Jacqueline Ferrand and Her Oeuvre

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Communicated by Allyn Jackson

Jacqueline Ferrand was born in Alès, in the south of France, one hundred years ago, in 1918. At the time, she was one of the very few women ever admitted to the École Normale Supérieure (ENS), and she succeeded at the (male) *agrégation* in 1939. She immediately won a teaching assistant position at École Normale Supérieure de Jeunes Filles. The director of the school, Mrs. Cotton, was convinced that women should nourish the same intellectual ambition as men, and she reckoned that this brilliant young mathematician would raise the level of teaching at Jeunes Filles to the same level as the ENS. Several people witnessed the energy Jacqueline Ferrand devoted to this task despite poor material conditions. With the same energy, she began to do research, under Arnaud Denjoy's supervision from afar. After her thesis defense, on June 12, 1942, she was awarded the Girbal Barral Prize in 1943 and the Fondation Peccot Prize in 1946. Her career developed quickly: assistant professor in Bordeaux in 1943, professor in Caen in 1945, in Lille in 1948, in Paris from 1956 until her retirement in 1984.

First Works

Jacqueline Ferrand's thesis deals with boundary values of planar conformal mappings. Since the time of Bernhard Riemann, it has been known that every simply connected domain D' admits a conformal mapping f from the unit disk D, i.e. a geographic map in which angles are preserved. One may view f as an analytic function of one variable defined on the disk, which is a bijection from D onto D'. The issue is whether f has a limit at every boundary point. The answer is positive if the boundary of D' is smooth enough. The problem becomes hard if the boundary is irregular.

In 1913, Constantin Carathéodory made a decisive step when he introduced the notion of prime end. Carathéodory considers nested *cuts* in D'. These are nested sequences of subdomains cut by simple arcs joining two boundary points, and whose lengths tend to 0. Two sequences are equivalent if each can be nested in the other. A *prime end* is an equivalence class of nested cuts. This definition, which involves lengths, is not obviously a conformal invariant. Carathéodory gives a proof of invariance that relies on fine properties of holomorphic functions.

In her thesis, published in the *Annales de l'École Normale Supérieure de Paris* in 1942, Ferrand gives a new proof of conformal invariance of prime ends that highlights the role played by the area of a conformal map f in estimating lengths of images of almost all curves. Furthermore, the fact that f is open allows one to pass from almost all curves to all curves, and hence to estimate the modulus of continuity of f.

Precise estimates lead to sufficient conditions on the domain D' for the conformal mapping to admit limits along curves contained in the disk and having a higher order contact with the boundary of the disk. Under stronger assumptions, Ferrand shows, in a 1942 paper in the *Bulletin de la Société Mathématique de France*, the existence of *angular derivatives*

$$\lim_{z\to a}\frac{f(z)-a}{z-a}$$

when a is a boundary point.

For a conformal (or holomorphic) bijection, the area of the image is given by the Dirichlet integral

$$\int_D |f'(z)|^2 dz.$$

For a harmonic function u (the real part of a holomorphic function f), the Dirichlet integral replaces area and the maximum principle provides an ersatz for the openness of f. Methods developed for conformal mappings in her 1944 paper in the *SMF Bulletin* hence extend to the boundary study of harmonic and superharmonic functions.

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This article originally appeared in French in the Gazette des Mathématiciens, *number 141, July 2014. Pansu prepared this English version, which appears here with the kind permission of the Société Mathématique de France.*

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DOI: http://dx.doi.org/10.1090/noti1641

Jacqueline Ferrand, shown here in a photo dated 1988, was an outstanding French geometer whose works continue to influence research today. She wrote her last mathematical papers at the age of eighty and died in 2014, at the age of ninety-six.[1]

Preholomorphic Functions

In that same paper, Ferrand introduced a discretization of the notion of holomorphic function. Given $h > 0$, she replaces the plane $\mathbb{C}$ (or a bounded domain D' of $\mathbb{C}$) with the finite set Z_h of points of D' whose real and imaginary parts are integer multiples of h. Classically, one defines a function u on Z_h to be harmonic (Ferrand writes *preharmonic*) if

$$4u(z) = u(z+h) + u(z-h) + u(z+ih) + u(z-ih).$$

Discretizing holomorphicity equations is less classical. Ferrand calls *preholomorphic* a complex-valued function f that satisfies, for all $z \in Z_h$,

$$f(z+ih) - f(z+h) = i(f(z+h+ih) - f(z)).$$

The real and imaginary parts of f are preharmonic functions on sub-lattices

$$Z'_h = \{z \in Z_h; (Re(z) + Im(z))/h \text{ is even}\}$$

and

$$Z''_h = \{z \in Z_h; (Re(z) + Im(z))/h \text{ is odd}\}.$$

Conversely, every preharmonic function on Z_h is the real part of a preholomorphic function on Z_h. Ferrand shows that when the mesh h tends to 0, preholomorphic functions on Z_h converge to holomorphic functions on the domain D'. *A priori* estimates on the modulus of continuity play a crucial role again.

