OPINION

A Formula Goes to Court: Partisan Gerrymandering and the Efficiency Gap

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ABSTRACT. Recently, a proposal has been advanced to detect unconstitutional partisan gerrymandering with a simple formula called the *efficiency gap*. The efficiency gap is now working its way towards a possible landmark case in the Supreme Court. This note explores some of its mathematical properties in light of the fact that it reduces to a straight proportional comparison of votes to seats. Though we offer several critiques, we assess that *EG* can still be a useful component of a courtroom analysis. But a famous formula can take on a life of its own and this one will need to be watched closely.

Gerrymandering is drawing political boundary lines with an ulterior motive. This idea has great currency right now, with particular attention paid to manipulating shapes of US congressional and legislative districts in order to obtain a preferred outcome. Gerrymandering comes in many flavors, including *racial gerrymandering*, where a minority group is subject to dilution of its voting strength; *partisan gerrymandering*, where one political party controls the process and tries to exaggerate its own political dominance; and *incumbent gerrymandering*, where officials try to create safe seats for incumbents on both sides of the aisle. All kinds of gerrymandering use some of the same techniques, especially *packing*, where you seek to stuff your opponents into districts with very high percentages, and *cracking*, where you disperse your opponents into several districts in numbers too small to predominate. Both of these techniques generate wasted votes by your opponents and thus reduce your opponents' share of seats relative to their share of the votes cast.

In this note we will focus on partisan gerrymandering. The Supreme Court has heard cases on partisan gerrymandering three times,[1] and each time the justices have balked. They disagreed among themselves about whether partisan gerrymandering was within the Court's purview at all and about what standards might be used to detect it. In the most recent case, the swing vote belonged to Justice Anthony Kennedy, who wrote a lengthy opinion explaining why he was unsatisfied with the standards proposed by the plaintiffs to demonstrate that unconstitutional gerrymandering had occurred. Kennedy also gave some guidelines for what would be needed to satisfy him in the future. For the past ten years, legal scholars and political scientists have pored over Kennedy's opinion, trying to design a standard that he might find convincing.

In 2015, one such team, Nicholas Stephanopoulos and Eric McGhee [3], advanced a new idea to quantify partisan gerrymandering, tailored to Kennedy's guidelines. They propose a simple numerical score called the *efficiency gap* (*EG*) to detect and reject unfair congressional and legislative maps that are rigged to keep one party on top in a way that is unresponsive to voter preferences. *EG* can be computed based on voting data from a single

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[1] *Davis v. Bandemer (1986); Vieth v. Jubelirer (2004); LULAC v. Perry (2006)*

election: if the result exceeds a certain threshold, then the districting plan has been found to have discriminatory partisan effect.

Last November, for the first time in thirty years, a federal court invalidated a legislative map as an unconstitutional partisan gerrymander [4].[2] A centerpiece in the district court's ruling was the high efficiency gap in Wiconsin's 2012–2016 elections in favor of Republicans. Now the case is under appeal to the Supreme Court, and many observers are hoping that *EG* will finally provide a legally manageable standard for detecting partisan gerrymanders in time for the 2020 census. If this happens, it will signal a seismic shift in American politics.

In this note, we engage in the following exercise: first, we give a self-contained analysis of *EG* for a mathematical audience, describing what it measures and what it does not. Then, we examine how our findings relate to the original proposal, the press coverage, and the court decisions to date. We close by looking to the future career of this consequential formula.

What is the Efficiency Gap?

The US electoral scene is dominated by the use of geographically-defined districts in which one representative is chosen by a plurality vote. We will follow the *EG* literature by simplifying to the case of only two political parties, A and B. Let's say that our state has S congressional or legislative seats, and denote the set of districts by $\mathcal{D} = \{d_1, \dots, d_S\}$. In a particular election, write $\mathcal{D} = \mathcal{D}^A \sqcup \mathcal{D}^B$ where $\mathcal{D}^P$ is the subset of districts won by party $P \in \{A, B\}$. In what follows, if a value X_i^P is defined with respect to party P and district d_i, we will write

$$X^P = \sum_{i=1}^{S} X_i^P; \quad X_i = X_i^A + X_i^B; \quad \text{and} \quad X = X^A + X^B = \sum_{i=1}^{S} X_i.$$

Let S_i^P be 1 if party P won in district d_i and 0 if not, so that $S^P = |\mathcal{D}^P|$ is the number of seats won statewide by P. Let T_i^P be the number of votes cast in district d_i for party P, so that T_i is the voting turnout in district d_i and T^A is the total number of statewide votes for A. Later we'll make the further simplifying assumption that the districts have not just equal population but equal turnout: $T_i = T/S$.

