Omega})$ such that f is holomorphic in Ω and $f = g$ on $b\Omega$? The answer to this question highlights the differences between complex analysis in one and several variables.

When $n = 1$, we know that if such an f exists, then on Ω it must be given by the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

Such an f will always be holomorphic, but it may not have g as a boundary value. For example, if Ω is the unit disc and $g(e^{i\theta}) = e^{-i\theta}$, then the Cauchy Integral Formula gives us $f(z) = 0$ on Ω. Nevertheless, the case $n = 1$ is completely understood. The Plemelj jump formula for the Cauchy Integral Formula implies that g is the boundary value of a holomorphic function f if and only if $\int_{b\Omega} g(\zeta)\zeta^m d\zeta = 0$ for all nonnegative integers m.

The cases $n \ge 2$ require further analysis. We focus on the $n = 2$ case for expositional clarity. Suppose ρ is a C^1 defining function for Ω (i.e., $\rho < 0$ on Ω, $\rho > 0$ outside Ω, and $\nabla \rho \ne 0$ on $b\Omega$). Then the vector field

$$\bar{L} = \frac{\partial \rho}{\partial \bar{z}_2}\frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho}{\partial \bar{z}_1}\frac{\partial}{\partial \bar{z}_2}$$

is a tangent vector to $b\Omega$ (since $\bar{L}\rho = 0$ on $b\Omega$), so $\bar{L}g$ is well-defined. If f is holomorphic in a neighborhood of $b\Omega$, then the Cauchy–Riemann equations would tell us that $\bar{L}f = 0$ on $b\Omega$, so g can be the boundary value

Phillip Harrington and Andrew Raich are associate professors in the Department of Mathematical Sciences at the University of Arkansas. Their e-mail addresses are psharrin@uark.edu *and* araich@uark.edu.

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of a holomorphic function only if $\bar{L}g = 0$ on $b\Omega$. Such functions are called CR functions. Once again, if g is the boundary value of a holomorphic function f, then f is given by an integral formula, in this case the Bochner–Martinelli Formula $f(z) = \int_{b\Omega} g(\zeta)B(\zeta, z)$ where

$$B(\zeta, z) = \frac{\left((\bar{\zeta}_2 - \bar{z}_2)d\bar{\zeta}_1 - (\bar{\zeta}_1 - \bar{z}_1)d\bar{\zeta}_2\right) \wedge d\zeta_1 \wedge d\zeta_2}{(2\pi i)^2 \, |\zeta - z|^4}.$$

In contrast to the Cauchy Integral Formula, the function f given by the Bochner–Martinelli Formula is not necessarily holomorphic (note that $B(\zeta, z)$ is not holomorphic in z). Fortunately, if g is a CR function and Ω is a bounded domain with C^1 boundary, then f is holomorphic and g is the boundary value of f. In contrast to the $n = 1$ case, there is no moment condition for g to satisfy. Instead, boundary values of holomorphic functions are characterized by the CR equation $\bar{L}g = 0$, just as holomorphic functions are characterized by the Cauchy–Riemann equations.

The CR structure reflects the ambient complex structure.

The complex structure on $\mathbb{C}^n$ can be identified with the decomposition of the complexified tangent space $\mathbb{C}T(\mathbb{C}^n)$ into Cauchy–Riemann derivatives of the form $\bar{L} = \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j}$, denoted $T^{0,1}(\mathbb{C}^n)$, and their conjugates, denoted $T^{1,0}(\mathbb{C}^n)$. Consequently, the complexified tangent space has an orthogonal decomposition

$$\mathbb{C}T(\mathbb{C}^n) = T^{1,0}(\mathbb{C}^n) \oplus T^{0,1}(\mathbb{C}^n).$$

The CR structure on a real submanifold M of $\mathbb{C}^n$ is the subspace $T^{0,1}(M) \subset T^{0,1}(\mathbb{C}^n)$ given by all tangential vector fields. If $T^{1,0}(M)$ is the conjugate of $T^{0,1}(M)$, then the complex tangent space of M is given by $T^{1,0}(M) \oplus T^{0,1}(M)$. A CR submanifold then is simply a real submanifold M of $\mathbb{C}^n$ on which the dimension of $T^{0,1}(M)$ is constant. If $\dim_{\mathbb{R}} M = 2\dim_{\mathbb{C}} T^{0,1}(M) + 1$, then we say that M is of hypersurface type. The boundary of a domain with C^1 boundary will always be a CR submanifold of hypersurface type. A CR function on M is any function $f \in C^1(M)$ satisfying $\bar{L}f = 0$ whenever $\bar{L} \in T^{1,0}(M)$. One active area of research is the characterization of CR mappings between CR submanifolds.