In 1955, Ferrand inferred a pretty and simple proof of the conformal mapping theorem for nonsimply connected domains. *Let D' be a planar domain whose boundary has at least one isolated connected component that is not a single point. Then there exists an essentially unique conformal mapping of D' onto a rectangle with segments parallel to one side removed.*

The notion of preholomorphic function has led to numerous developments.

Group Actions

During a visit to Institute for Advanced Study, Princeton, Ferrand investigated when a Lie algebra action integrates into a Lie group action. The upshot is a functional analytic characterization of completeness of vector fields, which she published in the *SMF Bulletin* in 1942.

This elegant result does not seem to be well known.

Let ξ be a locally Lipschitz vector field with vanishing divergence, on a manifold M equipped with a volume element ω. Then ξ is complete if and only if the differential operator $i\xi$ extends to a selfadjoint operator of $L^2(\omega)$. Assume that ξ generates a one-parameter group of isometries of M. This group is periodic if and only if the operator $i\xi$ has a closed image.

This elegant result does not seem to be well known.

Teaching Books

Ferrand's mathematical production decreased between 1958 and 1968. At this time, Jacqueline was the mother of four young children. She put a great deal of effort into her teaching and wrote a series of course notes that, with a lot of work, became textbooks. A basic differential geometry course was published by Masson in 1963. Her freshman courses were published by Armand Colin in 1964 and Dunod in 1967. In the 1970s Dunod published the series of books co-authored with Jean-Marie Arnaudiès, which covered the whole curriculum of the first two years of university (including exercises). It is still in use today in the *classes préparatoires aux grandes écoles*, which prepare students for the competitive examinations to enter the École Normale Supérieure.

[1] *This photo was taken at École Normale Supérieure and kindly provided to* Notices *by Ms. Ferrand's children.*

Riemannian Geometry

At the end of this period, Ferrand obtained her most famous results, leading to an invitation to the 1974 International Congress of Mathematicians in Vancouver. She solved a problem in Riemannian geometry posed by André Lichnerowicz in 1964, on which several partial answers had been published.

A conformal mapping of an open set of Euclidean space $\mathbb{R}^n$ is a diffeomorphism whose differential preserves angles. This notion extends to open sets of $\mathbb{R}^n$ equipped with a Riemannian metric, i.e. an inner product depending on position, to submanifolds of Euclidean spaces, and to abstract Riemannian manifolds. The prototype of a Riemannian manifold is the sphere

$$\{x \in \mathbb{R}^{n+1}; x_0^2 + \cdots + x_n^2 = 1\}.$$

Stereographic projection achieves a conformal diffeomorphism of the sphere, with one point deleted, onto Euclidean space. By transport of structure, Euclidean similitudes become conformal diffeomorphisms of the sphere. It follows that the group of conformal self-mappings of the sphere is noncompact, a result that Ferrand published in the *Mémoires de l'Académie Royale Belgique* in 1971.

Theorem 1. *If a compact Riemannian manifold M has a noncompact group of conformal self-mappings, then M is conformal to the sphere.*

The point is to estimate the modulus of continuity of a conformal mapping, in every dimension. Let us use the four-point conformal invariant Ferrand introduced in her 1973 paper in the *Journal of Differential Geometry*. The proof we are about to give in Euclidean n-space immediately extends to Riemannian manifolds.

Definition 2. *Let F_0, F_1 be disjoint compact connected subsets of $\mathbb{R}^n$. The capacity $\mathrm{cap}(F_0, F_1)$ is the infimum of integrals $\int |du|^n$ for all smooth functions u on $\mathbb{R}^n$ such that $u = 0$ on F_0 and $u = 1$ on F_1.*

Let x, y, z, t be 4 distinct points of $\mathbb{R}^n$. The *Ferrand invariant* $j(x, y, z, t)$ is the infimum of capacities of pairs (F_0, F_1) of compact connected sets such that F_0 contains x and z and F_1 contains y and t.

