WHAT IS…

a Rectifiable Set?

Editors

A *rectifiable set* is our best definition of a generalized k-dimensional surface in $\mathbb{R}^n$, perhaps with infinitely many singularities and infinite topological type, but still nice enough to do differential geometry. While smooth surfaces can be defined locally as images of nice smooth functions from the unit ball in $\mathbb{R}^k$ into $\mathbb{R}^n$, rectifiable sets are defined as countable unions of images of Lipschitz functions from arbitrary subsets of $\mathbb{R}^k$ into $\mathbb{R}^n$. A Lipschitz function f by definition satisfies

$$|f(y) - f(x)| \leq C|y - x|.$$

There is also a condition that a rectifiable set have finite k-dimensional measure (technically given by Hausdroff measure).

The simplest nonsmooth example of a Lipschitz function from $\mathbb{R}^1$ to $\mathbb{R}^2$ is $f(x) = (x, |x|)$ as in Figure 1; its image has a singularity at the origin. A 1-dimensional rectifiable set is a countable union of continuous curves of finite total length. A 2-dimensional rectifiable set can have dense singularities and infinite topological type, as in Figure 2.

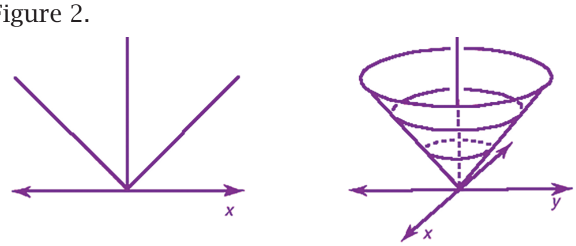

Figure 1. A rectifiable set, as the image of a Lipschitz function, can have sharp corner singularities.

Rademacher's Theorem says that a Lipschitz function is differentiable almost everywhere. The corresponding fact about rectifiable sets is the existence of an approximate tangent plane at almost every point p. *Approximate* means ignoring sets of density 0 at p. This fact means that rectifiable sets admit a measure theoretic generalization of differentiable geometry. It also means that you can integrate smooth differential forms over rectifiable sets and thus view them as the so-called *rectifiable currents* of geometric measure theory.

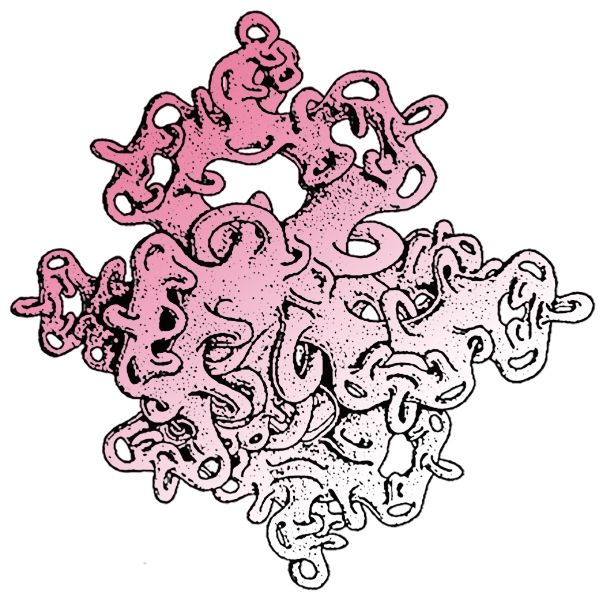

Figure 2. Adding infinitely many handles of finite total area to the sphere can produce a rectifiable set S with infinite topological type and a singular dense Cantor set of positive measure. Nevertheless, S has an approximate tangent plane at almost all points.

Besicovitch and Federer proved an amazing structure theorem, which says that every set of finite k-dimensional measure can be decomposed into a rectifiable set and a purely unrectifiable set. The purely unrectifiable set has the property that it is invisible from almost every direction, i.e., that its projection onto almost every k-plane

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has measure 0. Cantor sets provide examples of purely unrectifiable sets, as in Figure 3.

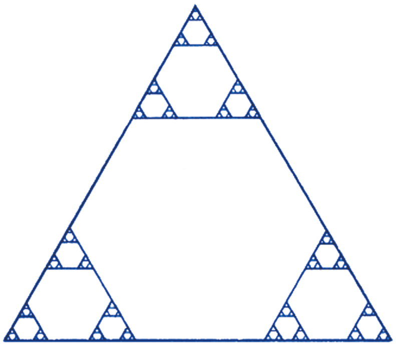

Figure 3. The pictured Cantor set, obtained by successively removing all but the three small corners of every triangle, is a purely unrectifiable set of unit 1-dimensional Hausdorff measure. Its projection onto almost every line has measure 0; a horizontal line is one exception.

Federer and Fleming provided an amazing compactness theorem for nice rectifiable sets with multiplicity in a ball in $\mathbb{R}^n$ in the context of geometric measure theory. By "nice" one means that in an appropriate sense the boundaries also are rectifiable. The only other hypothesis is a bound on the measures of the sets and of their boundaries. Then every sequence has a convergent subsequence in an appropriate weak topology, given by the so-called flat norm. For example, the flat distance between two close concentric spheres is the volume between them. In this norm, the exotic surface of Figure 4 and the flat disk are close together, because the region between them has small volume. This compactness theorem can be used to show that many problems have solutions. For example, every smooth closed curve in $\mathbb{R}^3$ is the boundary of a surface of least area.

Figure 4.This exotic surface with its long thin tentacles is close to the planar disc in the flat norm, because the region between them has small volume.

All of these notions and results have generalizations from $\mathbb{R}^n$ to nice Riemannian manifolds. Although solutions to geometric problems need not always be smooth, much current regularity theory focuses on proving them rectifiable.

References

[1] Morgan, Frank, *Geometric Measure Theory: A Beginner's Guide*, 5th ed., Academic Press, 2016.

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