WHAT IS...

Riemannian holonomy?

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Berger's classification of Riemannian holonomy groups is a strong organizing principle in differential geometry. It tells us about exceptional geometric structures, existing only in certain dimensions, that occupy central roles in physics, and generally serves as a road map for some research trends in algebraic and symplectic geometry.

To each Riemannian manifold (M, g) there are associated parallel transport maps P_γ that move vectors along a path $\gamma : I \to M$ such that the motion looks parallel from the metric's point of view. Fixing a point p in M the *holonomy group* is the group, written $\mathrm{Hol}_p(g)$, of parallel transport maps around loops based at p, as in Figure 1.

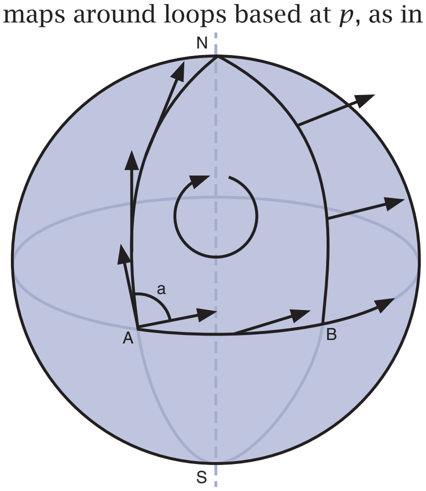

Figure 1. The holonomy group results from parallel-translating a vector around all loops from a fixed point A. On the round sphere, it includes all rotations, but on some manifolds, it is a smaller subgroup.

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If M is path-connected then $\mathrm{Hol}_p(g)$ is, up to conjugacy, independent of p. One should always regard the holonomy group as coming with a natural representation: the inclusion $\mathrm{Hol}_p(g) \to \mathrm{GL}(T_pM)$. In 1926 Élie Cartan observed that the holonomy group acts reducibly if and only if the metric is (locally) a product metric. This reduces the problem of classifying holonomy groups of Riemannian manifolds to the problem of classifying holonomy groups of irreducible Riemannian manifolds. Cartan also wrote down all the holonomy groups of so-called 'symmetric spaces.' A *symmetric space* is a Riemannian manifold M such that, at each $p \in M$, the geodesic reflection s_p is an isometry. Euclidean spaces $\mathbb{R}^n$, spheres S^n, and hyperbolic spaces $\mathbf{H}^n$ are all examples of symmetric spaces.

For (simply-connected) irreducible nonsymmetric Riemannian manifolds, Marcel Berger wrote down a list of all possible holonomy groups.

Theorem (Berger, 1955)**.** *Let (M, g) be a simply-connected, irreducible, nonsymmetric Riemannian manifold. Let $n = \dim M$. Then the holonomy group $\mathrm{Hol}(g)$ of (M, g) is either*

- $\mathrm{SO}(n)$,
- $\mathrm{U}(m)$ *with $n = 2m$ and $m \geq 2$,*
- $\mathrm{SU}(m)$ *with $n = 2m$ and $m \geq 2$,*
- $\mathrm{Sp}(m)$ *with $n = 4m$ and $m \geq 2$,*
- $Sp(m)Sp(1)$ *with $n = 4m$ and $m \geq 2$,*
- G_2 *with $n = 7$,*
- $\mathrm{Spin}(7)$ *with $n = 8$, or*
- $\mathrm{Spin}(9)$ *with $n = 16$.*

As it turns out, there are no irreducible nonsymmetric Riemannian manifolds with holonomy group equal to $\mathrm{Spin}(9)$. In 1968 Alexeevsky eliminated $\mathrm{Spin}(9)$ from this list. All other entries on Berger's list, however, do occur as the holonomy group of some irreducible nonsymmetric Riemannian manifold, although it took some time to realize this.

Manifolds with holonomy contained in $\mathrm{U}(m)$ are called *Kähler*, manifolds with holonomy contained in $\mathrm{SU}(m)$ are called *Calabi–Yau*, and those with holonomy contained in $\mathrm{Sp}(m)$ are called *hyperkähler*.

Berger's list, in a sense, echoes the classification of real normed division algebras.

Theorem (Dickson)**.** *There are exactly four real normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.*

Manifolds with special holonomy are important in physics.

Each group in Berger's list is a group whose elements are automorphisms/isometries of a vector space over some real divison algebra. For example $\mathrm{SO}(m)$ (resp. $\mathrm{SU}(m)$) is a group of automorphisms of $\mathbb{R}^m$ (resp. $\mathbb{C}^m$). In this sense, $\mathrm{Sp}(m)$ and $\mathrm{Sp}(m)\mathrm{Sp}(1)$ holonomies are quaternionic geometries, while G_2 holonomy and $\mathrm{Spin}(7)$ holonomy are octonionic geometries; G_2 is the group of automorphisms of $\mathbb{O}$ and $\mathrm{Spin}(7)$ is the group of isometries of the octonions $\mathbb{O}$ generated by left multiplication by unit length imaginary octonions.

Manifolds with special holonomy are important in physics. One reason is that a so-called "parallel spinor field" is required for the equations of supersymmetry to work. On a general Riemannian manifold, the parallel tensors determine the holonomy group. On a spin manifold, the holonomy group determines the parallel spinors. For this reason, manifolds with special holonomy groups (especially $\mathrm{SU}(3)$ and G_2) can be useful in theoretical physics.

Manifolds with holonomy $\mathrm{SU}(3)$, Calabi–Yau 3-folds, form so-called "string compactifications" in ten-dimensional supersymmetric string theories. This means that, in such a string theory, the universe is locally modeled on $\mathbb{R}^{(1,3)} \times X$, where $\mathbb{R}^{(1,3)}$ denotes Minkowski spacetime and X is a Calabi–Yau manifold of 6 real dimensions. In M-theory, the universe is supposed to have 11 dimensions and to be locally modelled on $\mathbb{R}^{(1,3)} \times X$, where X is a compact (singular) seven-dimensional manifold with holonomy G_2.

G_2 and $\mathrm{Spin}(7)$ manifolds are rather unlike Calabi–Yau manifolds, however, in that it is notoriously hard to write down examples. To illustrate the difficulty: finding a metric with holonomy G_2 amounts to solving a simultaneous system of forty-nine nonlinear PDEs. Bryant wrote down the first metrics with holonomy G_2 and with holonomy $\mathrm{Spin}(7)$ in 1987. These metrics were not complete. Later Bryant and Salamon constructed complete noncompact examples of manifolds with holonomy G_2 and $\mathrm{Spin}(7)$. In 1993 Joyce constructed the first examples of compact 7-manifolds with holonomy G_2 and of compact 8-manifolds with holonomy $\mathrm{Spin}(7)$. Since then other examples of compact manifolds with holonomy G_2 have been constructed by Corti–Haskins–Pacini–Nördstrom, Joyce–Karigiannis, and Kovalev.

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When not doing mathematics, **Jacob Gross** trains in acrobatics and in break dancing.