WHAT IS...

a Matroidal Family of Graphs?

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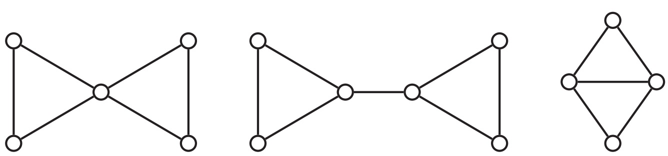

Figure 1. There are three types of bicycle graphs, including cycles as subgraphs. A graph and its cycles provide the simplest example of a matroid. A graph and its bicycles are another example.

In Figure 1, we have three graphs, called bicycles. Their cycles are the triangles and the quadrilateral of the third graph. Given any graph G, its cycles satisfy the following two conditions:

C1: No cycle contains properly another cycle;

C2: If x is an edge belonging to two distinct cycles, then there is a third cycle, in the union of those two, that does not contain x.

These are the defining axioms of a matroid. In the definition of a matroid on a set S, instead of *cycle* we use the word *circuit.*[1]

The edge sets of the cycles of a graph $G = (V, E)$ are the circuits of a matroid on the edge set E of the graph. This matroid is usually denoted by P_1. If we take as subgraphs the bicycles instead of the cycles, their edge sets are the circuits of another matroid on E, denoted P_2. These facts lead us to say that the set of graphs which are cycles, or the set of graphs which are bicycles, form *matroidal families of graphs*. As a general definition we have:

Definition. A *matroidal family of graphs* is a nonempty set P of finite, connected graphs such that, given an arbitrary graph G, the edge sets of subgraphs of G which are members of P are the circuits of a matroid on the edge set of G.

Unfortunately there is no simple generalization to "tricycles" or above. Besides the two examples of cycles and bicycles, P_1 and P_2, only one other nontrivial example P_3 was known: the circuits are the even cycles together with bicycles that have no even cycles. It was at this point that we [3] proved that for any other example of matroidal family no two circuits can be homeomorphic. Shortly afterwards, Thomas Andreae and Rüdiger Schmidt discovered new examples of matroidal families and, indeed, in every one of their examples, no two circuits are homeomorphic. Consequently, no two graphs are similar in their structure: neither "vertically" (one cannot contain another) nor "horizontally" (one cannot be homeomorphic to another). Nevertheless, in each one of those families, there are always two graphs, one of them being a minor of the other, as we are assured by the *minors theorem*, due to Neil Robertson and Paul Seymour, which says that in any infinite family of graphs, at least one of them is a minor of another one. Recall that a graph minor is obtained by deleting edges and vertices and contracting edges. These examples underline the *huge difference between subgraphs and minors*, even though they exhibit similar behaviors in some areas, such as planarity.

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[1] *The "Geometry of Matroids" (page X) gives a dual, equivalent definition of a matroid in which the distinguished subsets of S, called the independents, are the ones which do not contain a circuit.*

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We are now ready to describe the infinite matroidal families of graphs $P_{n,r}$ discovered by Andreae [1]: *With n and r integer numbers, $n \geq 0$ and $-2n+1 \leq r \leq 1$, the circuits consist of all edge sets of subgraphs $\Gamma = (V, E)$ with $|V(\Gamma)| \geq 2$ and $|E(\Gamma)| = n|V(\Gamma)|+r$, and which themselves have no such subgraphs.* For example, for $P_{1,0}$ the circuits are just the cycles and we recover P_1. For $P_{1,1}$, the circuits turn out to be the bicycles and we recover P_2. And for $P_{1,-1}$ we recover P_0, whose only member is the graph with just one edge.

The families discovered by Schmidt [2] are even more surprising. We need the concept of a *partly closed set* which is a set of graphs $Q \subseteq P_{n,r}$ and such that, if $X, Y \in Q$, $W = X \cup Y$, $X \neq W \neq Y$ and $|E(W)| = n|V(W)| + r + 1$, then, for each edge $a \in E(X \cap Y)$, there is $Z \in Q$ such that Z is isomorphic to a subgraph of $W - \{a\}$. As an example of a partly closed set, we may take the set $Q \subseteq P_{1,0}$ of the even cycles.

We now define Schmidt's families: *With a partly closed set $Q \subseteq P_{n,r}$, $r \leq 0$, and $P_{n,r+1,Q} = Q \cup P'$, where P' is the set of all graphs of $P_{n,r+1}$ which do not contain a subgraph isomorphic to a graph in Q, we get $P_{n,r+1,Q}$ as a matroidal family.* For example, taking Q to be the set of the even cycles, $P_{1,1,Q} = Q \cup P'$ where P' is the set of bicycles with no even cycle, we recover P_3. Taking Q to be the empty set, which is trivially closed, we recover Andreae's matroidal families: in fact, $P_{n,r+1} = P_{n,r+1,\emptyset}$.

Concerning the matroidal families discovered by Schmidt, his most unexpected result is that *the number of such families is uncountably infinite.* A question remains to challenge our readers: Are there other matroidal families of graphs?

References

[1] Thomas Andreae, Matroidal families of finite, connected, nonhomeomorphic graphs exist, *Journal of Graph Theory* **2** (1978), 149–153. MR493394

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[3] J. M. S. Simões-Pereira, *Matroidal Families of Graphs*, in Matroid Applications (Neil White, editor), Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press (1992), 91–105. MR1165541

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ABOUT THE AUTHOR

J. M. S. Simões-Pereira's area of research is discrete mathematics. In his spare time, he enjoys playing piano, grooming his cats, meeting with friends, and reading and speaking languages (besides Portuguese and English, French, German, Spanish, Italian and a bit of Russian).