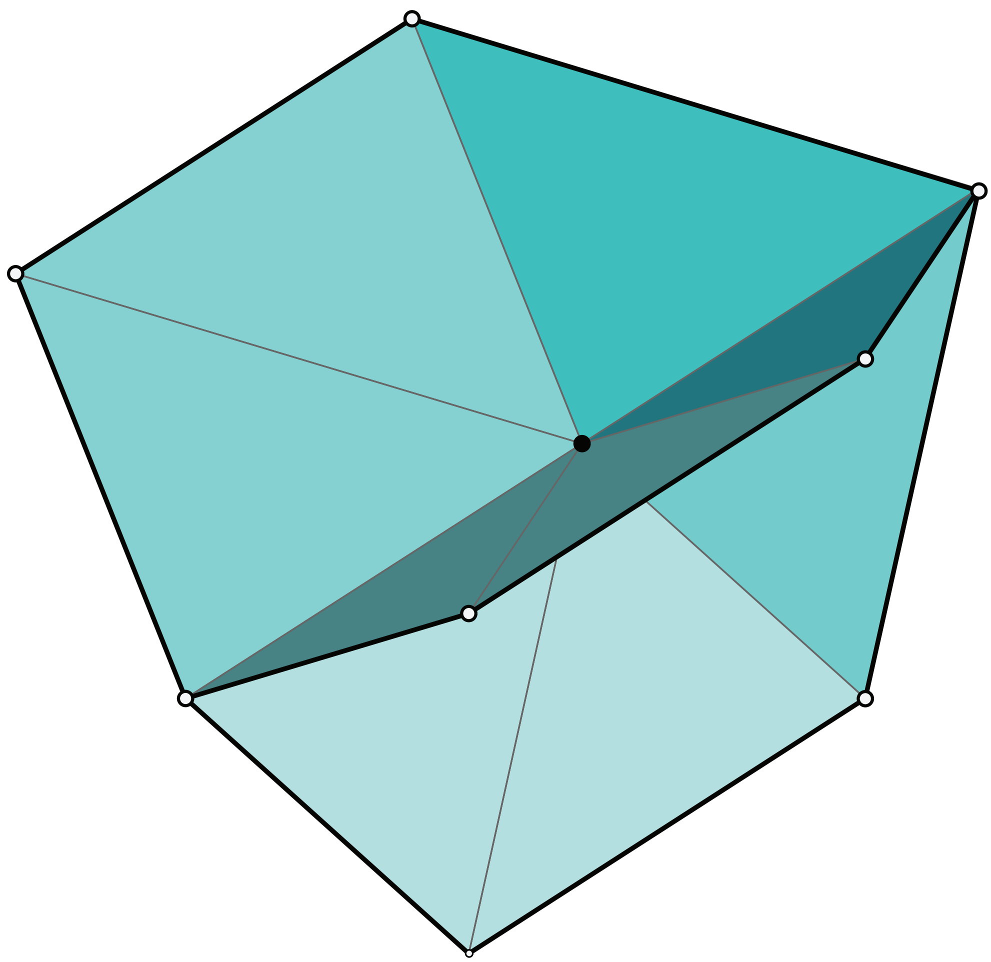
The Geometry of Matroids



Federico Ardila

Introduction

Matroid theory is a combinatorial theory of independence which has its origins in linear algebra and graph theory and turns out to have deep connections with many other fields. There are natural notions of independence in linear algebra, graph theory, matching theory, the theory of field extensions, and the theory of routings, among others. Matroids capture the combinatorial essence that those notions share.

Federico Ardila is professor of mathematics at San Francisco State University and profesor adjunto at Universidad de los Andes in Colombia. His email address is `federico@sfsu.edu`.

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Gian-Carlo Rota, who helped lay the foundations of the field and was one of its most energetic ambassadors, rejected the "ineffably cacophonous" name of *matroids*. He proposed calling them *combinatorial geometries* instead.[1] This alternative name never really caught on, but the geometric roots of the field have since grown much deeper, bearing many new fruits.

The geometric approach to matroid theory has recently led to the solution of long-standing questions and to the development of fascinating mathematics at the intersection of combinatorics, algebra, and geometry. This article is a selection of some recent successes, stemming from three geometric models of matroids.

[1]*It was tempting to call this note "The geometry of geometries."*

Definitions

Matroids were defined independently in the 1930s by Nakasawa and Whitney. A *matroid* $M = (E, \mathcal{I})$ consists of a finite set E and a collection $\mathcal{I}$ of subsets of E, called the *independent sets*, such that

(I-1) $\emptyset \in \mathcal{I}$.
(I-2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.
(I-3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J - I$ such that $I \cup j \in \mathcal{I}$.

We will assume that every singleton $\{e\}$ is independent.

Thanks to (I-2), it is enough to list the collection $\mathcal{B}$ of maximal independent sets; these are called the *bases* of M. By (I-3), they have the same size $r = r(M)$, which we call the *rank* of the matroid. Our running example will be the matroid with

$$E = abcde, \qquad \mathcal{B} = \{abc, abd, abe, acd, ace\}, \tag{1}$$

omitting brackets for easier readability. See Figure 1.