With this notation, we can write $\tau = \frac{T^A - T^B}{T}$ and $\sigma = \frac{S^A - S^B}{S}$ for the statewide *vote lean* favoring A and the statewide *seat lean* favoring A. Most people's intuitions about fairness would incline towards a districting plan that approximates *proportionality*, or $\sigma \approx \tau$. Historically, the courts have sometimes recognized rough proportionality as a virtue, but have explicitly rejected strict proportionality as a standard.[3]

Wasted votes, in the *EG* formulation, are any votes cast for the losing side or votes cast for the winning side in excess of the 50 percent needed to win. That is, the number of votes wasted by A-voters in district d_i is

$$W_i^A = \begin{cases} T_i^A - \frac{T_i}{2}, & d_i \in \mathcal{D}^A, \\ T_i^A, & d_i \in \mathcal{D}^B \end{cases} = T_i^A - S_i^A \cdot \frac{T_i}{2}.$$

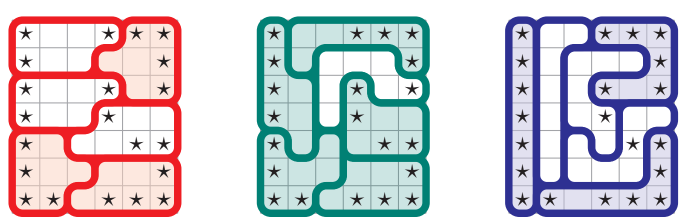
Quick observation: the total number of wasted votes in a district, $W_i = W_i^A + W_i^B$, is always half of the turnout T_i. The question is how the wasted votes are distributed. If (nearly) all the wasted votes belong to the winning side, it's a packed district. If (nearly) all the wasted votes belong to the losing side, it's a competitive district. And if there are several adjacent districts where most of the wasted votes are on the losing side, then it may be a cracked plan.

We can now define the *efficiency gap* associated with districting plan $\mathcal{D}$:

$$EG = \sum_{i=1}^{S} \frac{W_i^A - W_i^B}{T} = \frac{W^A - W^B}{T}.$$

Thus EG is a signed measure of how much more vote share is wasted by supporters of party A than B. If EG is large and positive, the districting plan is deemed unfair to A. On the other hand, $EG \approx 0$ indicates a fair plan, inasmuch as the two parties waste about an equal number of votes. Stephanopoulos–McGhee write that this definition "captures, in a single tidy number, all of the packing and cracking decisions that go into a district plan."

For a toy example, consider the figure below, which shows three possible districting plans for the same distribution of voters. Each box represents a voter; A-voters are marked $\star$ and B-voters are blank. Since A-voters make up half of the population, intuitively we expect a "fair" plan to have $S^A = S^B$. Plans I and III both do this and it is easy to check that they have $EG = 0$. In contrast, Plan II gives A five of the six seats by packing some B-voters into one district and cracking the rest. This gerrymander is successfully detected: $EG = -1/3$, unfairly favoring Party A. Note that there is a lot of packing and cracking in Plan III as well, but this is not penalized by EG because it happens symmetrically to voters of both parties.



Plan I: $EG = 0$ Plan II: $EG = -1/3$ Plan III: $EG = 0$

[2] *This happened only once before, in Davis v. Bandemer (1982), which was later overturned by the Supreme Court.*

[3] *The Supreme Court has a complicated attitude to proportionality. See, for instance, Gaffney v. Cummings (1973), Davis v. Bandemer (1986), and Johnson v. De Grandy (1994). However, it was clearly stated in the Bandemer plurality opinion that "the mere lack of proportional representation will not be sufficient to prove unconstitutional discrimination."*

Of course, in real life, no districting plan will have an efficiency gap of exactly 0. How high is too high? Stephanopoulos and McGhee argue that $EG = .08$ corresponds to a historically robust threshold for unacceptable partisan gerrymandering.[4] In the Wisconsin case, for example, the plaintiffs demonstrated that the last three elections for the State Assembly had efficiency gaps between 0.1 and 0.13 in favor of Republicans. Since it was a Republican legislature that had drawn the map, and there was plentiful evidence that they had intentionally done so to disfavor Democrats, the court ruled that the plan was an unconstitutional partisan gerrymander. QED.