Using an *a priori* estimate of the modulus of continuity of functions u that minimize $\int |du|^n$ (a nonlinear n-dimensional generalization of her thesis work), Ferrand shows that this infimum is positive. In fact (see the 1991 paper of Ferrand et al. in *Journal d'Analyse Mathématique*, completed by her 1996 paper in the *Pacific Journal of Mathematics*), when z and t are fixed,

$$d(x, y) = j(x, y, z, t)^{1/(1-n)}$$

is a distance that defines the usual topology of $\mathbb{R}^n \setminus \{z, t\}$ and that tends to infinity if, when y is fixed, x tends to z.

To give an idea of how the invariant j is used, let us show that, in an arbitrary Riemannian manifold, the group G of conformal mappings that fix three distinct points y, z, and t is compact. Indeed, G acts isometrically for metric d and fixes y. Therefore, every sequence of elements of G has a subsequence that C^0 converges to a homeomorphism. There remains to show that the limit is a conformal diffeomorphism and that the convergence is in C^∞. This is a nonlinear elliptic regularity theorem, due to Frederick Gehring and Yuri Reshetnyak in the Euclidean case, and extended by Ferrand in 1976 to general Riemannian manifolds.

More can be drawn from the j invariant. If a sequence f_k of conformal mappings diverges, then for all sequences x_k, z_k, t_k, if $f_k(z_k)$ converges to z and $f_k(t_k)$ converges to $t \neq z$, then $f_k(x_k)$ converges to z or to t. This means that at most two limits are possible, z and t. If $z \neq t$, then, up to extracting a subsequence, f_k maps the complement of every neighborhood of z into arbitrarily small neighborhoods of t. This implies that the manifold M is simply connected and that its metric is conformally flat, hence M is conformal to the sphere.

Quasiconformal Mappings

How did Ferrand discover her four-point invariants?

The idea of using a conformally invariant metric goes back to Lichnerowicz in 1964. Lichnerowicz uses constant scalar curvature metrics. In the 1960s, this approach was limited due to the incomplete solution of Hideiko Yamabe's problem: existence and uniqueness of a constant scalar curvature metric, conformal to a given Riemannian metric. The essential result obtained since—an analytic analogue of Ferrand's 1971 result mentioned above—is due to Richard Schoen (cf. his contribution to the Do Carmo Jubilee, 1988). Let (M, g) be a compact Riemannian manifold that is not conformal to the standard sphere. Then the set of constant scalar curvature metrics conformal to g is compact, up to scaling.

The idea of a naturally invariant metric has been developed with great success by Shoshichi Kobayashi in the holomorphic category (cf. his 1970 book). Although it is likely that Kobayashi's construction may have influenced Ferrand, it differs in an essential way from Ferrand's. The exact transcription of Kobayashi's definition in the conformal category (necessarily limited to conformally flat manifolds) was carried out by Ravi Kulkarni and Ulrich Pinkall in 1994.

Ferrand's inspiration is rather to be found in the theory of quasiconformal mappings launched by Herbert Grötsch in 1928. Here is the definition given by Lars Ahlfors in 1930. A *quadrilateral* is a planar domain delimited by a Jordan curve carrying four marked points. Such a domain admits a conformal mapping onto a rectangle that maps marked points to vertices. This rectangle is unique up to a similitude. The ratio of lengths of two consecutive sides is a conformal invariant of the quadrilateral, called its *modulus*. A homeomorphism f between planar domains is called *K-quasiconformal* if, for every quadrilateral Q,

$$K^{-1}\mathrm{modulus}(Q) \leq \mathrm{modulus}(f(Q)) \leq K\mathrm{modulus}(Q).$$

Note that the modulus of a rectangle is exactly half of the capacity of two opposite sides. In two dimensions, Ferrand's definition is therefore very close to Ahlfors's.

A homeomorphism between planar domains is quasiconformal if and only if it maps small balls to small domains of bounded eccentricity (cf. Jussi Väisälä's 1971 book). This notion makes sense for general metric spaces. Misha Gromov, following G. D. Mostow and Grigorii Margulis, has highlighted the role of quasiconformal mappings in geometric group theory: the ideal boundary of a hyperbolic group has a quasiconformal structure. This has motivated works by Juha Heinonen, Pekka Koskela, Adam Koranyi, and Hans–Martin Reimann (1995), where the regularity of quasiconformal mappings is investigated on larger and larger classes of metric spaces. In these works, the key point remains to estimate Ferrand's invariant. To some extent, the notion of a *Loewner space*, abstracted by Heinonen, characterizes spaces whose Ferrand invariant is nontrivial. Ferrand's invariant plays a crucial role in work by Marc Bourdon and Hervé Pajot (2000), who establish the quasi-isometric rigidity of certain hyperbolic buildings, and in Mario Bonk and Bruce Kleiner's 2002 characterization of the standard sphere up to quasisymmetry.