Let us now discuss the two most important motivating examples of matroids; there are many others.

Vector Configurations

Let $\mathbb{F}$ be a field, let E be a set of vectors in a vector space over $\mathbb{F}$, and let $\mathcal{I}$ be the collection of linearly independent subsets of E. Then $(E, \mathcal{I})$ is a *linear* matroid (over $\mathbb{F}$).

Graphs

Let E be the set of edges of a graph G and let $\mathcal{I}$ be the collection of forests of G, that is, the subsets of E containing no cycle. Then $(E, \mathcal{I})$ is a *graphical* matroid.

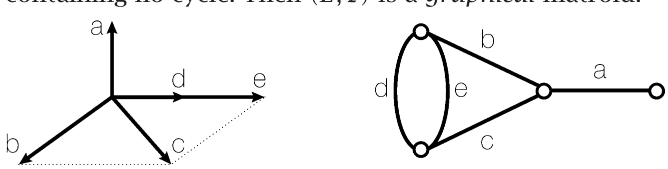

Figure 1. A linear and a graphical representation of the matroid of (1) with $\mathcal{B} = \{abc, abd, abe, acd, ace\}$.

There are several natural operations on matroids. For $S \subseteq E$, the *restriction* $M|S$ and the *contraction* M/S are matroids on the ground sets S and $E - S$, respectively, with independent sets

$$\begin{aligned} \mathcal{I}|S &= \{I \subseteq S \,:\, I \in \mathcal{I}\}, \\ \mathcal{I}/S &= \{I \subseteq E - S \,:\, I \cup I_S \in \mathcal{I}\} \end{aligned}$$

for any maximal independent subset I_S of S. When M is a linear matroid in a vector space V, $M|S$ and M/S are the linear matroids on S and $E - S$ that M determines on the vector spaces $\operatorname{span}(S)$ and $V/\operatorname{span}(S)$, respectively.

The *direct sum* $M_1 \oplus M_2$ of two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ on disjoint ground sets is the matroid on $E_1 \cup E_2$ with independent sets

$$\mathcal{I}_1 \oplus \mathcal{I}_2 = \{I_1 \cup I_2 \,:\, I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}.$$

Every matroid decomposes uniquely as a direct sum of its *connected components*.

Finally, the *orthogonal matroid* of M, denoted $M^\perp$, is the matroid on E with bases

$$\mathcal{B}^\perp = \{E - B \,:\, B \in \mathcal{B}\}.$$

Remarkably, this simple notion simultaneously generalizes orthogonal complements and dual graphs. If M is the matroid for the columns of a matrix whose rowspan is $U \subseteq V$, then $M^\perp$ is the matroid for the columns of any matrix whose rowspan is $U^\perp$. If M is the matroid for a planar graph G, drawn on the plane without edge intersections, then $M^\perp$ is the matroid for the dual graph $G^\perp$, whose vertices and edges correspond to the faces and edges of G, respectively, as shown in Figure 2.

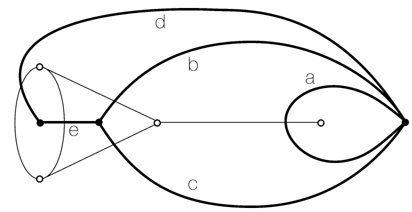

Figure 2. The planar graph of Figure 1 and its dual graph, whose set of bases is $\mathcal{B}^\perp = \{bd, be, cd, ce, de\}$.

Enumerative Invariants

Two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are *isomorphic* if there is a *relabeling* bijection $\phi : E_1 \to E_2$ that maps $\mathcal{I}_1$ to $\mathcal{I}_2$. A *matroid invariant* is a function f on matroids such that $f(M_1) = f(M_2)$ whenever M_1 and M_2 are isomorphic. Let us introduce a few important examples.

The f-vector and the h-vector

The independent sets of M form a simplicial complex $\mathcal{I}$ by (I-2); its f*-vector* counts the number $f_k(M)$ of independent sets of M of size $k+1$ for each k. The h*-vector* of M, defined by

$$\sum_{k=0}^{r} f_{k-1}(q-1)^{r-k} = \sum_{k=0}^{r} h_k q^{r-k},$$

stores this information more compactly. For example, the matroid of (1) has

$$f(M) = (1, 5, 9, 5), \qquad h(M) = (1, 2, 2, 0).$$

The Characteristic Polynomial

We define the *rank function* $r : 2^E \to \mathbb{Z}$ of a matroid M by

$$r(A) = \text{ largest size of an independent subset of } A,$$

for $A \subseteq E$. Let $r = r(M) = r(E)$ be the rank of M. When M is a linear matroid, $r(A) = \dim \operatorname{span}(A)$. The *characteristic polynomial* of M is

$$\chi_M(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{r(M) - r(A)}.$$

The sequence $w(M)$ of *Whitney numbers of the first kind* is defined by $\chi_M(q) = w_0 q^r - w_1 q^{r-1} + \cdots + (-1)^r w_r q^0$. For example, the matroid of (1) has

$$w(M) = (1, 4, 5, 2).$$

The characteristic polynomial of a matroid is one of its most fundamental invariants. For graphical and linear matroids, it has the following interpretations.