What Does the Efficiency Gap Actually Measure?

But wait! The *EG* formula turns out to simplify quite a bit, in a way that has bearing on our understanding of what it measures. Let's proceed with the assumption of equal voter turnout: $T_i = T/S$.[5] In this case, we get

$$W^A = \sum W_i^A = T^A - S^A \cdot \frac{T}{2S},$$

and thus

$$EG = \frac{T^A - T^B}{T} - \frac{1}{2}\frac{S^A - S^B}{S} = \tau - \frac{1}{2}\sigma.$$

That is: the efficiency gap is just the statewide vote lean favoring A minus half of the statewide seat lean favoring A.[6] It has nothing at all to do with how the voters are distributed among districts, per se. As long as the seat total comes out a certain way, as in Plan III shown above, *EG* does not penalize packing or cracking, or for that matter bizarrely-shaped districts—and indeed, it sometimes incentivizes them, as we will see below.

In its simplified form ($\tau - \frac{1}{2}\sigma$), we can see that *EG* has numerous potentially undesirable properties.

- **Penalizes proportionality.** If Party A has 60 percent of the statewide vote and 60 percent of the seats, *EG* rates this as an unacceptable gerrymander in favor of Party B! ($\tau - \frac{\sigma}{2} = 0.2 - 0.1 > 0.08$.) This is because $EG = 0 \iff \sigma = 2\tau$. That is, the intuitive idea that representation should be proportional to vote share is replaced by the conflicting principle that the seat lean should be *twice* the vote lean.
- **Volatile in competitive races.** *EG* behaves very erratically if there are districts with competitive races, because a genuinely close outcome will produce lopsided vote wastage, but it is unpredictable which side this falls on.[7] If, for instance, all districts are competitive but a last-minute trend pushes voters to one side systematically, then the *plan itself* will be rated as a gerrymander.
- **Fetishizes three-to-one landslide districts.** We've seen that *EG* is not sensitive to any changes in packing and cracking that preserve σ (i.e., that preserve the overall seat outcome). But if anything, *EG* rewards a certain level of district-by-district packing. Recall that every district has a total of 50 percent vote wastage. It immediately follows that the only way to share that fairly in a single district is to have 25 percent on each side, which is a 75–25 vote split. So the only districts viewed by *EG* to be perfectly neutral are highly non-competitive districts, and any plan made up entirely of these landslide districts will be judged perfectly fair.
- **Breaks down in edge cases.** Despite the fact that 75 percent is an artificial sweet spot, 80 percent statewide vote share breaks *EG* completely. If A controls more than 79 percent of the vote in a state, then $\tau > .79 - .21 = .58$. In order to get $\tau - \sigma/2 < .08$, we must have $\sigma > 1$, which is impossible. Thus, in this circumstance *EG* will identify absolutely *any* districting plan as a partisan gerrymander in favor of B.
- **Nongranular.** We have seen that *EG* does no more and no less than compare seats to votes by a double-proportionality standard. This has the added consequence that in states with a small number of districts, the way that *EG* depends on a districting plan is extremely nongranular: for a given vote split, there are only S possible values of the the efficiency gap, as the majority party's seat total ranges from 1 to S. For a particular voting split, a small state may have no outcome at all with a permissibly small *EG*. This makes the score far too coarse for use in many races; twenty one states currently have four or fewer Congressional districts.[8]

How is This Playing Out in Court?