Gromov has raised the question of whether quasiconformal geometry survives in infinite dimensions. This motivates the search for dimension-independent estimates on Ferrand's invariant.

Ferrand's invariant exploits the conformal invariance of integrals $\int |du|^n$. In a sense that we will make precise, these integrals entirely determine the conformal structure. Following Halsey Royden, Ferrand defines a family of commutative Banach algebras attached to an n-dimensional Riemannian manifold M. Given $p > 1$, let $A_p(M)$ denote the space of continuous bounded functions on M whose distributional first derivatives have integrable pth powers. $A_p(M)$ is a Banach algebra for the norm $\|u\|_\infty + \|du\|_p$.

Ferrand goes one step further.

In a 1973 paper in the *Duke Mathematical Journal*, Ferrand shows that *if $p = n$, every isomorphism $A_n(M) \to A_n(N)$ is induced by a quasiconformal mapping $M \to N$. However, if $p \neq n$, every isomorphism $A_p(M) \to A_p(N)$ is induced by a bi-Lipschitz homeomorphism.* Vladimir Goldshtein and Mati Rubin (1995) have a result that goes in the same direction. Bourdon (2007) has extended these ideas to ideal boundaries of hyperbolic metric spaces.

Ferrand goes one step further. She characterizes maps $M \to N$ that map $A_p(N)$ to $A_p(M)$, for $p > n$. The case when $p < n$ was studied by Goldshtein, Leonid Gurov, and Alexandr Romanov in 1995.

Finite Type Geometric Structures

Lichnerowicz's problem extends to noncompact Riemannian manifolds. Say that a group G of conformal mappings of a Riemannian manifold M is *inessential* if it preserves a Riemannian metric g' conformal to g. The problem becomes: show that a noncompact Riemannian manifold whose conformal group is essential is conformal to Euclidean space. Dmitry Alekseevski published a solution in 1972. Only in 1992 did Robert Zimmer and Karl Gutschera find a serious gap in Alekseevski's argument.

Here is the context that led Zimmer to study Lichnerowicz's problem. In his 1986 ICM lecture, Zimmer studies actions of noncompact groups on compact manifolds, from the point of view of dynamical systems. Every manifold admits an ergodic action of $\mathbb{R}$. However, only very special compact manifolds admit ergodic actions of a semi-simple Lie group (cf. François Labourie's lecture at the 1998 ICM). These groups bear geometry within themselves (Felix Klein's Erlangen program is not far) and carry it to manifolds on which they act. This restricts possibilities for invariant geometric structures.