Graphs

If M is the matroid of a connected graph G, then $q\,\chi_M(q)$ is the *chromatic polynomial* of G; it counts the colorings of the vertices of G with q given colors such that no two neighbors have the same color.

Hyperplane Arrangements

Suppose M is the matroid of nonzero vectors $v_1, \dots, v_n \in \mathbb{F}^d$, and consider the arrangement $\mathcal{A}$ of hyperplanes

$$H_i : v_i \cdot x = 0, \qquad 1 \le i \le n,$$

and its complement $V(\mathcal{A}) = \mathbb{F}^d - (H_1 \cup \cdots \cup H_n)$. Depending on the underlying field, $\chi_M(q)$ stores different information about $V(\mathcal{A})$:

(a) ($\mathbb{F} = \mathbb{F}_q$) $V(\mathcal{A})$ consists of $\chi_M(q)$ points.
(b) ($\mathbb{F} = \mathbb{R}$) $V(\mathcal{A})$ consists of $|\chi_M(-1)|$ regions.
(c) ($\mathbb{F} = \mathbb{C}$) The Poincaré polynomial of $V(\mathcal{A})$

$$\sum_{k\ge 0} \operatorname{rank} H^k(V(\mathcal{A}), \mathbb{Z}) q^k = (-1)^d \chi_M(-1/q).$$

Geometric Model 1. Matroid Polytopes

A crucial insight into the geometry of matroids came from two seemingly unrelated places: combinatorial optimization and algebraic geometry. From both points of view, it is natural to model a matroid in terms of the following polytope.

Definition 1 (Edmonds, 1970)**.** Let M be a matroid on the ground set E. The *matroid polytope*

$$P_M = \operatorname{conv}\{e_B : B \text{ is a basis of } M\},$$

where $\{e_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$, and we write $e_B = e_{b_1} + \cdots + e_{b_r}$ for $B = \{b_1, \dots, b_r\}$.

Figure 3 shows the matroid polytope for example (1).

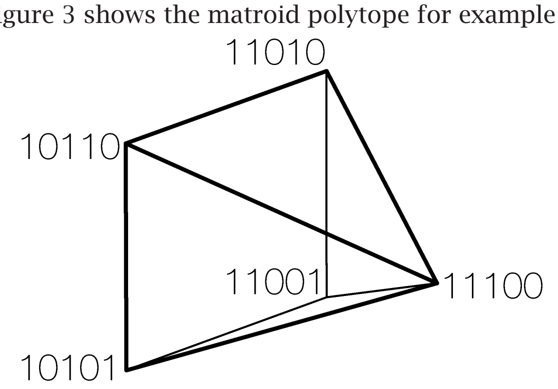

Figure 3. The matroid polytope for our sample matroid (1)**. The vertices exhibit which triplets form bases.**

Combinatorial Optimization

The central question of combinatorial optimization is the following: Given a family $\mathcal{B}$ of combinatorial objects and a *cost function* $c : \mathcal{B} \to \mathbb{R}$, find the object(s) B in $\mathcal{B}$ for which the cost $c(B)$ is minimized. To do this, one often looks for a polytope $P_{\mathcal{B}} \subset \mathbb{R}^d$ modeling the family $\mathcal{B}$ and a linear function f on $\mathbb{R}^d$ such that

- $P_{\mathcal{B}}$ has a vertex v_B for each object $B \in \mathcal{B}$, and
- $c(B) = f(v_B)$ for each $B \in \mathcal{B}$.

If one can do this, then the optimal object(s) B corresponds to the vertices of the face of the polytope $P_{\mathcal{B}}$ where the linear function f is minimized. This simple, beautiful idea is the foundation of *linear programming*. There are many techniques to optimize f, whose efficiency depends on the complexity of the polytope $P_{\mathcal{B}}$.

Edmonds observed that, given a matroid M and a cost function $c : E \to \mathbb{R}$ on its ground set, the bases $B = \{b_1, \dots, b_r\}$ of M of minimum cost $c(B) := c(b_1) + \cdots + c(b_r)$ can be found via linear programming on the matroid polytope P_M.

As a sample application, Edmonds used these ideas to solve the *matroid intersection problem* for matroids M and N on the same ground set. This problem asks us to find the size of the largest set which is independent in both M and N.

Algebraic Geometry

Instead of studying the r-dimensional subspaces of $\mathbb{C}^n$ one at a time, it is often useful to study them all at once. They can be conveniently organized into the space of r-subspaces of $\mathbb{C}^n$ called the *Grassmannian* $\mathrm{Gr}(r, n)$; each point of $\mathrm{Gr}(r, n)$ represents an r-subspace of $\mathbb{C}^n$.