We return to our motivating example in $\mathbb{C}^2$. Observe that the real and imaginary parts of $\bar{L}$ are linearly independent tangential vector fields. However, the tangent space of the boundary of a domain in $\mathbb{C}^2$ must have three real dimensions, so there must exist a third vector field T to complete the basis. For example, we can use

$$T = i \sum_{j=1}^{2} \left(\frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} - \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} \right).$$

Notice that whenever a CR submanifold is of hypersurface type, there must always exist a unique (up to a choice of orientation) real tangential vector field that is orthogonal to the complex tangent space. The richness of CR manifolds lies in the interplay between the CR structure, which reflects the ambient complex structure, and this remaining direction, which acts like a totally real vector field.

The model example of a CR submanifold is the boundary M of the Siegel upper half space

$$\Omega = \{z \in \mathbb{C}^2 : \mathrm{Im} z_2 > |z_1|^2\}.$$

We choose $\bar{L} = \frac{\partial}{\partial \bar{z}_1} - 2iz_1 \frac{\partial}{\partial \bar{z}_2}$ to represent the CR equations and $T = \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \bar{z}_2}$ to represent the totally real direction. In this setting, $\bar{L}$ is also known as the Lewy operator in honor of Lewy's result showing local nonsolvability of $\bar{L}$. This result stands in stark contrast to the real case where the Malgrange–Ehrenpreis Theorem tells us that any partial differential operator with real constant coefficients is locally solvable. Lewy showed that in order for $\bar{L}u = f$ to be locally solvable on M when f is a real function of $\mathrm{Re}\, z_2$, it must be the case that f is real-analytic. We note that the boundary of the Siegel upper half space also admits a group structure making it isomorphic to the Heisenberg group, but this useful tool is outside the scope of this article.

We can reformulate the notion of a CR manifold without complexification of the tangent bundle. If we write the coordinates of $\mathbb{C}^n$ by $(z_1, \dots, z_n)$ where $z_j = x_j + iy_j$, then the complex structure is denoted by J and acts on vector fields via

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \quad \text{and} \quad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j},$$

where $j \in \{1, \dots, n\}$. The map J has two eigenvalues: i and $-i$. In the complexified tangent bundle, the eigenvectors corresponding to i are linear combinations of $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$, and the eigenvectors corresponding to $-i$ are linear combinations of $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. Even without complexifying the tangent bundle, we can see that J is analogous to multiplication by $\pm i$ since $J^2 = -I$, where I is the identity map. If $M \subset \mathbb{C}^n$ and $p \in M$, the tangent space at p is denoted $T_p(M)$, and the holomophic tangent space at p, $H_p(M)$, is defined by

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

A smooth submanifold of $\mathbb{C}^n$ is an embedded CR manifold exactly when $\dim_{\mathbb{R}} H_p(M)$ is independent of p. This formulation of a CR manifold gives us a clean way to find many examples (and nonexamples). For example, all affine subspaces in $\mathbb{C}^n$ are CR submanifolds, while the manifold

$$M = \{(z_1, z_2) : x_2 = 0 \text{ and } y_2 = |z_1|^2\}$$

is not, since $\dim_{\mathbb{R}} H_{(0,p_2)}(M) = 2$, but $\dim_{\mathbb{R}} H_p(M) = 0$ if $p_1 \neq 0$. Returning to our initial example S, the equator of the unit sphere, one can check [Bog91, Example 1, p. 99] that $\dim_{\mathbb{R}} H_{(1,0,\dots,0)}(M) = 2n - 2$ and $\dim_{\mathbb{R}} H_{(0,1,0,\dots,0)}(M) = 2n - 4$, so that M is not a CR submanifold of $\mathbb{C}^n$. However, it turns out that the only bad points of M are $(\pm 1, 0, \dots, 0)$, and the manifold

$\tilde{M} = M \setminus \{(\pm 1, 0, \dots, 0)\}$ is a (noncompact) CR submanifold of $\mathbb{C}^n$.

Problems on CR manifolds can be approached from many different directions.

We have highlighted only the most basic aspects of the theory of CR manifolds; namely, we motivated the study of CR manifolds by considering boundary values of holomorphic functions, and we presented two formulations of the definition of CR manifolds to provide a wealth of examples. Problems on CR manifolds can be approached from many different directions, and we encourage the reader to seek out [Bog91] or [CS01] for a unified and in-depth discussion.

References

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Phillip Harrington

ABOUT THE AUTHORS

Phil Harrington's area of research is partial differential equations in several complex variables, particularly the $\bar{\partial}$-Neumann problem. In his spare time, he enjoys hiking and reading.

Andrew Raich

Andy Raich's area of research is several complex variables and harmonic analysis, where he focuses on $\bar{\partial}$-problems, broadly construed. When he can find time, he enjoys running and biking in the beautiful hills around Fayetteville, AR.

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