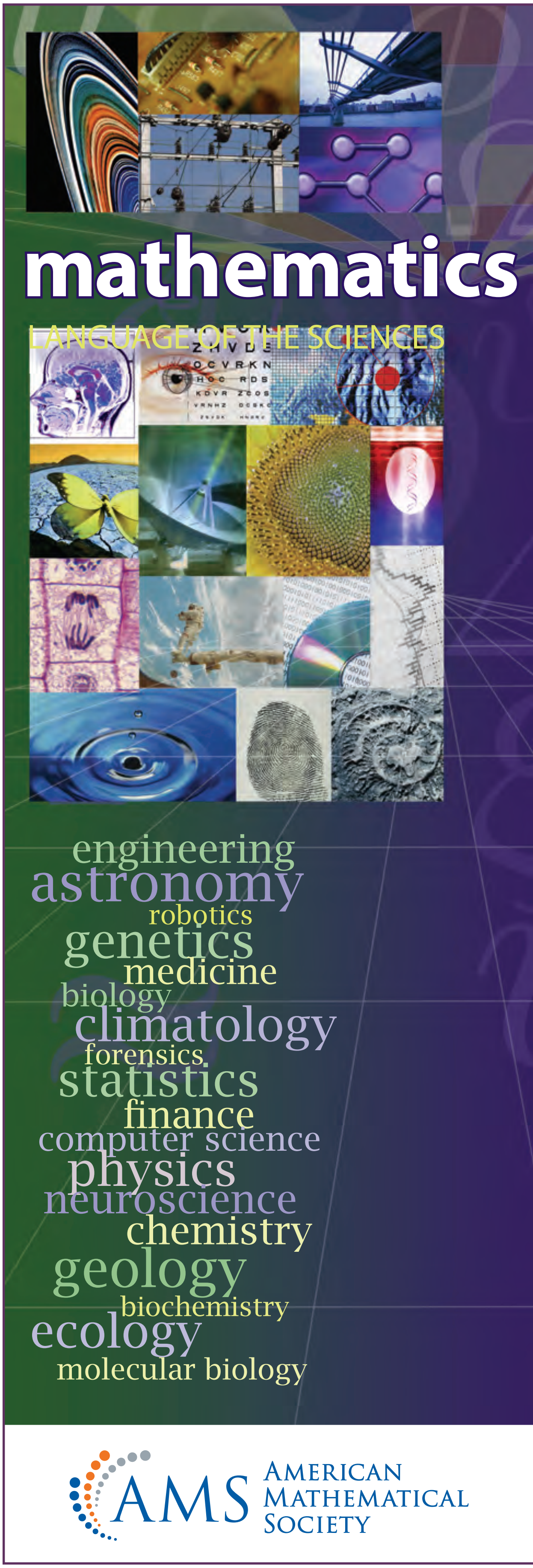
Let us call an *order r geometric structure* on a manifold M the datum of a reduction of the fibre bundle of order r-jets to an algebraic subgroup of the group $Gl_n(r)$ of r-jets of diffeomorphisms fixing the origin in $\mathbb{R}^n$. A geometric structure is *of finite type* if every automorphism is determined by its finite jet at some point. For instance, (pseudo-)Riemannian metrics, conformal structures, and projective structures have finite type. Symplectic or complex structures have infinite type.

It is a fascinating problem to decide which manifolds equipped with finite type geometric structures admit noncompact automorphism groups. An important step was made by Gromov in 1988. He shows that if the automorphism group has a dense orbit, this orbit is open. The sought-for objects are thus homogeneous almost everywhere. For instance, it follows that the isometry group of a compact manifold equipped with a real analytic Lorentzian metric is always compact (Giusi d'Ambra 1988). For conformal structures, the problem has been solved in Ferrand's 1971 paper in the *Mémoires de l'Académie Royale Belgique*. Generalizations of this beautiful result have been given by Charles Frances.

Here is the assertion from Alekseevski's paper that attracted Zimmer's attention. If G is a closed group of automorphisms of a manifold M equipped with a finite type geometric structure, and if point stabilizers are compact, then G acts properly on M. In this generality, this statement is wrong. In the special case of a conformal structure, it is equivalent to the noncompact Lichnerowicz conjecture, which was finally solved in a 1996 paper in *Mathematische Annalen* by Ferrand, who also solved its quasiconformal version in a paper that same year in *Journal d'Analyse Mathématique*. The solution relies on a three-point invariant, obtained from the four-point invariant by letting one point tend to infinity. Other invariants obtained by passing to the limit are discussed in a paper from 1999. After this last text, completed by a survey of the history of Lichnerowicz's conjecture, Ferrand deliberately laid down her mathematical feather, at the age of 80. She passed away on April 26th, 2014, in Sceaux, near Paris.

Conclusion

Ferrand's works have had a significant influence on several branches of mathematics. However, they are better known abroad than in her home country. She did not try to found



a school. As she modestly said, she hesitated to draw young mathematicians along paths she thought were not sufficiently promising. Ferrand has had collaborations abroad, especially in Finland where she was held in great esteem. Nevertheless, her intellectual route was mainly a lonely one. The value of her work has been fully recognized only when the mathematical mainstream came close to her. This happened in 1942 at the time of her thesis, in 1969 with Lichnerowicz's problem, and again in 1996. The energy she devoted, at the age of 80, to proving the theorem that provides a definite answer to Lichnerowicz's question, deserves admiration.

> *Her intellectual route was mainly a lonely one.*

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Photo of Jacqueline Ferrand courtesy Jean Lelong.
Photo of Pierre Pansu courtesy of Angela Pasquale.

ABOUT THE AUTHOR

Pierre Pansu is a differential geometer. He first met Jacqueline Ferrand at Arthur Besse's Paris seminar in the early 1980s. His own work being influenced by Ferrand's, he kept in touch with her until 1999.

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