A choice of a coordinate system on $\mathbb{C}^n$ gives rise to the *Plücker embedding* of

$$\mathrm{Gr}(r, n) \overset{p}{\hookrightarrow} \mathbb{P}\mathbb{C}^{\binom{n}{r}-1}$$

as follows. For an r-subspace $V \subset \mathbb{C}^n$, choose an $r \times n$ matrix A with $V = \operatorname{rowspan}(A)$. Then for each of the $\binom{n}{r}$ r-subsets B of $[n]$ let

$$p_B(V) := \det(A_B)$$

be the determinant of the $r \times r$ submatrix A_B of A whose columns are given by the subset B. Although there are many different choices for the matrix A, they can be obtained from one another by elementary row operations, which only change the *Plücker vector* $p(V)$ by multiplication by a global constant. Therefore $p(V)$ is well defined as an element of projective space. The map p provides a realization of the Grassmannian as a smooth projective variety.

The torus $\mathbb{T} = (\mathbb{C} - \{0\})^n$ acts on $\mathbb{C}^n$ by stretching the n coordinate axes, thus inducing an action of $\mathbb{T}$ on $\mathrm{Gr}(r, n)$. This action gives rise to a *moment map* $\mu : \mathrm{Gr}(r, n) \to \mathbb{R}^n$ given by

$$\mu(V)_i = \frac{\sum_{B \ni i} |\det(A_B)|^2}{\sum_B |\det(A_B)|^2} \quad \text{for } 1 \le i \le n.$$

Now consider the trajectory $\mathbb{T} \cdot V$ of the r-subspace $V \in \mathrm{Gr}(r, n)$ as the torus $\mathbb{T}$ acts on it, and take its closure. Where does the resulting toric variety $\overline{\mathbb{T} \cdot V} \subset \mathrm{Gr}(r, n)$ go under the moment map? Precisely to the matroid polytope!

Define the *matroid $M(V)$ of the subspace $V \subset \mathbb{C}^n$* to be the matroid of the columns of A; its bases B correspond to the nonzero Plücker coordinates $p_B(V)$. Gelfand, Goresky, MacPherson, and Serganova showed that

$$\mu(\overline{\mathbb{T}\cdot V}) = P_{M(V)}.$$

Thus matroid polytopes arise naturally in this algebro-geometric setting as well.

As a sample application, the degree of $\overline{\mathbb{T}\cdot V} \subset \mathbb{PC}^{\binom{n}{r}-1}$ is then given by the volume of the matroid polytope $P_{M(V)}$. Ardila, Benedetti, and Doker used this to find a purely combinatorial formula for $\deg(\overline{\mathbb{T}\cdot V})$ in terms of the matroid $M(V)$.[2]

A Geometric Characterization of Matroids

In most contexts where polytopes arise, it is advantageous if they happen to have a nice structure. For example, in optimization, the edges of the polytope are crucial to various algorithms for linear programming. In geometry, they control the GKM presentation of the equivariant cohomology of the Grassmannian.

Matroid polytopes have the following beautiful combinatorial characterization, which was discovered in the context of toric geometry.

Theorem 2 (Gelfand–Goresky–MacPherson–Serganova, 1987)**.** *A collection $\mathcal{B}$ of subsets of $[n]$ is the set of bases of a matroid if and only if every edge of the polytope*

$$P_{\mathcal{B}} := conv\{e_B \,:\, B \in \mathcal{B}\} \subset \mathbb{R}^n$$

is a translate of $e_i - e_j$ for some i, j.

This makes matroid polytopes a very useful model for matroids. In fact, one could *define* a matroid to be a subpolytope of the cube $[0,1]^n$ that uses only these vectors as edges. Notice that from this polytopal point of view, even if one cares only about linear matroids, all matroids are equally natural. Matroid theory provides the correct level of generality.

The theorem above shows that in matroid theory, a central role is played by one of the most important vector configurations in mathematics, the *root system* for the special linear group SL_n:

$$A_{n-1} = \{e_i - e_j \,:\, 1 \le i, j \le n\},$$

as shown in Figure 4 for $n = 4$. From this point of view, it is natural to extend this construction to other Lie groups. The resulting theory of *Coxeter matroids*, introduced by Gelfand and Serganova, is ripe for further combinatorial exploration.

[2] *This was the subject of Carolina Benedetti and Jeff Doker's final project for the first course offered by the SFSU-Colombia Combinatorics Initiative in 2007, as described in* [2].