So far we have highlighted, with the help of simple algebra, several grave limitations of *EG* as a stand-alone metric. If the Supreme Court were to enshrine the efficiency gap as a dispositive indicator of partisan gerrymandering, it would be sure to produce false positives as well as false negatives with respect to any common-sense understanding of political unfairness. But has anyone actually proposed

[4] *They propose a somewhat different standard of two excess seats for congressional districting plans.*

[5] *Dropping the equal-turnout assumption only makes matters worse in the critiques of EG that follow, because this corresponds to a weighting of terms that is harder to interpret and defend. Without the equal-turnout simplification, EG is affected as follows: if there is lower average turnout in the districts $\mathcal{D}^A$, then maintaining a low EG requires A to get more seats than party B would have with the same vote share.*

[6] *This can also be written in terms of seat share and vote share rather than seat lean and vote lean:*
$EG = 2(T^A/T) - (S^A/S) - 1/2$.

[7] *In a current preprint, Cover* [2, p. 34] *argues persuasively that this provides a stark disincentive for even an honest mapmaker to draw a competitive plan.*

[8] *Cho* [1] *discusses nongranularity at length.*

that it be used that way? Press headlines have touted *EG* as "a formula to end gerrymandering" and a simple way to "end gerrymandering once and for all," trumpeting that "this is how to get rid of gerrymandered districts."[9] But a brief review of the political science literature and court documents shows circumspect writing and multifaceted analysis at all stages: the role of *EG* in the legal landscape is much more complex.

Original Framing

First and importantly, Stephanopoulos and McGhee propose a doctrinal test for partisan gerrymandering in which *EG* can be considered only if several other conditions are met. They address concerns about volatility by requiring evidence of stability: plaintiffs must perform a "sensitivity analysis" to show that their disadvantage would persist under a modest statewide swing to one party or the other and must demonstrate the gap's likely durability over time. Nongranularity is indirectly acknowledged: in their historical analyses, the authors only considered elections with eight or more seats. And edge cases, in the form of extremely lopsided elections, are simply dismissed as never occurring at the statewide level.

Defending the idealization of double-proportionality is a heavier lift. The fact that using *EG* commits you to condemning proportional outcomes is actually observed in a footnote [3, p. 18, note 107], but Stephanopoulos and McGhee defend it as reasonable on the grounds of quantifying an appropriate "seat bonus" for the majority party. On this view, the efficiency gap quantifies the level of bonus that *should* be enjoyed by the winner: a party with 60 percent of the vote should have 70 percent of the seats, a party with 70 percent of the vote should have 90 percent of the seats, etc. We will return to this point below.

The District Court

Many objections to the usefulness of *EG* were introduced in the Wisconsin case in the form of expert reports and testimony for the defense. The court decision endorses a three-pronged test expanding on the proposal described above by requiring proof of discriminatory intent.[10] The majority opinion argues that this protects against false positives: "If a nonpartisan or bipartisan plan displays a high *EG*, the remaining components of the analysis will prevent a finding of a constitutional violation." Thus, for instance, a proportional plan would not be thrown out on the grounds of high *EG*, because the plaintiffs would not be able to demonstrate improper partisan intent. The court affirms *EG* not as a conclusive indicator, but only as persuasive "corroborative evidence of an aggressive partisan gerrymander."

The Supreme Court

The strategy that the Wisconsin plaintiffs will use at the next level is outlined in their motion to appeal. While praising the usefulness and diminishing the critiques of *EG*, this document does emphasize its limited role: "To be clear, Appellees do not ask the Court to endorse any particular measure of partisan asymmetry or any particular technique for demonstrating durability. The [district court] did not do so, nor need the [Supreme Court] in order to affirm. Rather, Appellees advocate the same course of action the Court has followed in other redistricting contexts involving discriminatory effects: namely, the articulation of a standard whose precise contours are filled in through subsequent litigation."

And Beyond

There is a philosophical issue at the heart of this analysis. Can a formula be said to "measure" quantities that are used to compute it, or only those to which it is numerically sensitive? Wasted votes are apparent inputs into *EG*, but their local contributions come out in the wash, and only *deviation from double-proportionality of seats to votes* remains. Setting double-proportionality as a target contravenes the common-sense preference for proportionality, an ideal that even the Supreme Court has recognized despite its insistence that it is not constitutionally required. While political scientists have found evidence of a hyper-proportional seat bonus in real elections, we have seen no persuasive case for the slippage from an empirical description to a specific normative standard (that is, from what *is* the case to what *should be* the case). We assess this as the most serious indictment of the *EG* formula.