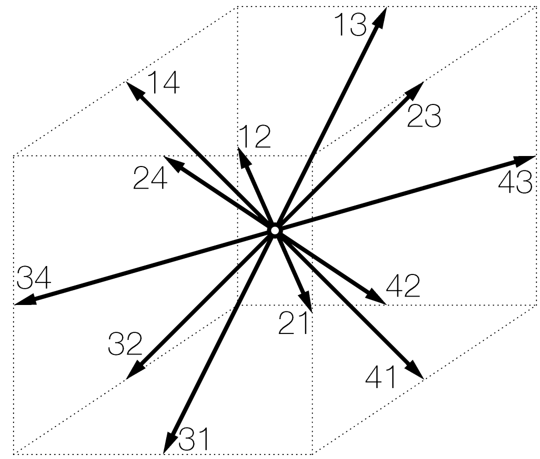

Figure 4. The root system $A_3 = \{e_i - e_j \,:\, 1 \le i, j \le 4\}$, where $e_i - e_j$ is denoted ij. Root systems play an essential role in matroid theory, as demonstrated by Theorem 2.

Hopf Algebra

Joni and Rota showed that many combinatorial families have natural *merging* and *breaking* operations that give them the structure of a Hopf algebra, with many useful consequences. In particular, in the 1970s and 1980s, Joni–Rota and Schmitt defined the *Hopf algebra of matroids* $\mathbb{M}$ as the span of the set of matroids modulo isomorphism, with the product $\cdot : \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}$ and coproduct $\Delta : \mathbb{M} \to \mathbb{M} \otimes \mathbb{M}$ given by

$$\begin{aligned} M \cdot N &:= M \oplus N && \text{for matroids } M \text{ and } N, \\ \Delta(M) &:= \sum_{S \subseteq E} (M|S) \otimes (M/S) && \text{for a matroid } M \text{ on } E. \end{aligned}$$

For $\mathbb{M}$ to be a Hopf algebra, we require an *antipode map* S, which is the Hopf-theoretic analogue of an inverse. General results of Schmitt and Takeuchi show that this map exists.

The antipode S is a fundamental ingredient of a Hopf algebra, so it is important to find an efficient formula for it. For the Hopf algebra of matroids $\mathbb{M}$, this was only resolved recently, thanks to the new insight that the matroid polytope plays an essential role. An important preliminary observation, which readily follows from Theorem 2, is that every face of a matroid polytope is itself a matroid polytope.

Theorem 3 (Aguiar–Ardila, 2017)**.** *The antipode of the Hopf algebra of matroids $\mathbb{M}$ is given by*

$$S(M) = \sum_{P_N \text{ face of } P_M} (-1)^{c(N)} N$$

for any matroid M, where $c(N)$ denotes the number of connected components of N.

This formula is the best possible: it involves no cancellation. It has the unexpected consequence that matroid polytopes are also algebraic in nature. In the Hopf algebraic structure of matroids, matroid polytopes are fundamental.

Geometric Model 2. Bergman Fans

We now introduce a second geometric model of matroids, coming from tropical geometry. The *flats* of M are an important ingredient; these are the subsets $F \subseteq E$ such that $r(F \cup e) > r(F)$ for all $e \notin F$. We say F is *proper* if it does not have rank 0 or r. The *lattice of flats* of M, denoted L_M, is the set of flats, partially ordered by inclusion, as shown in Figure 5.

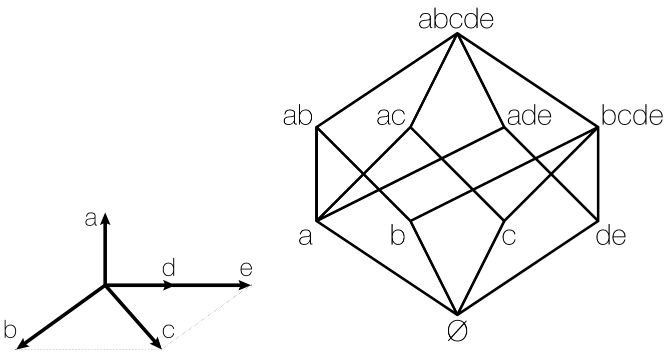

Figure 5. The lattice of flats of our sample matroid (1)**.**

When M is the matroid of a vector configuration E in a vector space V, the flats of M are the (subsets of E contained in the) subspaces spanned by E.

Tropical Geometry

Tropicalization is a powerful technique that turns an algebraic variety V into a simpler, piecewise linear space $\operatorname{Trop} V$ that still contains geometric information about V. Tropical geometry answers questions in algebraic geometry by translating them into polyhedral questions that can be approached combinatorially [4].

An important early success of the theory was Mikhalkin's 2005 tropical computation of the *Gromov-Witten invariants of* $\mathbb{CP}^2$, which count the plane curves of degree d and genus g passing through $3d + 1 - g$ general points. Since then, many new results in classical algebraic geometry have been obtained through tropical techniques.

Tropical varieties are simpler than algebraic varieties, but they are still very intricate. An important example to understand is that of linear spaces. What is the tropicalization of a linear subspace V of $\mathbb{C}^n$? Sturmfels realized that the answer depends only on the matroid of V. It can be described as follows.

Definition/Theorem 4 (Ardila–Klivans, 2006)**.**

(1) The *Bergman fan* Σ_M of a matroid M on E is the polyhedral complex in $\mathbb{R}^E / \langle e_E \rangle$ consisting of the cones

$$\sigma_{\mathcal{F}} = \operatorname{cone}\{e_F \,:\, F \in \mathcal{F}\}$$

for each flag $\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_l\}$ of proper flats of M. Here $e_F := e_{f_1} + \cdots + e_{f_k}$ for $F = \{f_1, \ldots, f_k\}$.

(2) The tropicalization of a linear subspace V of $\mathbb{C}^n$ is the Bergman fan of its matroid:

$$\operatorname{Trop} V = \Sigma_{M(V)}.$$

(3) The Bergman fan Σ_M is a cone over a wedge of w_r spheres of dimension $r - 2$, where w_r is the last Whitney number of the first kind.

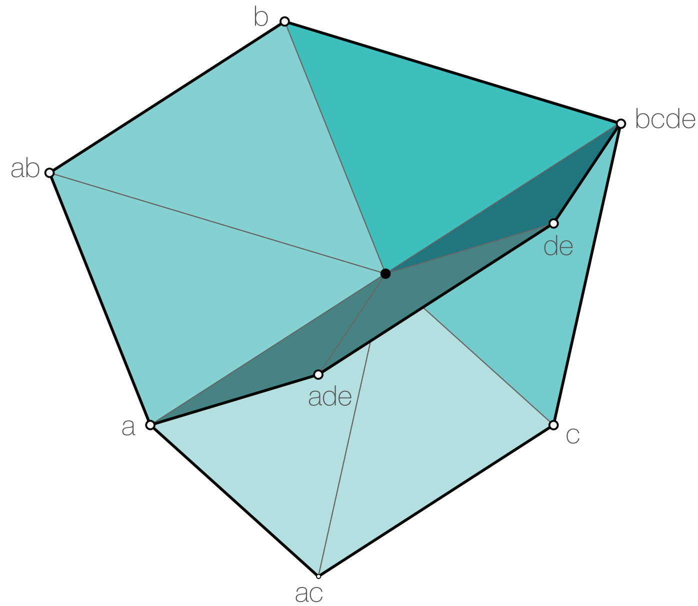

Figure 6. The Bergman fan of our sample matroid (1) **is modeled after the lattice of flats of Figure 5. It has 8 rays and 9 facets. It is a cone over a wedge of** $w_3 = 2$ **circles.**

A Geometric Characterization of Bergman Fans

Tropical varieties have a natural notion of *degree*, analogous to the notion of the degree of an algebraic variety. We have the following remarkable characterization.

Theorem 5 (Fink, 2013)**.** *A tropical variety has degree* 1 *if and only if it is the Bergman fan of a matroid.*

We conclude that Bergman fans are also excellent models for matroids. In fact, one could *define* a matroid to be a tropical variety of degree 1; this is the tropical analogue of a linear space. Notice that, although Σ_M only arises via tropicalization when M is a linear matroid, one should really consider the Bergman fans of all matroids; they are equally natural from the tropical point of view. Again, matroid theory really provides the correct level of generality.

The theorems above explain the important role that matroids play in tropical geometry. On the one hand, they provide a useful testing ground, providing hints for the kinds of general results that may be possible and the sorts of difficulties that one should expect. On the other hand, they are fundamental building blocks; for instance, in analogy with the classical definition of a manifold, a *tropical manifold* is a tropical variety that locally looks like a (Bergman fan of a) matroid, as in Figure 7.

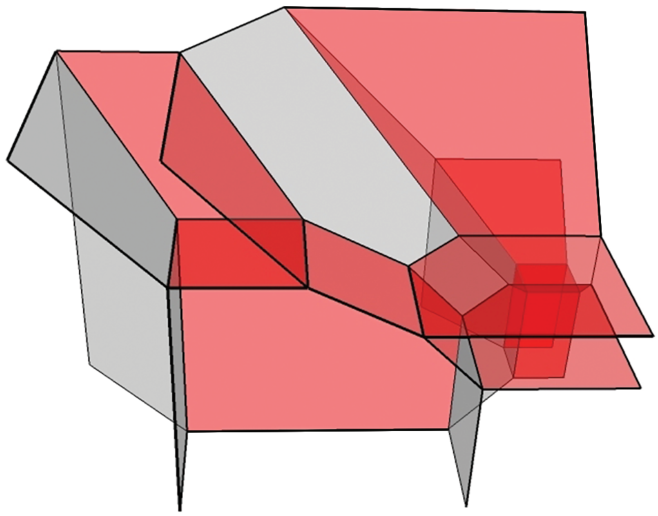

Figure 7. A tropical manifold is a tropical variety that locally looks like a (Bergman fan of a) matroid.

The Chow Ring and Hodge Theory

The *Chow ring* of the Bergman fan Σ_M is defined to be

$$A^*(\Sigma_M) := \mathbb{R}[x_F : F \text{ proper flat of } M]/(I_M + J_M),$$

where

$$I_M = \langle x_{F_1}x_{F_2} : F_1 \subsetneq F_2 \text{ and } F_1 \supsetneq F_2 \rangle,$$
$$J_M = \left\langle \sum_{F \ni i} x_F - \sum_{F \ni j} x_F : i, j \in E \right\rangle.$$

This ring has a natural geometric interpretation when M is linear over $\mathbb{C}$: Feichtner and Yuzvinsky proved that $A^*(\Sigma_M)$ is the Chow ring of De Concini and Procesi's *wonderful compactification* of the complement of a hyperplane arrangement.