Our evaluation suggests a suite of particular circumstances in which *EG* could be a useful component of a broader analysis[11]—and it seems that Wisconsin's State Assembly districts currently present such a case. However, a mathematician could well argue that a formula which is only used to lend numerical corroboration in special and extreme cases is a formula of limited usefulness indeed. This caution has to be weighed against the plain fact that courts have found no manageable standard to date for even the most extreme partisan abuses in redistricting.

To a great extent, critiques of *EG* are mitigated by the limited circumstances in which courts will apply it and

[9] *See coverage in the* Detroit Free Press, New Republic, *and* Washington Post.

[10] *First, plaintiffs must establish the intent to discriminate on the basis of political affiliation. Then, discriminatory effect must be established, and EG can be used to that end. And finally, defendants must fail to provide justification of the necessity of the plan on other legitimate legislative grounds.*

[11] *To use EG, we'd want a state with enough congressional districts for EG to be sufficiently granular, a close enough overall partisan preference split that double-proportionality does not predict outlandish seat margins, few enough uncontested districts to require a minimum of counterfactual speculation in incorporating them in to the numbers, an appropriate balance of competitive districts to pass sensitivity analysis, and an egregious enough partisan power grab to still show up.*

the fact that it may well be endorsed only as a first draft of a general standard that will be refined over time. But major concerns remain. Courts will have safeguards in place, but *EG* is already in play outside of courts, and there is a real risk that it may come to stand in as an operationalized *definition* of partisan gerrymandering. We are seeing hints of this effect not only in the overheated popular press coverage but in more scholarly work as well.[12]

Legal scholars believe that *EG* will appeal to the courts because of its simple, one-shot construction with no technical machinery. As we have seen, the simplicity is actually illusory: a lot of care, including further statistical testing and modeling, is required to use *EG* responsibly. Moreover, *EG* comes on the scene at a time when having a single formula is becoming less important. One of the most promising directions in the detection of gerrymandering is the use of computational methods that can take multiple indicators into account simultaneously. For instance, various teams of researchers have developed sampling tools for comparing a proposed plan against an algorithmically generated batch of alternate plans.[13] Typically these samplers can incorporate both equality constraints and inequality constraints, encoding both legal requirements and preferences. The core idea of *EG* can be coded into a sampling analysis by, for instance, replacing the use of $\tau - \frac{1}{2}\sigma$ as a score with bounds that constrain the deviation from proportionality rather than prescribing it. Alternatively, the seats-to-votes proportionality could be used as an evaluation axis. A plan like the Wisconsin legislative map could be taken as input, and hundreds of thousands of alternate maps would be randomly generated that meet the relevant legal requirements and are scored at least as well by legitimate districting criteria. If a districting plan had a seat share to vote share ratio of $(S^A/S)/(T^A/T) = 1.25$ in a particular election and 95 percent of computer-generated alternatives had ratios of 1.05–1.15 with the same data, then we'd have excellent evidence of excessive partisan skew. Use of sampling comparisons is improving quickly as increased computing power lets the algorithms visit more of the space of possible plans. As the ability to study large samples of neutrally-generated maps rapidly becomes both sophisticated and practicable, the hoped-for breakthrough in adjudicating gerrymanders, partisan and otherwise, may be coming within close reach after all.

The Wisconsin plaintiffs are not asking the court to enshrine *EG* as the one true measure of partisan gerrymandering, but only to accept it as a starting point in building a test to show when entrenched partisan advantage has risen to the level of vote dilution of political opponents. We hope that the Supreme Court agrees with them in a decision that leaves room for *EG* to pave the way for refined metrics and methods in the years to come.

[12] *One such example is* Extreme Maps, *a research report put out by the Brennan Center at NYU, which uses EG without any caveats to assess the current state of partisan gerrymandering in the US.*

[13] *See for instance multiple papers with various co-authors by Jowei Chen and by Jonathan Mattingly, whose teams use Markov chain Monte Carlo, or by Wendy Cho and Yan Liu, who use genetic and evolutionary algorithms.*

References

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