Surprisingly, $A^*(\Sigma_M)$ behaves as nicely as the cohomology ring of a smooth projective variety. This is one of the most celebrated recent results in matroid theory, since it provided the tools to prove several long-standing conjectures, as we now briefly explain.

Theorem 6 (Adiprasito–Huh–Katz, 2015)**.** *The Chow ring $A^*(\Sigma_M)$ of the Bergman fan of a matroid M satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations.*

The inspiration for this theorem is geometric, coming from the Grothendieck standard conjectures on algebraic cycles. The statement and proof are combinatorial. For further details and a precise statement, see [1], [3].

Let us focus on a comparatively small but very powerful consequence. The Chow ring $A^*(\Sigma_M)$ is graded of degree $r-1$, and there is an isomorphism $\deg : A^{r-1} \to \mathbb{R}$ characterized by the property that $\deg(F_1 \cdots F_{r-1}) = 1$ for any full flag $F_1 \subsetneq \cdots \subsetneq F_{r-1}$ of proper flats. Say a function $c : 2^E \to \mathbb{R}$ is *submodular* if $c_\varnothing = c_E = 0$ and $c_A + c_B \geq c_{A\cup B} + c_{A \cap B}$ for any $A, B \subseteq E$, and let

$$\overline{K}(M) = \left\{ \sum_{F \text{ flat}} c_F x_F : c \text{ submodular} \right\} \subset A^1(\Sigma_M).$$

The Hodge–Riemann relations imply that for any $L_1, \ldots, L_{r-3}, a, b \in \overline{K}(M)$, if we write $L = L_1 \cdots L_{r-3}$, we have

$$\deg(La^2)\deg(Lb^2) \leq \deg(Lab)^2. \tag{2}$$

Unimodality and Log-Concavity

We say a sequence $a_0, a_1, \ldots, a_r$ of nonnegative integers is *unimodal* if there is an index $0 \leq m \leq r$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_r,$$

and, more strongly, it is *log-concave* if for all $1 \leq i \leq r-1$,

$$a_{i-1}a_{i+1} \leq a_i^2.$$

It is *flawless* if we have

$$a_i \leq a_{s-i}$$

for all $1 \leq i \leq \frac{s}{2}$, where s is the largest index with $a_s \neq 0$.

Many sequences in mathematics have these properties, but proving it is often very difficult. Aside from their intrinsic interest, these kinds of questions have been a source of fresh mathematics, because their solutions have often required a fundamentally new construction or connection and have given rise to unforeseen structural results about the objects of interest.

For matroids, this Hodge theory provides such a connection. Consider the elements of the Chow ring $A^*(\Sigma_M)$,

$$\alpha = \alpha_i = \sum_{F \ni i} x_F, \qquad \beta = \beta_i = \sum_{F \not\ni i} x_F,$$

which are independent of i and lie in the cone $\overline{K}(M)$. A clever combinatorial computation in $A^*(\Sigma_M)$ shows that

$$\deg(\alpha^k \beta^{r-1-k}) = |\text{ coeff. of } q^k \text{ in } \chi_M(q)/(q-1)|.$$

As k varies, this sequence of degrees is log-concave by (2). In turn, by elementary arguments, this implies the following theorems, which were conjectured by Rota, Heron, Mason, and Welsh in the 1970s and 1980s.

Theorem 7 (Adiprasito–Huh–Katz, 2015)**.** *For any matroid M of rank r, the following sequences, defined in "Enumerative Invariants," are unimodal and log-concave:*

- *the Whitney numbers of the first kind $w(M)$, and*
- *the f-vector $f(M)$.*

Geometric Model 3. Conormal Fans

We now introduce another polyhedral model of M that leads to stronger inequalities for matroid invariants. We say that a flag $\mathcal{F} = \{F_1 \subseteq \cdots \subseteq F_l\}$ of nonempty flats of M and a flag $\mathcal{G} = \{G_1 \supseteq \cdots \supseteq G_l\}$ of nonempty flats of $M^\perp$ of the same length are *compatible* if

$$\bigcap_{i=1}^{l}(F_i \cup G_i) = E, \qquad \bigcup_{i=1}^{l}(F_i \cap G_i) \neq E.$$

All maximal compatible pairs have length $n-2$.

Definition 8 (Ardila–Denham–Huh, 2017)**.** The *conormal fan* $\Sigma_{M,M^\perp}$ of a matroid M is the polyhedral complex in $\mathbb{R}^E/\langle e_E\rangle \times \mathbb{R}^E/\langle e_E\rangle$ consisting of the cones

$$\sigma_{\mathcal{F},\mathcal{G}} = \text{cone}\{e_{F_i} + f_{G_i} : 1 \leq i \leq l\}$$

for each compatible pair of flags $(\mathcal{F}, \mathcal{G})$. Here $\{e_i : i \in E\}$ and $\{f_i : i \in E\}$ are the standard bases for two copies of $\mathbb{R}^E$.

It would be interesting to find an intrinsic characterization of conormal fans of matroids, in analogy with Theorems 2 and 5.

The Chow Ring and Hodge Theory

Consider the polynomial ring with variables $x_{F,G}$ where F and G are nonempty flats of M and $M^\perp$ respectively, not both E, such that $F \cup G = E$. When it is defined, we write $x_{\mathcal{F},\mathcal{G}} = x_{F_1,G_1} \cdots x_{F_l,G_l}$ for flags $\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_l\}$ and $\mathcal{G} = \{G_1 \supsetneq \cdots \supsetneq G_l\}$. We also need the special elements

$$a_i = \sum_{E \neq F \ni i} x_{F,G}, \quad a_i' = \sum_{E \neq G \ni i} x_{F,G}, \quad d_i = \sum_{F \cap G \ni i} x_{F,G}.$$

We define the *Chow ring of the conormal fan* of M to be

$$A^*(\Sigma_{M,M^\perp}) := \mathbb{R}[x_{F,G}]/(I_{M,M^\perp} + J_{M,M^\perp}),$$

where

$$I_{M,M^\perp} = \langle x_{\mathcal{F},\mathcal{G}} : \mathcal{F} \text{ and } \mathcal{G} \text{ are not compatible} \rangle,$$
$$J_{M,M^\perp} = \left\langle a_i - a_j, a_i' - a_j' : i, j \in E \right\rangle.$$

The Chow ring of the conormal fan behaves as nicely as the Chow ring of the Bergman fan, though proving it requires significant additional work.

Theorem 9 (Ardila–Denham–Huh, 2017)**.** *The Chow ring $A^*(\Sigma_{M,M^\perp})$ of the conormal fan of a matroid satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations.*

This Chow ring $A^*(\Sigma_{M,M^\perp})$ has degree $n-2$, and there is an isomorphism $\deg : A^{n-2} \to \mathbb{R}$ characterized by the property that $\deg(x_{\mathcal{F},\mathcal{G}}) = 1$ for any maximal pair of compatible flags $\mathcal{F}$ and $\mathcal{G}$. The inequality (2) is still satisfied for elements of a suitable cone $\overline{K}(M, M^\perp)$.

Unimodality, Log-Concavity, and Flawlessness

We now apply (2) to the elements $a = a_i$ and $d = d_i$ of the Chow ring $A^*(\Sigma_{M,M^\perp})$, which are independent of i and lie in the relevant cone $\overline{K}(M, M^\perp)$. A subtle combinatorial argument shows that

$$\deg(a^k d^{n-2-k}) = |\text{ coeff. of } q^{k+1} \text{ in } \chi_M(q+1)|.$$

As k varies, this sequence of coefficients is the h-vector of the *broken circuit complex* $\overline{BC}_<(M)$. This is the collection of subsets of $E - \min E$ that do not contain a *broken circuit*; that is, a set of the form $C - \min C$ for a minimal dependent set C. The broken circuit complex depends on a choice of a linear order $<$ on E, but its h-vector is independent of $<$.

The inequalities (2) for the Chow ring $A_{M,M^\perp}$ then imply the following theorems, which were conjectured by Brylawski and Dawson in the 1980s.

Theorem 10 (Ardila–Denham–Huh, 2017)**.** *For any matroid M of rank r, the following sequences, defined in "Enumerative Invariants," are unimodal and log-concave:*

- *the h-vector of the broken circuit complex, and*
- *the h-vector $h(M)$.*

Theorem 10 is significantly stronger than Theorem 7. By work of Juhnke-Kubitzke and Le, it also implies a 2003 conjecture of Swartz: These h-vectors are flawless.

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A version of this survey including a full list of references is at: `math.sfsu.edu/federico/matroidsnotices.pdf`.

References

[1] K. Adiprasito, J. Huh, and E. Katz, Hodge theory of matroids, *Notices Amer. Math. Soc.* **64**(1), 26–30. MR3586249
[2] F. Ardila-Mantilla, Todos cuentan: Cultivating diversity in combinatorics, *Notices Amer. Math. Soc.* **63**(10), 1164–1170.
[3] M. Baker, Hodge theory in combinatorics, *Bull. Amer. Math. Soc. (N.S.)* **55**(1), 57–80. MR3737210
[4] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015. MR3287221

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Federico Ardila

ABOUT THE AUTHOR

Federico Ardila works in combinatorics and its connections to other areas of mathematics and applications. He also strives to build joyful, empowering, and equitable mathematical spaces. He has advised over forty thesis students in the US and Colombia; more than half of his US students are members of underrepresented groups and more than half are women. Outside of work, he is often playing fútbol or DJing with his wife May-Li, and Colectivo La Pelanga.